What can amplitudes teach us about cosmological correlators, the wavefunction and the de Sitter S-matrix?

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Primordial Cosmo = QFT in curved spacetime / Quantum Gravity

- On large scales (>> Mpc) cosmological surveys measure QFT correlators of metric fluctuation

$$\langle \prod_a^n \delta(\mathbf{k}_a) \rangle \sim \left[\prod_a^n \Delta^{(\delta)}(\mathbf{k}_a) \right] \langle \prod_a^n \zeta(\mathbf{k}_a) \rangle$$

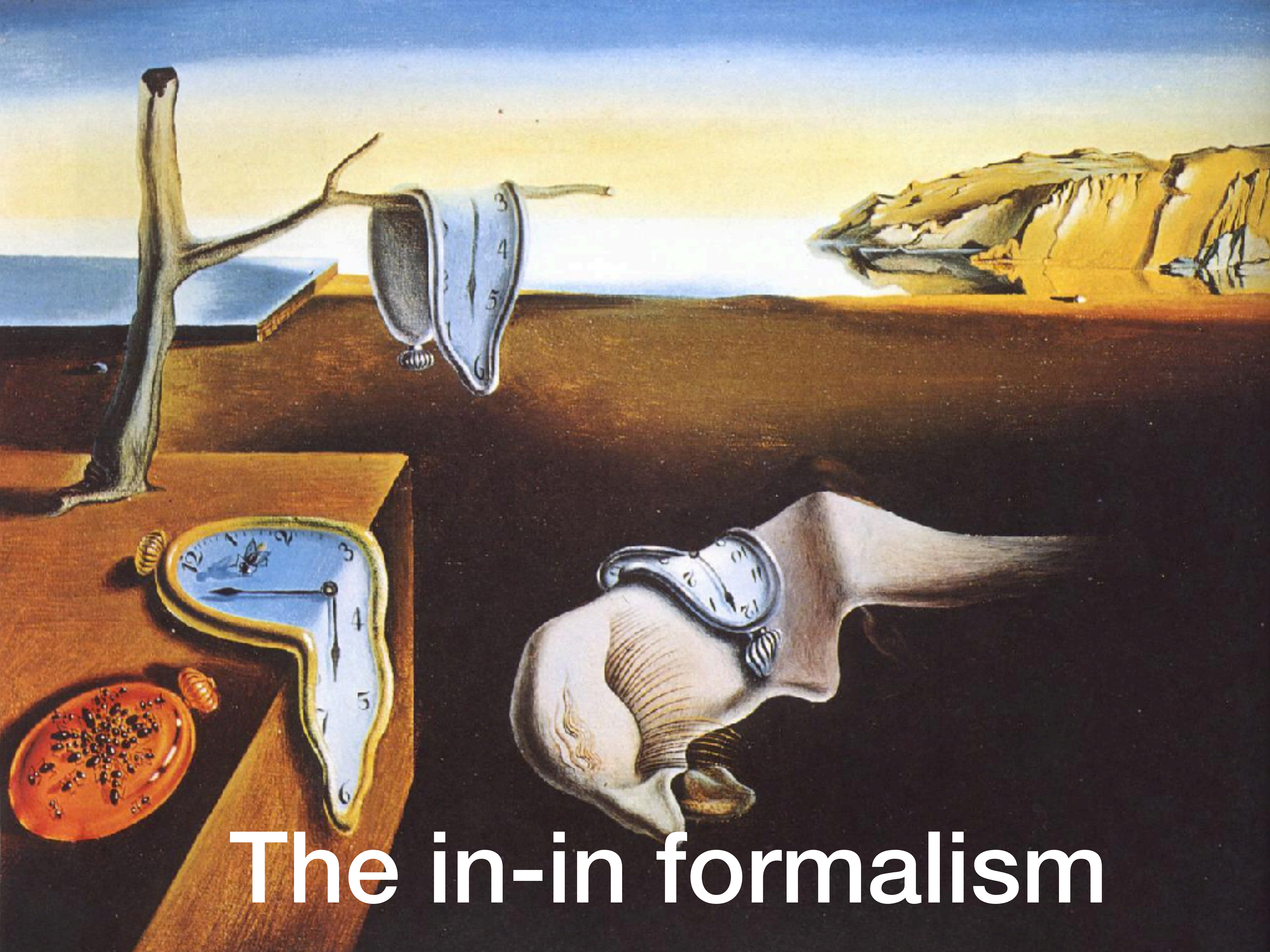
CMB temperature
density of galaxies
dark matter, ...

QFT / QG
in de Sitter

- Gravitational floor of non-Gaussianity in single-clock inflation: $f_{NL}^{eq} \gtrsim 10^{-2}$
- The goal of primordial cosmology is to understand QFT and QG in (approximately, asymptotically) de Sitter

Aspirations

- We want to learn about fundamental physics from cosmo:
  - *New degrees of freedom and their interactions*: Inflation requires at least one degree of freedom and three energy scales beyond the standard model.
  - The laws of gravity at short distances/high energies: probe GR and beyond at high energies
  - QFT in FLRW/de Sitter: which theories are consistent?
  - When does QFT on curved space-time break down and we need quantum gravity in dS?



The in-in formalism

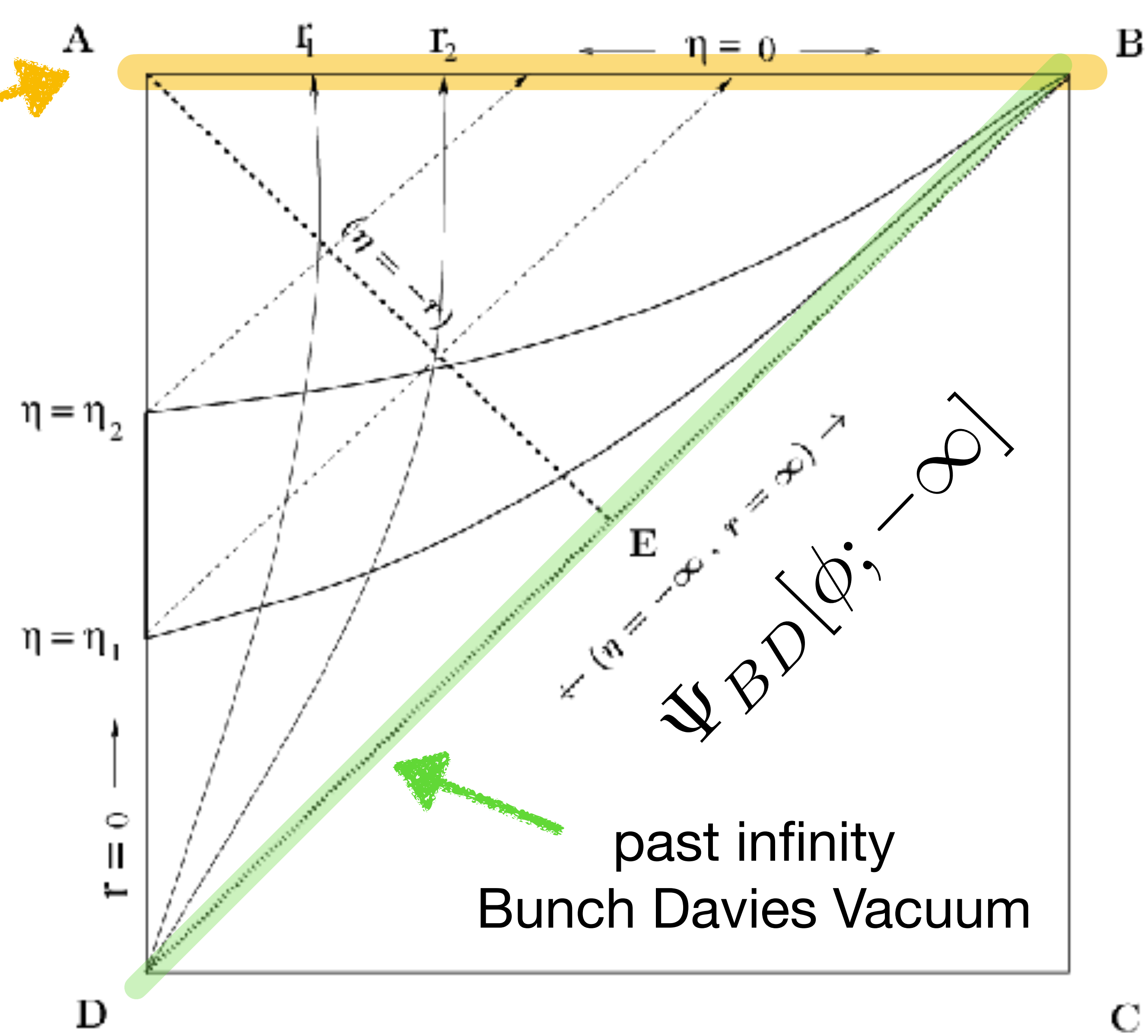
Penrose diagram

- We work in the Poincare' patch (half of dS)

$$ds^2 = -dt^2 + a^2 dx^2 = a^2(-d\eta^2 + dx^2)$$

$$\boxed{\Psi[\phi; \eta \to 0]}$$

The future (conformal) boundary can be thought as the reheating surface after inflation and determines the statistics of *LSS and CMB observations*



Correlators

- The observables of cosmology are correlators of the product of equal-time local operators $\mathscr{O}$ at $\eta \to 0$

$$\lim_{\eta\to 0} \langle\Omega| \prod_{a=1}^{n} \mathcal{O}(\mathbf{k}_a, \eta) |\Omega\rangle \equiv \langle \prod_{a=1}^{n} \mathcal{O}(\mathbf{k}_a) \rangle \equiv \langle \mathcal{O}^n \rangle \, .$$

- they are usually computed in the interaction picture. For *closed systems* in a *pure state* we have

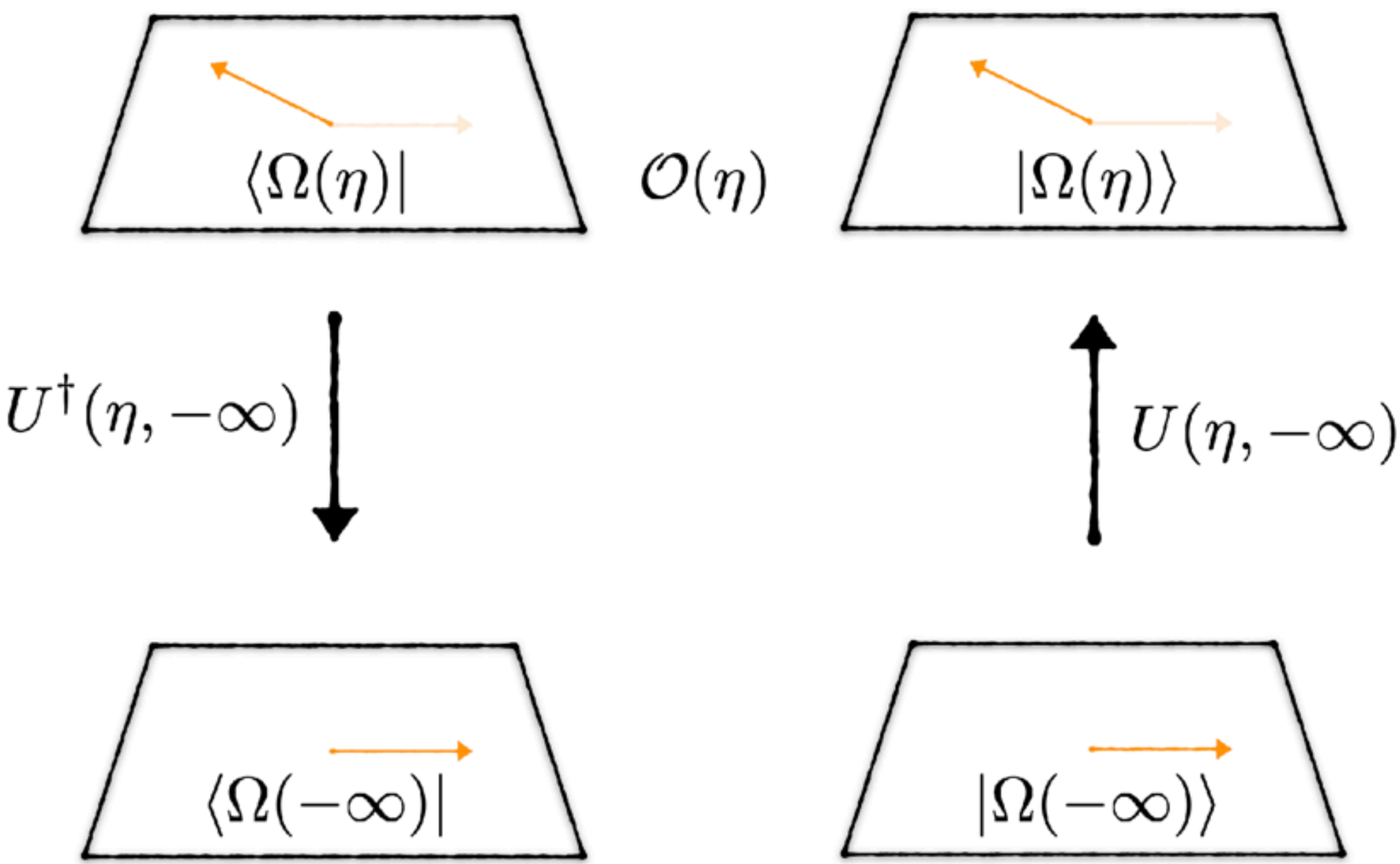
$$\langle \mathcal{O}(\eta) \rangle = \langle 0| \left[\bar{T} e^{\left(i \int_{-\infty(1+i\epsilon)}^{\eta} d\eta' H_{\text{int}}(\eta') \right)} \right] \mathcal{O}_I(\eta) \left[T e^{\left(-i \int_{-\infty(1-i\epsilon)}^{\eta} d\eta' H_{\text{int}}(\eta') \right)} \right] |0\rangle$$

Bunch-Davies

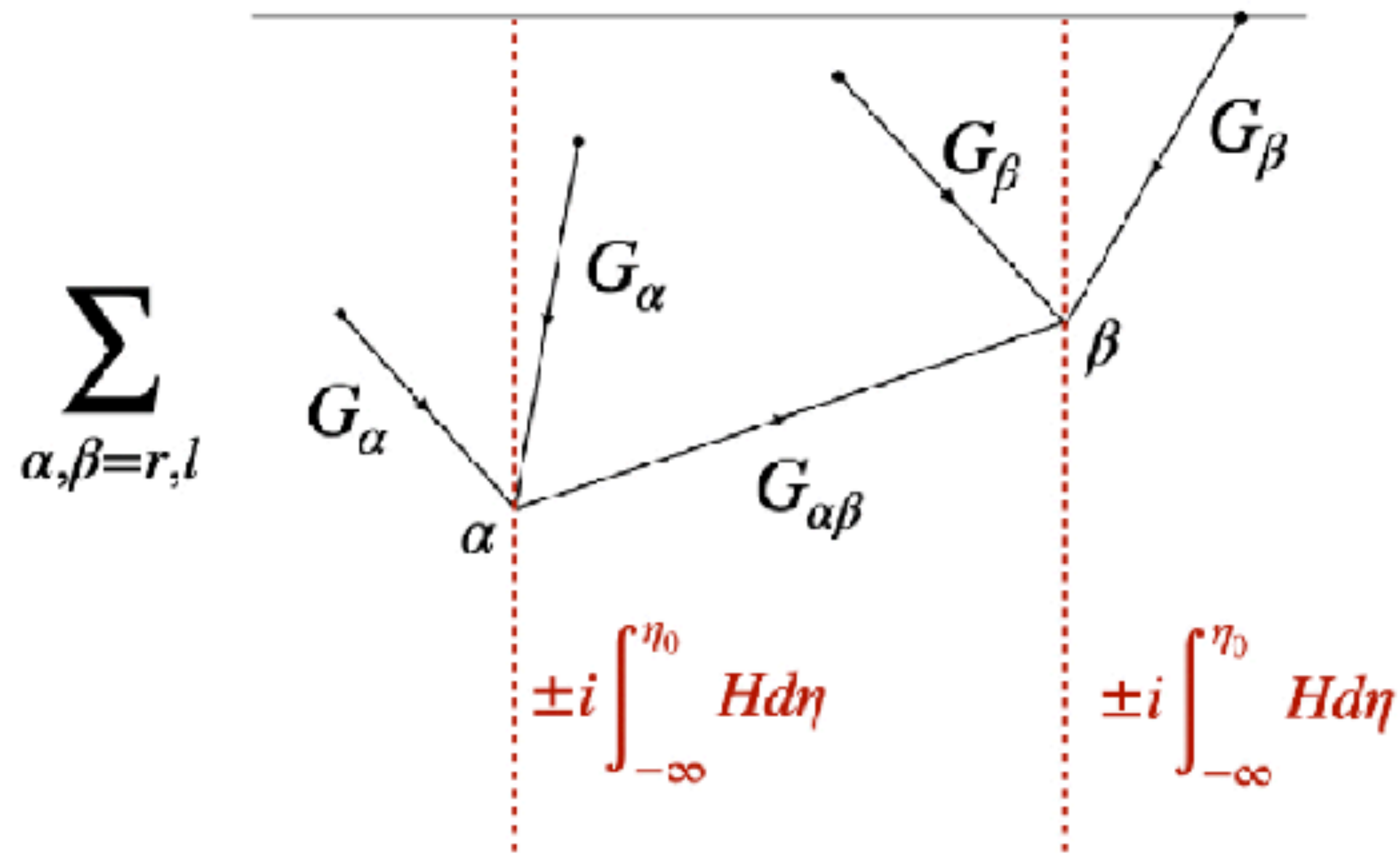
time evolution operator

- We can compute this in perturbation theory with ad hoc Feynman rules

In-in correlators



In-in Feynman rules



$$
\begin{aligned}
\bullet - \bullet = G_{rr}(\eta_1,\eta_2,p) &= \langle 0|\, T\phi(\eta_1,\mathbf{p})\phi(\eta_2,\mathbf{p}')\,|0\rangle \\
&= f_p(\eta_1)f_p^*(\eta_2)\theta(\eta_1-\eta_2) + f_p^*(\eta_1)f_p(\eta_2)\theta(\eta_2-\eta_1) \\
\circ - \bullet = G_{lr}(\eta_1,\eta_2,p) &= \langle 0|\, \phi(\eta_1,\mathbf{p})\phi(\eta_2,\mathbf{p}')\,|0\rangle = f_p(\eta_1)f_p^*(\eta_2) \\
\bullet - \circ = G_{rl}(\eta_1,\eta_2,p) &= \langle 0|\, \phi(\eta_2,\mathbf{p}')\phi(\eta_1,\mathbf{p})\,|0\rangle = G_{lr}^*(\eta_1,\eta_2,p) \\
\circ - \circ = G_{ll}(\eta_1,\eta_2,p) &= \langle 0|\, \bar{T}\phi(\eta_1,\mathbf{p})\phi(\eta_2,\mathbf{p}')\,|0\rangle = G_{rr}^*(\eta_1,\eta_2,p) \\
&= f_p^*(\eta_1)f_p(\eta_2)\theta(\eta_1-\eta_2) + f_p(\eta_1)f_p^*(\eta_2)\theta(\eta_2-\eta_1)\,,
\end{aligned}
$$

In-in Feynman rules

- Each external line is

$$\bullet\!- = G_r(\eta, p) = f_p(\eta_0) f_p^*(\eta)\,, \qquad\qquad \circ\!- = G_l(\eta, p) = f_p^*(\eta_0) f_p(\eta)\,.$$

- Diagrams with “left” ↔ “right” are complex conjugate of each other (so you need only to compute half of them)

- Left (right) vertices are i (-i) times the coupling constant, the vertex factor including derivatives and an time integral

$$\int d\eta \sqrt{-g} = \int d\eta (\eta H)^{-4}$$

- Notice that even tree-level diagrams with V vertices require performing V nested time integrals (for amplitudes in Minkowski these are all energy conserving delta functions)

Types of integral

- We encounter the following types of (IR-finite) integrals *at tree-level*

$$\int_{-\infty}^{0} d\eta^V \, \text{Poly}(\eta, k_a) \sum_{A,B \in V} \theta(\eta_A - \eta_B) \prod_a H_{\nu_a}^{(1)}(-k_a \eta) \qquad \text{(de Sitter, general masses)}$$

$$\int_{-\infty}^{0} d\eta^V \, \text{Poly}(\eta, k_a) \sum_{A,B \in V} \theta(\eta_A - \eta_B) \, e^{i \sum_a^n (\pm k_a)\eta_a} \qquad \text{(de Sitter, } m = 0, 2H^2\text{)}$$

$$\int_{-\infty}^{0} dt^V \, e^{i \sum_a^n (\pm \Omega_a) t_a} \sum_{A,B \in V} \theta(t_A - t_B) \qquad \text{(Minkowski, any mass)}$$

- At loop level we find $\int \frac{d^3k}{\prod_b \sqrt{k_b^2 + m_b^2}}$ of the above

- We can do all tree-level in Mink and dS+($m = 0$). Loop integrals are barely explored even in Minkowski (handful of papers (3?))

Some difficulties

- The in-in formalism for cosmological correlators has been used and studies extensively *in the past 20 year* since [Maldacena ’02; Weinberg ’05]
- One encounters a few difficulties:
  - A diagram with V vertices has 2^V possibilities to label vertices
  - Each contributions is a nested time integral of Hankel functions
  - Mixing of Wightman and Feynman propagators and their complex conjugates makes it hard to import amplitudes results and technology
  - General consequences of unitarity, locality, causality are obscured

Different approaches

I

The field-theoretic wavefunction

[much recent work from many authors…]

QFT in de Sitter
Cosmo correlators

II

in-in = in-out
de Sitter S-matrix

[based on 2402.05999 with Yaniv Donath]

III

de Sitter
S matrix

[based 2402.05999 with Yaniv Donath and work in progress]

The Analytic Wavefunction

I

The wavefunction

- The *field theoretic wavefunction* is the projection of the quantum state $|\Psi\rangle$ of the system onto eigenstates $|\phi\rangle$ of the field operators, $\hat{\phi}(x,\eta)|\phi\rangle = \phi(x,\eta)|\phi\rangle$, namely $\Psi[\phi,\eta] \equiv \langle\phi|\Psi,\eta\rangle$

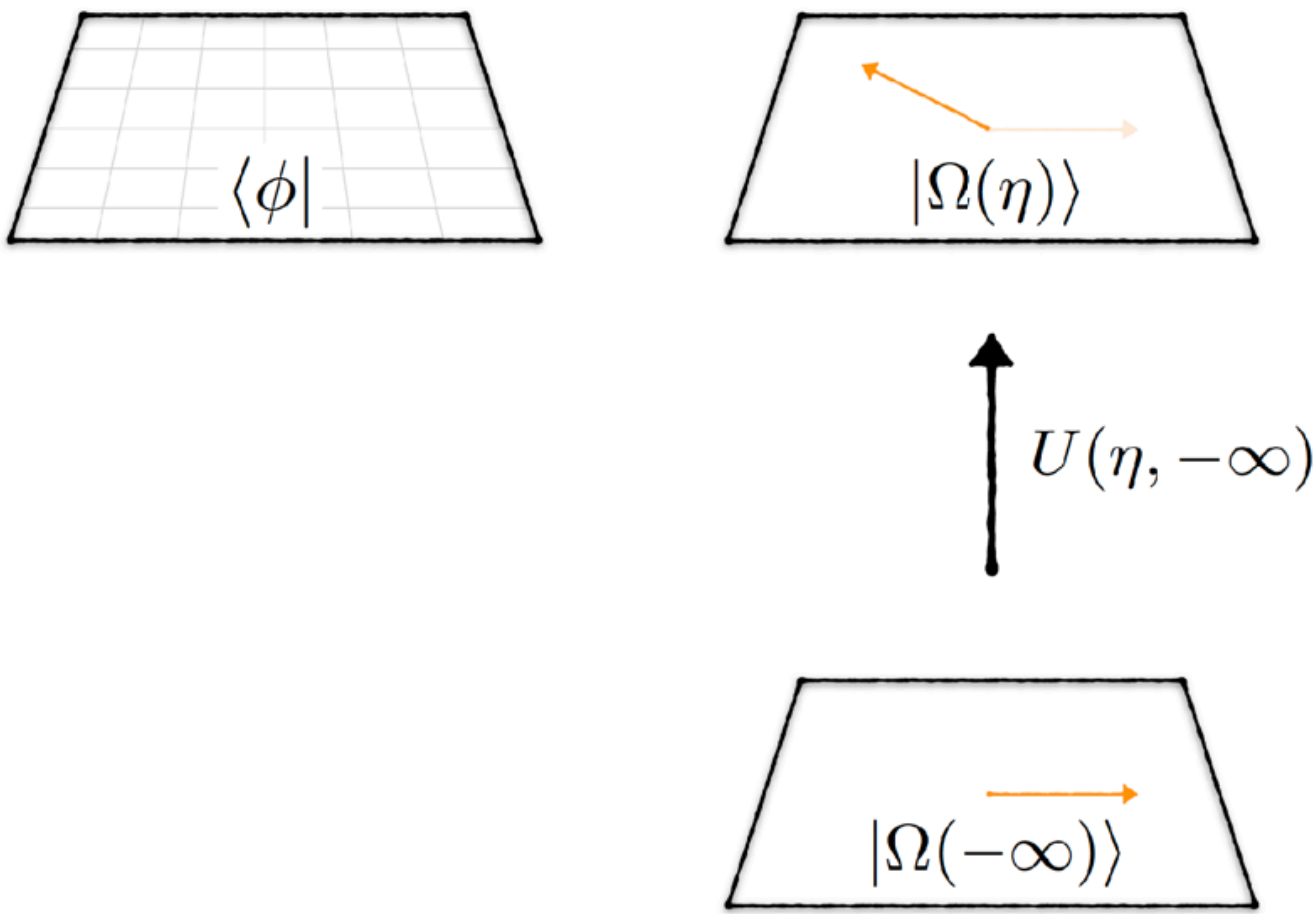
- It is a functional of the all fields in the theory (including the metric) at some time. It can be written in terms of wavefunction coefficients ψ_n

$$\Psi[\phi,\eta] = \exp\left[\sum_{n=2}^{\infty}\int_{\mathbf{k}} \psi_n(\mathbf{k}_1,\ldots,\mathbf{k}_n;\eta)\prod_a^n \phi(\mathbf{k}_a)\right]$$

- We Taylor expand $\log\Psi$ as opposed to Ψ itself so that the ψ_n are computed in terms of connected diagrams

- The field theoretic wavefunction coincides with the large volume limit of the wavefunction of the universe in canonical quantum gravity, which solves the Wheeler de Witt equation.

The wavefunction



From Ψ to correlators

- All probabilities can be computed from Ψ as in QM

$$\langle \mathcal{O} \rangle = \int d\phi \Psi^* \mathcal{O} \Psi$$

- The Ψ_n are closely related to *cosmological correlators*, which determine the statistics of the Cosmic Microwave Background and Large Scale Structures. For example

$$\langle \phi_{\mathbf{p}}(\eta_0)\phi_{-\mathbf{p}}(\eta_0)\rangle' = \frac{1}{2\,\mathrm{Re}\,\psi_2(p)},$$

$$\left\langle \prod_{a=1}^{3} \phi_{\mathbf{p}_a}(\eta_0) \right\rangle' = -2 \prod_{a=1}^{3} \frac{1}{2\,\mathrm{Re}\,\psi_2'(p_a)} \left[\psi_3(\mathbf{p}) + \psi_3(-\mathbf{p})\right],$$

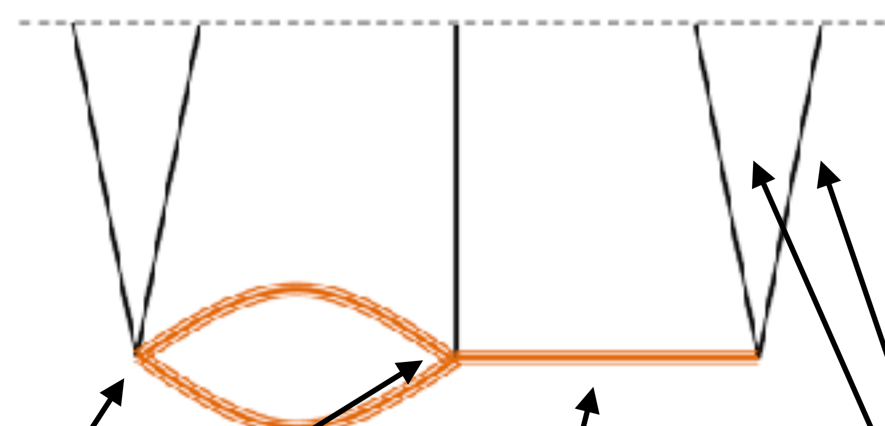
$$\left\langle \prod_{a=1}^{4} \phi_{\mathbf{p}_a}(\eta_0) \right\rangle' = -2 \prod_{a=1}^{4} \frac{1}{2\,\mathrm{Re}\,\psi_2(p_a)} \left[\left[\psi_4(\mathbf{p}) + \psi_4(-\mathbf{p})\right]\right.$$
$$\left. - \frac{\left[\psi_3(\mathbf{p}_1,\mathbf{p}_2,-\mathbf{s}) + \psi_3(-\mathbf{p}_1,-\mathbf{p}_2,-\mathbf{s})\right]\left[\psi_3(\mathbf{p}_3,\mathbf{p}_4,\mathbf{s}) + \psi_3(-\mathbf{p}_3,-\mathbf{p}_4,-\mathbf{s})\right]}{\mathrm{Re}\,\psi_2'(s)} - t - u\right].$$

Feynman Diagrams

- Given a model, the wavefunction can be computed perturbatively from a path integral

$$\Psi[\phi(\mathbf{x});\eta_0] = \int_{\text{vacuum}}^{\phi(\mathbf{x},\eta_0)} [\mathcal{D}\phi]\, e^{iS[\phi]}$$

with the following Feynman rules



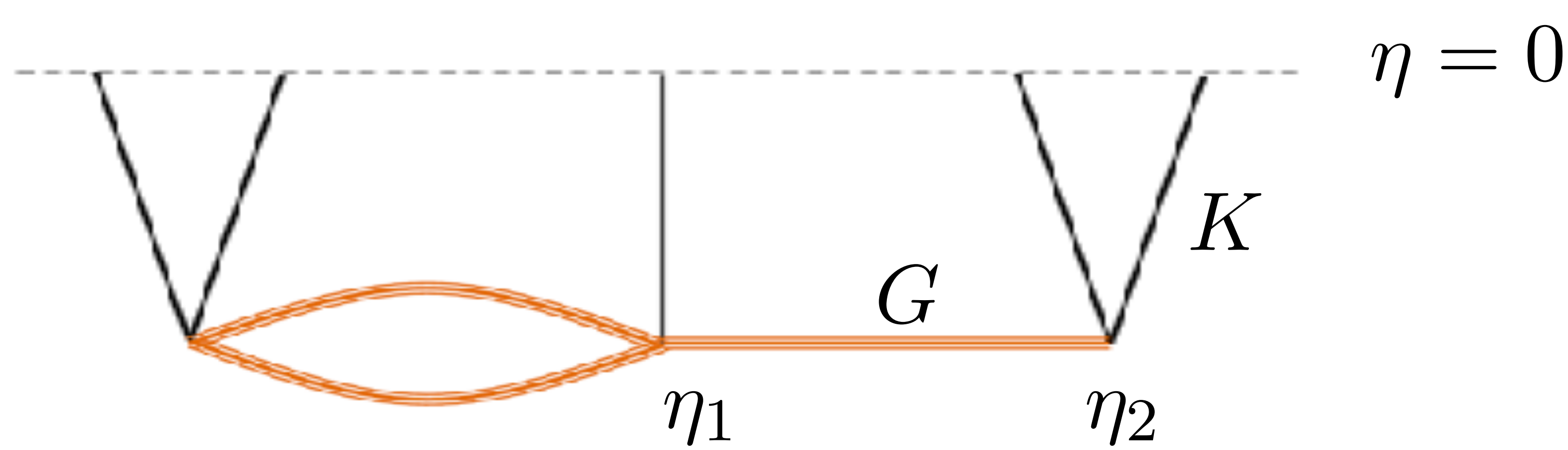
$$\psi_n = \left[\prod_A^V \int d\eta_A\, F_A\right] \left[\prod_m^I G(p)\right] \left[\prod_a^n K(k_a)\right]$$

Propagators

- Simples examples are the massless scalars (or gravitons) in dS $K = (1 - ik\eta)e^{ik\eta}$ or in Mink $K = e^{i\omega t}$

- Bulk-bulk propagator G is *Feynman propagator + boundary term* $G = \langle T\phi\phi\rangle + G_\partial = G_F + G_\partial$ where G_∂ solves the homogeneous E.o.M and ensures G vanishes at $\eta = \eta_0$.

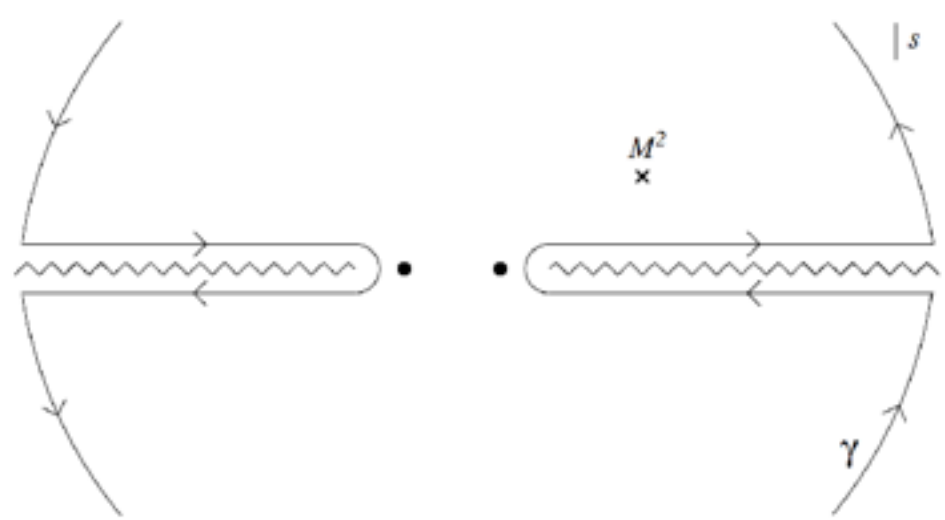
- The boundary term G_∂ is the root case of all differences with amplitudes, e.g. in the analytic structure and in cutting rules



Analyticity and causality

- There is a well-known connection between causality and analyticity, which leads to powerful UV/IR sum rules, analogous to the Kramers Kronig “dispersion relation”.

- Ex: operators commuting outside the light cone implies the 2-to-2 amplitude is analytic in Mandelstam s at fixed t



- What is the analytic structure of wavefunction coefficients?

- Here are some results [Goodhew, Lee, Melville & Pajer ’22]

Off-shell wavefunction

- Analytic in what?! We need to go off-shell.

- Off-shell wavefunction coefficients are the F-transform of amputated (i.e. acted on eq. of motions for all fields), connected in-out Green's functions

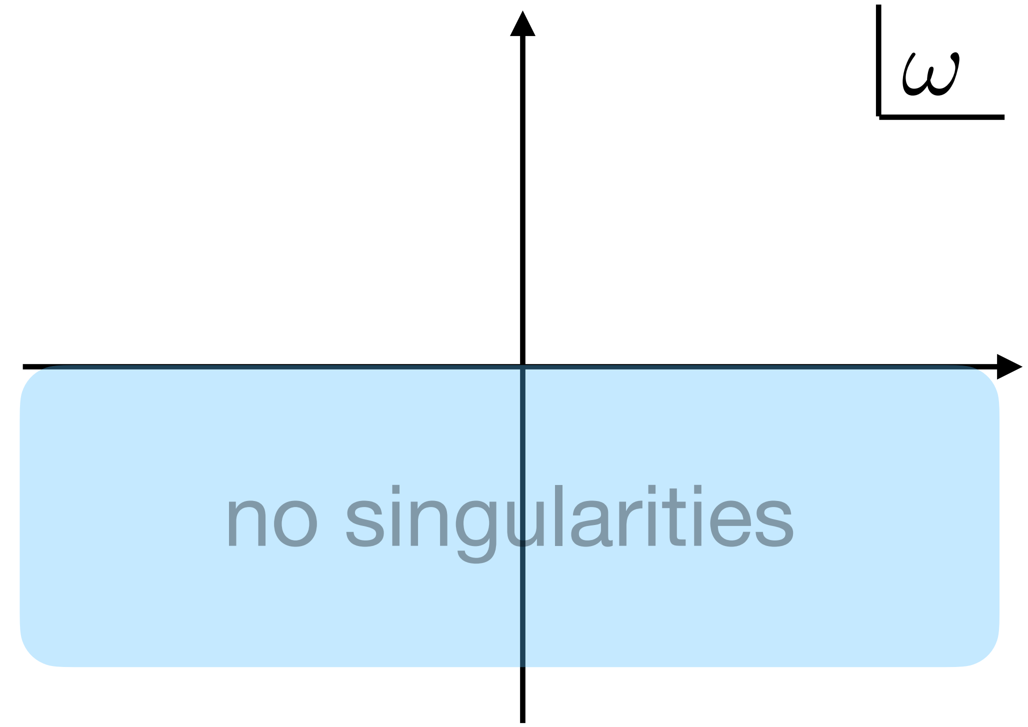
$$\psi_n\left(\{\omega\},\{\mathbf{k}\}\right)=\left[\prod_{j=1}^{n}\int_{-\infty}^{0}d\eta_j\,K_\nu(\omega_a,\eta_a)\right]\,G^{\text{amp.con.}}_{\mathbf{k}_1\ldots\mathbf{k}_n}(t_1,...,t_n)$$

$$\langle\phi(0)=0|\,T\prod_{a}^{n}\hat{\phi}_{\mathbf{k}}(t_a)|\Omega_{\text{in}}\rangle_{con}=G_{\mathbf{k}_1\ldots\mathbf{k}_n}(t_1,...,t_n)\;\delta_D^{(3)}\left(\sum_{a}^{n}\mathbf{k}_a\right)$$

- where K are mode functions in any FLRW spacetime with Bunch-Davies initial conditions.

- This is construction is non-perturbatively and reminiscent of LSZ.

Analyticity



$$\psi_n(\omega) \sim \int_{-\infty}^{0} dt\, e^{i\omega t}\, G(t)$$

- Time integral for $\Psi[\phi,\eta_0]$ stops at η_0 because of causality

- Then ψ_n are analytic in ω in the lower-half complex plane because the integral is even more convergent. *This is true non-perturbatively*

- *If* Hermitian analyticity is valid non-perturbatively, this extends to the upper-half plane $\psi_n(\omega^*)=\psi_n^*(\omega)$

- *Singularities only on the negative real axis*

The energy-conservation condition

- Results below are proven in Minkowski with some comments on dS.

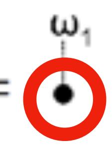
- *The energy conservation condition states that the location of singularities of a wavefunction coefficient corresponds to the vanishing of the partial energy of a connected sub diagram* [Agui-Salcedo, Melville, Lee & EP ’22]

- At tree level all these singularities are simple poles in Minkowski (and higher order poles of massless scalars in de Sitter)

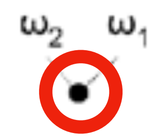
- All residues of partial energy singularities are fixed by unitarity [Jazayeri, EP & Stefanyszyn ’21] in the form of the Cosmological Optical Theorem [Goodhew, Jazayeri, EP ’20]

Tree-level examples

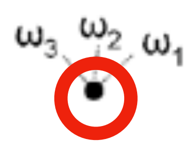
- Total energy singularities

$$\psi_1(\omega_1) = $$ 

$$= \frac{F_1(\omega_1)}{\omega_1},$$

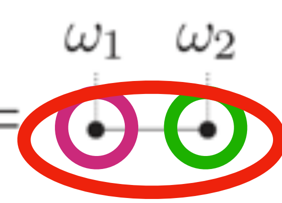
$$\psi_2(\omega_1, \omega_2; \mathbf{k}_1) = $$ 

$$= \frac{F_2(\omega_1, \omega_2, \mathbf{k}_1)}{\omega_1 + \omega_2},$$

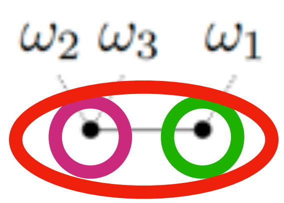
$$\psi_3(\omega_1, \omega_2, \omega_3; \mathbf{k}_1, \mathbf{k}_2) = $$ 

$$= \frac{F_3(\omega_1, \omega_2, \omega_3; \mathbf{k}_1, \mathbf{k}_2)}{\omega_1 + \omega_2 + \omega_3}.$$

- Partial energy singularities

$$\psi_2(\omega_1, \omega_2; \mathbf{k}) = $$ 

$$= \frac{F_L(\omega_1; \mathbf{k}) F_R(\omega_2; \mathbf{k})}{\underline{(\omega_1 + \Omega_k)}\,\underline{(\omega_2 + \Omega_k)}\,\underline{(\omega_1 + \omega_2)}}.$$

$$\psi_3 = $$ 

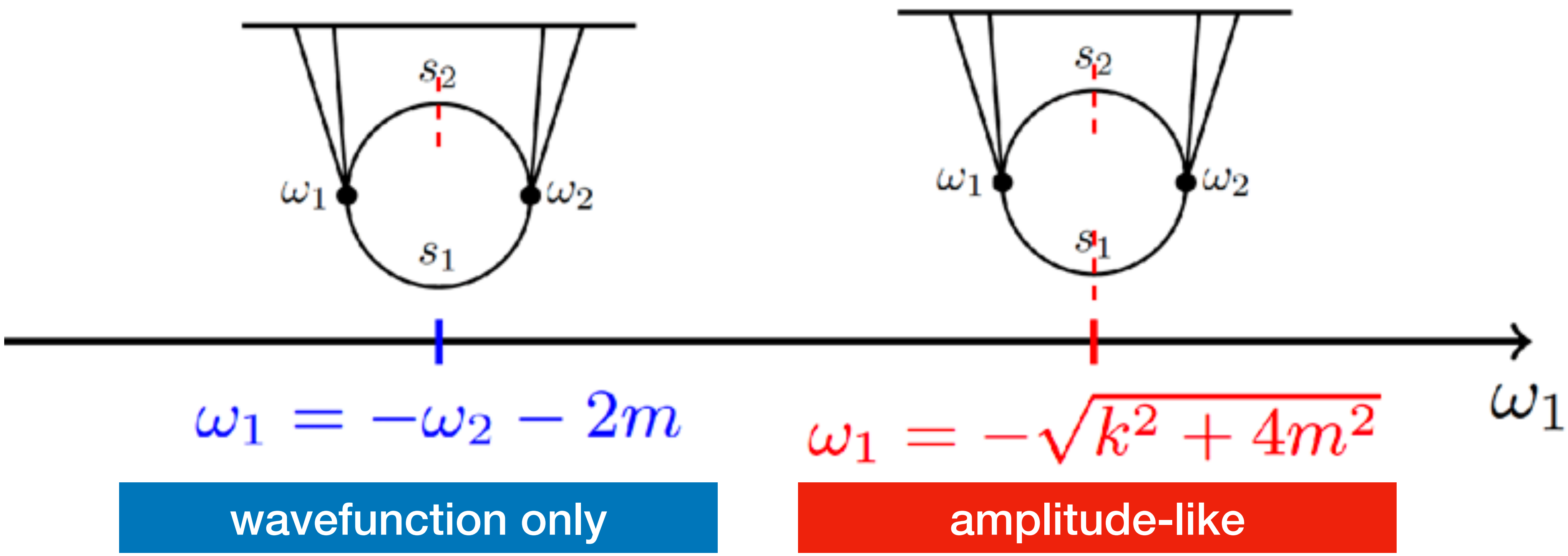
$$= \frac{\tilde{F}_L(\omega_2, \omega_3; \mathbf{k}_1) \tilde{F}_R(\omega_1; \mathbf{k}_1)}{\underline{(\omega_1 + \Omega_{k_1})}\,\underline{(\omega_2 + \omega_3 + \Omega_{k_1})}\,\underline{(\omega_1 + \omega_2 + \omega_3)}}$$

Loop level examples

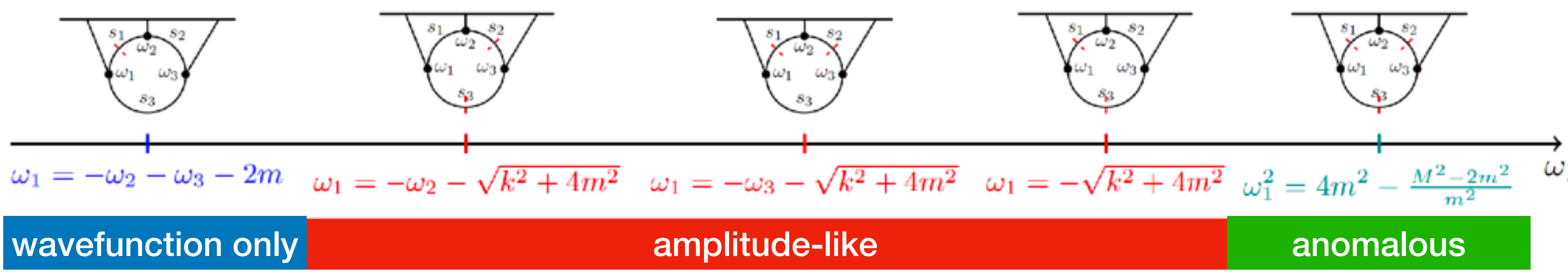
- At loop level
  - for massless particles every pole becomes a branch point
  - for massive theories for each pole there is an infinite series of branch points at successively more negative ω
- Recalling that the wavefunction propagator is $G = G_F + G_\partial$, the singularities of ψ_n can be classified into two classes [Lee 23]:
  - Amplitude-like singularities have analogous singularities in amplitude Feynman diagrams and correspond to cutting internal lines and putting them on-shell. These come form $G_F \subset G$
  - Wavefunction-only singularities don't have any analogue in amplitudes and correspond to cutting a single line. These come from all the $G_\partial \subset G$

Loop example

- A simple example is [Lee 23]

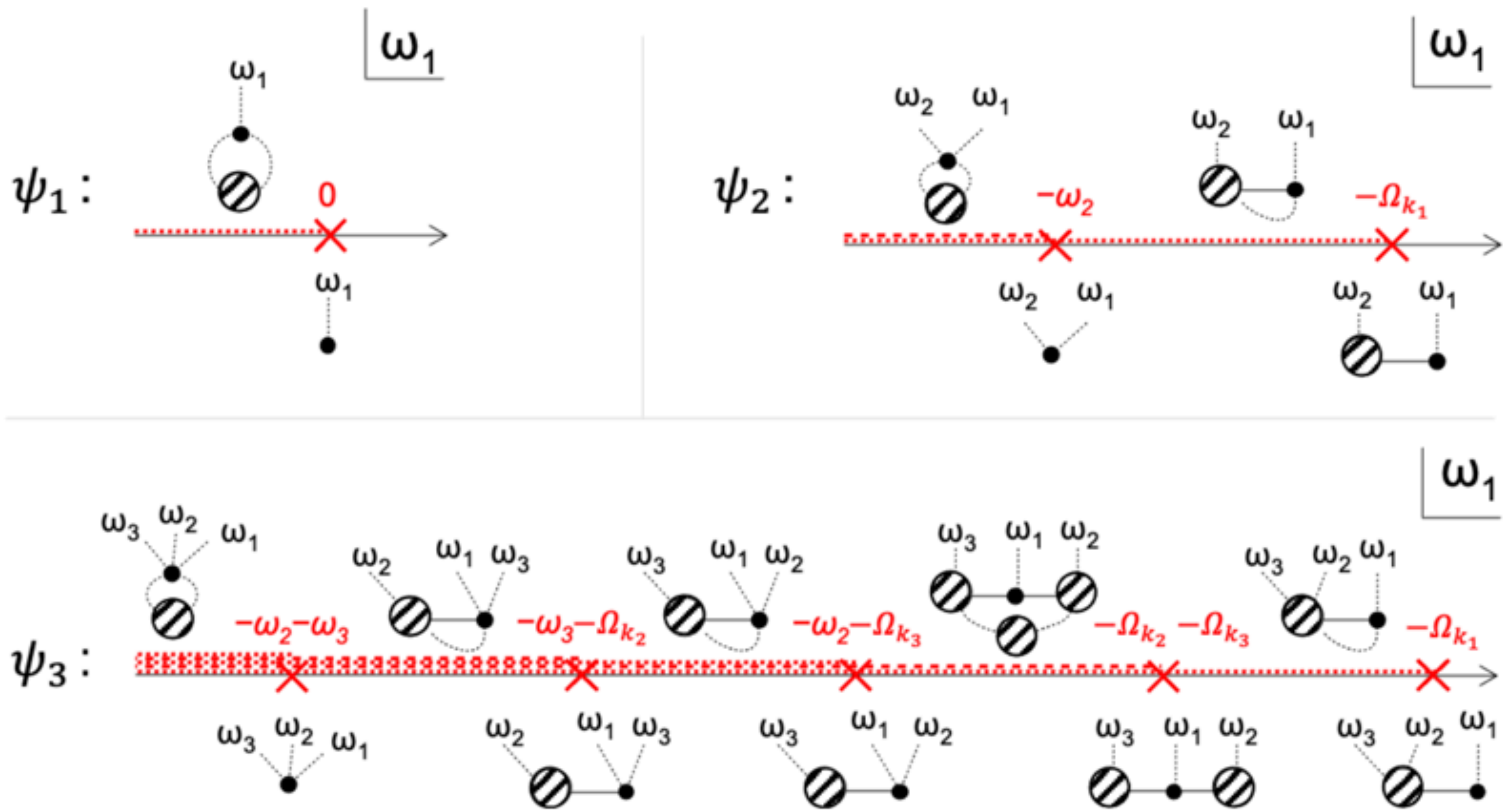


- Another example displays anomalous thresholds in ψ_n



Normal thresholds

- In summary, singularities in ψ_n occur only on the negative real ω axis where the energy of a perturbative subdiagram vanishes (energy-conservation condition)



The surprise

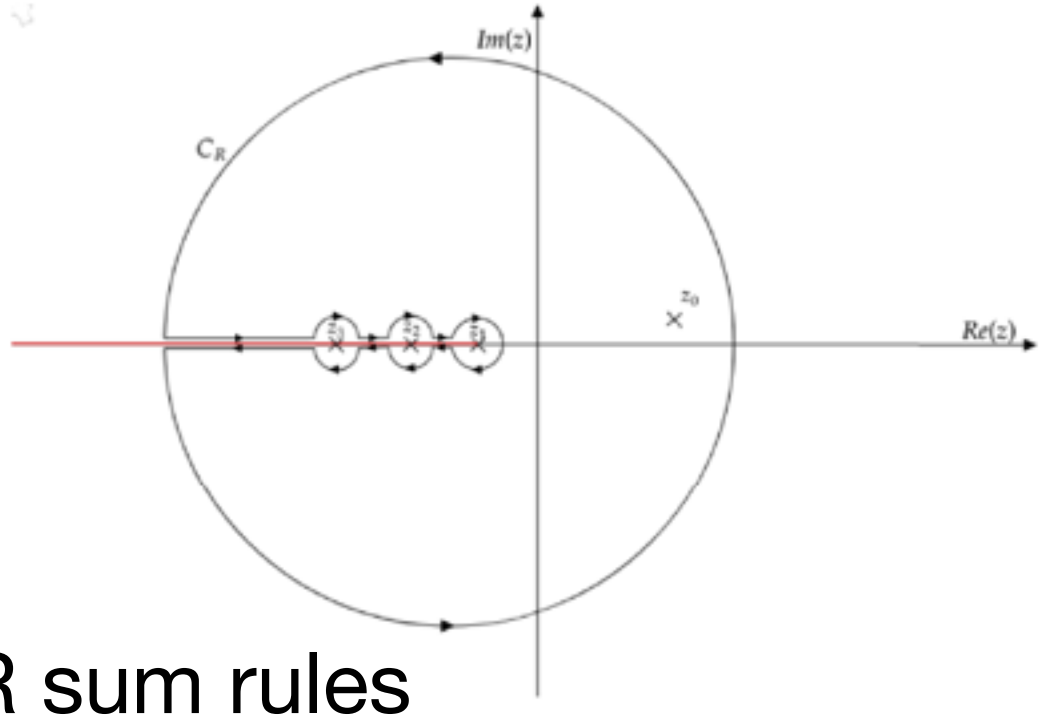
- We introduce the wavefunction a somewhat simpler object to study that correlator, but what we observe are correlators

- It turns out that in the few cases studied, the wavefunction-only singularities cancel exactly when computing correlators!

- Some of these cancellations can be understood in analogy to the so-called KLN “theorem” for amplitudes [Agui-Salcedo & Melville 23]

- We will see another interpretation of this result

UV/IR sum rules



- By Cauchy's theorem we can write UV/IR sum rules

$$\omega_T \psi_{EFT}(\omega_i, \mathbf{k}_j) = \int_{-\infty}^{0} \frac{d\omega}{2\pi i} \frac{\mathrm{disc}(\omega_T \psi_{\mathrm{UV}}(\omega, \omega_{i\neq 1}, \mathbf{k}_j))}{\omega - \omega_1} + \operatorname*{Res}_{\infty}(\frac{\omega_T \psi_{\mathrm{UV}}(\omega, \omega_{i\neq 1}, \mathbf{k}_j)}{\omega - \omega_1}).$$

- The LHS can computed in a low-energy EFT. The RHS depends on the full UV theory.

- This fixes all Wilson coefficients in the EFT, including total derivatives and terms proportional to the eq of motion.


Unitarity

Unitary time evolution

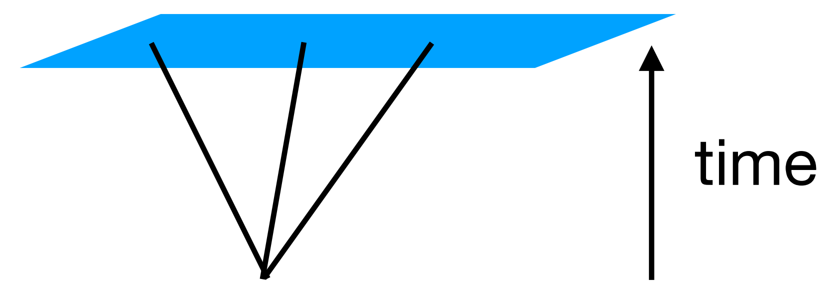
- In Quantum Mechanics we compute probabilities, which must be between 0 and 1 to make sense
- This requires the positive norm of states in the Hilbert space *and* Unitary time evolution, $UU^\dagger=1$. Colloquially this is the *conservation of probabilities*
- The consequences of unitarity for particle physics amplitudes were discover over *60 years ago*: the Optical theorem and Cutkosky Cutting Rules.
- In cosmology we don't see the time evolution, so how can we see it's unitary?!

The Cosmological Optical Theorem (COT) [Goodhew, Jazayeri, EP ’20]

- From unitarity, $UU^\dagger=1$, we found infinitely many relations.

- The simplest applies to contact n-point functions

$$\psi_n(\{\omega\},\{\mathbf{k}\})+\psi_n^*(\{-\omega\},\{-\mathbf{k}\})=0$$



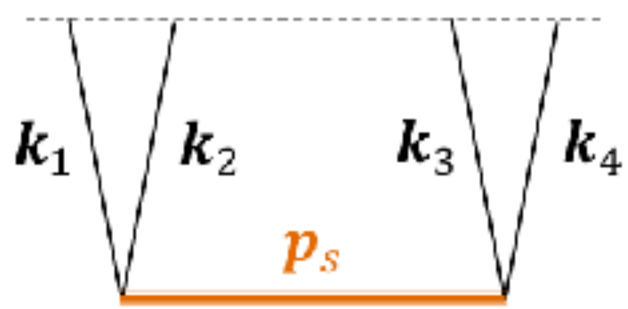
- It follows from unitarity time evolution, but the equation does not involve time! Time “emerges” at boundary as in holography…

- This is a Cosmological Optical Theorem (COT) and can be interpreted as fixing a “discontinuity”

$$\mathrm{Disc}\psi_n=\psi_n(\{\omega\},\{\mathbf{k}\})+\psi_n^*(\{-\omega\},\{-\mathbf{k}\})=0$$

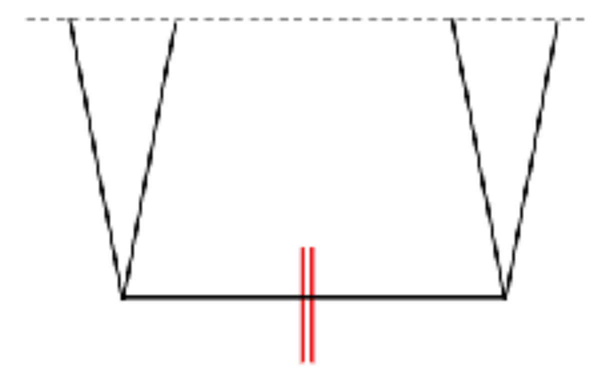
Exchange diagrams

- The next simplest case is a 4-particle exchange diagram (trispectrum). The Cosmo Optical Theorem (COT) is



$i\,\mathrm{Disc}_{\boldsymbol{p}_s}\left[i\psi^{(s)}_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{k}_4}\right]$

$=$



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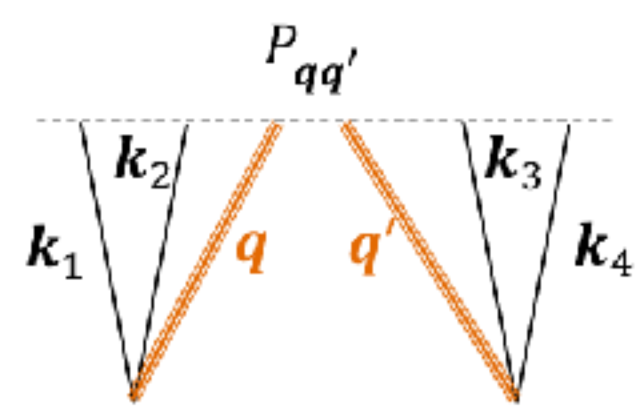
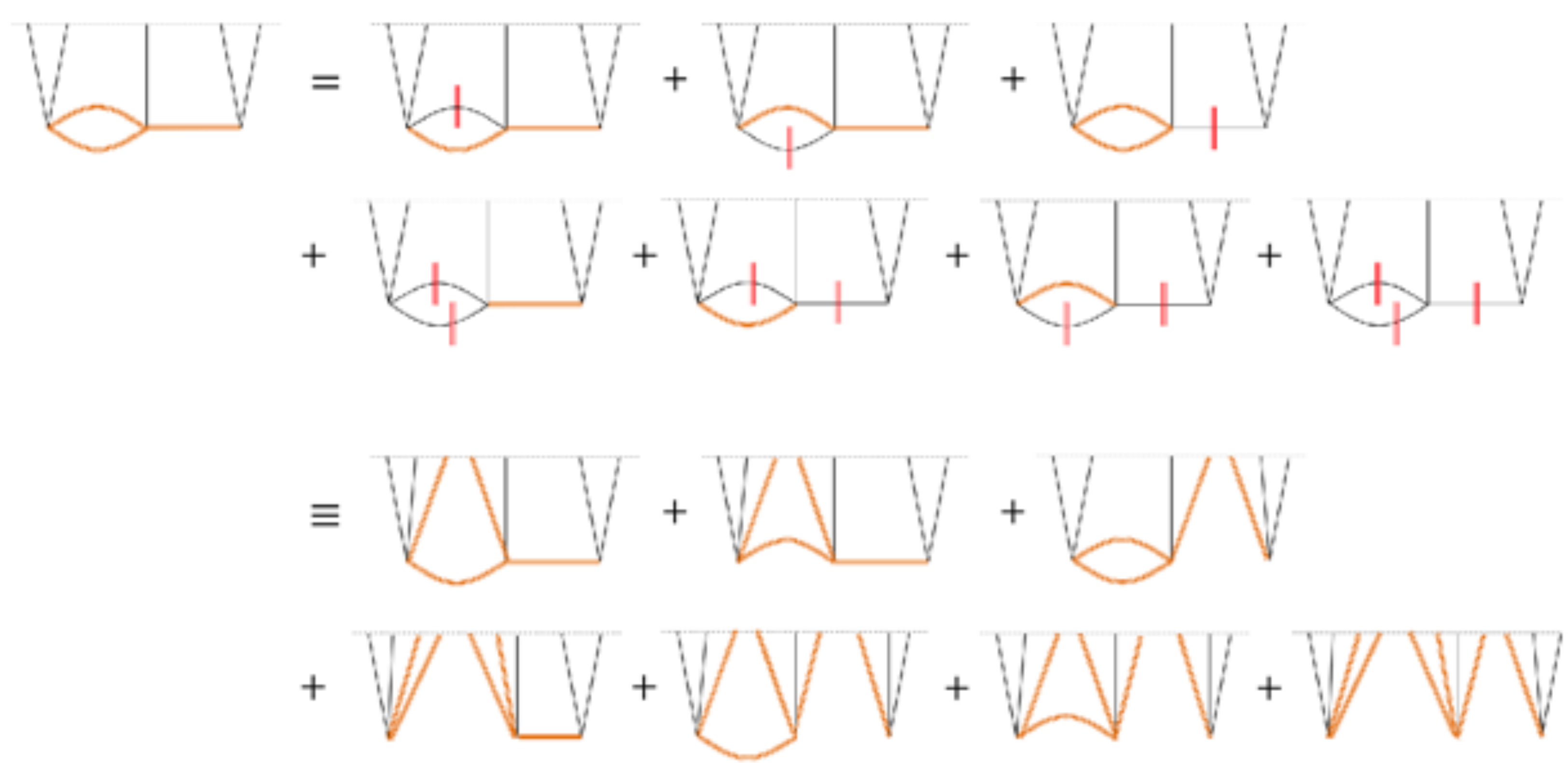


$i\,\mathrm{Disc}_{\boldsymbol{q}}\left[i\psi_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{q}}\right]\;P_{\boldsymbol{q}\boldsymbol{q}'}\;\;i\,\mathrm{Disc}_{\boldsymbol{q}'}\left[i\psi_{\boldsymbol{q}'\boldsymbol{k}_3\boldsymbol{k}_4}\right]$

General diagrams

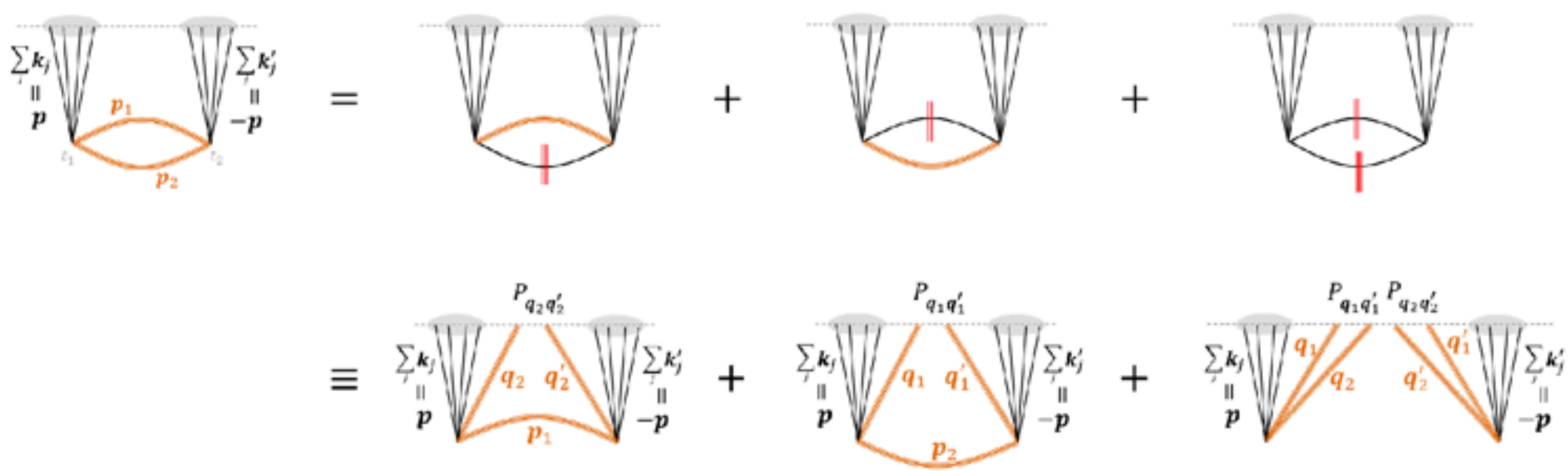
- These relations are valid to *all order in perturbation theory to any number of loops for fields of any mass and spin and arbitrary interactions (around any FLRW admitting a Bunch Davies initial condition)* [Goodhew, Jazayeri & EP '21; Melville & EP '21]

$$i \underset{\substack{\text{internal}\\ \text{lines}}}{\text{disc}} \left[i\,\psi^{(D)} \right] = \sum_{\text{cuts}} \left[\prod_{\substack{\text{cut}\\ \text{momenta}}} \int P \right] \prod_{\text{subdiagrams}} (-i) \underset{\substack{\text{internal \&}\\ \text{cut lines}}}{\text{disc}} \left[i\,\psi^{(\text{subdiagram})} \right] ,$$



Loop corrections

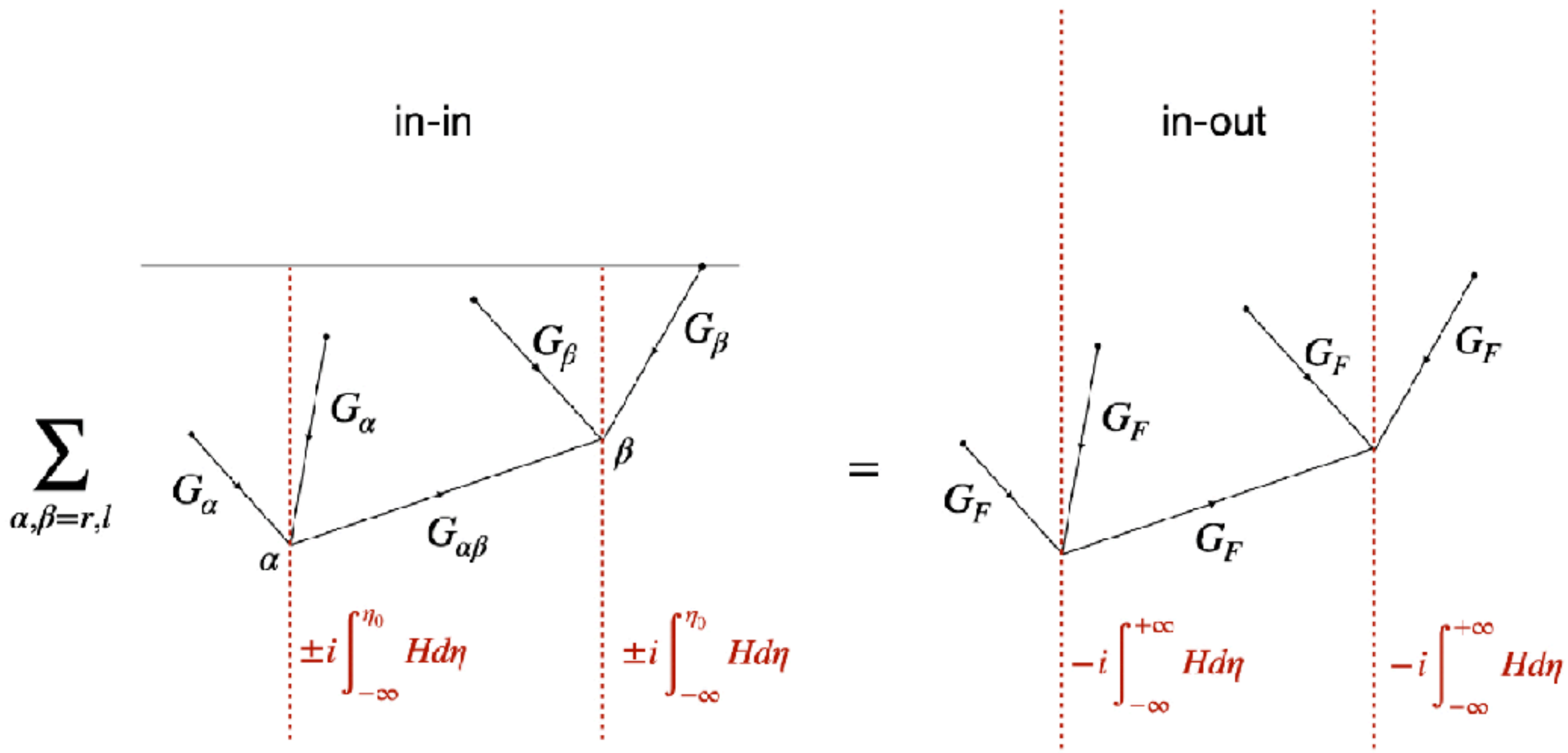
- Unitarity gives us also *loop corrections*! For example we compute the leading 1-loop corrections for the power spectrum in the EFT of inflation, from tree-level results.



$$i\mathrm{Disc}\left[i\psi^{\text{1-loop}}_{\mathbf{k}_1\mathbf{k}_2}\right] = \frac{H^2}{f_\pi^4}\,\frac{ik^3}{480\pi}\,\frac{(1-c_s^2)^2}{c_s^4}\left[(4\tilde{c}_3+9+6c_s^2)^2+15^2\right]$$

in-in = in-out

Main message

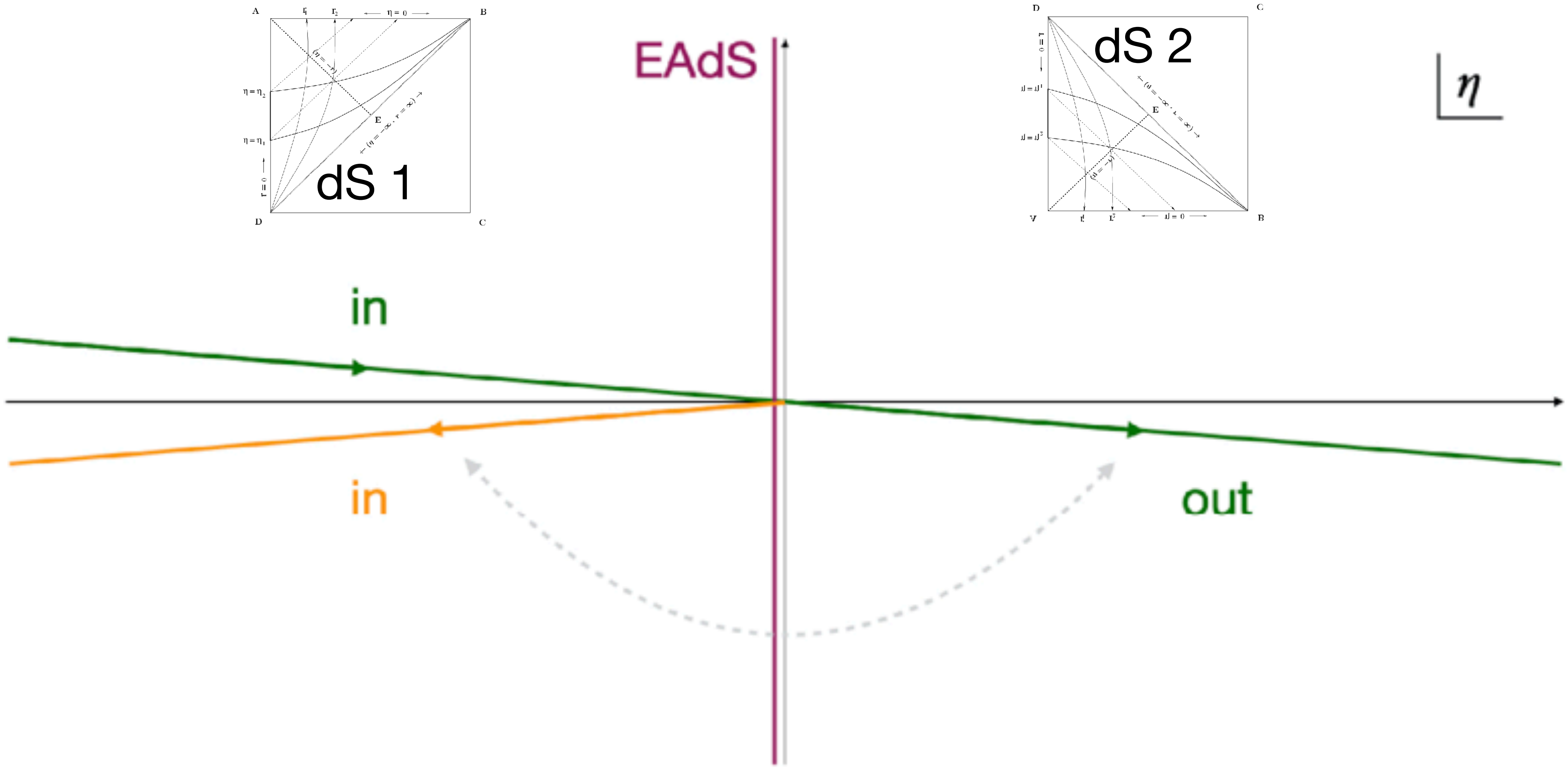


- Cosmological correlators in de Sitter (and Minkowski) can be computed using the *in-out formalism*, using only the familiar time-ordered (Feynman) propagator

- Assumptions: interactions are IR finite; evolution is unitary (closed system in Bunch Davies); any number of fields of any spin and mass.

- This leads to significant simplifications practically (many applications: new recursion relations, cutting rules, pole bagging) and conceptually (S-matrix technology, dS S-matrix, non-perturbative optical theorem)

Main message, take 2



- in-in contour can be deformed into in-out by adding a *second spacetime* (contracting Poincare' patch) that prepares the bra

- We get a straight contour, just like in (Euclidean)AdS and in Minkowski amplitudes

In-in correlators

- Let's slightly generalise the definition of an in-in correlator allowing for un-equal time inside a time ordering

$$B_{\text{in-in}} \equiv \langle 0 | \, \bar{T} \left[e^{+i \int_{-\infty(1+i\epsilon)}^{t_0} H_{\text{int}} dt} \right] T \left[\mathcal{O}(\{t, \mathbf{x}\}) e^{-i \int_{-\infty(1-i\epsilon)}^{t_0} H_{\text{int}} dt} \right] |0\rangle'$$

- The $i\epsilon$ rotation of the countour selected the Fock vacuum as initial state (Bunch-Davies state) in the infinite past by turning off interactions adiabatically

- time t_0 is *any* time after all operator insertions

In-out correlators

- We define an in-out correlator in dS/Mink as the following object

$$B_{\text{in-out}} \equiv \frac{\langle 0|\, T\left[\mathcal{O}(\{t,\mathbf{x}\})e^{-i\int_{-\infty(1-i\epsilon)}^{+\infty(1-i\epsilon)} H_{\text{int}}dt}\right]|0\rangle'}{\langle 0|\, T\left[e^{-i\int_{-\infty(1-i\epsilon)}^{+\infty(1-i\epsilon)} H_{\text{int}}dt}\right]|0\rangle'}$$

- Time integral goes over $-\infty < \eta < +\infty$, encompassing the standard expanding Poincare' patch, $-\infty < \eta < 0$, and an extra contracting Poincare' patch, $0 < \eta < +\infty$.

- The $i\epsilon$ rotation of the contour turns off interactions adiabatically at the past and future null "boundary"

- Denominator removes the vacuum-to-vacuum bubbles so: $\langle 1 \rangle_{\text{in-out}} = 1$

In-in = in-out

- Claim: for all IR-finite interactions, for which the time integral converges around $\eta = 0$, we have $B_{\text{in-in}} = B_{\text{in-out}}$

- This is a known fact in Minkowski. Here we claim it applies to de Sitter too (and probably to any accelerated FLRW but we haven't checked yet)

- We'll provide a formal argument and some explicit checks

A formal argument

- A formal argument relies on the observation that infinite time evolution changes the ground state only by a phase

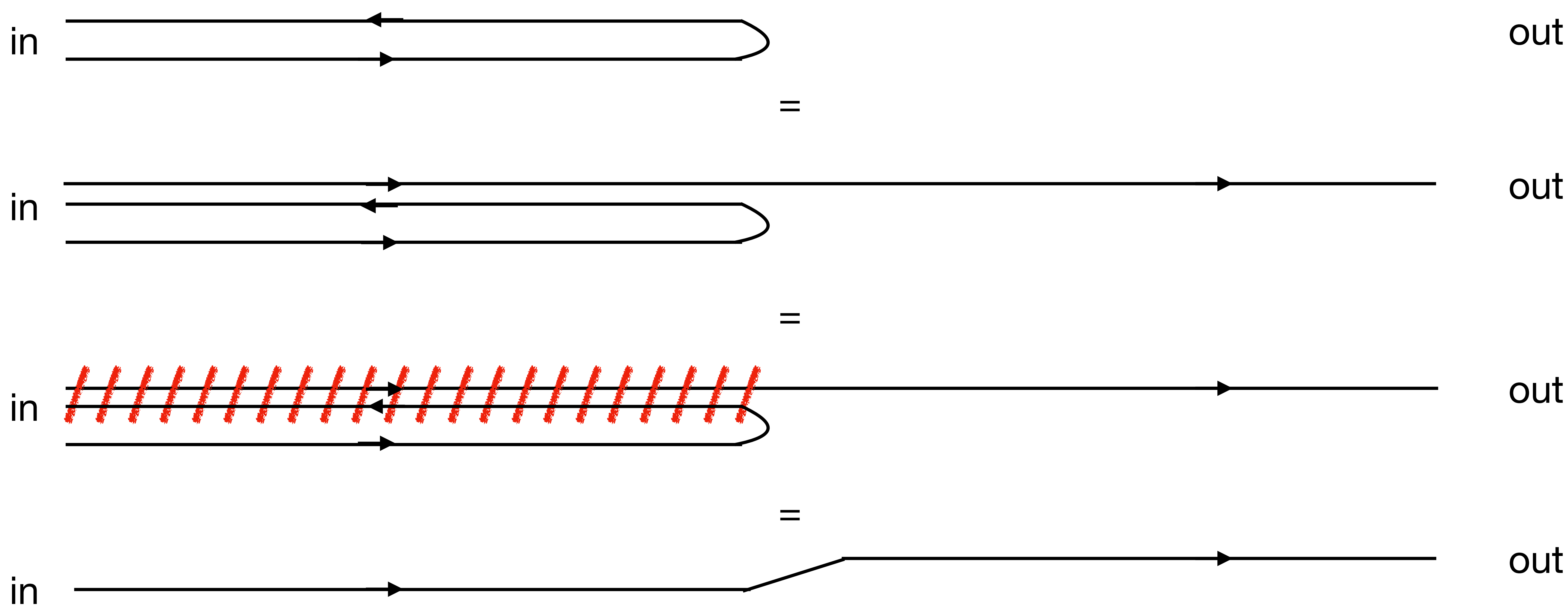
$$U(+\infty,-\infty)\left|0\right\rangle = \left|0\right\rangle \left\langle 0\right| U(+\infty,-\infty)\left|0\right\rangle$$

- This can be checked to all orders in perturbation theory by projecting on any excited state: the result is a derivative of a delta function of energy conservation and has zero support on physical perturbations. Then

$$\begin{aligned}
B_{\text{in-out}} &= \frac{\langle T\left[\prod_a^n \phi(t_a) e^{-i\int_{-\infty_-}^{+\infty_-} H_{\text{int}}dt}\right]\rangle'}{\langle T\left[e^{-i\int_{-\infty_-}^{+\infty_-} H_{\text{int}}dt}\right]\rangle'} \\
&= \langle U^\dagger(+\infty_+,-\infty_+) T\left[\prod_a^n \phi(t_a) e^{-i\int_{-\infty_-}^{+\infty_-} H_{\text{int}}dt}\right]\rangle' \\
&= \langle \bar{T}\left[e^{+i\int_{-\infty_+}^{+\infty_+} H_{\text{int}}dt}\right] T\left[e^{-i\int_{t_0}^{+\infty_-} H_{\text{int}}dt}\right] T\left[\prod_a^n \phi(t_a) e^{-i\int_{-\infty_-}^{t_0} H_{\text{int}}dt}\right]\rangle' \\
&= \langle \bar{T}\left[e^{+i\int_{-\infty_+}^{t_0} H_{\text{int}}dt}\right] U^\dagger(+\infty,t_0)U(+\infty,t_0) T\left[\prod_a^n \phi(t_a) e^{-i\int_{-\infty_-}^{t_0} H_{\text{int}}dt}\right]\rangle' \\
&= B_{\text{in-in}}\,,
\end{aligned}$$

A formal argument, graphically

- In terms of path integral contours this is simply



Applications

Correlator cutting rules

- In dS and Minkowski we can derive cutting rules for correlators following the derivation of Cutkosky cutting rules for amplitudes (with the additional complication of external legs)

- The starting point is the field identity

$$\sum_{r=0}^{n}(-1)^r \sum_{\sigma\in\Pi(r,n-r)} \bar{T}\left[\mathcal{O}_{\sigma(1)}(t_{\sigma(1)})...\mathcal{O}_{\sigma(r)}(t_{\sigma(r)})\right] T\left[(\mathcal{O}_{\sigma(r+1)}(t_{\sigma(r+1)})...\mathcal{O}_{\sigma(n)}(t_{\sigma(n)})\right] = 0 .$$

- This leads to infinitely many propagator identities, which in turn become *correlator cutting rules*

- These appear to be equivalent to the wavefunction cutting rules of , the advantage is that one works directly with observables, i.e. correlators, rather than with the more primitive wavefunction

Propagator identities

- We use the identity to re-write in-out correlators

$$\mathcal{O}_1 = \phi(\mathbf{x}_1, t_0)^m\,, \qquad \mathcal{O}_2 = \phi(\mathbf{x}_2, t_0)^{n-m}\,, \qquad \mathcal{O}_3 = H_{\rm int}(t)\,,$$

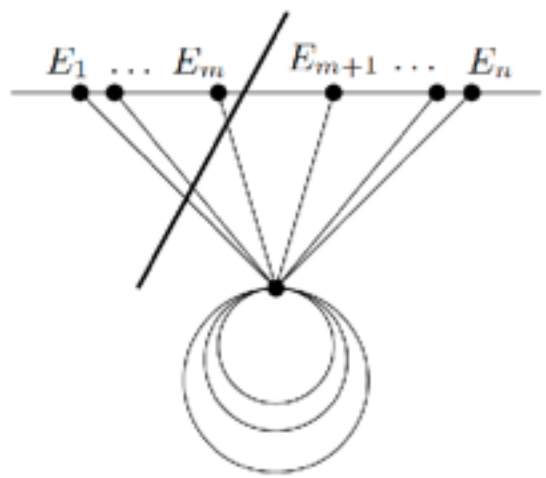
$$\mathrm{Im}\left\{\langle T[\phi^n H_{\rm int}]\rangle - \langle \phi^m T[\phi^{n-m} H_{\rm int}]\rangle - \langle \phi^{n-m} T[\phi^m H_{\rm int}]\rangle\right\} \simeq 0,$$

- Changing integration variables and using the properties of the Feynman propagator this becomes

$$\begin{aligned} B_n^c(\{E_i\}_{i=1}^n) - \frac{1}{2}\left[B_n^c(\{E_i\}_{i=1}^n) + (-1)^m B_n^c(\{-E_i\}_{i=1}^m, \{E_i\}_{i=m+1}^n)\right] \\ -\frac{1}{2}\left[B_n^c(\{E_i\}_{i=1}^n) + (-1)^{n-m} B_n^c(\{E_i\}_{i=1}^m, \{-E_i\}_{i=m+1}^n)\right] = 0 \end{aligned}$$

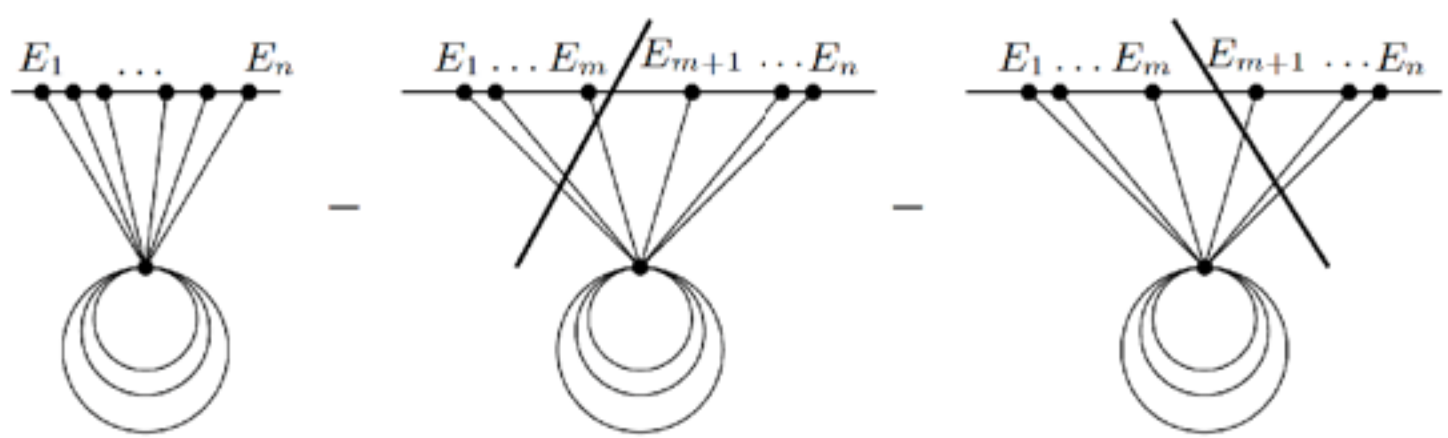
Contact diagrams

- There is a handy graphic notation



$$= \frac{1}{2}\left[B_n^c(\{E_i\}_{i=1}^n) + (-1)^m B_n^c(\{-E_i\}_{i=1}^m, \{E_i\}_{i=m+1}^n)\right].$$

- The contact identity becomes

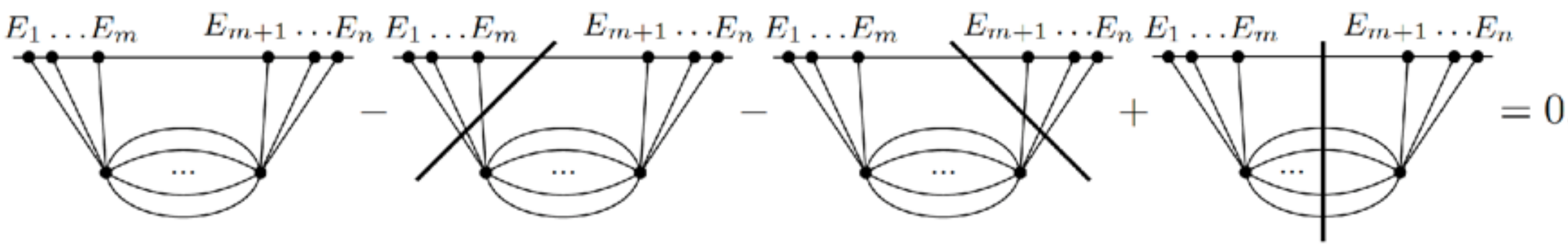


$$= 0.$$

$$B_n^c(\{E_i\}_{i=1}^n) + (-1)^n B_n^c(\{-E_i\}_{i=1}^n) = 0.$$

Correlator cutting rules: contact diagrams

- For exchange diagrams



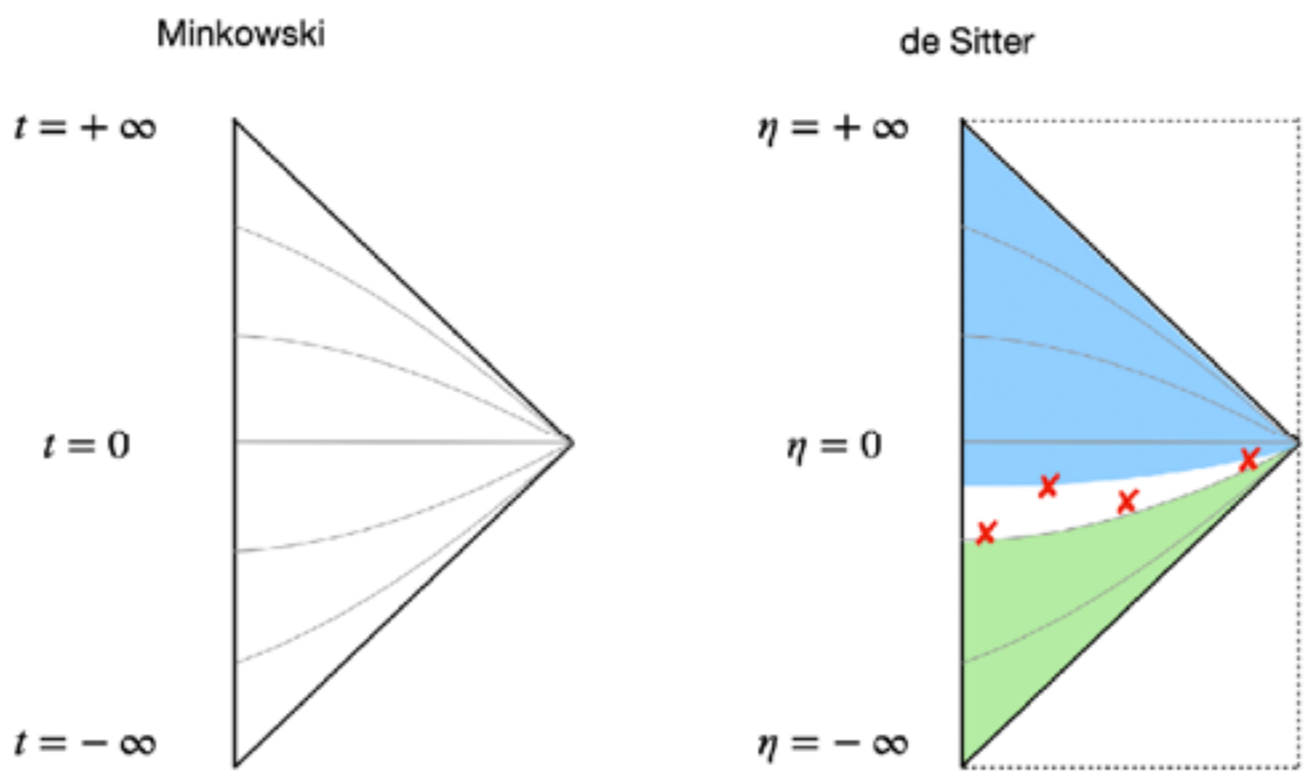
$$
\begin{aligned}
&B_n^{\mathrm{ex},s}(\{E_i\}_{i=1}^n) + (-1)^n B_n^{\mathrm{ex},s}(\{-E_i\}_{i=1}^n) = \qquad (5\\
&= 2\int_{\vec{p}_1\ldots\vec{p}_{L+1}} \frac{B^{\mathrm{c,cut}}_{m,L+1}(\{E_i\}_{i=1}^m, \{y_i\}_{i=1}^{L+1})\, B^{\mathrm{c,cut}}_{n-m,L+1}(\{E_i\}_{i=m+1}^n, \{y_i\}_{i=1}^{L+1})}{\prod_{i=1}^{L+1} P(y_i)}.
\end{aligned}
$$

- We haven’t yes found a good combinatorial structure to write the most general correlator cutting rule

Scattering in de Sitter

III

dS scattering



- The in-out formalism suggests a natural definition of a scattering matrix in "extended" de Sitter

$$S_{n,n'} = \langle n' | \, U(+\infty, -\infty) \, |n\rangle = \langle n' | \, T e^{-i\int_{-\infty}^{+\infty} H_{\rm int}(\eta) d\eta} \, |n\rangle$$

- In and out states are tensor products of unitary dS irrupts

$$|n\rangle = \bigotimes_a^n |\Delta_a, \mathbf{k}_a, s_a, \sigma_a\rangle$$

Conceptual problems?

- Previous proposals of a dS scattering matrix are in [Marolf Morrison & Srednicki ’12; Melville & Pimentel ’23]

- Common criticisms and difficulties:

  - IR divergences prevent a dS S-matrix. Possible, we assume that derivatives and massive field cancel IR divergences (as e.g. for a shift-symmetric massless field)

  - Particles are unstable so no asymptotic states. Our $i\epsilon$ prescription turns on/off interaction adiabatically at $\eta = \pm\infty$

  - blue-shifted particles near null infinity (the “big bang”) lead to large backreaction. This is a coordinate artefact. In global coordinates a particle can cross the “big bang”

  - particle creation prevents an out state at $\eta = 0$. Possibly, but we work at $\eta = \pm\infty$

dS amplitudes

- Since S-matrix elements don’t have only an energy conserving delta function, we define amplitudes by

$$S_{n,n'} = \langle n', +\infty | n, -\infty \rangle$$

- Using a “relativistic normalization” we define

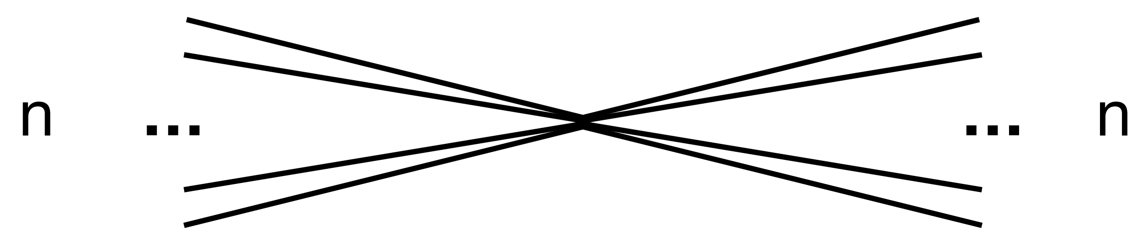
$$|\Delta, \mathbf{k}\rangle = \sqrt{2|\mathbf{k}|} a^\dagger_{\mathbf{k}} |0\rangle$$

- Notice we don’t factor out the “energy conserving” Dirac delta

$$\langle f | U(+\infty, -\infty) - 1 | i \rangle = i(2\pi)^4 \delta^{(3)}(\mathbf{k}_{\rm in} - \mathbf{k}_{\rm out}) A_{if} \,,$$

Contact dS amplitudes

- Let's compute the simplest process: contact scattering of n conformally coupled scalars ($m^2 = 2H^2$) ($n + n' = 4$ is Minkowski amplitude)

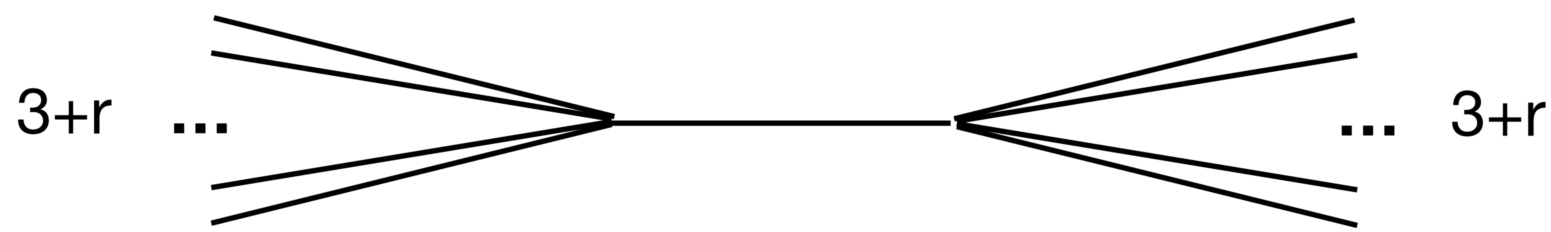


$$A_{nn'} = -\lambda\,(-iH\partial_{E_T})^{n+n'-4}\delta(E_T)\,.$$

$$E_T = -\sum_a^n |\mathbf{k}_a| + \sum_b^{n'} |\mathbf{k}_b|$$

Exchange diagram

- For an exchange diagram of cc-scalars mediated by a cc-scalar we find ($r = 1$ is just Minkowski amplitude)



$$A_{3+r,3+r} = \frac{\lambda^2 H^{2r}}{2} \sum_{l=0}^{r} b_l \frac{(k_{\rm in} - E_{\rm in})^{1+r-l} + (k_{\rm in} + E_{\rm in})^{1+r-l}}{2k_{\rm in}(-E_{\rm in}^2 + k_{\rm in}^2)^{1+r-l}} \partial^{r+l} \delta(E_{\rm in} - E_{\rm out})$$

$$E_{\rm in} = \sum_{a=1}^{3+r} |\mathbf{k}_a| \,, \qquad E_{\rm out} \equiv \sum_{a=4+r}^{6+2r} |\mathbf{k}_a| \,,$$

$$\mathbf{k}_{\rm in} = \sum_{a=1}^{3+r} \mathbf{k}_a \qquad \mathbf{k}_{\rm out} \equiv \sum_{a=4+r}^{6+2r} \mathbf{k}_a$$

The optical theorem

- Claim: these dS amplitudes satisfy the standard generalised optical theorem (non perturbative, usual derivation)

$$A_{if} - A_{fi}^* = i \sum_X \int d\Pi_X \, (2\pi)^4 \delta^{(3)}(\mathbf{k}_{\rm in} - \mathbf{k}_X) A_{iX} A_{fX}^*$$

- Because of our “symmetric” definition of in and out states, the right-hand side above is positive in the forward limit!

- We hope to use this to obtain de Sitter positivity bounds

Non-trivial check

- The optical theorem is satisfied somewhat non-trivially. For example, for $r = 1$ we have $4 \to 4$ scattering

$$\mathrm{LHS} = \frac{\lambda^2 H^2}{2} 2i\,\mathrm{Im}\left[-\frac{1}{(-E_{\mathrm{in}}^2 + k_{\mathrm{in}}^2)}\delta''(E_{\mathrm{in}} - E_{\mathrm{out}}) + \frac{k_{\mathrm{in}}^2 + E_{\mathrm{in}}^2}{k_{\mathrm{in}}(E_{\mathrm{in}}^2 - k_{\mathrm{in}}^2)^2}\delta'(E_{\mathrm{in}} - E_{\mathrm{out}})\right]$$

$$\mathrm{RHS} = i\int \frac{dk_X^3}{(2\pi)^3)} \frac{1}{2E_X}(2\pi)^4 \delta^{(3)}(\mathbf{k}_{\mathrm{in}} - \mathbf{k}_X)|A_{4,1}|^2$$
$$= 2\pi i \lambda^2 H^2 \delta'(E_{\mathrm{out}} - E_{\mathrm{in}}) \frac{\delta'(E_{\mathrm{in}} - k_{\mathrm{in}})}{2E_{\mathrm{in}}}\,.$$

- They seem pretty different. But integrating by part and using the standard $i\epsilon$ prescription to shift the energy pole leads to a perfect match

Outlook

- We have correlators, the wavefunction and dS amplitudes, all computed by Feynman diagrams with a combination of time-ordered (Feynman) and not-ordered (Wightman) propagators

- There are only a dozen papers studying the general singularity structure, Landau analysis and master integral. It's a great time to have a large impact.

On analytic properties of vertex parts in quantum field theory

L.D. Landau (Moscow, Inst. Phys. Problems)
Jul, 1959

12 pages
Part of Collected Papers of L.D. Landau
Published in: *Nucl.Phys.* 13 (1959) 1, 181-192, *Sov.Phys.JETP* 10 (1960) 1, 45-50, *Zh.Eksp.Teor.Fiz.* 37 (1959) 1, 62-70
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