A Singular Approach to Feynman Integration

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Main idea: reconstruct the result of a Feynman integration from the knowledge of singularities. Precisely, rewrite it in terms of iterated integrals.
We will need more than just the *location* of singularities. For integrals of polylogarithmic type it will be possible to go pretty far.

Why write an integral in terms of other integrals?

- Fewer integrations (half!). Conjecturally, number-theoretical optimum of simplicity.
- Cancellations are "obvious" in the iterated integral form.
- Asymptotic expansions around singularities are simpler in iterated integral form.
- Monodromies around singularity loci are simpler in iterated integral form.

Precursors:

- BCFW recursion relations at tree level: reconstruct the tree level amplitude from multiparticle pole singularities.
- Steinmann relations revived by [Bartels, Lipatov & Sabio Vera: 0802.2065], [Brower, Nastase & Schnitzer: 0801.3891] and used extensively by [Dixon et al.]
- Pham–Steinmann relations, introduced by [Pham] and used by [Hannesdóttir, McLeod, Schwartz, CV]

Singularities of Feynman-type integrals

Theorem [Landau, Leray][see also Pham] An integral of type

$$I(t) = \int_\Gamma \frac{N(x,t)d^n x}{s_1(x,t)\cdots s_m(x,t)}$$

is analytic in t except perhaps at the values of t for which we can simultaneously solve the following equations

$$s_e = 0, \qquad \text{for } e \in \mathscr{E} \subset \{1,\ldots,m\}, \tag{1}$$

$$d\ell(t) = \sum_{e\in\mathscr{E}} \alpha_e ds_e(x,t), \tag{2}$$

for α_e not all vanishing.
These are called "on-shell equations" and "Landau loop equations".

The most familiar (but not the only) way to understand these singularities is to group the denominators using Feynman's formula

$$\frac{1}{s_{i_1}(x,t)\cdots s_{i_m}(x,t)} = (m-1)! \int_\Delta \frac{d^{m-1}\alpha}{(\alpha_{i_1}s_{i_1}(x,t)+\cdots+\alpha_{i_m}s_{i_m}(x,t))^m},$$

where Δ is a simplex defined by $\alpha_e \geq 0$ and $\sum_{e\in\mathscr{E}} \alpha_e = 1$.
Then we define

$$F_{\mathscr{E}}(\alpha,x,t) = \sum_{e\in\mathscr{E}} \alpha_e s_e(x,t)$$

and look at its critical points

$$\partial_{\alpha_e} F_{\mathscr{E}} = s_e(x,t) = 0, \qquad (3)$$

$$d_x F_{\mathscr{E}} = \sum_{e\in\mathscr{E}} \alpha_e d_x s_e(x,t) = 0 \qquad (4)$$

Note that both the on-shell conditions *and* the Landau loop equations have the same origin: critical point conditions on a function $F_{\mathscr{E}}$.

This way of thinking about the necessary conditions for the singularities is due to Pham (influenced by René Thom).

A subtle but important distinction from the usual way the equations are presented: it is possible to have $\alpha_e = 0$ while at the same time $s_e(x, t) = 0$. Normally this is understood as either $\alpha_e = 0$ or $s_e(x, t) = 0$.

The singularity corresponds to a subgraph whose edges are in the set $\mathscr{E}$.

Generically the critical points (α^*, x^*) are isolated (F_I is a Morse function) and its Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 F_{\mathscr{E}}}{\partial x \partial x} & \frac{\partial^2 F_{\mathscr{E}}}{\partial \alpha \partial x} \\ \frac{\partial^2 F_{\mathscr{E}}}{\partial \alpha \partial x} & 0 \end{pmatrix}$$

is definite (positive or negative) at the critical point then we have the simplest situation (sometimes called a "simple pinch").

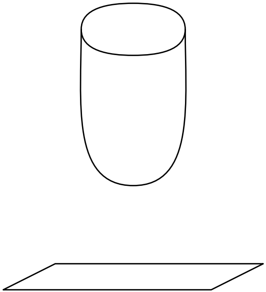


Figure: A cartoon of $F_{\mathscr{E}}$ in the space of coordinates (α, x).

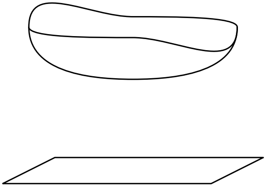


Figure: A cartoon of a $F_{\mathscr{E}}$ with a "shallow" direction.

The singularities arise for the values of t such that $F_{\mathscr{E}}(\alpha^*(t), x^*(t), t) = 0$. Generically (but not always) a codimension one variety.

Cutkosky's theorem

Given a subset $\mathscr{E}$ of propagators such that the Landau equations have a simple pinch solution we have a singularity at a hypersurface L defined by $\ell(t) = 0$ where $\ell(t) = F_I(\alpha^*(t), x^*(t), t)$. When this is a branch cut singularity, the monodromy around (a smooth point) of L is given by:

$$A_L(t) = (1 - M_L)I(t) = (-2\pi i)^m \int \frac{N(x,t) \prod_{e \in \mathscr{E}} \delta(s_e(x,t))}{\prod_{e \notin \mathscr{E}} s_e(x,t)}.$$

Here the arguments $s_e(x,t)$ are real. The *real* hypersurfaces $S_e(\mathbb{R})$ defined by $s_e = 0$ may have multiple branches so we need to select the ones compatible with external data (energy conditions sometimes written as $\delta_+(s_e(x,t))$).

Absorption integrals

An integral such as $A_L(t)$ is called an *absorption integral*. It is not a priori clear what analytical properties $A_L(t)$ could have. The integral definition uses reality in an essential way. Can we analytically continue?

The answer is *yes* and this can be seen in several ways:

- Using Cutkosky's representation which he used to prove his theorem.
- Using a construction due to Pham and Leray, which proceeds through Picard–Lefschetz theorem, Leray coboundaries, Leray formula for residues and a Poincaré duality.

Some restrictions apply in the second case whose correspondent in the first method is not immediately clear.

Pham's theorem on singularities of absorption integrals

Pham: Since $A_L(t)$ can be analytically continued, what singularities can it have?
Answer: It can have singularities of two types. The first type involves a superset $\mathscr{E}' \supset \mathscr{E}$ (new propagators are added to the pinch). The second type involves a completely new set $\overline{\mathscr{E}}$ of propagators producing a pinch (there are strong constraints on when this can happen).
The first type are called "hierarchical principle" singularities. The second type are Pham–Steinmann singularities.

Intersection of Landau loci

Principal Landau loci can intersect transversally or tangentially. What happens when taking monodromies in the neighborhood of the intersection?

Theorem (Pham): If the Landau locus L'_0 is above the threshold for the Landau locus L_0 and is tangent to it, then in the neighborhood of the intersection (and away from other Landau loci), we have

$$\mathrm{Disc}_{L'_0} \mathrm{Disc}_{L_0} A = \mathrm{Disc}_{L'_0} A, \tag{5}$$

where Disc_{L_0} is the discontinuity around the Landau locus L_0 and $\mathrm{Disc}_{L'_0}$ is the discontinuity around the Landau locus L'_0. The discontinuities are taken around the *effective* parts (non-negative α) of the Landau loci.

Tangential contraction diagram

$$
\begin{tikzcd}
& & & \ker \kappa_0 \arrow[dd, tail] \\
& \ker \kappa_0' \arrow[urr, two heads] \arrow[dr, tail] & & \\
\ker \kappa' \arrow[ur, tail] \arrow[rr, tail] & & G' \arrow[r, "\kappa'"] \arrow[dr, two heads, "\kappa_0'"] & G \arrow[d, two heads, "\kappa_0"] \\
& & & G_0
\end{tikzcd}
$$

Transversal intersection

Finally, this is the generalization of Steinmann relations.
Theorem (Pham) If the Landau loci L' and L'' intersect transversally and effectively (fibered product, see below), then in a neighborhood of the intersection (and away from other singularities) we have

$$\mathrm{Disc}_{L'}\,\mathrm{Disc}_{L''}\,A = \mathrm{Disc}_{L''}\,\mathrm{Disc}_{L'}\,A \tag{6}$$

where $\mathrm{Disc}_{L'}$ is the discontinuity around the branch cut ending at the Landau locus L' and similarly for L''.

Transversal contraction diagram

$$
\begin{tikzcd}
\ker\kappa' \arrow[rr, tail, shift left] \arrow[drr, tail] \arrow[ddr, tail] & & \ker\kappa \arrow[ll, two heads, shift left] \arrow[rr, two heads, shift left] \arrow[d, tail] & & \ker\kappa'' \arrow[ll, tail, shift left] \arrow[dll, tail] \arrow[ddl, tail] \\
& & G \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, two heads, "\kappa"] & & \\
& G' \arrow[dr, two heads, "\kappa'"] & & G'' \arrow[dl, two heads, "\kappa''"] & \\
& & G_0 & &
\end{tikzcd}
$$

Sketch of proof (hierarchical case)

We can think of the $\prod_{e\in\mathscr{E}} \delta(s_e(x,t))$ in the numerator as defining a new variety where the new contour of integration lives. If we are away from the Landau locus of the pinch determined by the edges in $\mathscr{E}$, the intersection is transverse.

We can group the extra propagators in $\mathscr{E}' \setminus \mathscr{E}$ as usual using the Feynman formula and this defines a denominator

$$F_{\mathscr{E}'\setminus\mathscr{E}}(\alpha,x,t) = \sum_{e\in\mathscr{E}'\setminus\mathscr{E}} \alpha_e s_e(x,t).$$

We need to find the critical points of this function, subject to the constraints $s_e(x,t)=0$ from the constraints in the numerators.

Proof sketch (continued)

We can impose these constraints via Lagrange multipliers so we need to find the critical points of

$$F_{\mathscr{E}'}(\alpha, x, t) = \sum_{e \in \mathscr{E}'} \alpha_e s_e(x, t),$$

which gives the same equations as for the Landau locus corresponding to the subgraph of edges $\mathscr{E}'$.
The difference is that the α_e for $e \in \mathscr{E}$ do not have to satisfy a positivity condition anymore.
For the critical points to be minima/maxima we need a *bordered Hessian* to be positive/negative.
A less appealing alternative is to solve (parametrize rationally) the on-shell constraints explicitly, which should always be possible for polylogarithmic integrals.

Iterated integrals

Singularities of iterated integrals

Theorem (Goncharov, arXiv:0103059, Prop. 2.4)

Consider an iterated integral with forms $\omega_1, \ldots, \omega_l$, such that the form ω_p has a pole along a codimension one variety S and no other forms have a singularity there. Next, consider two paths $\gamma_\pm$ with the same end points and such that they go around S in opposite ways such that $\gamma_+\gamma_-^{-1}$ goes around S in the counter-clockwise orientation. Then, we have

$$\int_{\gamma_+} \omega_1 \circ \cdots \circ \omega_l - \int_{\gamma_-} \omega_1 \circ \cdots \circ \omega_l = 2\pi i \,\mathrm{res}\, \omega_p \int_{\gamma'} \omega_1 \circ \cdots \circ \omega_{p-1} \int_{\gamma''} \omega_{p+1} \circ \cdots \circ \omega_l, \quad (7)$$

where γ' is the initial section of the path until S and γ'' is the final section of the path γ starting at S and ending at the end-point of γ.

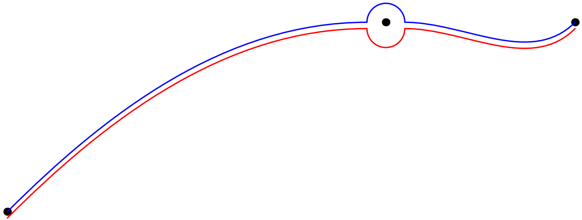


Figure: Difference of contours for iterated integrals.

For logarithmic singularities take the residue.
For square root singularities, subtract the value obtained by replacing $\sqrt{\bullet} \to -\sqrt{\bullet}$. Sometimes this yields zero even when the symbol letters contain square roots (Galois symmetry). Will show examples below.

Examples, bubble in two dimensions

$$I = \frac{1}{\sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}}$$
$$\Big(\log(\sqrt{s-(m_1+m_2)^2}-\sqrt{s-(m_1-m_2)^2})-$$
$$\log(\sqrt{s-(m_1+m_2)^2}+\sqrt{s-(m_1-m_2)^2})\Big). \quad (8)$$

- the prefactor can be computed algebraically (jacobian)
- logarithmic singularities at $m_e^2 = 0$
- $\mathrm{Disc}_{m_1^2=0}\,\mathrm{Disc}_{m_2^2=0}\, I = 0$ (tadpole Pham-Steinmann).
- no singularity under $\sqrt{s-(m_1-m_2)^2} \to -\sqrt{s-(m_1-m_2)^2}$.

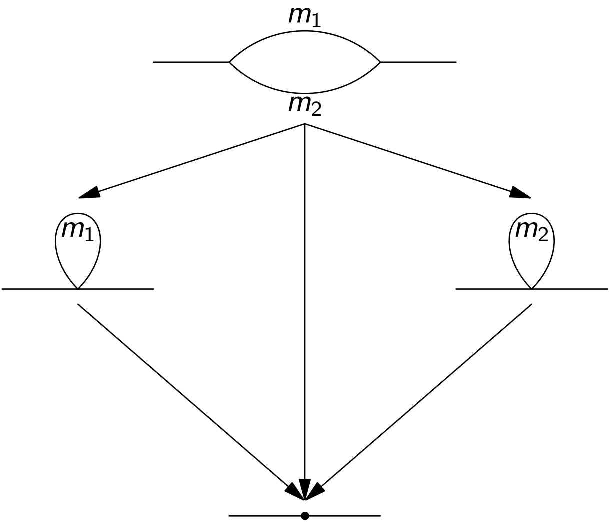


Figure: Contractions for bubble integral.

Examples, bubble in three dimensions

$$I = \frac{1}{\sqrt{s}}\Big(\log(\sqrt{m_1^2}+\sqrt{m_2^2}+\sqrt{s}) - \log(\sqrt{m_1^2}+\sqrt{m_2^2}-\sqrt{s})\Big). \quad (9)$$

- Second type singularity at $s=0$, invisible on the physical sheet. Therefore, invariance under $\sqrt{s}\to-\sqrt{s}$.
- Square root singularities at $m_1^2=0$ and $m_2^2=0$.
- $\mathrm{Disc}_{m_1^2=0}\,\mathrm{Disc}_{m_2^2=0}\, I = 0$.
- $s=(m_1+m_2)^2$ singularity on the physical sheet, $s=(m_1-m_2)^2$ only accessible after analytic continuation $m_2^2\to e^{2\pi i}m_2^2$.

Second type singularities

The $s = 0$ singularity for the bubble in three dimensions is a second type singularity (pinch happening at infinity).
To analyze it, we can do an inversion in the dual coordinate x_0, $x_0 \to \frac{x_0}{x_0^2}$. We have

$$d^D x_0 \to \frac{d^D x_0}{(x_0^2)^D}, \tag{10}$$

$$(x_i - x_0)^2 \to \frac{1}{x_0^2}(1 - 2x_0 \cdot x_i + x_0^2 x_i^2). \tag{11}$$

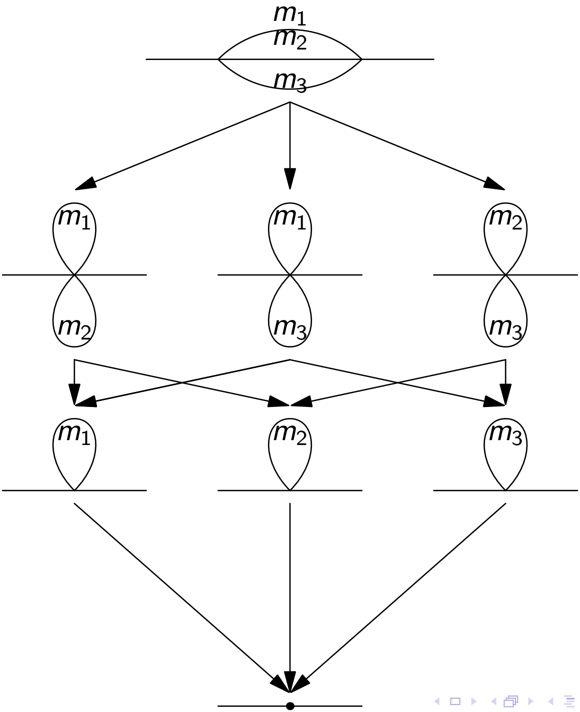
If $D \neq 2$ (not dual conformal invariant) then we have an *extra* denominator $(x_0^2)^{D-2}$. Then apply the usual treatment.
For higher loops, can treat mixed singularities by inverting only in a subset of dual points.

Example, sunrise in two dimensions at $p^2 = 0$

$$I = \frac{1}{r_{+++}r_{-++}r_{+-+}r_{++-}}\Bigg(\left[m_1 \,\Big|\, \frac{r_{+++}r_{-++} - ir_{+-+}r_{++-}}{r_{+++}r_{-++} + ir_{+-+}r_{++-}}\right] + \left[m_2 \,\Big|\, \frac{r_{+++}r_{+-+} - ir_{-++}r_{++-}}{r_{+++}r_{+-+} - ir_{-++}r_{++-}}\right] + \left[m_3 \,\Big|\, \frac{r_{+++}r_{++-} - ir_{-++}r_{+-+}}{r_{+++}r_{++-} + ir_{-++}r_{+-+}}\right]\Bigg), \quad (12)$$

where $m_{s_1,s_2,s_3} = \sqrt{s_1 m_1 + s_2 m_2 + s_3 m_3}$.

- Under $m_e \to e^{\pi i} m_e$ the m_{s_1,s_2,s_3} get permuted so that their contribution cancels.
- Logarithmic (first entry) $m_e^2 = 0$ singularity (tadpole Landau diagram).
- After $m_1^2 \to e^{2\pi i} m_1^2$ monodromy $\mathrm{Disc}_{m_1^2=0}\, I$ has logarithmic singularities at $(r_{+++}r_{-++} - ir_{+-+}r_{++-})(r_{+++}r_{-++} - ir_{+-+}r_{++-}) = 4m_2^2 m_3^2 = 0$. Double tadpole singularity. See also [Abreu, Britto, Duhr, Gardi].
- $\mathrm{Disc}_{m_1^2=0}\, I$ also has square root singularities from the sunrise Landau diagram, but only subset compatible with $\alpha > 0$. Same mechanism as for the bubble integral.


m_1
m_2
m_3
m_1
m_2
m_1
m_3
m_2
m_3
m_1
m_2
m_3

Vanishing Hessian example

The sunrise Landau singularity at $p^2 = 0$ has a vanishing Hessian (it is proportional to p^2, the only Lorentz-invariant kinematics dependence it can have).
Toy example:

$$F(\epsilon) = \int_{\mathbb{R}^2} \frac{dxdy}{\epsilon + x^2 + y^4} = \frac{\pi}{2} B(\frac{1}{4}, \frac{1}{4}) \epsilon^{-\frac{1}{4}}. \tag{13}$$

At the critical point $(x, y) = (0, 0)$ this has a degenerate Hessian matrix

$$\left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right).$$

In general, keep terms of cubic order in (α, k), the highest power in Feynman parametrization. Catastropy theory or tropical analysis. Can obtain $\frac{1}{3}$ and $\frac{1}{4}$ exponents.

Examples, massless

Integrals with massless propagators *always* have pinches, for all values of external kinematics (permanent pinches). This also happens for *all* higher-loop integrals in α-space [Boyling].
Not possible to formulate the integral as a pairing between homology and cohomology before resolving the singularities at the tip of the light-cone.

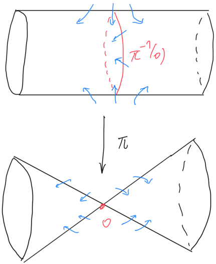


Figure: Blow-up of lightcone.

Blow-up

For a massless propagator q^2 we have the on-shell condition $(q^0)^2 - \vec{q}^2 = 0$.
The blow-up is a change of coordinates

$$q^0 = \rho, \qquad \vec{q} = \rho \vec{y},$$

$$\pi(\rho, \vec{y}) = (\rho, \rho \vec{y}) = (q^0, \vec{q}).$$

For $\rho \neq 0$ the change of coordinates is one-to-one. But $\pi^{-1}(0, \vec{0}) = (0, \vec{y})$, with $\vec{y}^2 = 1$.
The on-shell condition becomes $\rho^2(1 - \vec{y}^2) = 0$. An extra denominator in the integrand.

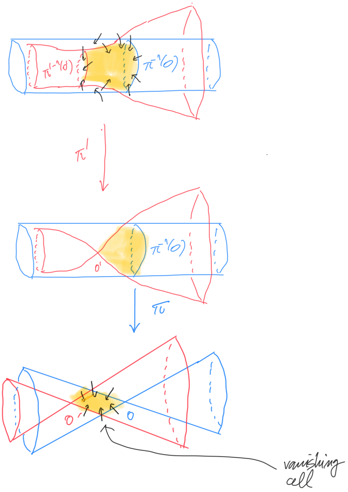


Figure: Blow-up of lightcone.

Deformation instead of blow-up

For a massless bubble at the Landau locus $p^2 = 0$ we have $q_1 = zp$ and $q_2 = (1-z)p$ for $z \in [0,1]$ so we don't have a simple pinch. Collinear singularities.
Instead of studying the problem at $p^2 = 0$, study its deformation $p^2 = \epsilon$ and take the limit $\epsilon \to 0$. While $\epsilon \neq 0$ the Hessian is non-degenerate but vanishes in the limit $\epsilon \to 0$. We have $\det H \sim \epsilon^\nu$ for some computable ν.

Massless bubble

$$I = \int \frac{d^D q_1}{q_1^2 q_2^2} = \int \frac{d\alpha d^D q_1}{(\alpha q_1^2 + (1-\alpha) q_2^2)^2}.$$

Analyze critical points of $F(\alpha, q_1) = \alpha q_1^2 + (1-\alpha)(p - q_1)^2$

$$0 = \frac{\partial F}{\partial \alpha} = q_1^2 - (p - q_1)^2, \tag{14}$$

$$0 = \frac{\partial F}{\partial q_1} = 2q_1 - 2(1-\alpha)p \tag{15}$$

to find $\alpha^* = \frac{1}{2}$ and $q_1^2 = \frac{1}{2}p$. We have a *simple pinch*! We have $F^* = F(\alpha^*, q_1^*) = -\frac{1}{4}\epsilon$.

$$H = \begin{pmatrix} \eta_{\mu\nu} & \frac{p_\mu}{2} \\ \frac{p_\nu}{2} & 0 \end{pmatrix}$$

So $\det H = (-1)^D \frac{p^2}{4}$. Therefore the full integral behaves as

$$I \sim \epsilon^{\frac{D-4}{2}}.$$

Asymptotic expansion (Landau exponent) constrains location in the symbol [Hannesdóttir, McLeod, Schwartz, CV]. Extension of this result for square roots.
We have analyzed a number of other mixed massive-massless integrals and we always obtain agreement with existing computations.

Remaining questions

- Regularization?
- How to deal with elliptic and Calabi-Yau integrals?

Thank you!