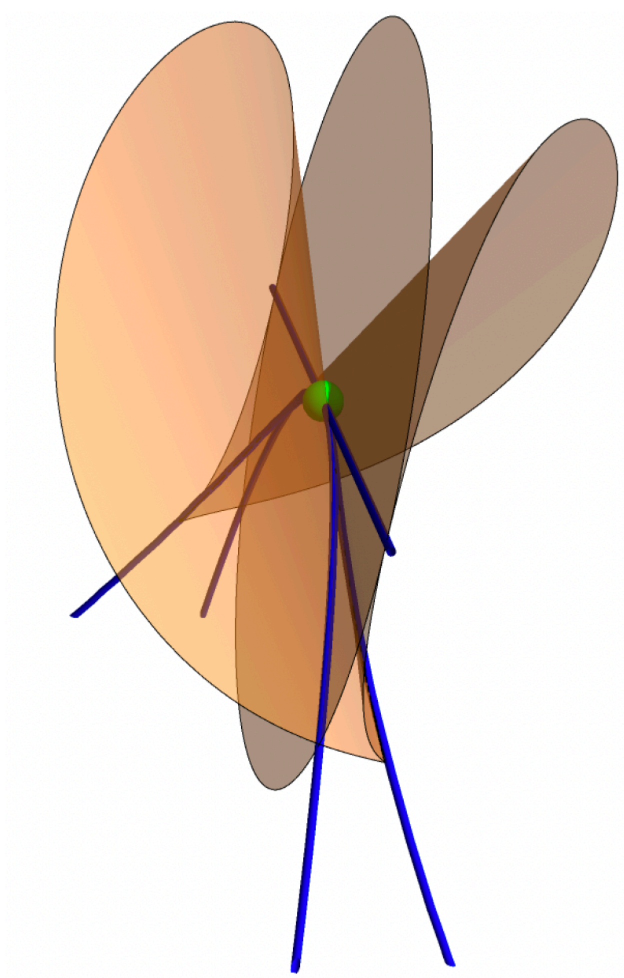
Max-Planck-Institut für Mathematik in den Naturwissenschaften

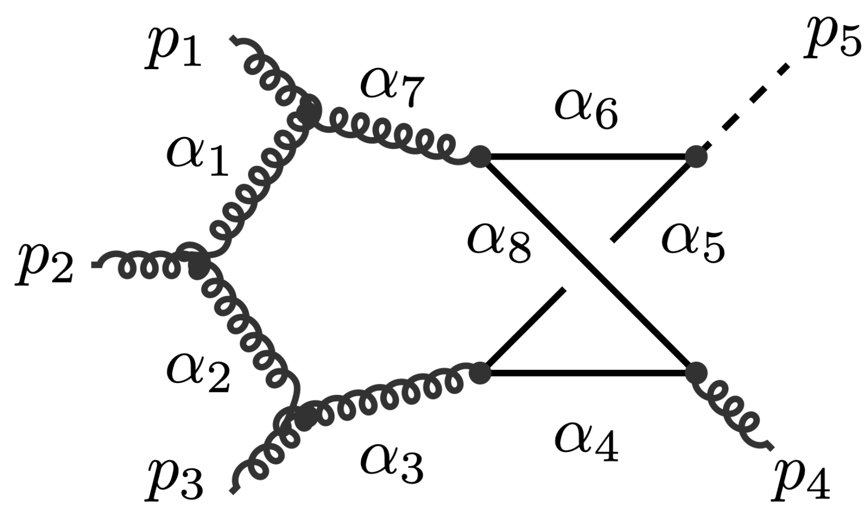
Euler discriminants

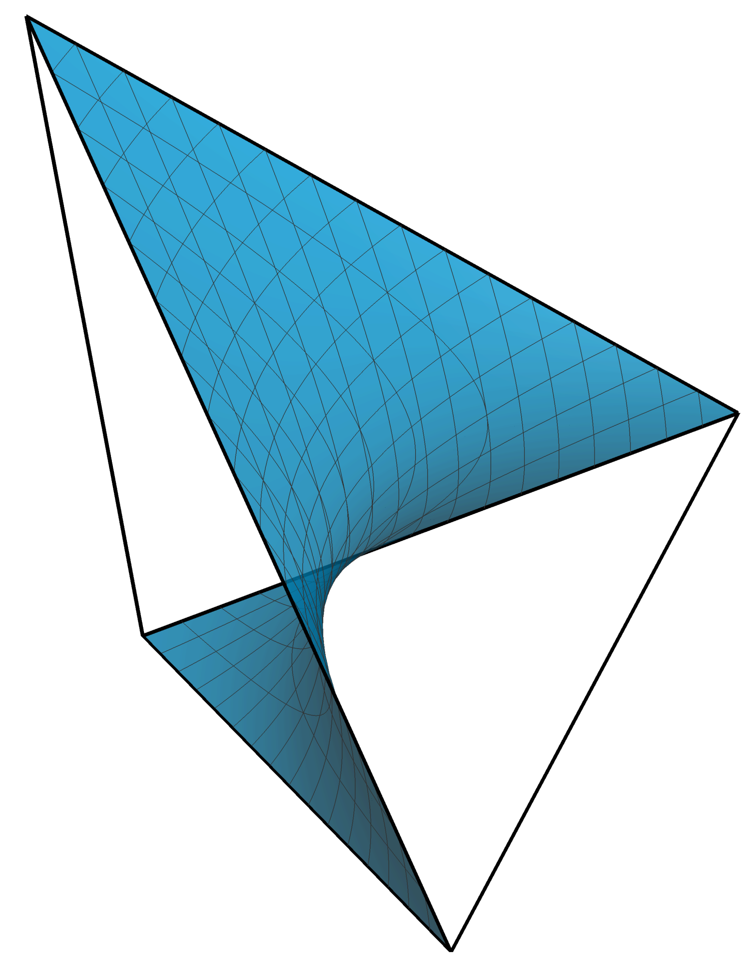
in physics and statistics

Simon Telen

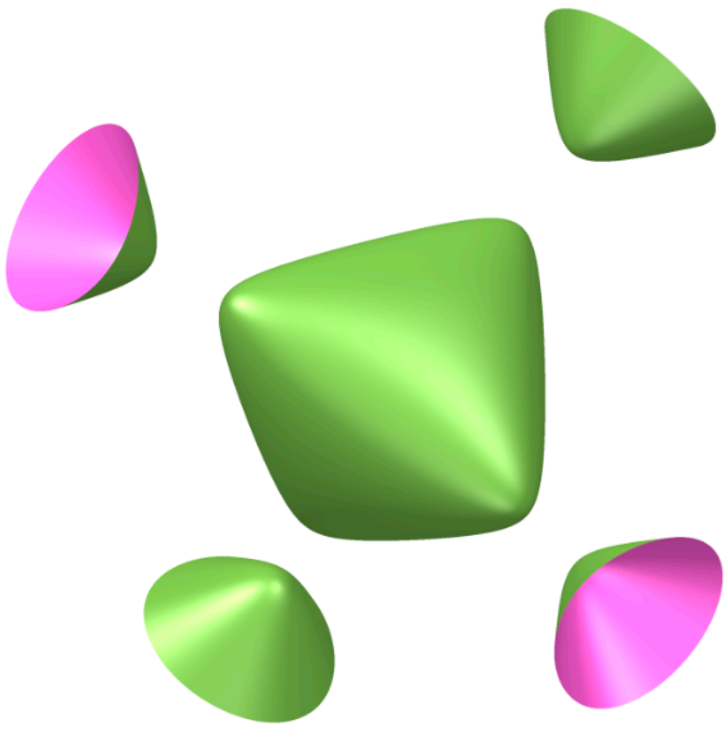
New Frontiers in Landau Analysis
The Higgs Centre for Theoretical Physics, Edinburgh
April 25, 2024



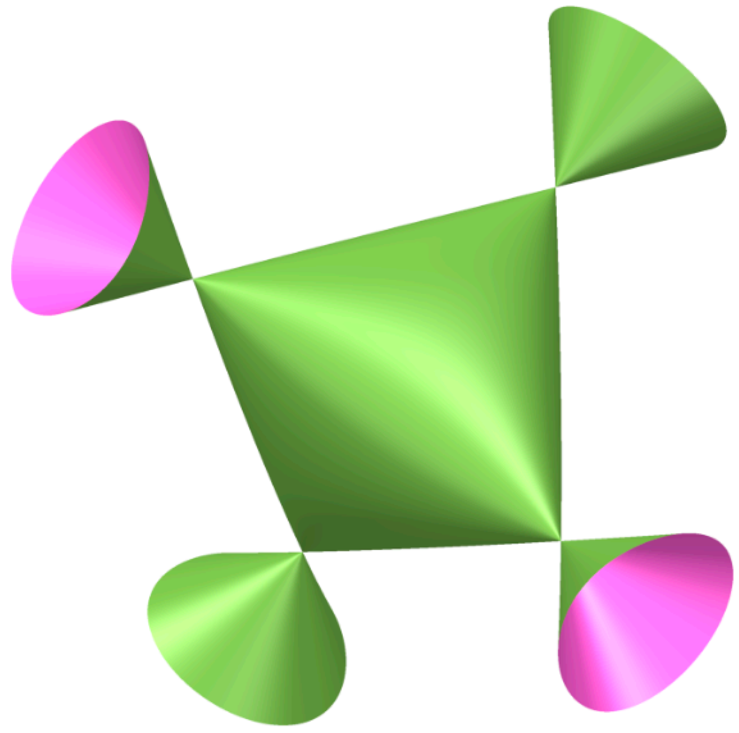




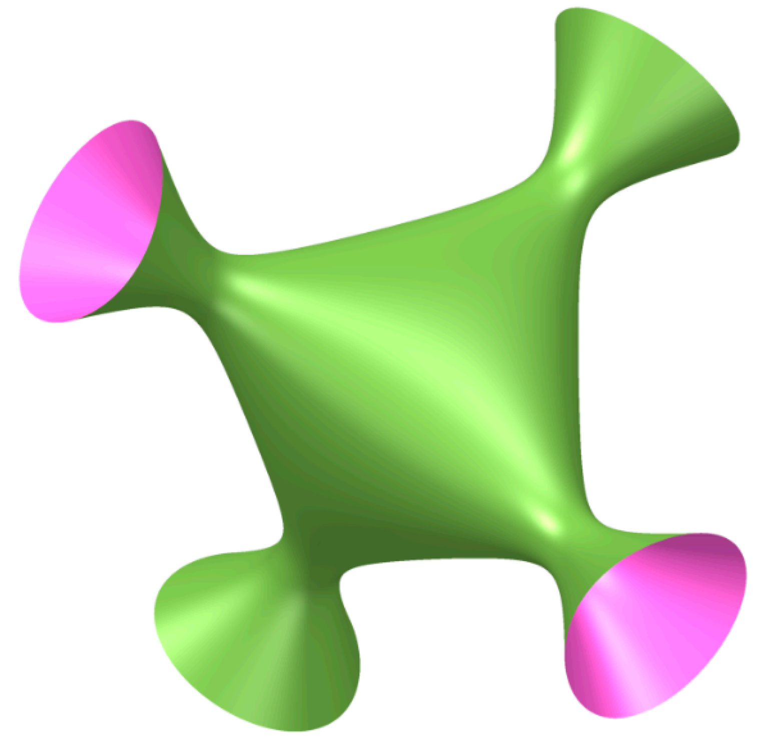
Discriminants



$X^2 + Y^2 + Z^2 + 2XYZ - 0.9 = 0$



$X^2 + Y^2 + Z^2 + 2XYZ - 1 = 0$



$X^2 + Y^2 + Z^2 + 2XYZ - 1.1 = 0$

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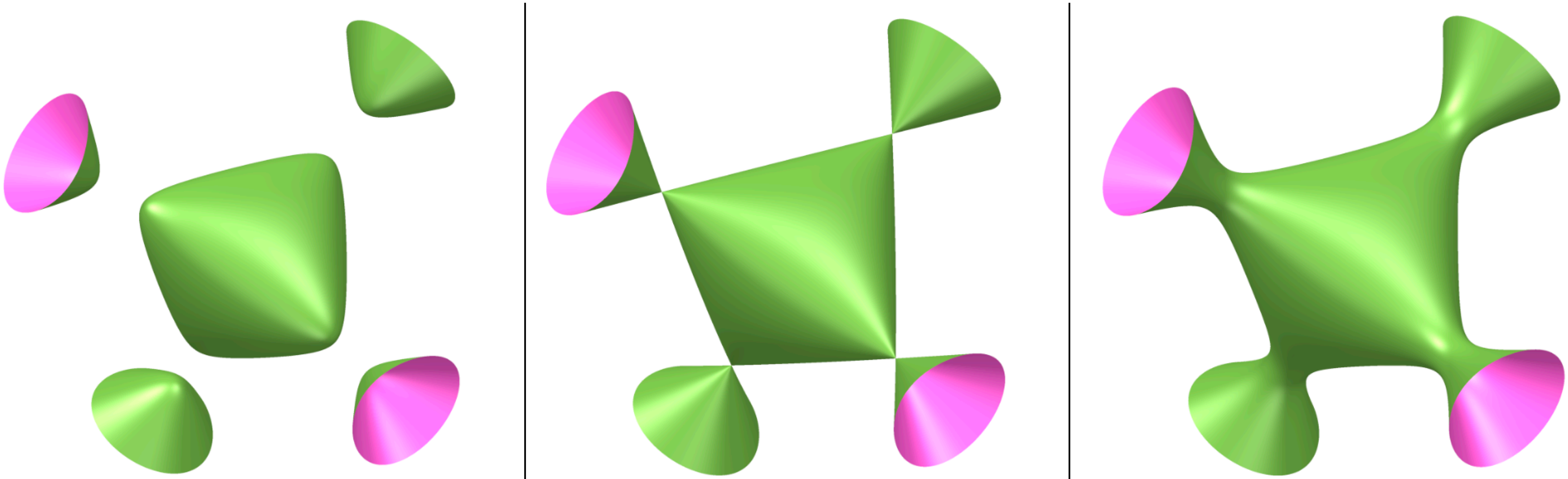

$X^2 + Y^2 + Z^2 + 2XYZ - 0.9 = 0$ | $X^2 + Y^2 + Z^2 + 2XYZ - 1 = 0$ | $X^2 + Y^2 + Z^2 + 2XYZ - 1.1 = 0$

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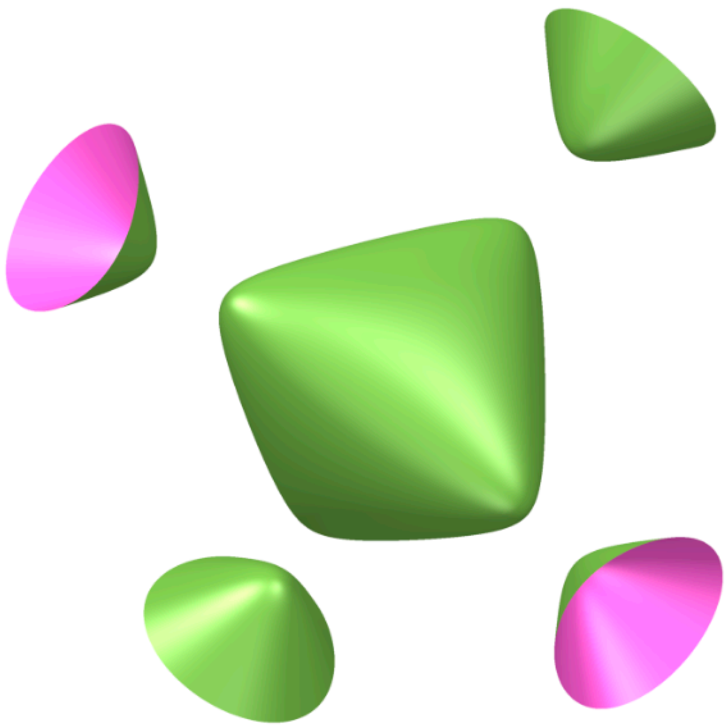

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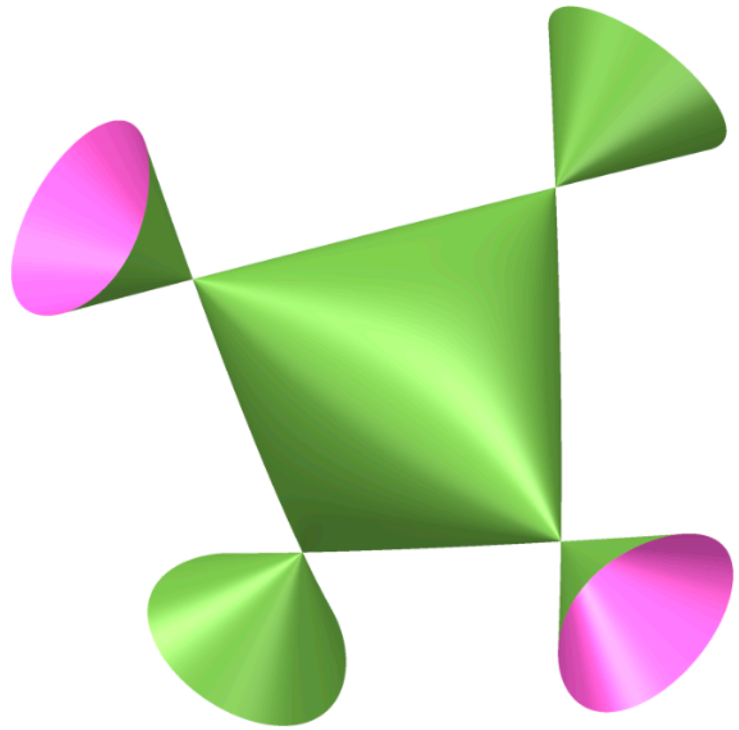

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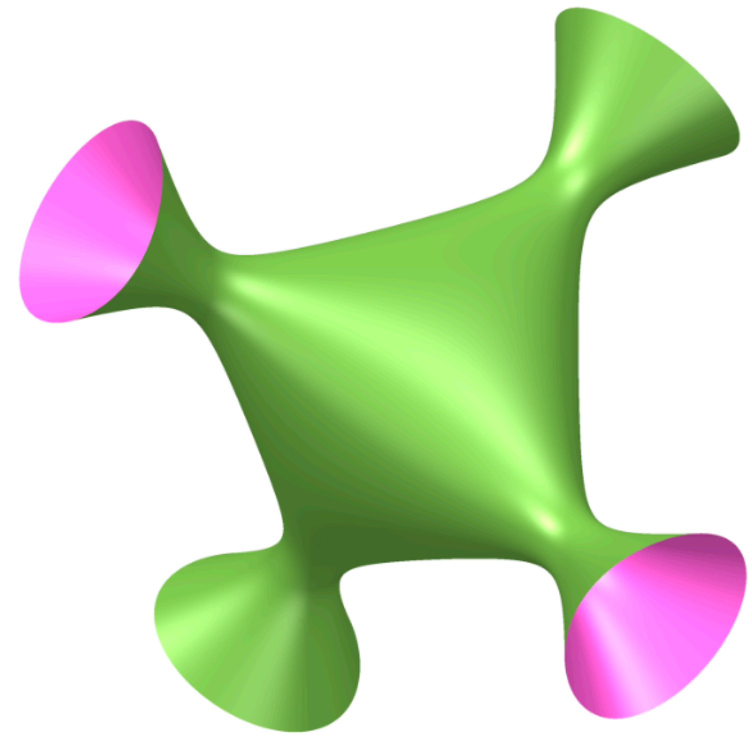

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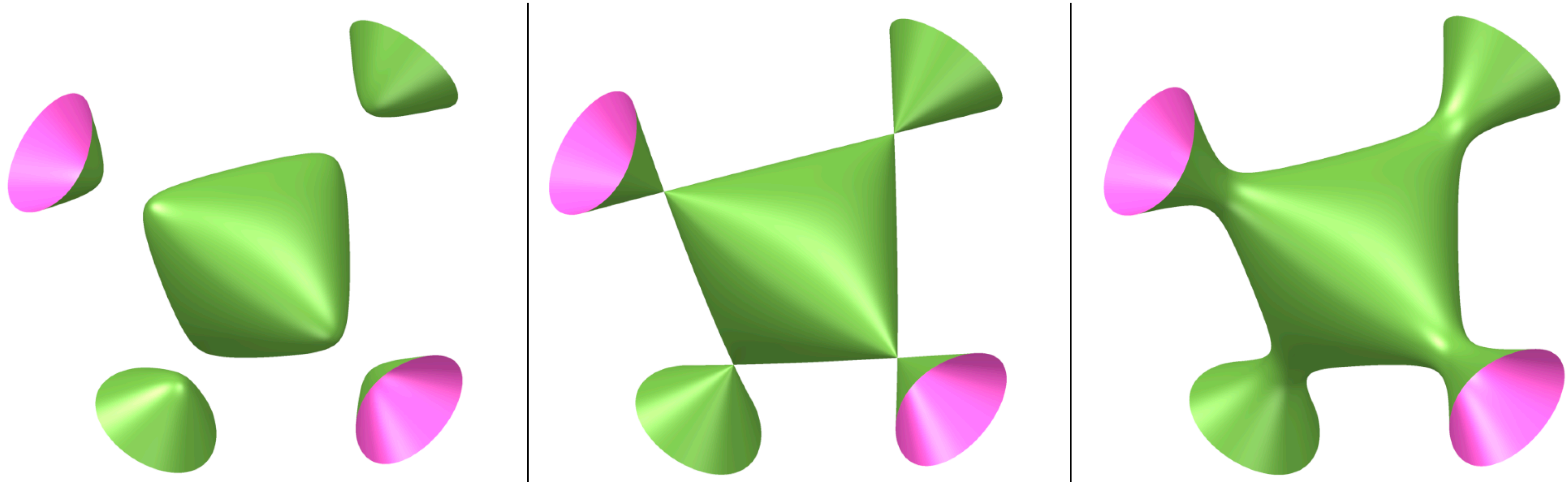

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Discriminants are everywhere, and they can be computed more often than you think

Euler integrals

$$\mathcal{I}_\Gamma(z) \;=\; \int_\Gamma \,(z_1\, x^{m_1} + z_2\, x^{m_2} + \cdots + z_s\, x^{m_s})^{\mu}\; x_1^{\nu_1}\,\cdots\, x_n^{\nu_n}\,\frac{\mathrm{d}x_1}{x_1}\wedge\cdots\wedge\frac{\mathrm{d}x_n}{x_n}$$

$$f_A(x;z) \;=\; z_1\, x^{m_1} + z_2\, x^{m_2} + \cdots + z_s\, x^{m_s}$$

$$A \;=\; \begin{pmatrix} m_1 & m_2 & \cdots & m_s \\ 1 & 1 & \cdots & 1 \end{pmatrix} \;\in\; \mathbb{Z}^{(n+1)\times s}$$

$\mu \in \mathbb{C}$, $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{C}^n$

Γ is a twisted cycle on $X_z = (\mathbb{C}^*)^n \backslash V_{A,z}$, where $V_{A,z} \;=\; V_{(\mathbb{C}^*)^n}(f_A(x;z))$

Generalized Euler Integrals and
A-Hypergeometric Functions

I. M. GELFAND,* M. M. KAPRANOV,† AND A. V. ZELEVINSKY†

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Generalized Euler Integrals and
A-Hypergeometric Functions

I. M. Gelfand,* M. M. Kapranov,† and A. V. Zelevinsky†

In physics, these are Feynman integrals:

$$z_1\, x_1x_2 + z_2\, x_1x_3 + z_3\, x_2x_3 + z_4\, x_1^2x_2 + z_5\, x_1^2x_3 + z_6\, x_2^2x_3 + z_7\, x_1x_2^2 + z_8\, x_1x_3^2 + z_9\, x_2x_3^2 + z_{10}\, x_1x_2x_3$$

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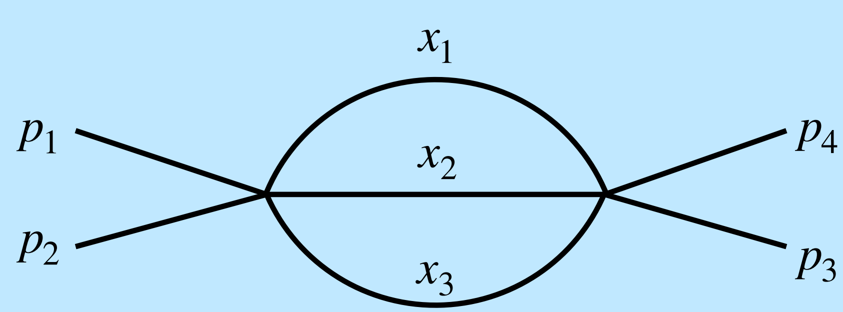

G_{el'fand}-K_{apranov}-Z_{elevinsky} systems

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The matrix $A \in \mathbb{Z}^{(n+1)\times s}$ defines a projective toric variety $\mathscr{X}_A \subset \mathbb{P}^{s-1}$

$$I(\mathscr{X}_A) \; = \; \langle \partial_z^u - \partial_z^v \; : \; u, v \in \mathbb{N}^s, \; A \cdot (u-v) = 0 \rangle$$

here $\partial_z^u \; = \; \partial_{z_1}^{u_1} \partial_{z_2}^{u_2} \cdots \partial_{z_s}^{u_s}$

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The GKZ system or A-hypergeometric system of differential equations for $A, (-\mu, \nu)$ is

$$P(\partial_{z_1}, \partial_{z_2} \ldots, \partial_{z_s}) \bullet F(z) \;=\; 0 \quad \forall P \in I(\mathcal{X}_A), \qquad \left[A \cdot \begin{pmatrix} z_1\partial_{z_1} \\ z_2\partial_{z_2} \\ \vdots \\ z_s\partial_{z_s} \end{pmatrix} + \begin{pmatrix} -\mu \\ \nu_1 \\ \vdots \\ \nu_n \end{pmatrix} \right] \bullet F(z) \;=\; 0$$

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Theorem (GKZ). Assuming that the parameters μ, ν are non-resonant, the local solutions of the A-hypergeometric system are $\mathcal{I}_\Gamma(z)$, for all twisted cycles Γ.

Counting solutions

The number of linearly independent functions $\mathscr{I}_\Gamma(z)$ in a neighbourhood of $z^* \in \mathbb{C}^s$

= the dimension of the space of local solutions of a GKZ system

= the number of "master integrals"

= the dimension of the n-th twisted (co)homology of $X_{z^*} = (\mathbb{C}^*)^n \backslash V_{A,z^*}$

= the signed topological Euler characteristic of V_{A,z^*}

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[Submitted on 26 Dec 2017 (v1), last revised 16 Mar 2018 (this version, v2)]

Feynman integral relations from parametric annihilators

Thomas Bitoun, Christian Bogner, Rene Pascal Klausen, Erik Panzer

[Submitted on 9 Oct 2018 (v1), last revised 5 Mar 2019 (this version, v2)]

Feynman Integrals and Intersection Theory

Pierpaolo Mastrolia, Sebastian Mizera

Sci|Post SciPost Phys. Lect. Notes 75 (2023)

Four lectures on Euler integrals

Saiei-Jaeyeong Matsubara-Heo[1*], Sebastian Mizera[2†] and Simon Telen[3‡]

A-discriminants

For which values of z is $\chi(V_{A,z})$ different from its generic value?

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We start with $z \in \mathbb{C}^s$ such that $V_{A,z} = V_{(\mathbb{C}^*)^n}(f_A(x;z))$ is a singular hypersurface

$$Y_A \;=\; \{(x,z) \in (\mathbb{C}^*)^n \times \mathbb{C}^s \;:\; f_A(x;z) \;=\; \partial_x f_A(x;z) \;=\; 0\}$$

"Landau equations", "pinch singularities"

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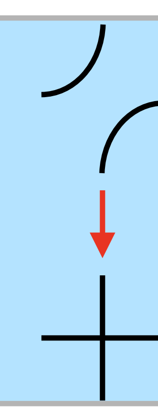
$\Delta_A = z_1 z_3 - z_2 z_4$



Principal A-determinants

… are built from A-discriminants

$$f_A(x;z) \;=\; z_{00} + z_{01}\,x_1 + z_{02}\,x_2 + z_{11}\,x_1^2 + z_{12}\,x_1x_2 + z_{22}\,x_2^2$$

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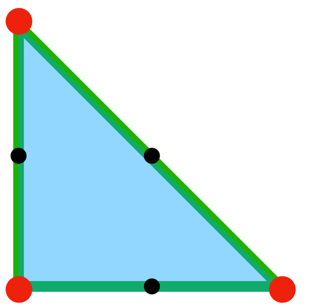

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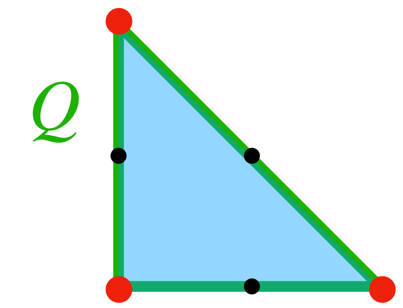

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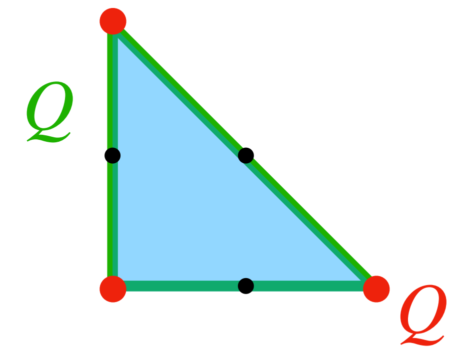

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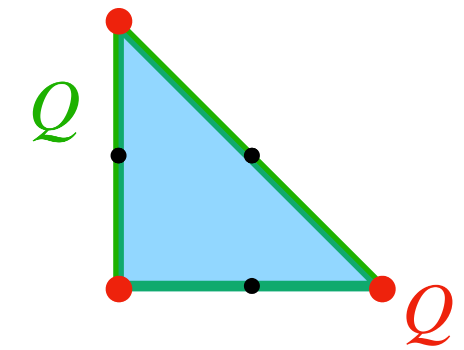

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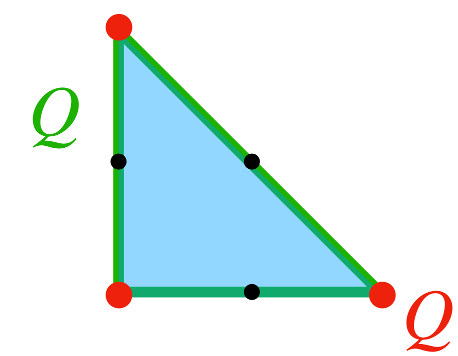

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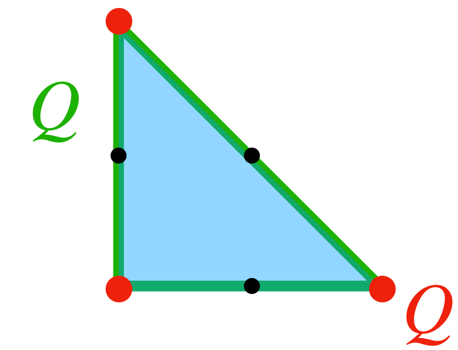

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$$E_A = z_{00} \cdot z_{11} \cdot z_{22} \cdot (z_{01}^2 - 4z_{00}z_{11}) \cdot (z_{02}^2 - 4z_{00}z_{22}) \cdot (z_{12}^2 - 4z_{11}z_{22}) \cdot \det M(z)$$

They are computable via elimination!

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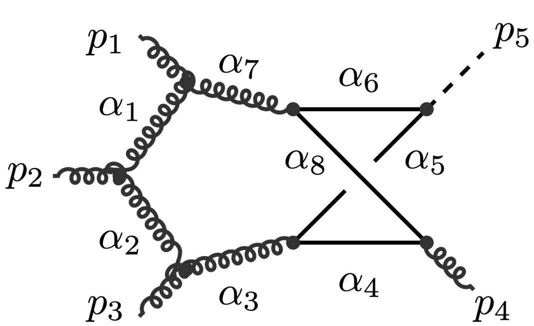
Landau analysis: singularities of Feynman integrals. These are specialized GKZ integrals



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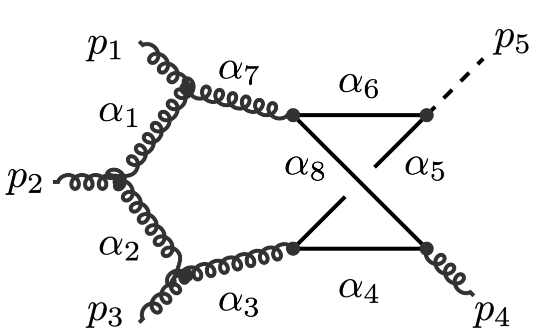

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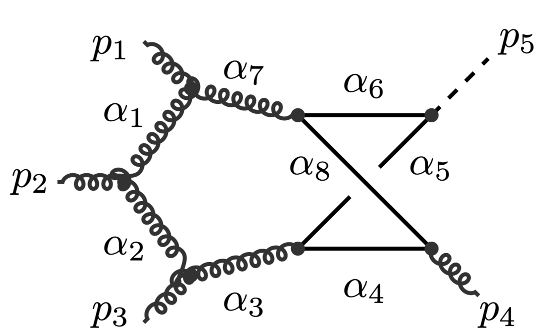

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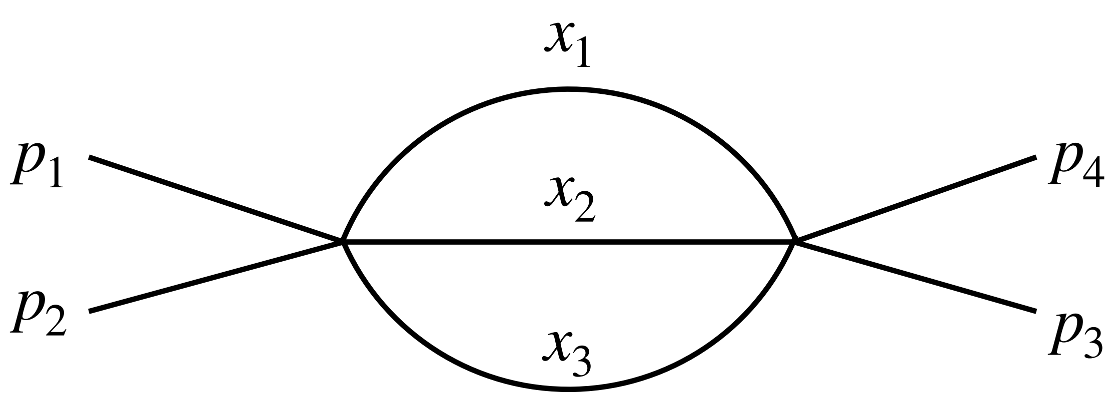
Coefficients z are restricted to lie in a linear subspace $\mathcal{K} \subset \mathbb{C}^s$, the kinematic space

Sunrise problem



$$\mathscr{I}_\Gamma(z) \;=\; \int_\Gamma \left[(1 - \sum_{i=1}^{3} m_i x_i)(x_1 x_2 + x_1 x_3 + x_2 x_3) + s x_1 x_2 x_3\right]^\mu x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3} \, \frac{\mathrm{d}x_1}{x_1} \wedge \frac{\mathrm{d}x_2}{x_2} \wedge \frac{\mathrm{d}x_3}{x_3}$$

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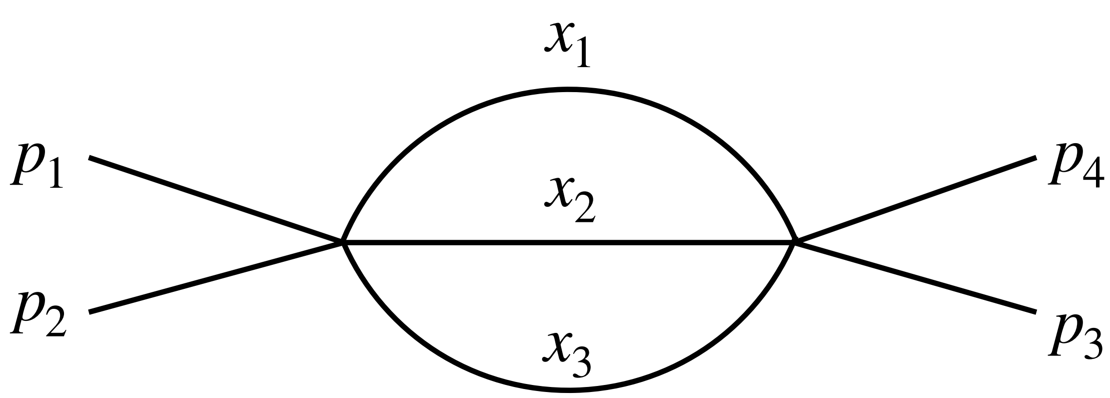

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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$$z_1\, x_1x_2 + z_2\, x_1x_3 + z_3\, x_2x_3 + z_4\, x_1^2x_2 + z_5\, x_1^2x_3 + z_6\, x_2^2x_3 + z_7\, x_1x_2^2 + z_8\, x_1x_3^2 + z_9\, x_2x_3^2 + z_{10}\, x_1x_2x_3$$

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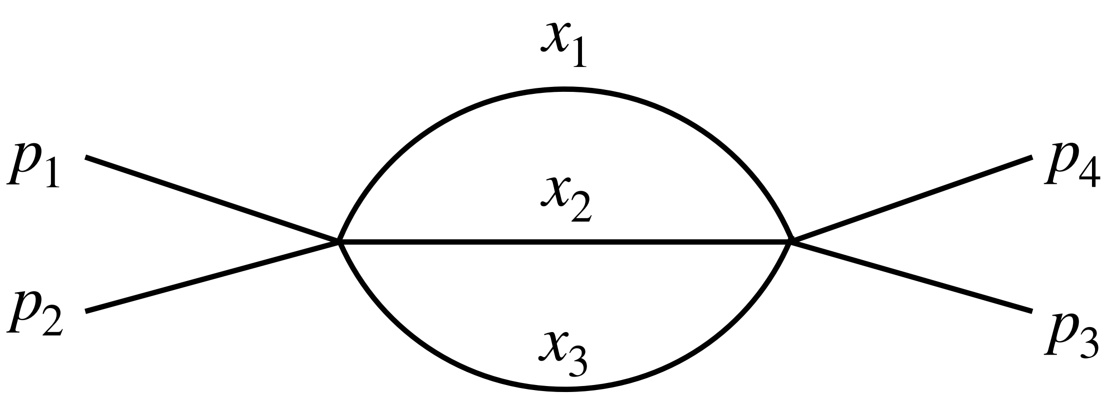

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restrict to the kinematic space $\mathscr{K}$

$$(z_1, \ldots, z_{10}) = (1, 1, 1, -\mathsf{m}_1, -\mathsf{m}_1, -\mathsf{m}_2, -\mathsf{m}_2, -\mathsf{m}_3, -\mathsf{m}_3, s - \mathsf{m}_1 - \mathsf{m}_2 - \mathsf{m}_3)$$

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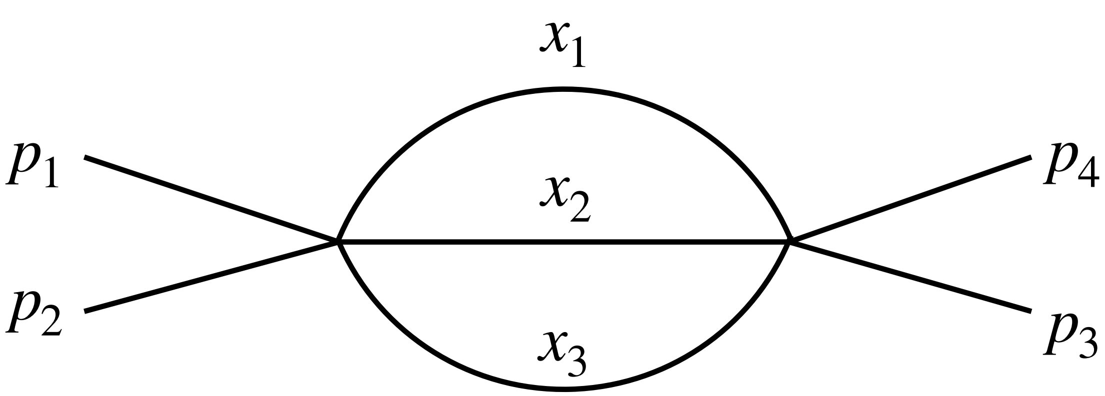

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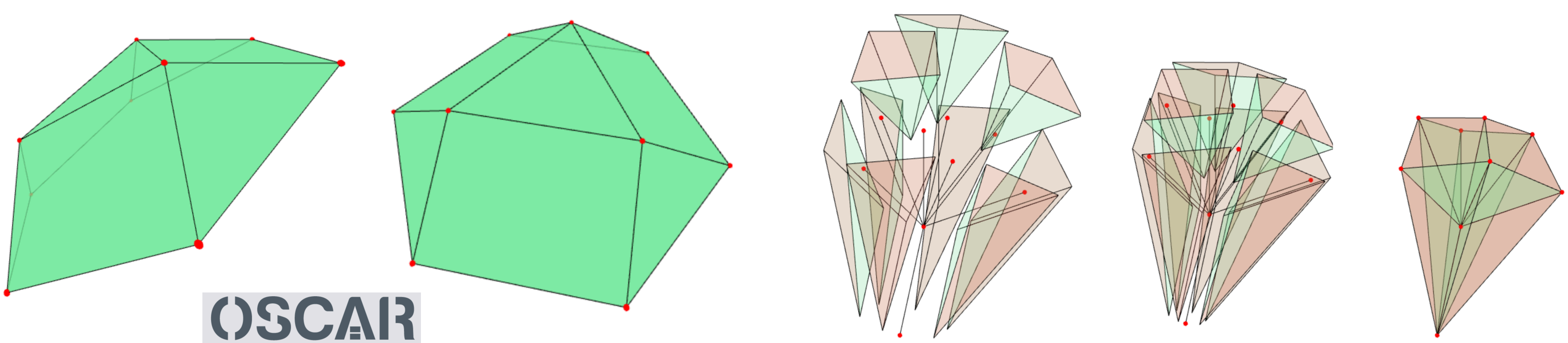
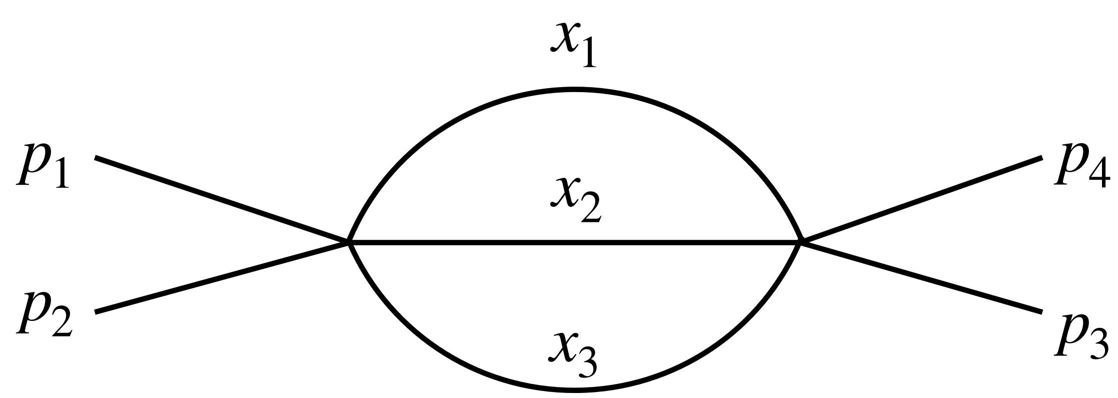


OSCAR SYMBOLIC TOOLS

Sunrise problem



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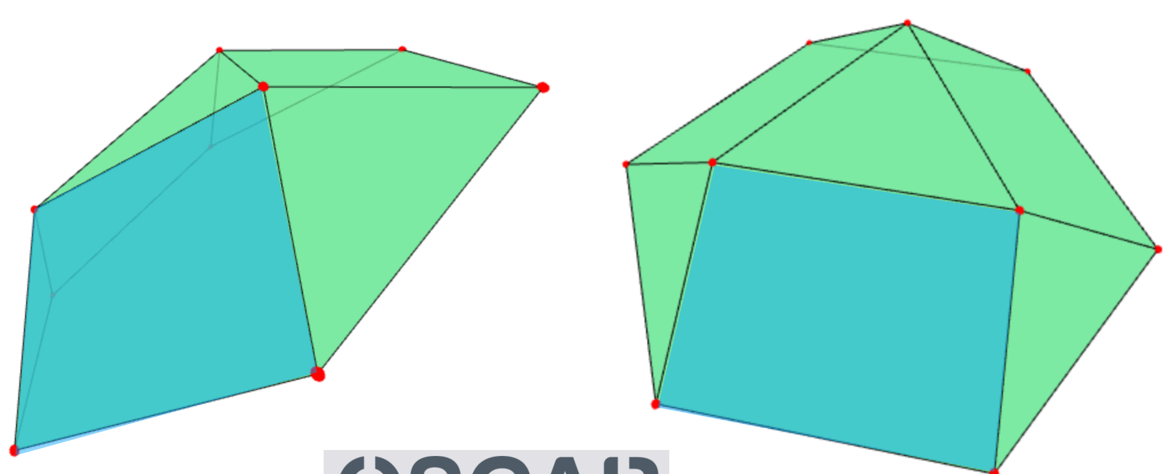

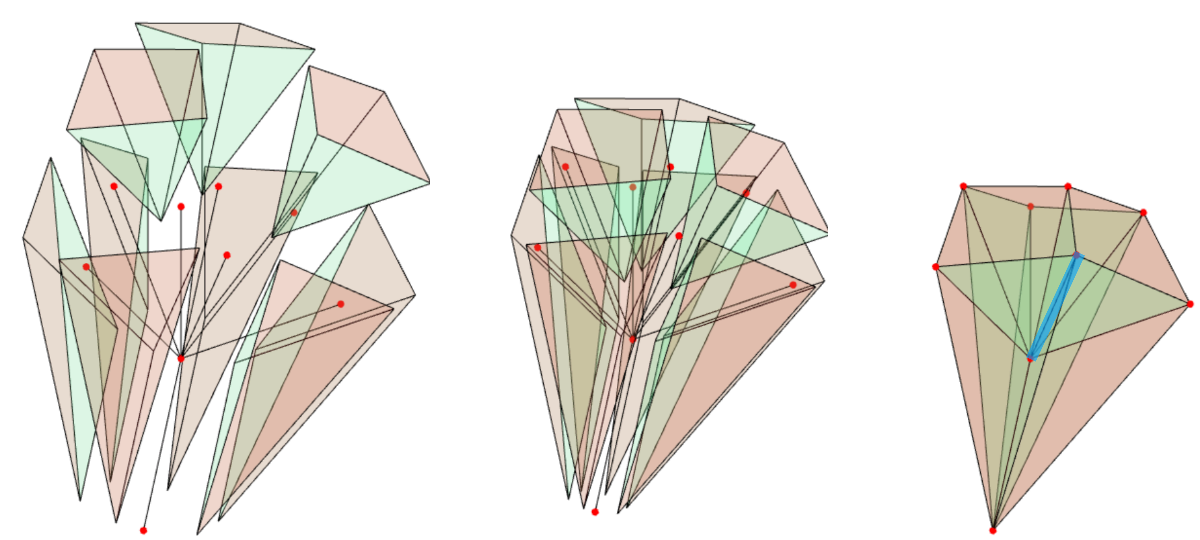

Euler discriminants

Let $\pi : \mathscr{V} \to \mathscr{Z}$ be a surjective map of irreducible quasi-projective $\mathbb{C}$-varieties

Let χ^* be the signed Euler characteristic of a generic fiber $V_{z^*} = \pi^{-1}(z^*)$

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The Euler discriminant of π is

$$\nabla_\chi(\pi) = \overline{\{z \in \mathscr{Z} \ : \ |\chi(V_z)| \neq \chi^*\}}$$

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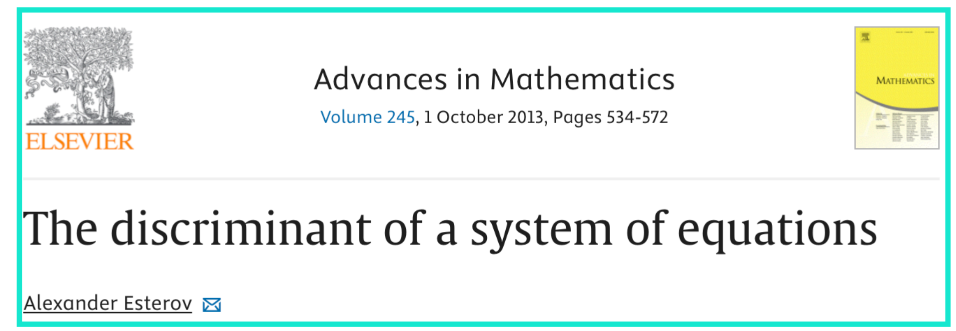
Advances in Mathematics
Volume 245, 1 October 2013, Pages 534-572
The discriminant of a system of equations
Alexander Esterov

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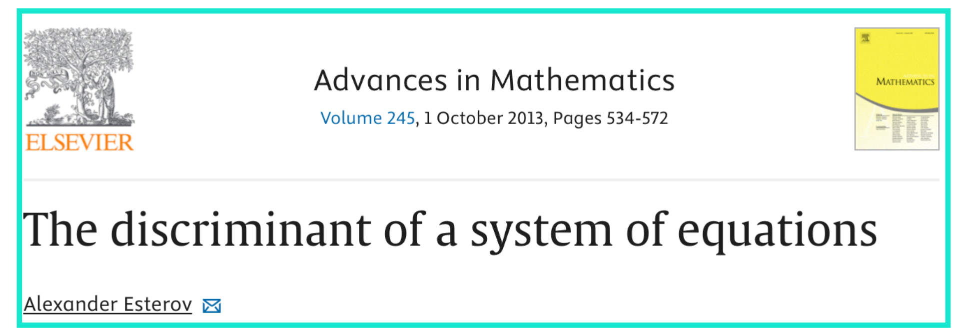
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The discriminant of a system of equations

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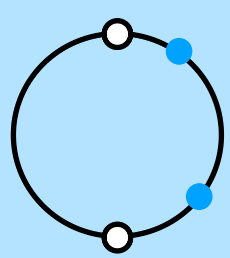
Example. $\mathscr{V} = \{(x,z) \in \mathbb{C}^* \times \mathbb{P}^2 \;:\; f_A(x;z) \;=\; z_3\, x^2 + z_2\, x + z_1 = 0\}, \quad \mathscr{Z} = \mathbb{P}^2$

$V_{z^*} =$  $\quad \chi^* = 2 \quad \nabla_\chi = \{E_A(z) = z_1 z_3 (z_2^2 - 4 z_1 z_3) = 0\}$

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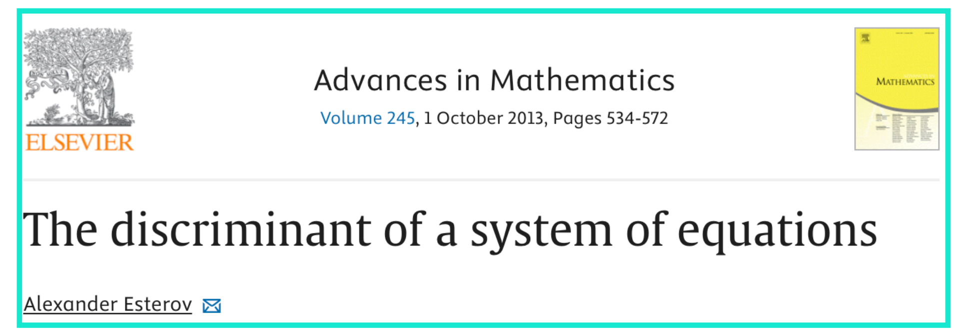

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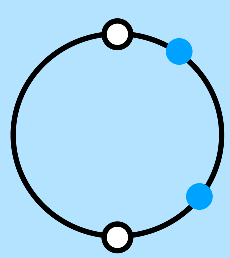
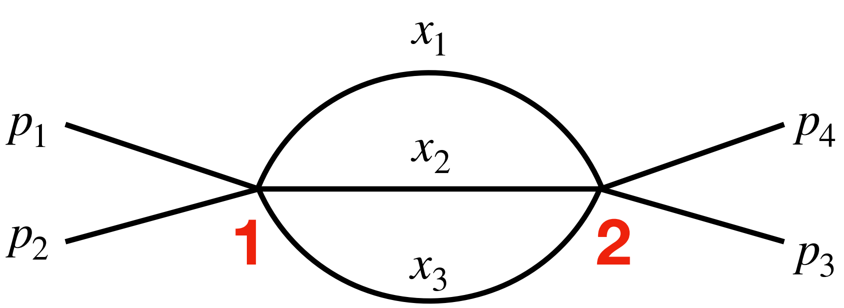
$V_{z^*} =$  $\quad \chi^* = 2 \quad \nabla_\chi = \{E_A(z) = z_1 z_3 (z_2^2 - 4 z_1 z_3) = 0\}$

Corollary. $\mathcal{V} = \{(x, z) \in (\mathbb{C}^*)^n \times \mathbb{P}^{s-1} \;:\; f_A(x; z) \;=\; 0\}, \quad \mathcal{Z} = \mathbb{P}^{s-1}$

$$\nabla_\chi = \{z \in \mathbb{P}^{s-1} \;:\; E_A(z) = 0\}$$

and no closure is needed

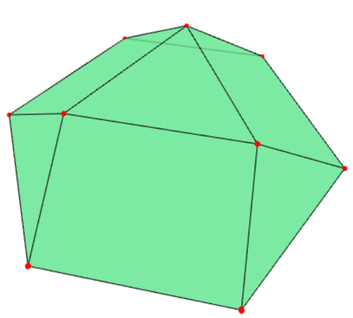
Sunrise solution: PLD.jl

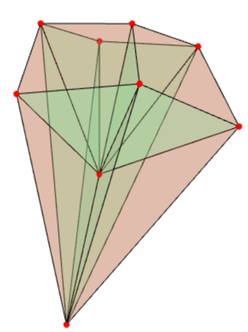


$$\mathscr{V} = \{(1 - \sum_{i=1}^{3} m_i x_i)(x_1x_2 + x_1x_3 + x_2x_3) + s x_1x_2x_3 = 0\}$$

$$\big\downarrow \pi$$

$$\mathscr{Z} = \mathbb{C}^4$$





Principal Landau Determinants

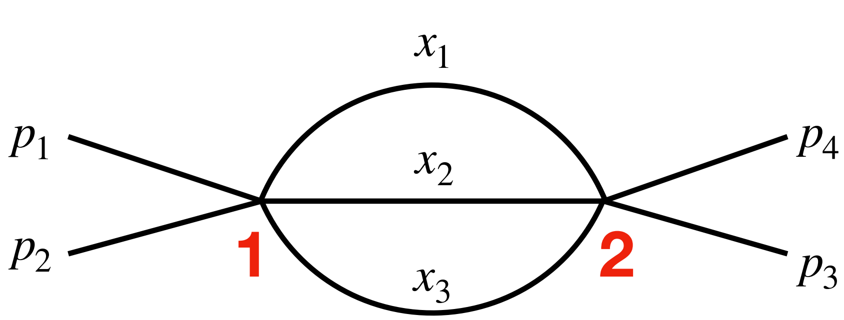
Claudia Fevola,[1] Sebastian Mizera,[2] Simon Telen[3]

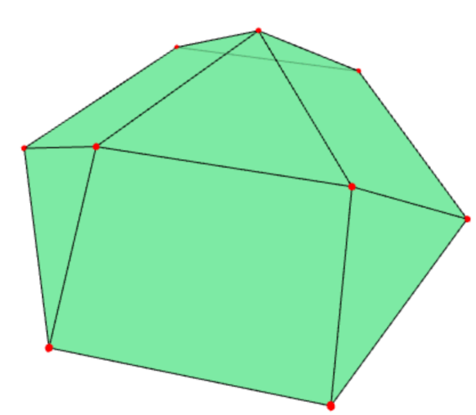
```
julia> edges = [[1,2],[1,2],[1,2]]; nodes = [1,1,2,2]; @var m[1:3] M[1:4];

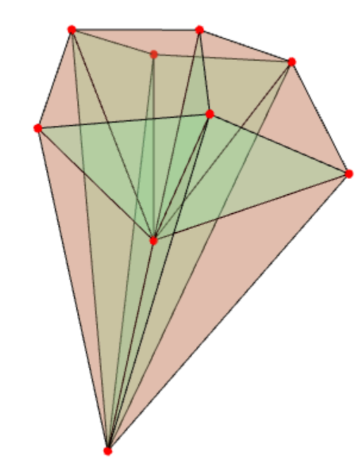
julia> getPLD(edges, nodes; internal_masses = m, external_masses = M, method = :num)
------- codim = 3, 9 faces
codim: 3, face: 1/9, weights: [-1, 0, 1], discriminant: m₁
New discriminants after codim 3, face 1/9. The list is: m₁
codim: 3, face: 2/9, weights: [-1, 1, 0], discriminant: m₁
codim: 3, face: 3/9, weights: [0, -1, 1], discriminant: m₂
New discriminants after codim 3, face 3/9. The list is: m₁, m₂
codim: 3, face: 4/9, weights: [2, 2, 4], discriminant: 1
New discriminants after codim 3, face 4/9. The list is: 1, m₁, m₂
codim: 3, face: 5/9, weights: [0, 1, -1], discriminant: m₃
New discriminants after codim 3, face 5/9. The list is: 1, m₁, m₂, m₃
codim: 3, face: 6/9, weights: [2, 4, 2], discriminant: 1
codim: 3, face: 7/9, weights: [1, -1, 0], discriminant: m₂
codim: 3, face: 8/9, weights: [1, 0, -1], discriminant: m₃
codim: 3, face: 9/9, weights: [4, 2, 2], discriminant: 1
Unique discriminants after codim 3: 1, m₁, m₂, m₃
------- codim = 2, 15 faces
codim: 2, face: 1/15, weights: [-1, 0, 0], discriminant: m₁
codim: 2, face: 2/15, weights: [-1, -1, 0], discriminant: 1
codim: 2, face: 3/15, weights: [0, 1, 2], discriminant: 1
codim: 2, face: 4/15, weights: [-1, 0, -1], discriminant: 1
codim: 2, face: 5/15, weights: [0, 2, 1], discriminant: 1
codim: 2, face: 6/15, weights: [1, 0, 2], discriminant: 1
codim: 2, face: 7/15, weights: [0, -1, 0], discriminant: m₂
codim: 2, face: 8/15, weights: [1, 2, 2], discriminant: 1
codim: 2, face: 9/15, weights: [2, 1, 2], discriminant: 1
codim: 2, face: 10/15, weights: [1, 2, 0], discriminant: 1
codim: 2, face: 11/15, weights: [0, 0, -1], discriminant: m₃
codim: 2, face: 12/15, weights: [2, 2, 1], discriminant: 1
codim: 2, face: 13/15, weights: [0, -1, -1], discriminant: 1
codim: 2, face: 14/15, weights: [2, 0, 1], discriminant: 1
codim: 2, face: 15/15, weights: [2, 1, 0], discriminant: 1
Unique discriminants after codim 2: 1, m₁, m₂, m₃
```

Sunrise solution: PLD.jl

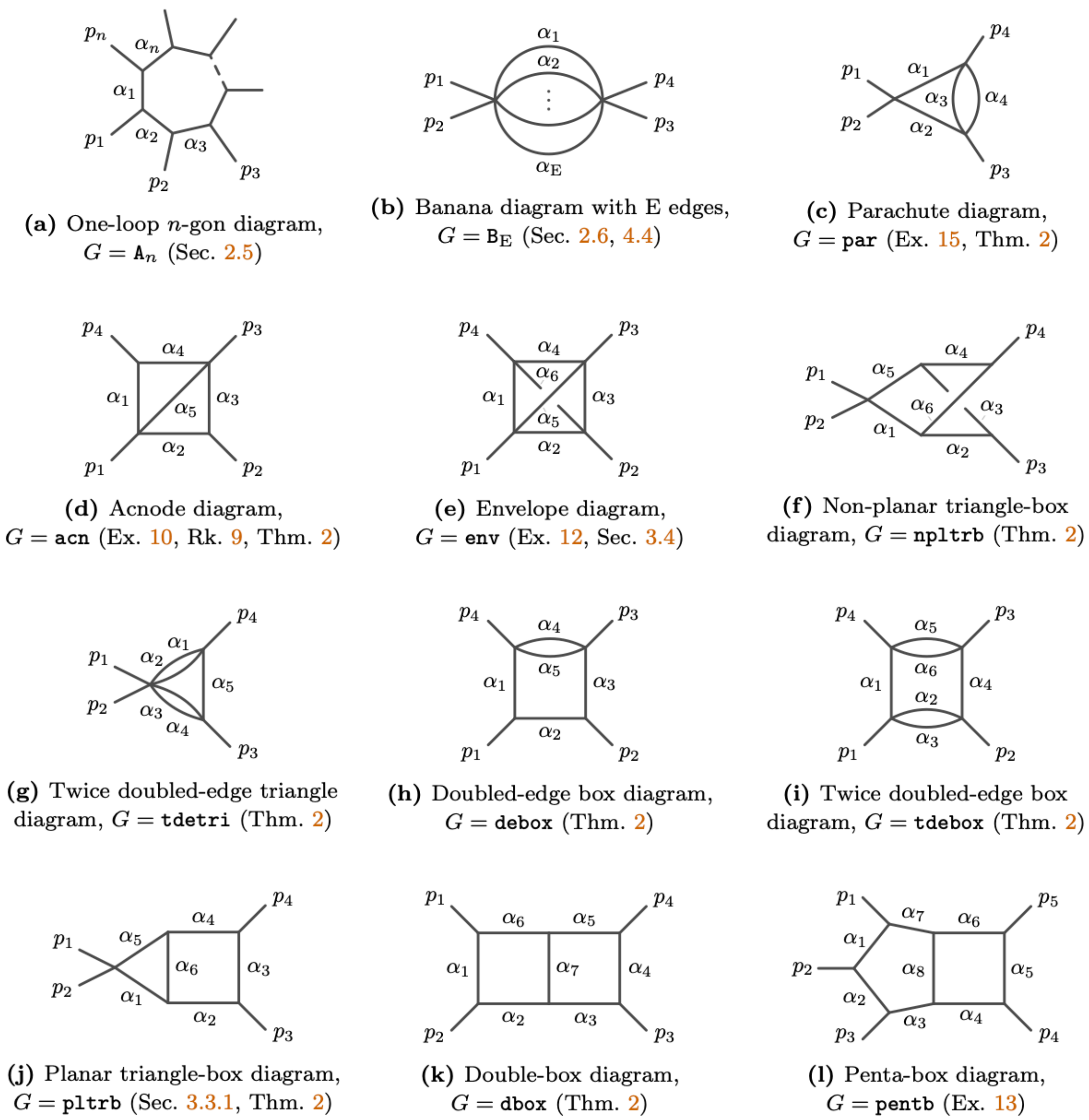
```
-------- codim = 1, 8 faces
codim: 1, face: 1/8, weights: [-1, -1, -1], discriminant: m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ +
4*m₁^2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*
m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s
^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, s
New discriminants after codim 1, face 1/8. The list is: 1, m₁, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*
m₃ + 4*m₁^2*m₂*s + 6*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s
 + 4*m₁*m₂*s^2 - 4*m₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m
₂^2*s^2 - 4*m₂*m₃^3 + 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, m₂, m₃, s
codim: 1, face: 2/8, weights: [0, 1, 1], discriminant: 1
codim: 1, face: 3/8, weights: [0, 0, 1], discriminant: 1
codim: 1, face: 4/8, weights: [0, 1, 0], discriminant: 1
codim: 1, face: 5/8, weights: [1, 0, 1], discriminant: 1
codim: 1, face: 6/8, weights: [1, 1, 1], discriminant: 1
codim: 1, face: 7/8, weights: [1, 1, 0], discriminant: 1
codim: 1, face: 8/8, weights: [1, 0, 0], discriminant: 1
Unique discriminants after codim 1: 1, m₁, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6
*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m
₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3
+ 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, m₂, m₃, s
-------- codim = 0, 1 faces
codim: 0, face: 1/1, weights: [0, 0, 0], discriminant: s
Unique discriminants after codim 0: 1, m₁, m₁^4 - 4*m₁^3*m₂ - 4*m₁^3*m₃ - 4*m₁^3*s + 6*m₁^2*m₂^2 + 4*m₁^2*m₂*m₃ + 4*m₁^2*m₂*s + 6
*m₁^2*m₃^2 + 4*m₁^2*m₃*s + 6*m₁^2*s^2 - 4*m₁*m₂^3 + 4*m₁*m₂^2*m₃ + 4*m₁*m₂^2*s + 4*m₁*m₂*m₃^2 - 40*m₁*m₂*m₃*s + 4*m₁*m₂*s^2 - 4*m
₁*m₃^3 + 4*m₁*m₃^2*s + 4*m₁*m₃*s^2 - 4*m₁*s^3 + m₂^4 - 4*m₂^3*m₃ - 4*m₂^3*s + 6*m₂^2*m₃^2 + 4*m₂^2*m₃*s + 6*m₂^2*s^2 - 4*m₂*m₃^3
+ 4*m₂*m₃^2*s + 4*m₂*m₃*s^2 - 4*m₂*s^3 + m₃^4 - 4*m₃^3*s + 6*m₃^2*s^2 - 4*m₃*s^3 + s^4, m₂, m₃, s
```

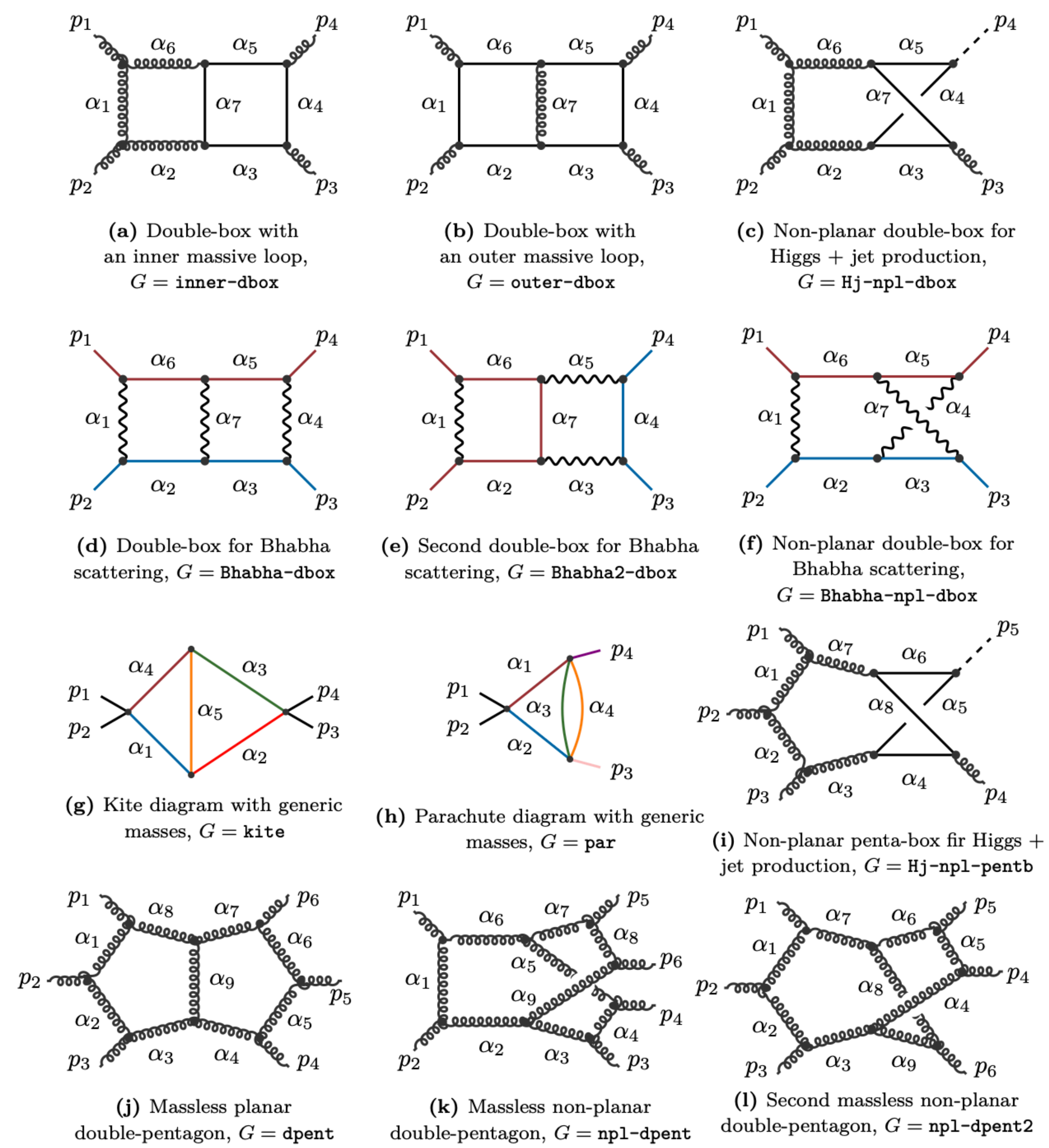






A zoo of examples





Landau Discriminants

Sebastian Mizera,[1] Simon Telen[2]

Principal Landau Determinants

Claudia Fevola,[1] Sebastian Mizera,[2] Simon Telen[3]

Related work

Computer Physics Communications
Volume 188, March 2015, Pages 148-166

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2019

Symbol Alphabets from the Landau Singular Locus

2023

Christoph Dlapa,[a] Martin Helmer,[b] Georgios Papathanasiou[a] and Felix Tellander[a]

Landau Singularities from Whitney Stratifications

Martin Helmer,[1,*] Georgios Papathanasiou,[2,3,†] and Felix Tellander[3,‡]

2024

Maximum likelihood estimation

A statistical model for a discrete random variable with s states is a subset of the probability simplex of dimension $s-1$

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Joint probability distribution of two binary random variables

	LOTR	HP
Red	p_{00}	p_{01}
White	p_{10}	p_{11}

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independence:

$$p_{\text{LOTR}} + p_{\text{HP}} = 1, \quad p_{\text{red}} + p_{\text{white}} = 1$$

$$p_{00} = p_{\text{LOTR}} \cdot p_{\text{red}}$$

$$= \begin{bmatrix} p_{\text{red}} \\ p_{\text{white}} \end{bmatrix} \cdot \begin{bmatrix} p_{\text{LOTR}} & p_{\text{HP}} \end{bmatrix}$$

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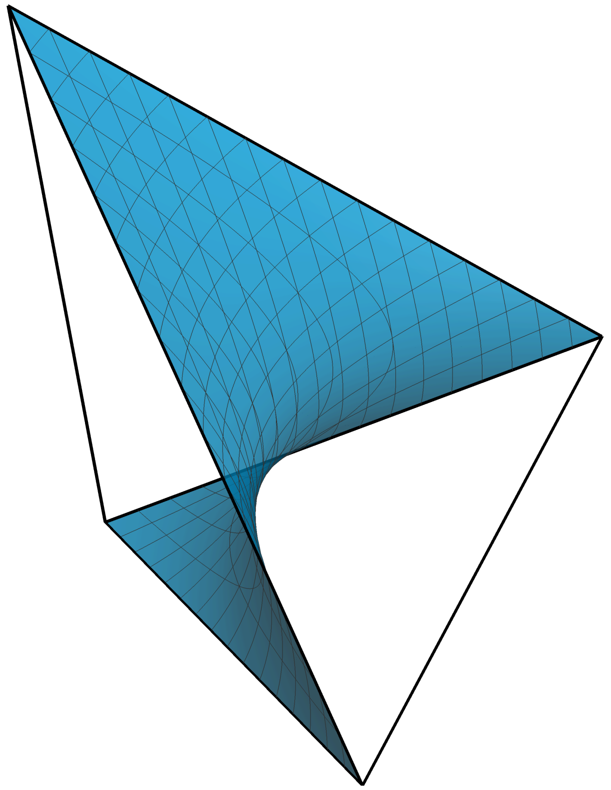

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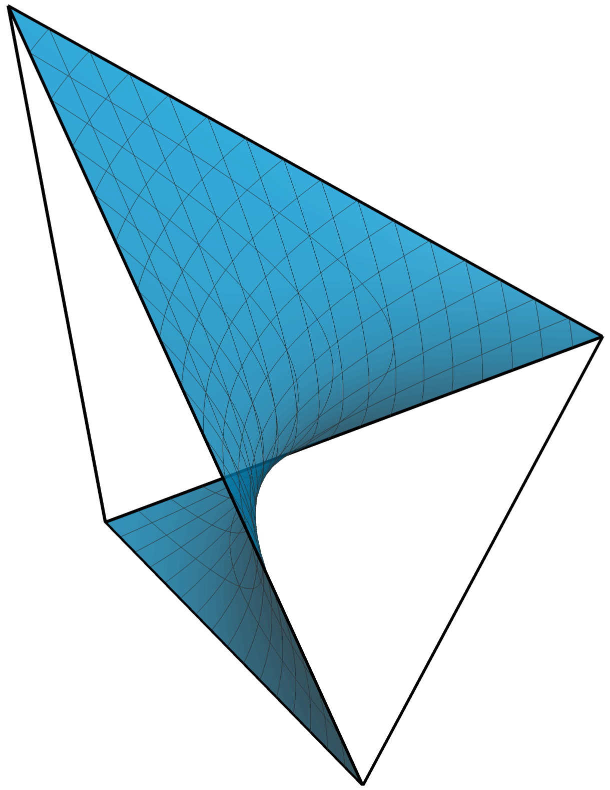

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This model is the intersection of the Segre quadric in $\mathbb{P}^3$ with the probability simplex $\mathbb{P}^3_{>0}$

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independence model of two binary random variables

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suppose that, in an experiment, we observe state ij a total amount of u_{ij} times

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MLE: infer $x = \frac{x_0}{x_1}$ and $y = \frac{y_0}{y_1}$ by maximizing $L_u = \log p_{00}^{u_{00}} p_{01}^{u_{01}} p_{10}^{u_{10}} p_{11}^{u_{11}}$

$$\frac{\partial L_u}{\partial x} = \frac{\partial L_u}{\partial y} = 0$$ → Homotopy Continuation.jl → $$x = \frac{u_{00} + u_{01}}{u_{00} + u_{01} + u_{10} + u_{11}}, \quad y = \frac{u_{00} + u_{10}}{u_{00} + u_{01} + u_{10} + u_{11}}$$

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The number of complex solutions for generic data u is called the maximum likelihood degree of the model

The maximum likelihood degree

Fabrizio Catanese, Serkan Hoşten, Amit Khetan, Bernd Sturmfels

American Journal of Mathematics, Volume 128, Number 3, June 2006, pp. 671-697 (Article)

Discrete exponential families

$$A = \begin{pmatrix} m_1 & m_2 & \cdots & m_s \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{Z}^{(n+1)\times s}$$

$$x \mapsto \left(\frac{z_1 x^{m_1}}{f_A}, \frac{z_2 x^{m_2}}{f_A}, \ldots, \frac{z_s x^{m_s}}{f_A}\right)$$

$$f_A(x;z) = z_1\, x^{m_1} + z_2\, x^{m_2} + \cdots + z_s\, x^{m_s}$$

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Theorem (Huh) The maximum likelihood degree of the discrete exponential family corresponding to A is the signed Euler characteristic of $V_{A,z} = \{x \in (\mathbb{C}^*)^n \ : \ f(x; z) = 0\}$

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Theorem (Amendola, Bliss, Burke, Gibbons, Helmer, Hoşten, Nash, Rodriguez, Smolkin)

$$|\chi(V_{A,z})| < \mathrm{vol}(\mathrm{conv}(A)) \iff E_A(z) = 0$$

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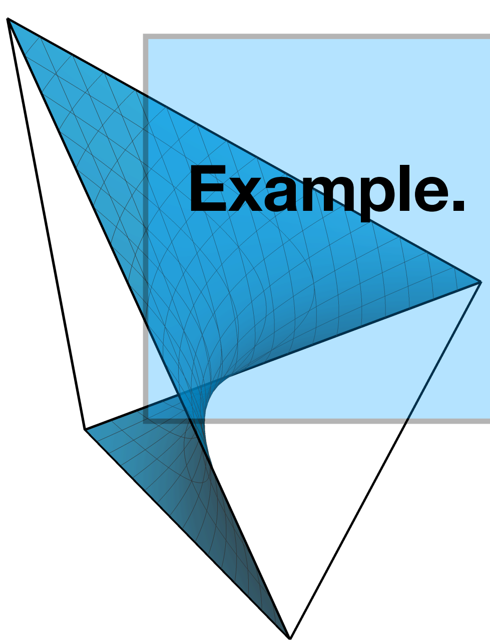
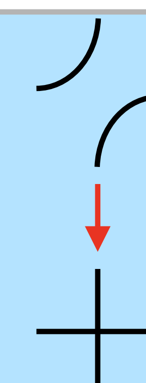


Example. $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $f_A(x;z) = z_1 + z_2 x_1 + z_3 x_2 + z_4 x_1 x_2$

$$\Delta_A = z_1 z_3 - z_2 z_4$$



Tossing a biased coin

A biased coin shows HEADS with probability x, TAILS with probability y

We toss five times and count the number of HEADS

$$x \mapsto (\frac{y^5}{f_A}, \frac{5xy^4}{f_A}, \frac{10x^2y^3}{f_A}, \frac{10x^3y^2}{f_A}, \frac{5x^4y}{f_A}, \frac{x^5}{f_A}), \quad f_A(x, y; z*) = y^5 + 5xy^4 + 10x^2y^3 + 10x^3y^2 + 5x^4y + x^5$$

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SciPost — SciPost Phys. Lect. Notes 75 (2023)

Four lectures on Euler integrals

Saiei-Jaeyeong Matsubara-Heo[1*], Sebastian Mizera[2†] and Simon Telen[3‡]

```
using HomotopyContinuation
@var x s nu
f = 1 + 5*x + 10*x^2 + 10*x^3 + 5*x^4 + x^5
L = nu*log(x) - s*log(f)
F = System([differentiate(L,x)], parameters = [s;nu])
monres = monodromy_solve(F)
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5 solutions

Euler stratification

Let $\pi : \mathscr{V} \to \mathscr{Z}$ be a surjective map of irreducible quasi-projective $\mathbb{C}$-varieties

The Euler discriminant is $\nabla_\chi(\pi) \ = \ \overline{\{z \in \mathscr{Z} \ : \ |\chi(V_z)| \neq \chi^*\}}$

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An Euler stratification of π is a partially ordered finite set $\mathscr{S}$ of quasi-projective subvarieties (strata) of $\mathscr{Z}$ such that for any $S, S' \in \mathscr{S}$

- $S \cap S' = \varnothing$ when $S \neq S'$, and $\sqcup_{S \in \mathscr{S}} S = \mathscr{Z}$
- $\overline{S}$ is a union of strata
- $\chi(V_z) = \chi(\pi^{-1}(z))$ is constant for $z \in \mathscr{Z}$
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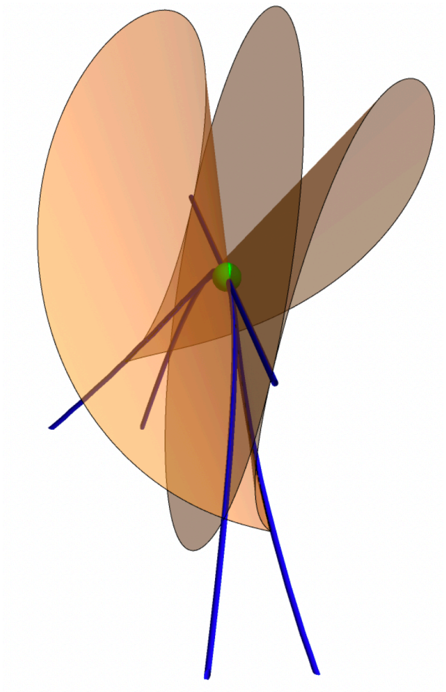
Favorite examples:

$$\mathscr{V} = \{(x, z) \in (\mathbb{C}^*)^n \times \mathbb{P}^{s-1} \,:\, f_A(x; z) = 0\}, \quad \mathscr{Z} = \mathbb{P}^{s-1}$$

$$\mathscr{V} = \{(x, z) \in \mathscr{X}_A \times \mathbb{P}^{s-1} \,:\, f_A(x; z) = 0\}, \quad \mathscr{Z} = \mathbb{P}^{s-1}$$

Points on the line

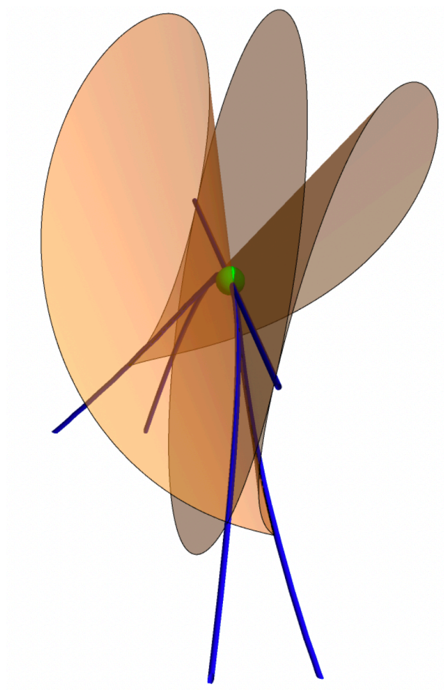
Euler stratification of $\pi : \{(x,z) \in \mathbb{C}^* \times \mathbb{P}^3 \ : \ z_1 + z_2 x + z_3 x^2 + z_4 x^3 = 0\} \longrightarrow \mathbb{P}^3$



$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Points on the line

Euler stratification of $\pi : \{(x,z) \in \mathbb{C}^* \times \mathbb{P}^3 \ : \ z_1 + z_2 x + z_3 x^2 + z_4 x^3 = 0\} \longrightarrow \mathbb{P}^3$



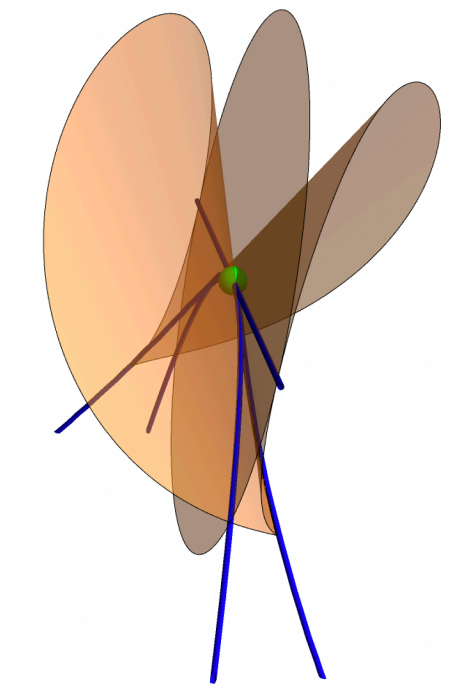
$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$E_A(z) = z_1 z_4 (z_3^2 z_2^2 - 4 z_4 z_2^3 - 4 z_3^3 z_1 + 18 z_1 z_2 z_3 z_4 - 27 z_4^2 z_1^2)$$

- dense stratum: $\chi = 3$
- principal A-determinant: $\chi = 2$
- singular locus: $\chi = 1$
- torus invariant points: $\chi = 0$

Points on the line

Euler stratification of $\pi : \{(x,z) \in \mathbb{C}^* \times \mathbb{P}^3 \; : \; z_1 + z_2x + z_3x^2 + z_4x^3 = 0\} \longrightarrow \mathbb{P}^3$



$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$E_A(z) = z_1z_4(z_3^2z_2^2 - 4z_4z_2^3 - 4z_3^3z_1 + 18z_1z_2z_3z_4 - 27z_4^2z_1^2)$$

- dense stratum: $\chi = 3$
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- singular locus: $\chi = 1$
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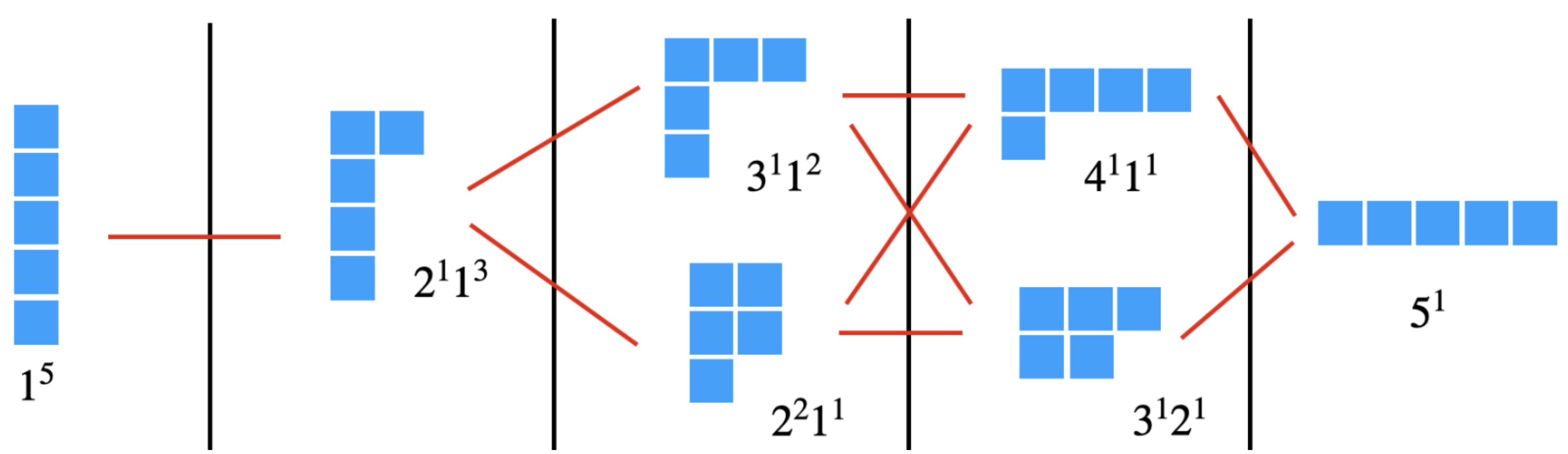
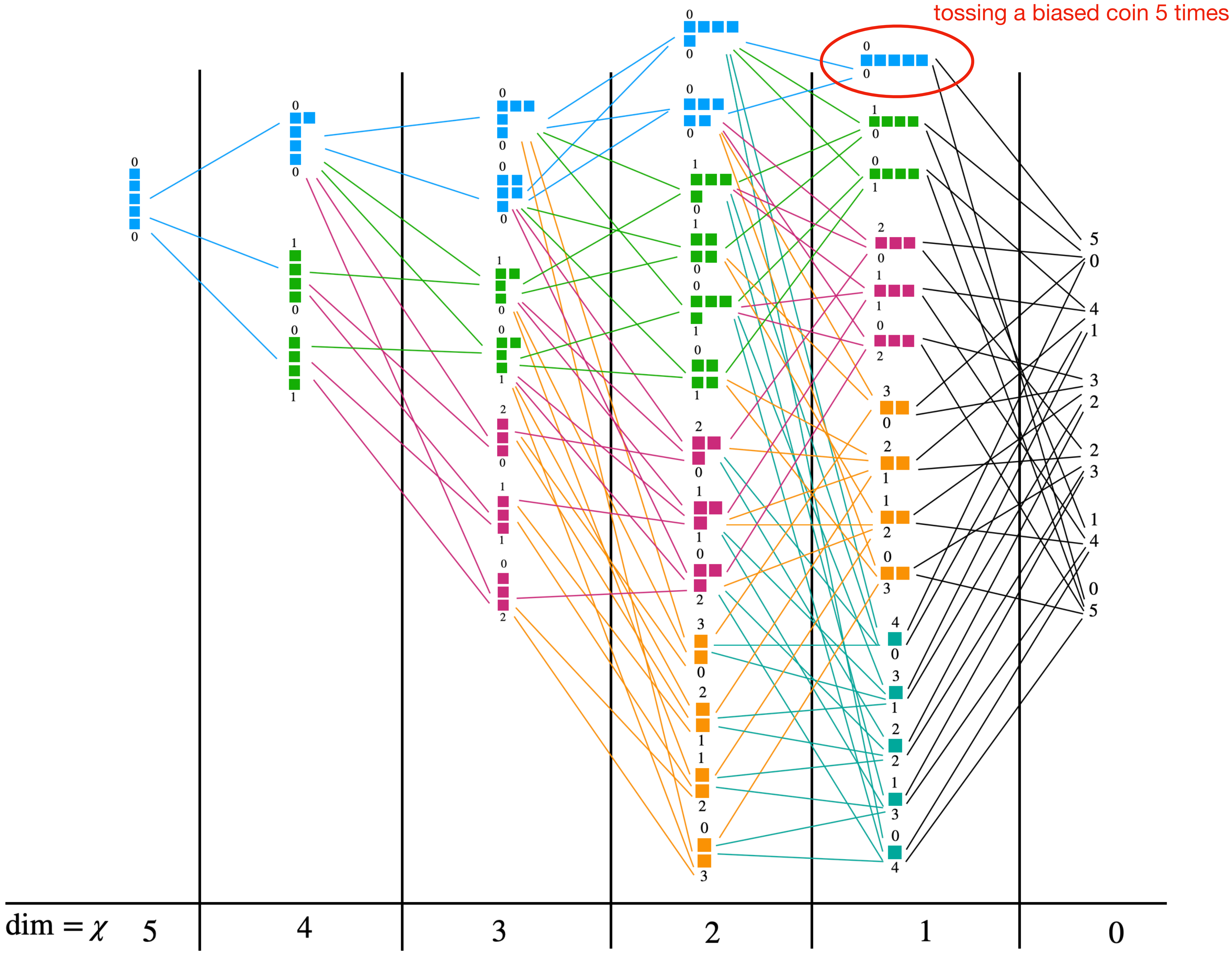
$$\pi : \{(x,z) \in \mathbb{P}^1 \times \mathbb{P}^5 \; : \; z_1 + z_2x + z_3x^2 + z_4x^3 + z_5x^4 + z_6x^5 = 0\} \longrightarrow \mathbb{P}^5$$

strata are indexed by partitions of 5 = Young diagrams using 5 boxes



Five points in $\mathbb{C}^*$

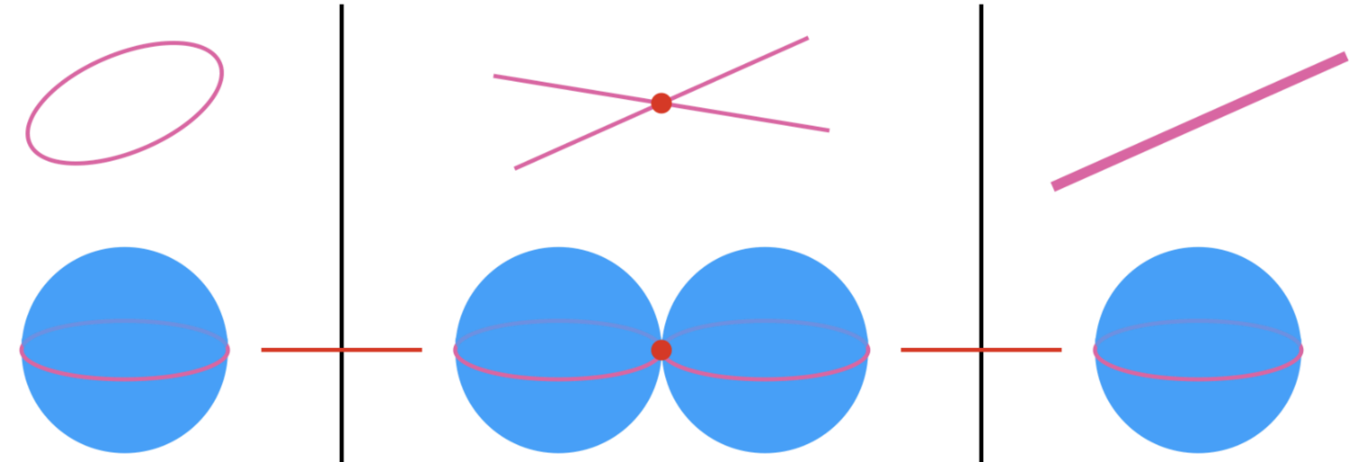


Euler stratifications for plane curves

ongoing with Maximilian Wiesmann

$$M(z) = \begin{pmatrix} 2z_{00} & z_{01} & z_{02} \\ z_{01} & 2z_{11} & z_{12} \\ z_{02} & z_{12} & 2z_{22} \end{pmatrix}$$

$\det M(z) = 0$ $\qquad$ $\operatorname{rank} M(z) = 1$

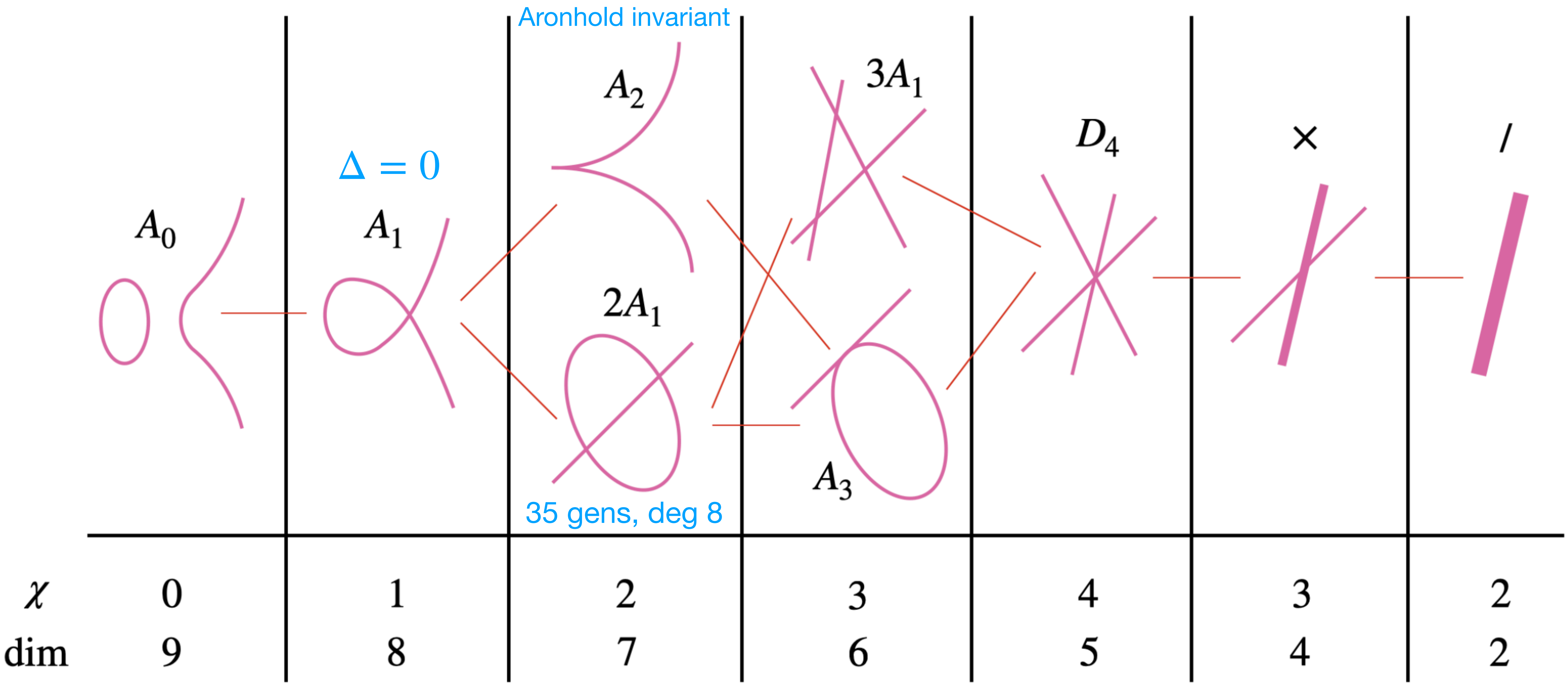


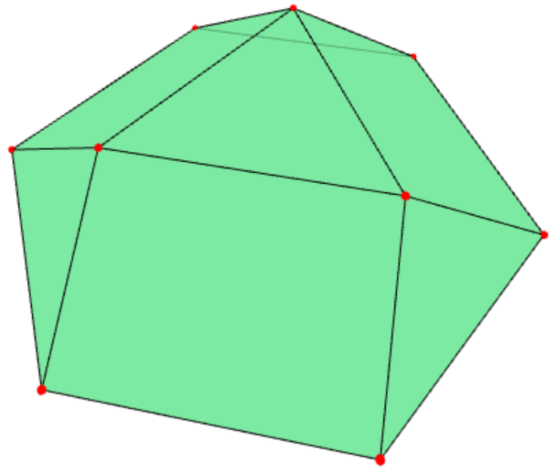
rank $M(z)$	3	2	1
χ	2	3	2
dim	5	4	2

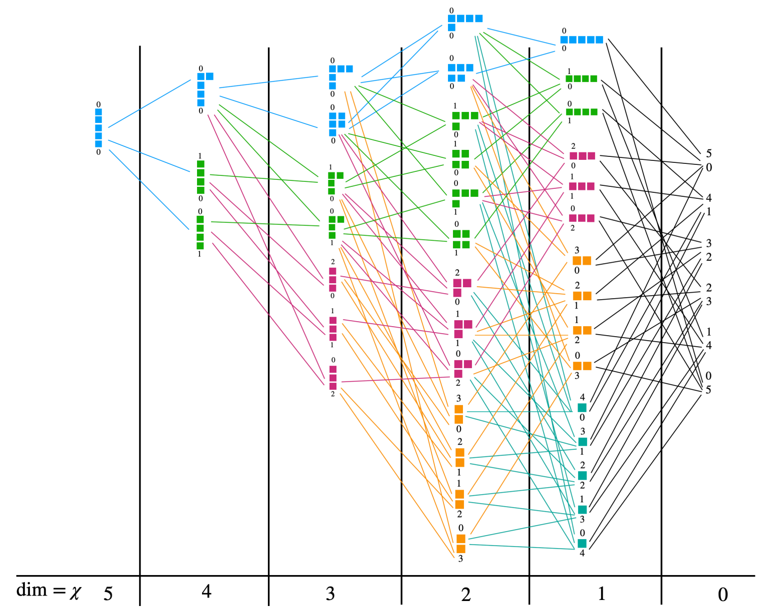
Euler stratifications for plane curves

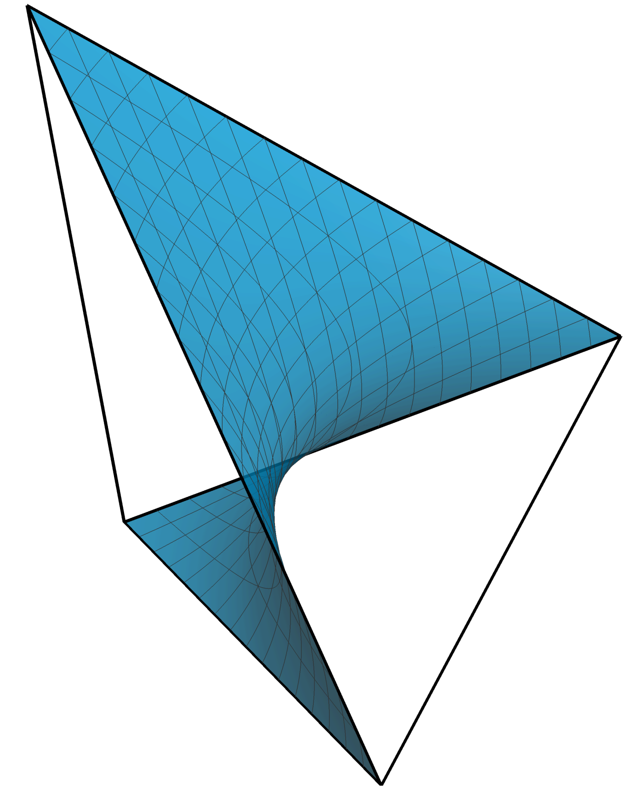
ongoing with Maximilian Wiesmann

$$
\begin{aligned}
&z_4^4 - 8\, z_3 z_4^2 z_5 + 16\, z_3^2 z_5^2 + 24\, z_2 z_4 z_5 z_6 - 48\, z_1 z_5^2 z_6 - 8\, z_2 z_4^2 z_7 - 16\, z_2 z_3 z_5 z_7 + 24\, z_1 z_4 z_5 z_7 + 16\, z_2^2 z_7^2 \\
&-48\, z_0 z_5 z_7^2 + 24\, z_2 z_3 z_4 z_8 - 8\, z_1 z_4^2 z_8 - 16\, z_1 z_3 z_5 z_8 - 48\, z_2^2 z_6 z_8 + 144\, z_0 z_5 z_6 z_8 - 16\, z_1 z_2 z_7 z_8 + 24\, z_0 z_4 z_7 z_8 + 16\, z_1^2 z_8^2 \\
&-48\, z_0 z_3 z_8^2 - 48\, z_2 z_3^2 z_9 + 24\, z_1 z_3 z_4 z_9 + 144\, z_1 z_2 z_6 z_9 - 216\, z_0 z_4 z_6 z_9 - 48\, z_1^2 z_7 z_9 + 144\, z_0 z_3 z_7 z_9
\end{aligned}
$$

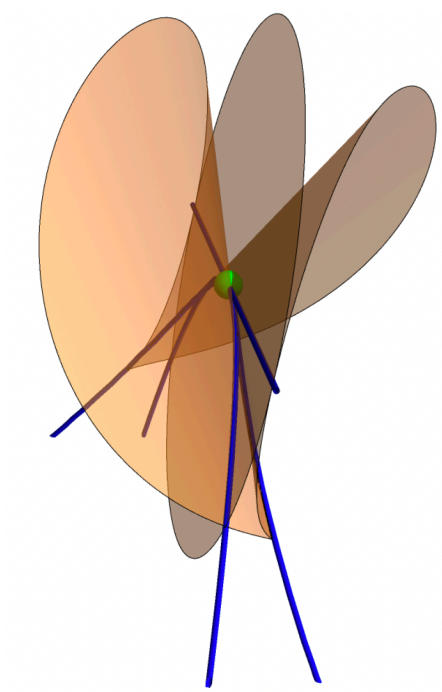


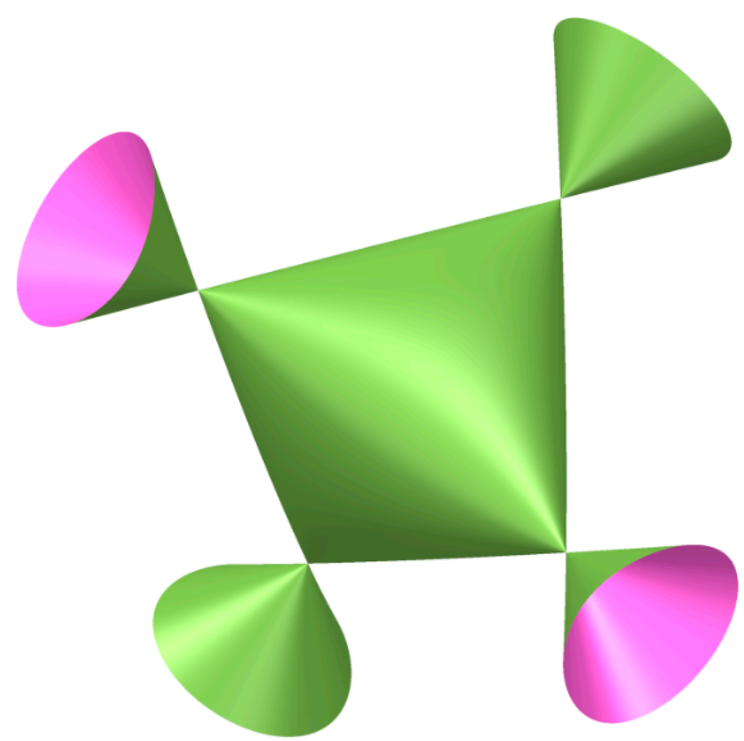


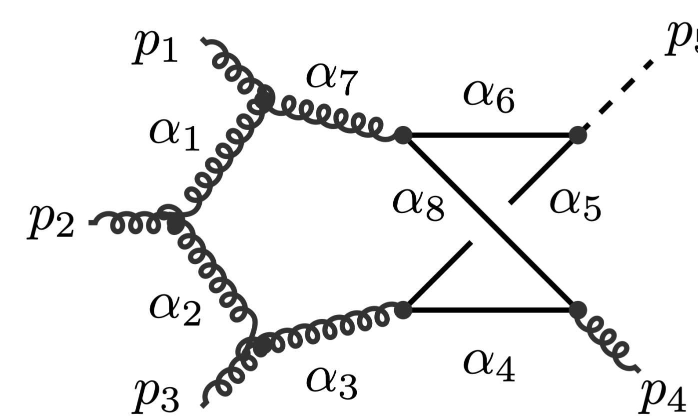

dim = χ
5
4
3
2
1
0



Thank you!






p_1
p_2
p_3
p_4
p_5
α_1
α_2
α_3
α_4
α_5
α_6
α_7
α_8

Euler integrals

Euler's Beta integral $\int_0^1 (1-x)^\mu\, x^\nu\, \frac{\mathrm{d}x}{x}$ converges for $\mathrm{Re}(\nu) \geq 0,\ \mathrm{Re}(\mu) \geq -1$

Euler integrals

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Its meromorphic extension to $\mathbb{C}^2$ is the Beta function

$$\mathrm{B}(\nu, 1+\mu) \ = \ \int_\Gamma (1-x)^\mu \, x^\nu \frac{\mathrm{d}x}{x}$$

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Here Γ is a twisted cycle on $\mathbb{C}^* \backslash \{1\}$

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Here Γ is a twisted cycle on $\mathbb{C}^* \backslash \{1\}$

A similar integral appears in Euler's integral formula for ${}_2F_1$:

$$\mathrm{B}(\nu, 1+\mu_1) \; {}_2F_1(-\mu_2, \nu, \mu_1+1+\nu \, ; \, z) \; = \; \int_\Gamma (1-x)^{\mu_1} (1-zx)^{\mu_2} \, x^\nu \, \frac{\mathrm{d}x}{x}$$

Sunrise problem

Sunrise problem

$$
\begin{aligned}
z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 &= 0 \\
z_2\,\alpha_3 + z_8\,\alpha_3^2 &= 0 \\
z_3\,\alpha_3 + z_9\,\alpha_3^2 &= 0 \\
z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 &= 0 \\
\alpha_1\alpha_2\alpha_3 y - 1 &= 0
\end{aligned}
\quad \Longrightarrow \quad \Delta_{A\cap Q} = z_2 z_9 - z_3 z_8 = 0
$$

Sunrise problem

$$
\begin{aligned}
z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 &= 0 \\
z_2\,\alpha_3 + z_8\,\alpha_3^2 &= 0 \\
z_3\,\alpha_3 + z_9\,\alpha_3^2 &= 0 \\
z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 &= 0 \\
\alpha_1\alpha_2\alpha_3 y - 1 &= 0
\end{aligned}
$$

$$\implies \quad \Delta_{A\cap Q} = z_2 z_9 - z_3 z_8 = 0 \qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0$$

$$(z_1, \ldots, z_{10}) = (1, \underline{1}, \underline{1}, -\mathsf{m}_1, -\mathsf{m}_1, -\mathsf{m}_2, -\mathsf{m}_2, \underline{-\mathsf{m}_3}, \underline{-\mathsf{m}_3}, s - \mathsf{m}_1 - \mathsf{m}_2 - \mathsf{m}_3)$$

Sunrise problem

$$z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 \;=\; 0$$
$$z_2\,\alpha_3 + z_8\,\alpha_3^2 \;=\; 0$$
$$z_3\,\alpha_3 + z_9\,\alpha_3^2 \;=\; 0$$
$$z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 \;=\; 0$$
$$\alpha_1\alpha_2\alpha_3 y - 1 \;=\; 0$$

$$\Longrightarrow \quad \Delta_{A\cap Q} \;=\; z_2 z_9 - z_3 z_8 \;=\; 0 \qquad\qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0$$

$$(z_1,\ldots,z_{10}) \;=\; (1,\underline{1},\underline{1},-\mathrm{m}_1,-\mathrm{m}_1,-\mathrm{m}_2,-\mathrm{m}_2,\underline{-\mathrm{m}_3},\underline{-\mathrm{m}_3},s-\mathrm{m}_1-\mathrm{m}_2-\mathrm{m}_3)$$

all parameters in $\mathscr{E}$ lie inside the principal A-determinant

$\Updownarrow$

the generic Euler characteristic on $\mathscr{E}$ is strictly smaller than $\mathrm{vol}(A)$

Sunrise problem

$$
\begin{aligned}
z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 &= 0\\
z_2\,\alpha_3 + z_8\,\alpha_3^2 &= 0\\
z_3\,\alpha_3 + z_9\,\alpha_3^2 &= 0\\
z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 &= 0\\
\alpha_1\alpha_2\alpha_3 y - 1 &= 0
\end{aligned}
\quad\Longrightarrow\quad \Delta_{A\cap Q} = z_2 z_9 - z_3 z_8 = 0 \qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0
$$

$$(z_1,\ldots,z_{10}) = (1, \underline{1}, \underline{1}, -\mathsf{m}_1, -\mathsf{m}_1, -\mathsf{m}_2, -\mathsf{m}_2, \underline{-\mathsf{m}_3}, \underline{-\mathsf{m}_3}, s-\mathsf{m}_1-\mathsf{m}_2-\mathsf{m}_3)$$

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G	$\mathcal{K}$	$\mathcal{E}^{(\mathsf{M}_i,0)}$	$\mathcal{E}^{(0,\mathsf{m}_e)}$	$\mathcal{E}^{(0,0)}$
$\mathtt{A}_4$	(15, 15)	(11, 11)	(11, 15)	(3, 3)
$\mathtt{B}_4$	(15, 35)	(1,1)	(15,35)	(1,1)
par	(19, 35)	(4, 8)	(13, 35)	(1, 3)
acn	(55, 136)	(20, 54)	(36, 136)	(3, 9)
env	(273, 1496)	(56, 262)	(181, 1496)	(10, 80)
npltrb	(116, 512)	(28, 252)	(77, 512)	(5, 61)
tdetri	(51, 201)	(4, 18)	(33, 201)	(1, 5)
debox	(43, 96)	(11, 33)	(31, 96)	(3, 10)
tdebox	(123, 705)	(11, 113)	(87, 705)	(3, 41)
pltrb	(81, 417)	(16, 201)	(61, 417)	(4, 80)
dbox	(227, 1422)	(75, 903)	(159, 1422)	(12, 238)
pentb	(543, 4279)	(228, 3148)	(430, 4279)	(62, 1186)

G	$\mathcal{E}$
inner-dbox	(43, 834)
outer-dbox	(64, 1302)
Hj-npl-dbox	(99, 1016)
Bhabha-dbox	(64, 774)
Bhabha2-dbox	(79, 910)
Bhabha-npl-dbox	(111, 936)
kite	(30, 136)
par	(19, 35)
Hj-npl-pentb	(330, 3144)
dpent	(281, 5511)
npl-dpent	(631, 5784)
npl-dpent2	(458, 5467)

Sunrise problem

$$z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 \;=\; 0$$
$$z_2\,\alpha_3 + z_8\,\alpha_3^2 \;=\; 0$$
$$z_3\,\alpha_3 + z_9\,\alpha_3^2 \;=\; 0$$
$$z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 \;=\; 0$$
$$\alpha_1\alpha_2\alpha_3 y - 1 \;=\; 0$$

$$\Longrightarrow \quad \Delta_{A\cap Q} \;=\; z_2 z_9 - z_3 z_8 \;=\; 0 \qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0$$

$$(z_1, \ldots, z_{10}) \;=\; (1, \underline{1}, \underline{1}, -\mathrm{m}_1, -\mathrm{m}_1, -\mathrm{m}_2, -\mathrm{m}_2, \underline{-\mathrm{m}_3}, \underline{-\mathrm{m}_3}, s - \mathrm{m}_1 - \mathrm{m}_2 - \mathrm{m}_3)$$

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G	$\mathcal{K}$	$\mathcal{E}^{(\mathrm{M}_i,0)}$	$\mathcal{E}^{(0,\mathrm{m}_e)}$	$\mathcal{E}^{(0,0)}$
$\mathtt{A}_4$	$(15,15)$	$(11,11)$	$(11,15)$	$(3,3)$
$\mathtt{B}_4$	$(15,35)$	$(1,1)$	$(15,35)$	$(1,1)$
par	$(19,35)$	$(4,8)$	$(13,35)$	$(1,3)$
acn	$(55,136)$	$(20,54)$	$(36,136)$	$(3,9)$
env	$(273,1496)$	$(56,262)$	$(181,1496)$	$(10,80)$
npltrb	$(116,512)$	$(28,252)$	$(77,512)$	$(5,61)$
tdetri	$(51,201)$	$(4,18)$	$(33,201)$	$(1,5)$
debox	$(43,96)$	$(11,33)$	$(31,96)$	$(3,10)$
tdebox	$(123,705)$	$(11,113)$	$(87,705)$	$(3,41)$
pltrb	$(81,417)$	$(16,201)$	$(61,417)$	$(4,80)$
dbox	$(227,1422)$	$(75,903)$	$(159,1422)$	$(12,238)$
pentb	$(543,4279)$	$(228,3148)$	$(430,4279)$	$(62,1186)$

G	$\mathcal{E}$
inner-dbox	$(43,834)$
outer-dbox	$(64,1302)$
Hj-npl-dbox	$(99,1016)$
Bhabha-dbox	$(64,774)$
Bhabha2-dbox	$(79,910)$
Bhabha-npl-dbox	$(111,936)$
kite	$(30,136)$
par	$(19,35)$
Hj-npl-pentb	$(330,3144)$
dpent	$(281,5511)$
npl-dpent	$(631,5784)$
npl-dpent2	$(458,5467)$

The Euler discriminant can usually **not** be obtained by restricting the principal A-determinant

Sunrise problem

$$z_2\,\alpha_1\alpha_3 + z_3\,\alpha_2\alpha_3 + z_8\,\alpha_1\alpha_3^2 + z_9\,\alpha_2\alpha_3^2 = 0$$
$$z_2\,\alpha_3 + z_8\,\alpha_3^2 = 0$$
$$z_3\,\alpha_3 + z_9\,\alpha_3^2 = 0$$
$$z_2\,\alpha_1 + z_3\,\alpha_2 + 2z_8\,\alpha_1\alpha_3 + 2z_9\,\alpha_2\alpha_3 = 0$$
$$\alpha_1\alpha_2\alpha_3 y - 1 = 0$$

$$\Longrightarrow \quad \Delta_{A\cap Q} = z_2 z_9 - z_3 z_8 = 0 \qquad (\Delta_{A\cap Q})_{|\mathscr{E}} = 0$$

$$(z_1,\ldots,z_{10}) = (1, \underline{1}, \underline{1}, -\mathrm{m}_1, -\mathrm{m}_1, -\mathrm{m}_2, -\mathrm{m}_2, \underline{-\mathrm{m}_3}, \underline{-\mathrm{m}_3}, s - \mathrm{m}_1 - \mathrm{m}_2 - \mathrm{m}_3)$$

all parameters in $\mathscr{E}$ lie inside the principal A-determinant

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the generic Euler characteristic on $\mathscr{E}$ is strictly smaller than $\mathrm{vol}(A)$

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acn	(55, 136)	(20, 54)	(36, 136)	(3, 9)
env	(273, 1496)	(56, 262)	(181, 1496)	(10, 80)
npltrb	(116, 512)	(28, 252)	(77, 512)	(5, 61)
tdetri	(51, 201)	(4, 18)	(33, 201)	(1, 5)
debox	(43, 96)	(11, 33)	(31, 96)	(3, 10)
tdebox	(123, 705)	(11, 113)	(87, 705)	(3, 41)
pltrb	(81, 417)	(16, 201)	(61, 417)	(4, 80)
dbox	(227, 1422)	(75, 903)	(159, 1422)	(12, 238)
pentb	(543, 4279)	(228, 3148)	(430, 4279)	(62, 1186)

G	$\mathcal{E}$
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outer-dbox	(64, 1302)
Hj-npl-dbox	(99, 1016)
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The Euler discriminant can usually **not** be obtained by restricting the principal A-determinant

The **principal Landau determinant** is a computable subset of the Euler discriminant, whose definition is inspired by GKZ