Scattering in lattice field theory and the role of Landau singularities

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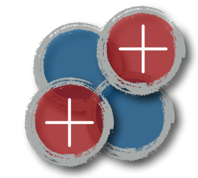
Recipe for strong force predictions

1. Lagrangian defining QCD
2. Formal / numerical machinery (lattice field theory)
3. A few experimental inputs (e.g. M_π, M_K, M_Ω)

$$\mathcal{L}_{\text{QCD}} = \sum_f \overline{\Psi}_f (i\not{D} - m_f)\Psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

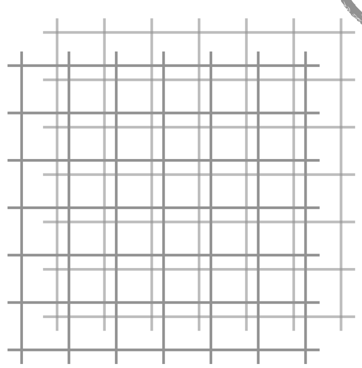


Wide range of precision pre-/post-dictions

$m_q^{\overline{\text{MS}}}(\mu)$ $\alpha_{\overline{\text{MS}}}(\mu)$

m_p

m_n f_π



g_A f_K

$(g-2)_\mu^{\text{HVP}}$ f_D f_B

FLAG 2021 Flavour Lattice Averaging Group

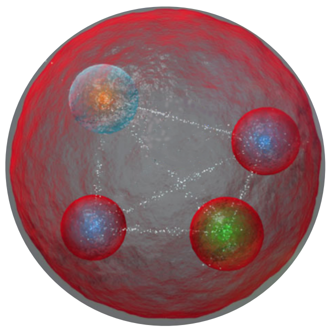
$\alpha^{(5)}_{\overline{\text{MS}}}(M_Z) = 0.1184(8)$ lattice average

$\alpha^{(5)}_{\overline{\text{MS}}}(M_Z) = 0.1176(11)$ PDG 20 (non-lattice)

Overwhelming evidence for QCD ✓ Tool for new-physics searches ✓

Multi-hadron observables

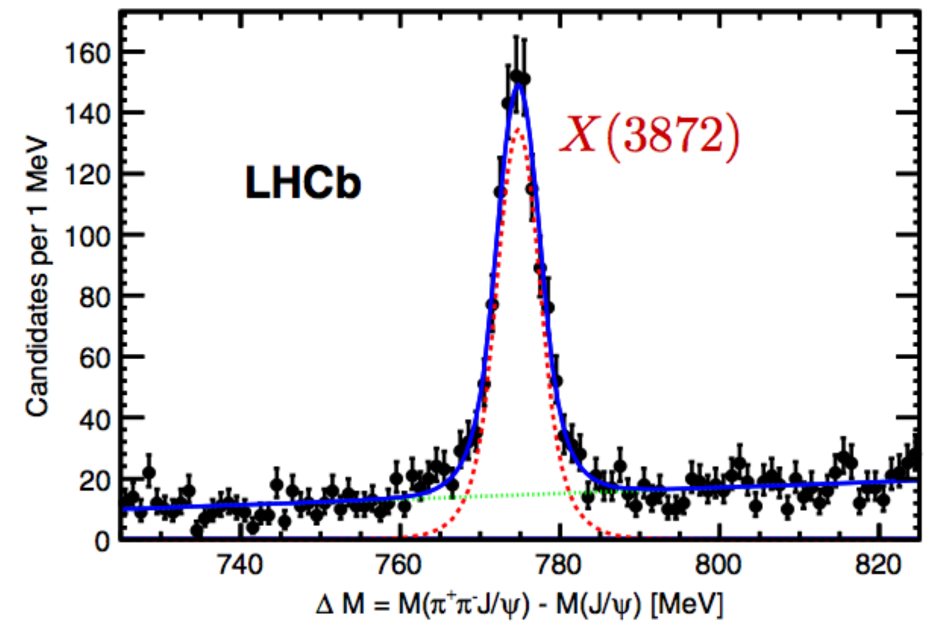
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$\sim |D^0 \overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle$?

• LHCb (PRD92, 2015) •



- Eletroweak, CP violation, resonant enhancement

CP violation in charm $D \to \pi\pi, K\overline{K}$

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}

• Soni (2017) •

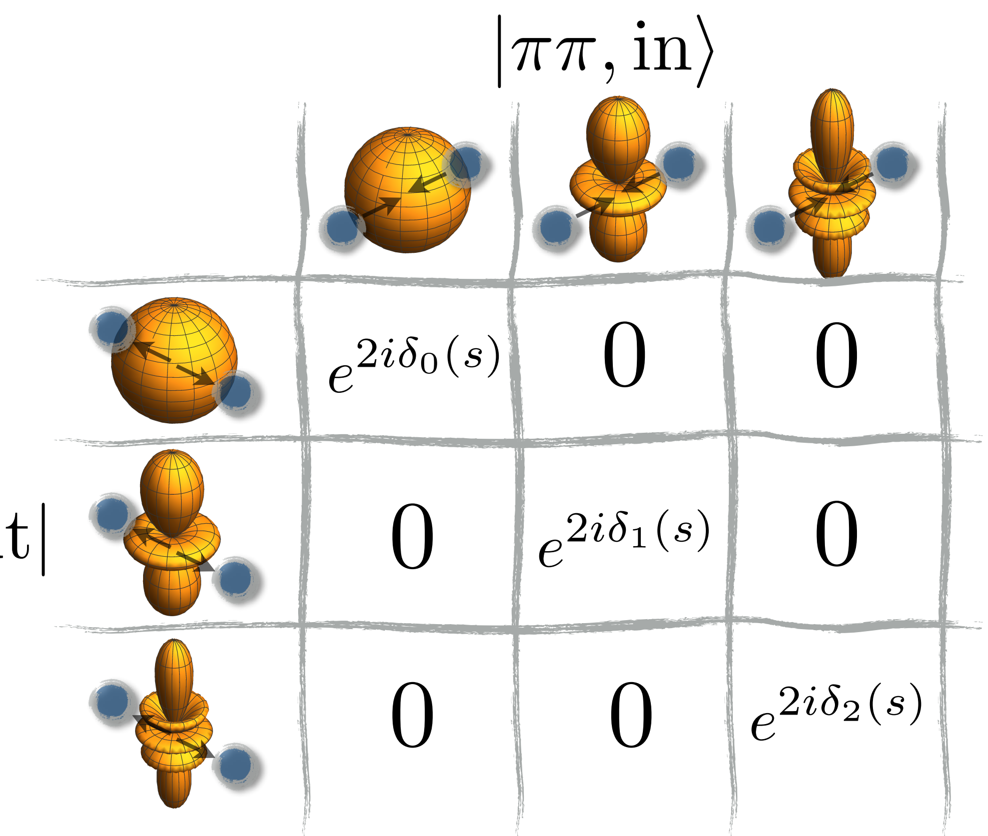
Resonant B decays $B \to K^* \ell\ell \to K\pi\, \ell\ell$

$|X\rangle, |\rho\rangle, |K^*\rangle, |f_0\rangle \notin$ **QCD Fock space**

QCD Fock space

- ❑ At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- ❑ Overlaps of multi-hadron ***asymptotic states*** → S matrix

$S(s) \equiv \langle \pi\pi, \mathrm{out}|$



depends on $s = E_{\mathrm{cm}}^2$
and angular variables

diagonal in angular momentum

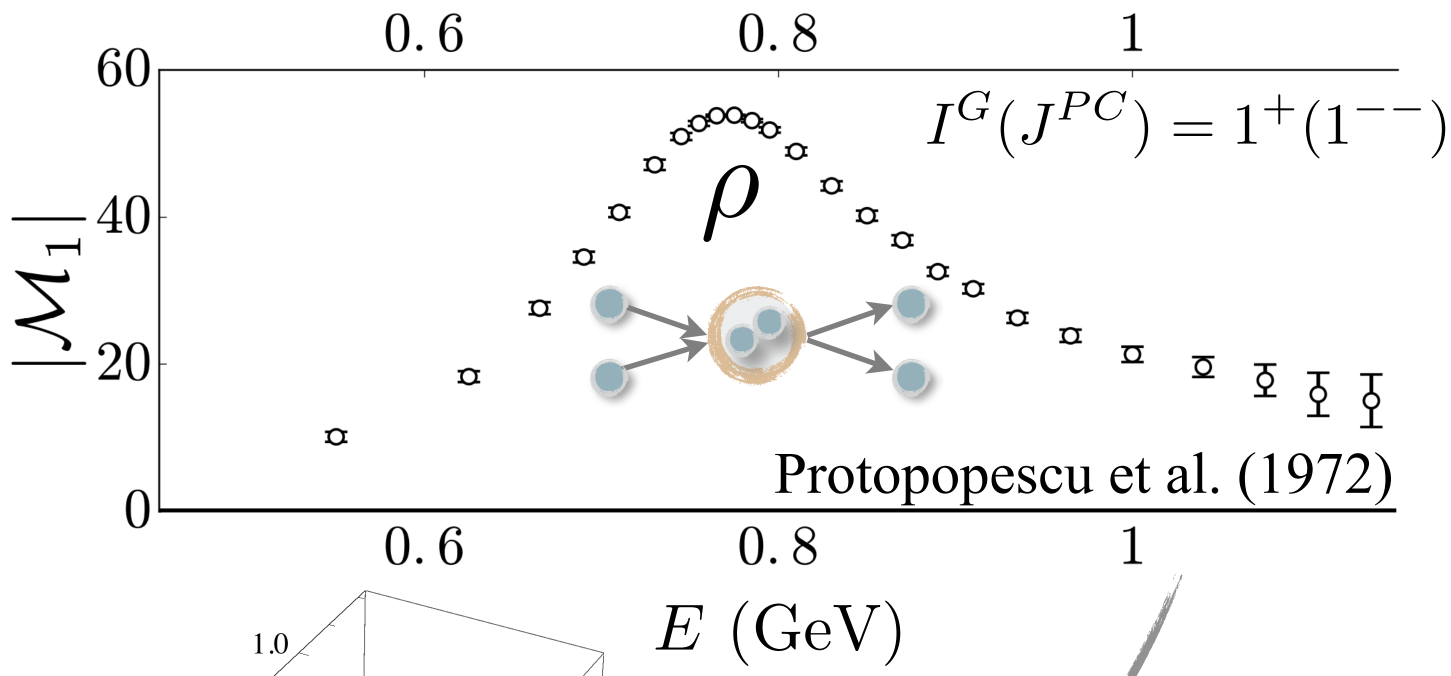
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

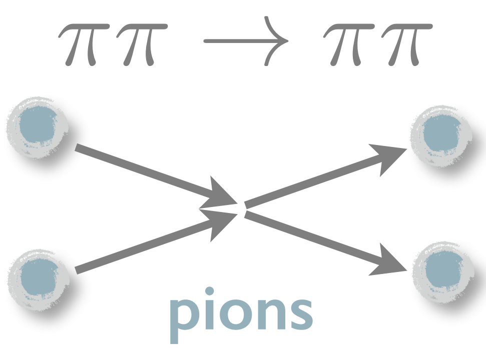
- ❑ ***An enormous space of information*** $|\pi\pi\pi\pi, \mathrm{in}\rangle \quad |K\overline{K}, \mathrm{in}\rangle \quad \cdots$

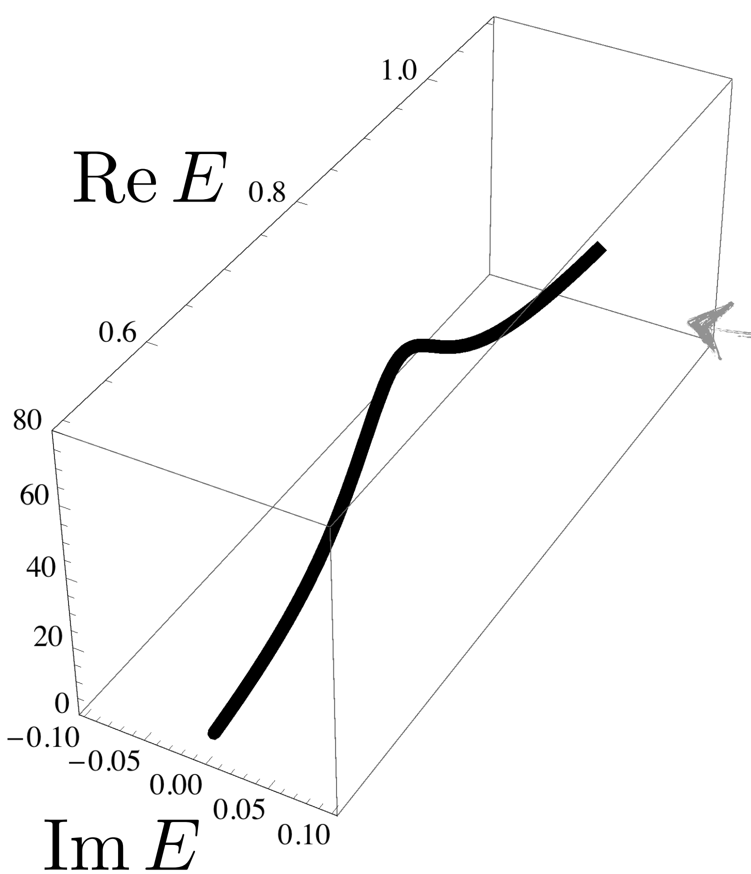
QCD resonances

- ☐ Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2\delta_\ell(s)$

scattering rate







Extend to the **complex plane**

QCD resonances

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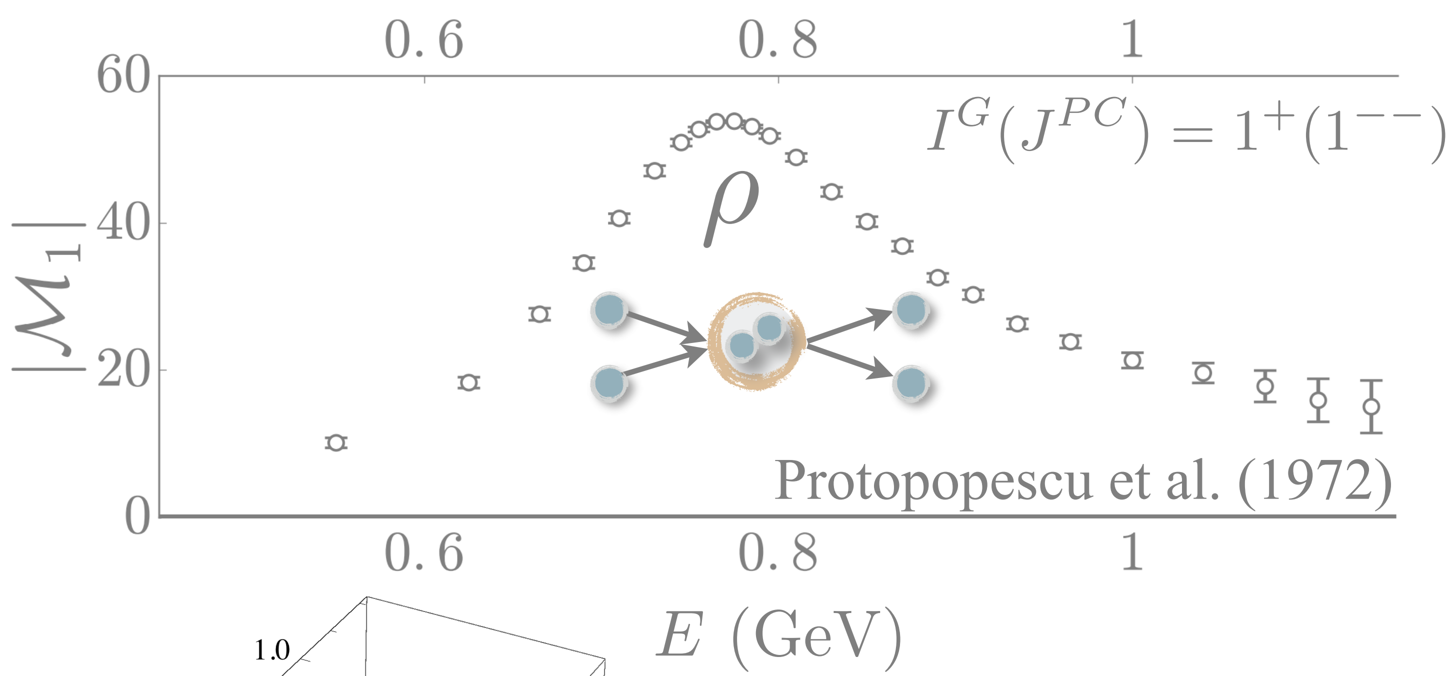
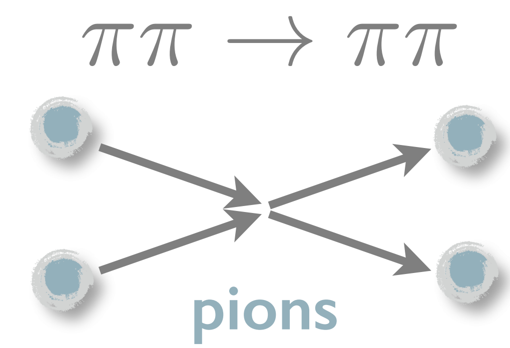
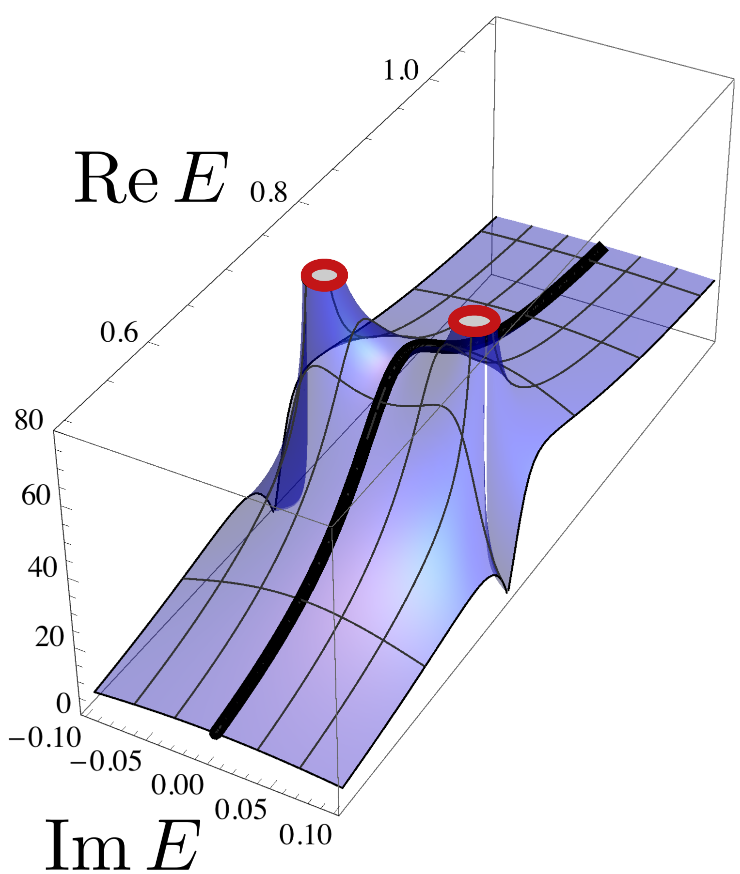
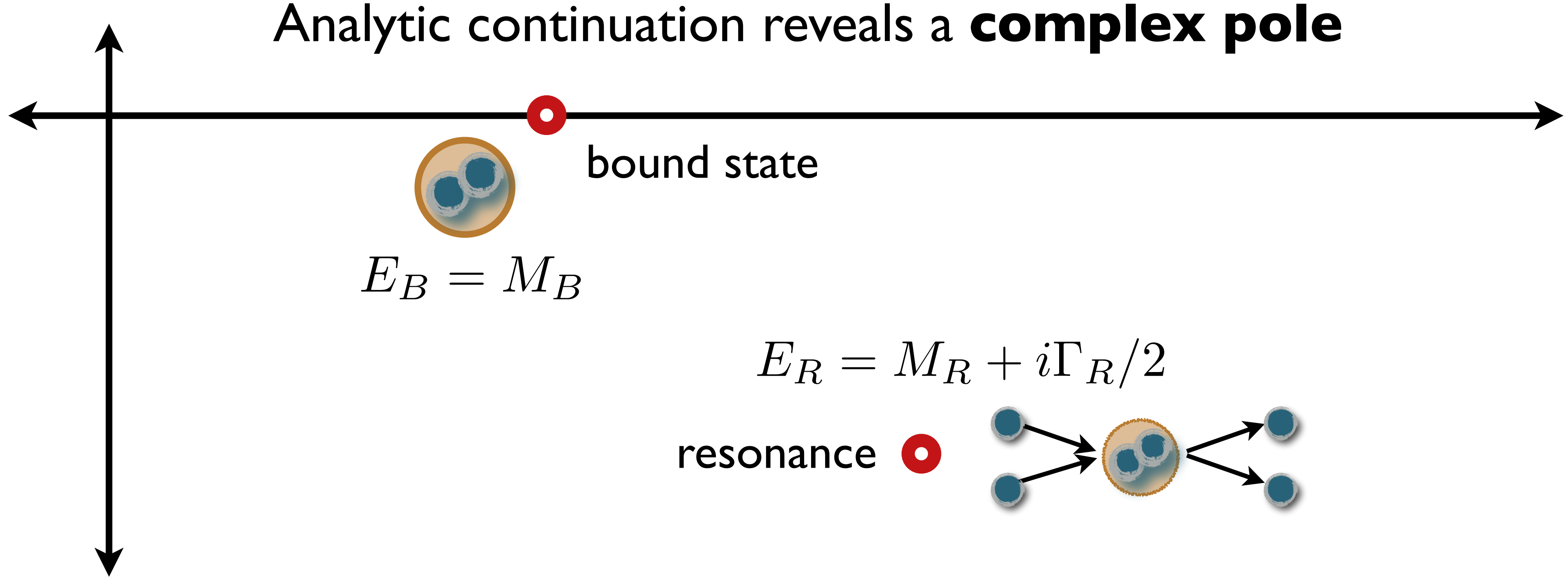




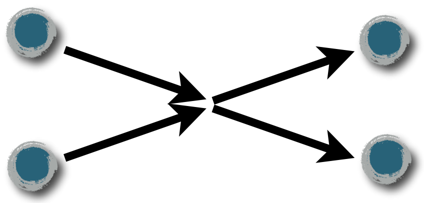


Analytic continuation reveals a **complex pole**



Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself



For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?

- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \mathrm{Im}\,\mathcal{M}_\ell(s)$$

where $\rho(s) = \dfrac{\sqrt{1-4m^2/s}}{32\pi}$ is the two-particle phase space

- Unique solution is…

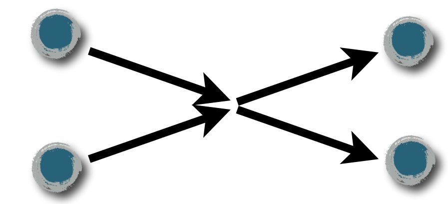
$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

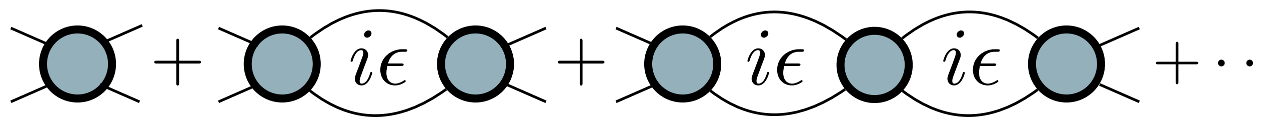
K matrix (short distance)

phase-space cut (long distance)

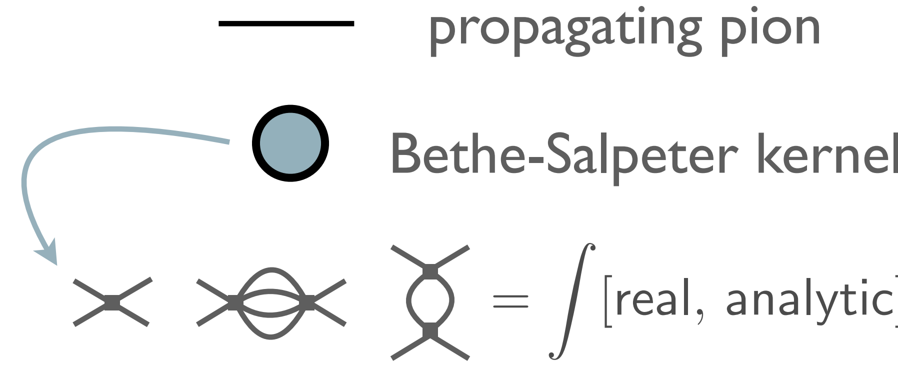
Key message: *The scattering amplitude has a square-root branch cut*

Analyticity (all orders diagrammatic)



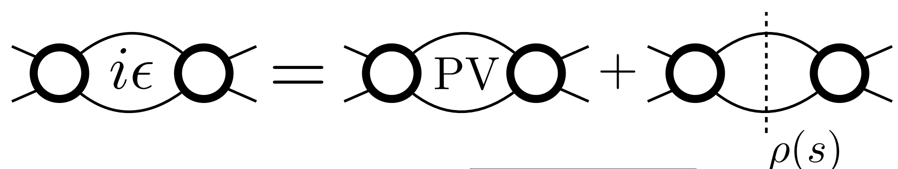
$\mathcal{M}(s) \equiv$ 

on-shell particles = singularities:
non-analytic for $(2m)^2 < s < (4m)^2$



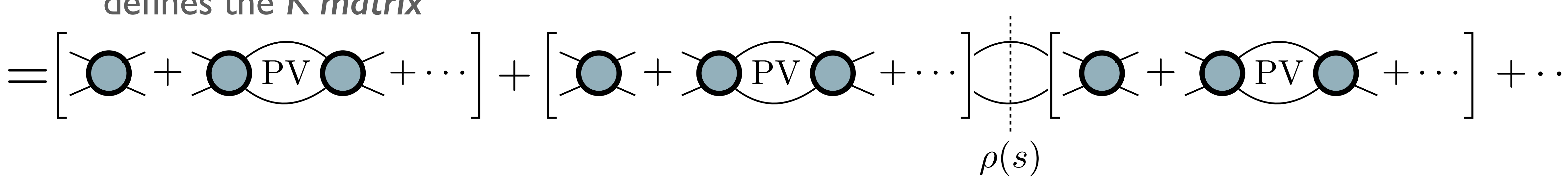
for $(2m)^2 < s < (4m)^2$

cutting rule



$\rho(s) \propto i\sqrt{s-(2m)^2}$

defines the *K* ***matrix***



$$= \mathcal{K}(s) + \mathcal{K}(s)\rho(s)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - \rho(s)}$$

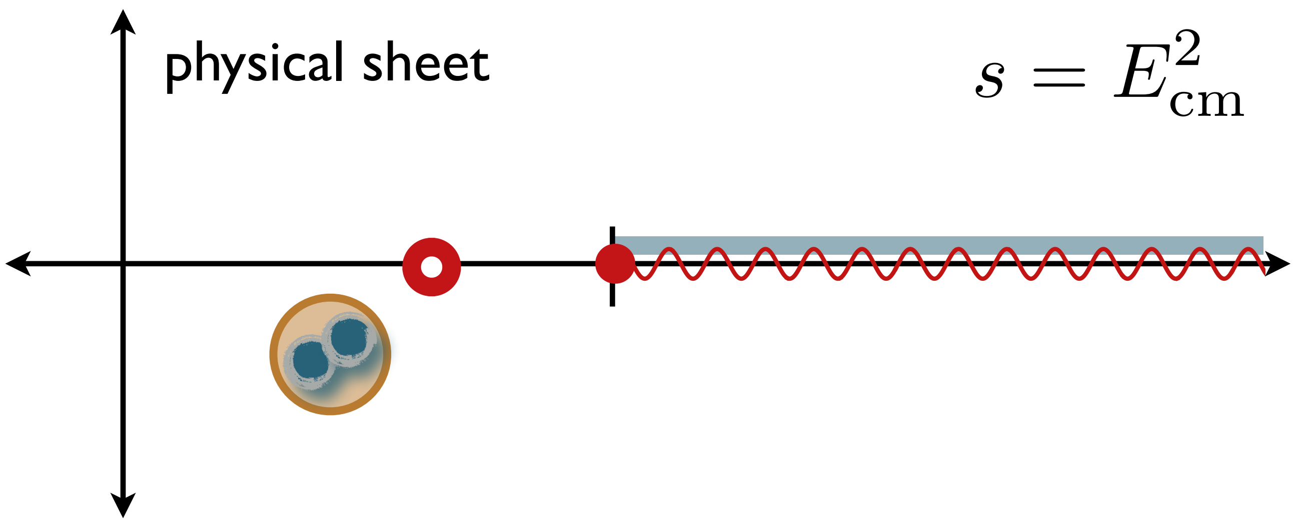
K matrix (short distance)

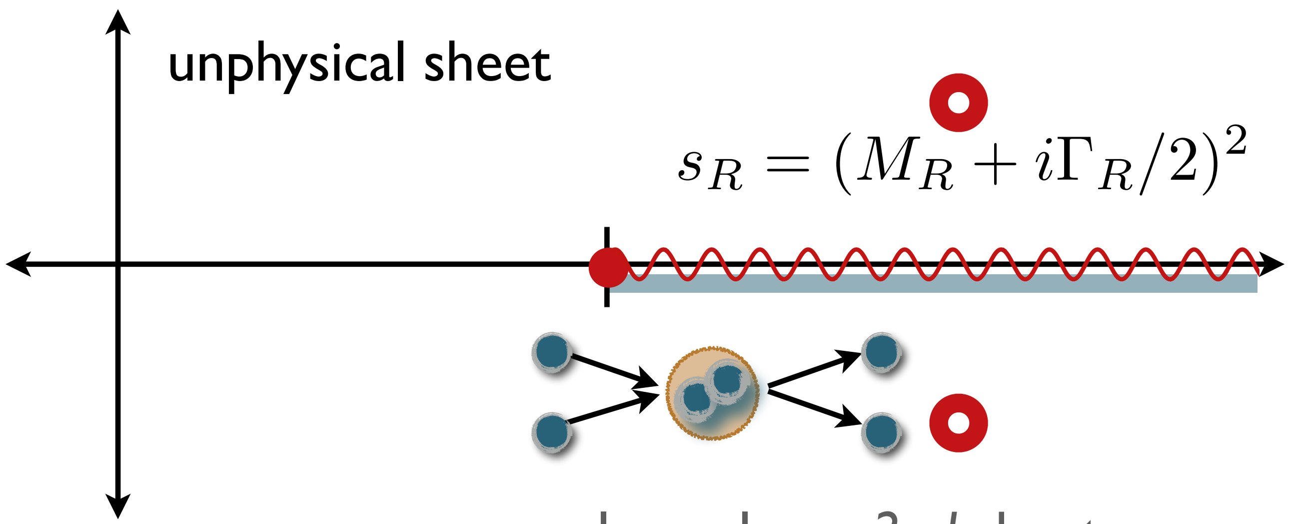
phase-space cut (long distance)

Cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - \rho(s)} \propto \frac{1}{p\cot\delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \qquad \rho(s) \propto i\sqrt{s-(2m)^2}$$

- ❑ Each channel generates a *square-root cut* → doubles the number of sheets





- ❑ Important lessons:

Details of analyticity = important for quantitative understanding

Possible to separate…
(i) long-distance kinematic singularities
(ii) short-distance/microscopic physics (depending on interaction details)

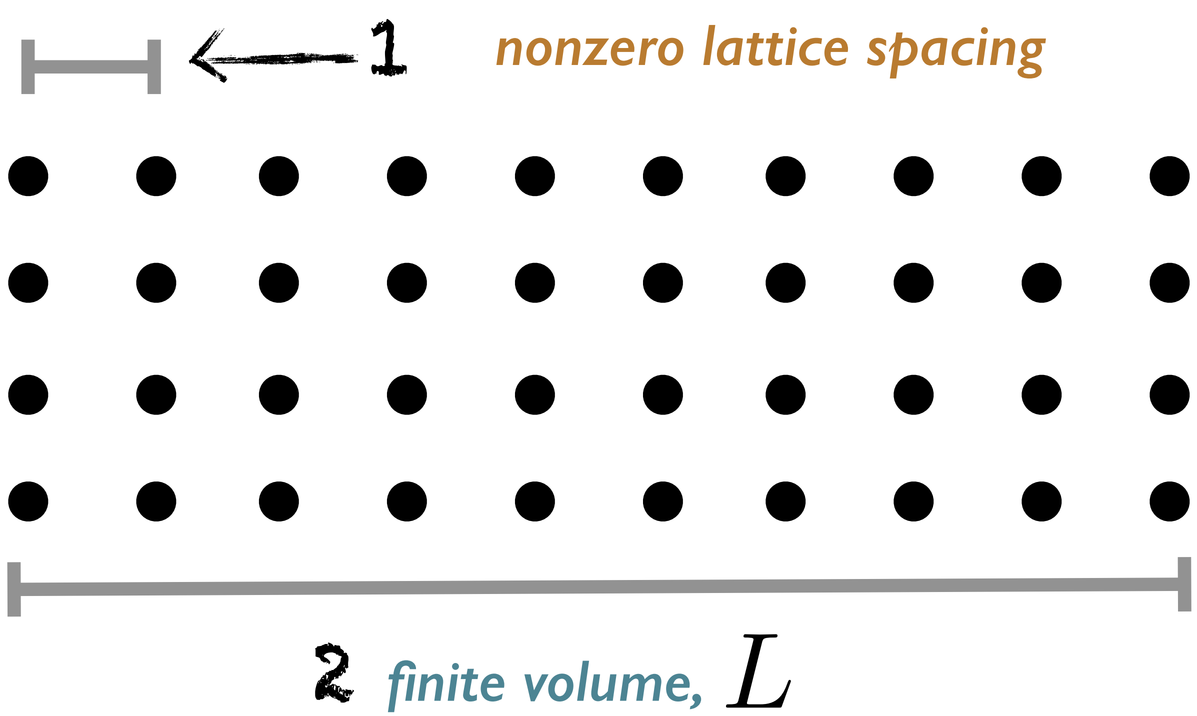
Microscopic physics *via Lattice QCD*

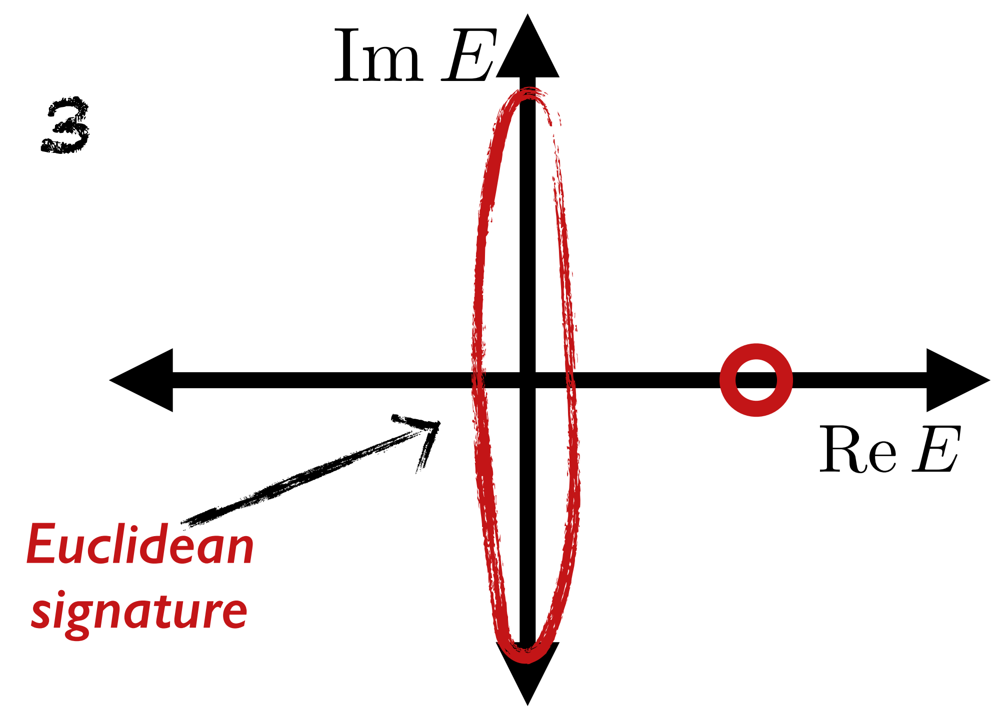
$$\text{observable} \quad = \int \mathcal{D}\phi \; e^{iS} \begin{bmatrix} \textbf{interpolator} \\ \textbf{for observable} \end{bmatrix}$$

Microscopic physics *via Lattice QCD*

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make *three modifications*



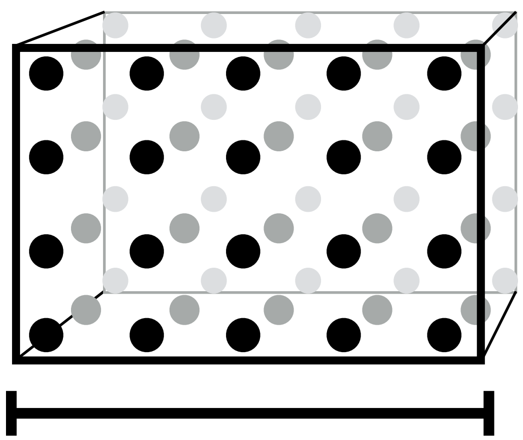


Also… $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

(but physical masses → increasingly common)

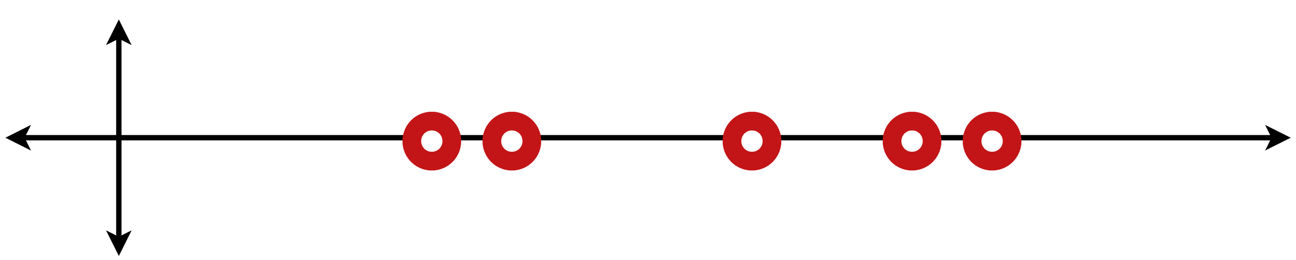
Difficulties for multi-hadron observables

- The ***Euclidean signature / imaginary time***…
  - ***Obscures*** real time evolution (that defines scattering)
  - ***Prevents*** normal LSZ (want $p_4^2 = -(p^2 + m^2)$, but we have only $p_4^2 > 0$)

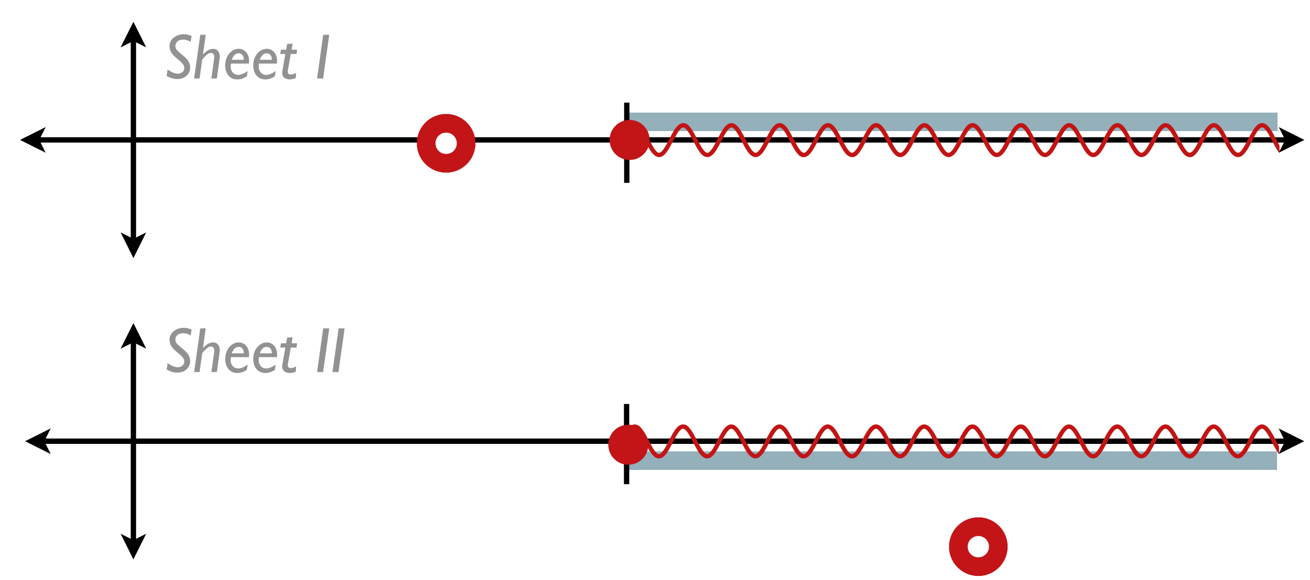


- The ***finite volume***…
  - ***Discretizes*** the spectrum
  - ***Eliminates*** the branch cuts and extra sheets
  - ***Hides*** the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure

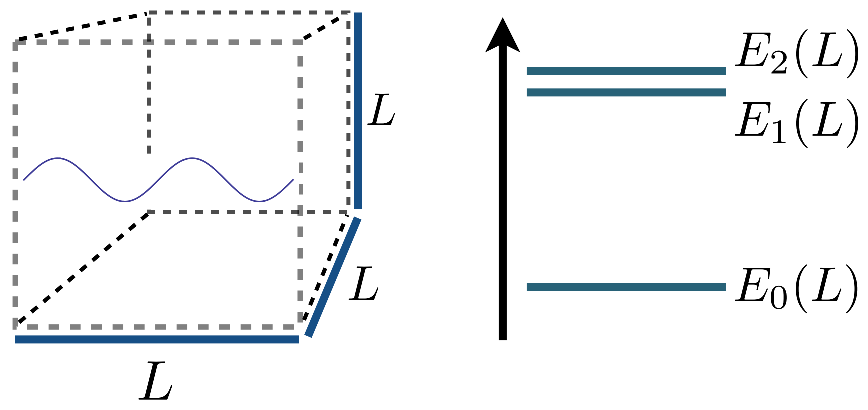


Importance of the finite volume

$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin$ **QCD Fock**

$|\pi\pi, \mathrm{out}\rangle, \quad |K\pi, \mathrm{out}\rangle, \quad \cdots \in$ **QCD Fock space (continuum of states)**

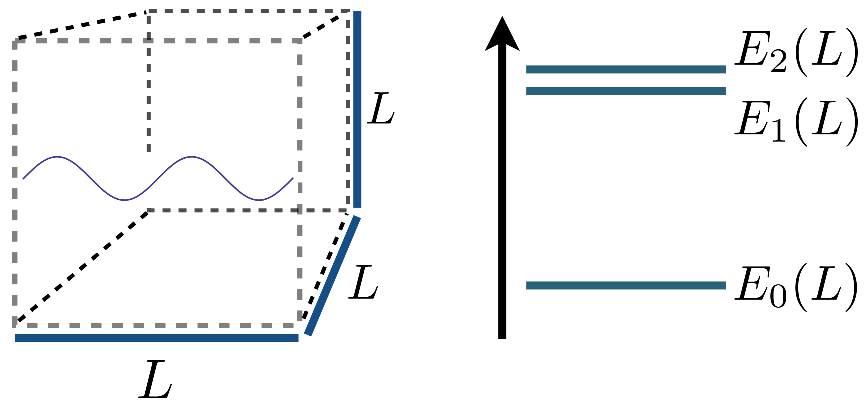
Relation is (highly) non-trivial



$\in$ **Discrete set of finite-volume states**

The finite-volume as a tool

- Finite-volume set-up



- **cubic**, spatial volume (extent L)
- **periodic**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- L is large enough to neglect $e^{-M_\pi L}$
- T and lattice also negligible

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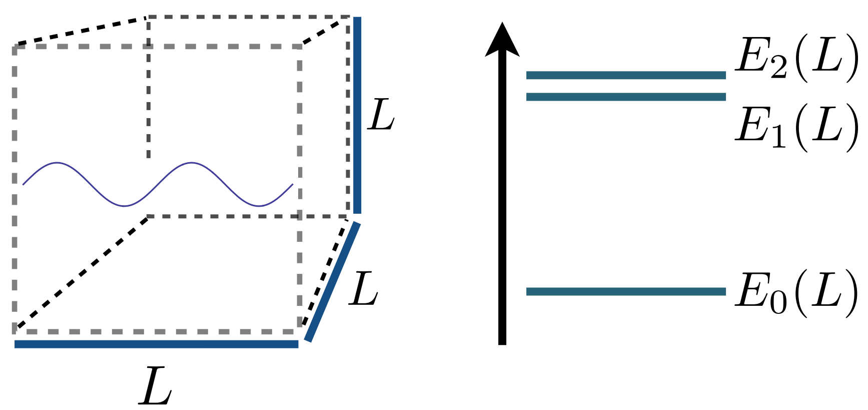

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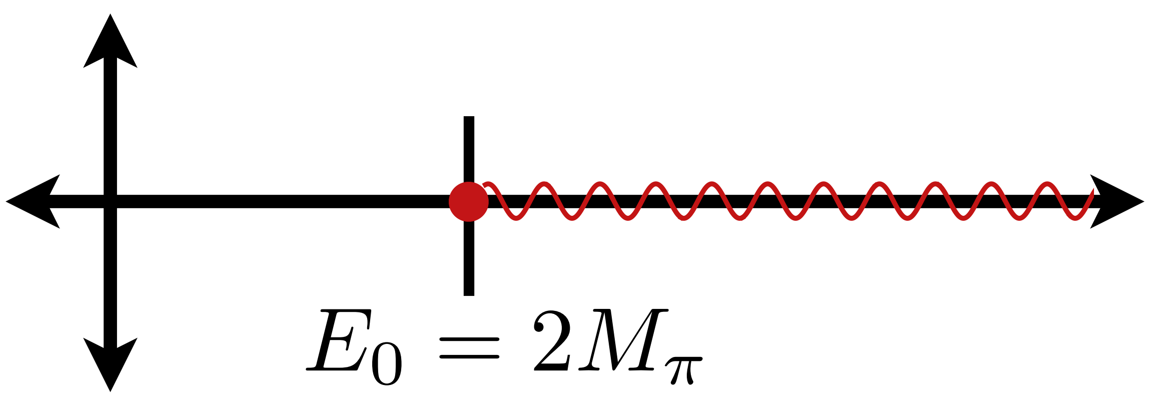
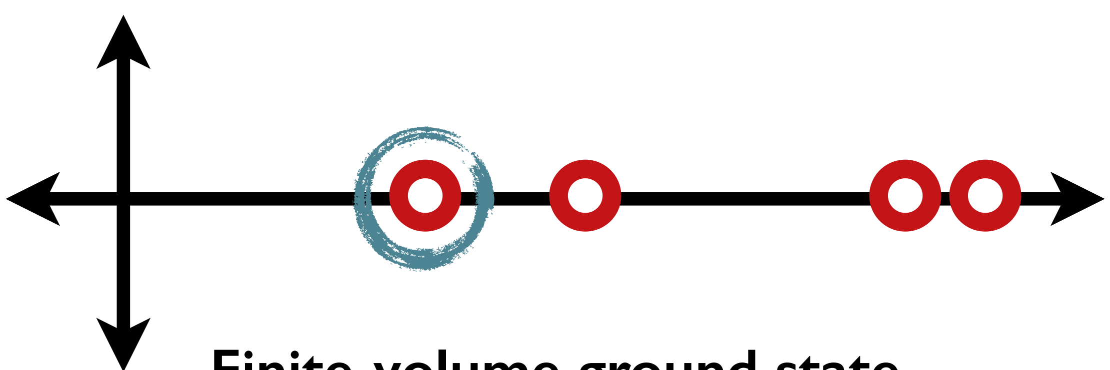
- Scattering leaves an *imprint* on finite-volume quantities





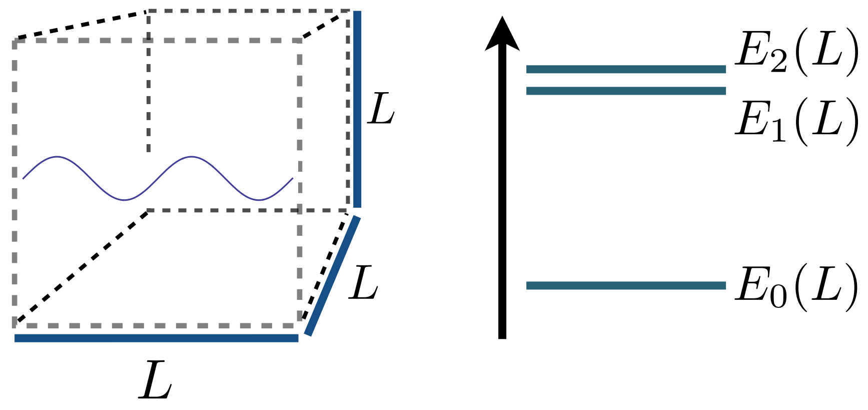
$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

The finite-volume as a tool

- Finite-volume set-up



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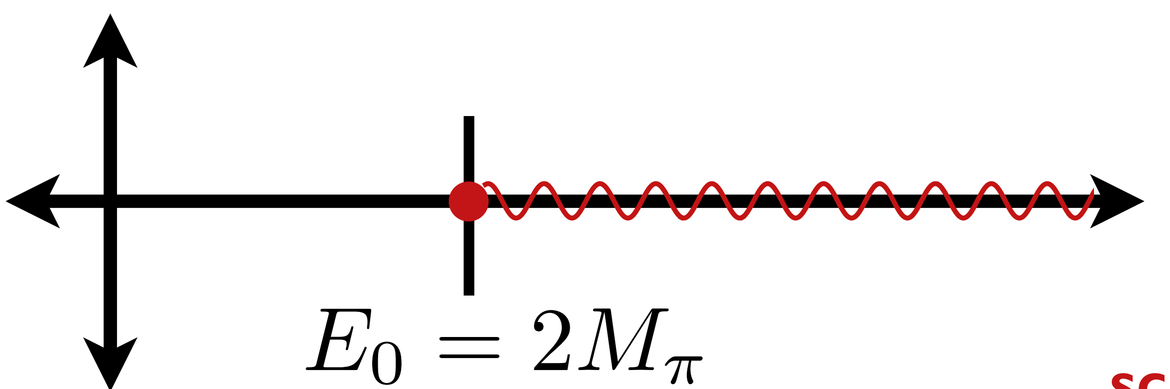
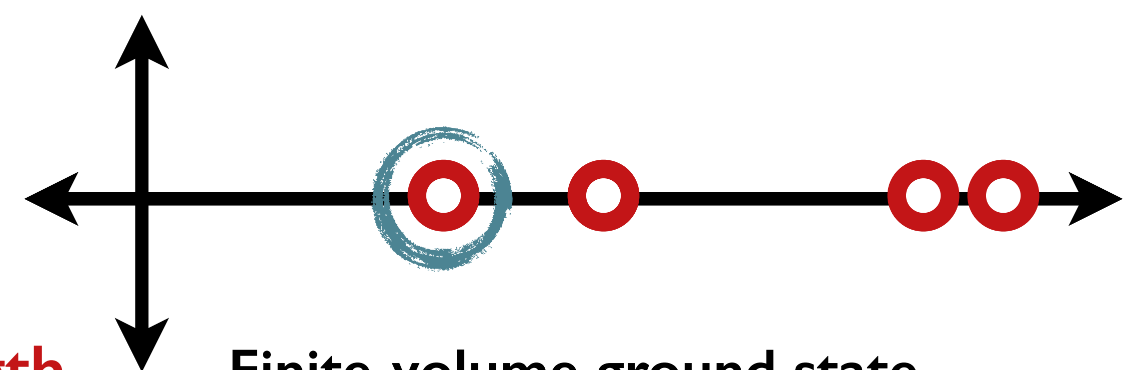


$$E_0 = 2M_\pi$$

Infinite-volume threshold

scattering length



Finite-volume ground state

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

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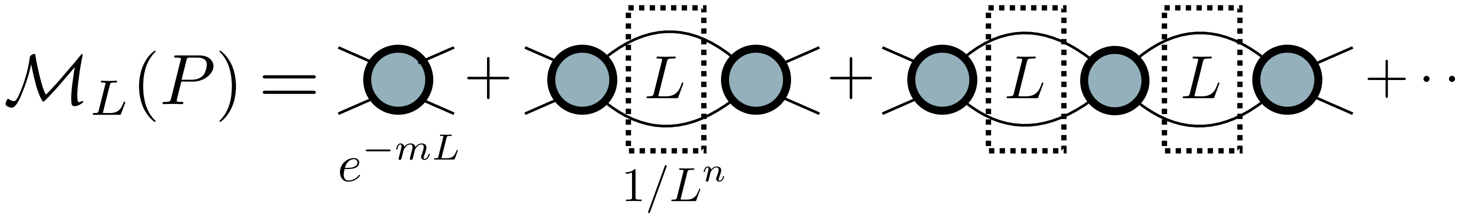
• Huang, Yang (1958) •

Derivation (all orders diagrammatic)

- Consider the finite-volume correlator:

$\mathcal{M}(s)$	$\mathcal{M}_L(P)$
probability amplitude	poles give f.v. spectrum



For two-particle energies $(2m)^2 < s < (4m)^2$, what is the *L* dependence?

— propagating pion

● Bethe-Salpeter kernel

⬚ $= \sum_{\mathbf{k}}$

• Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

Derivation (all orders diagrammatic)

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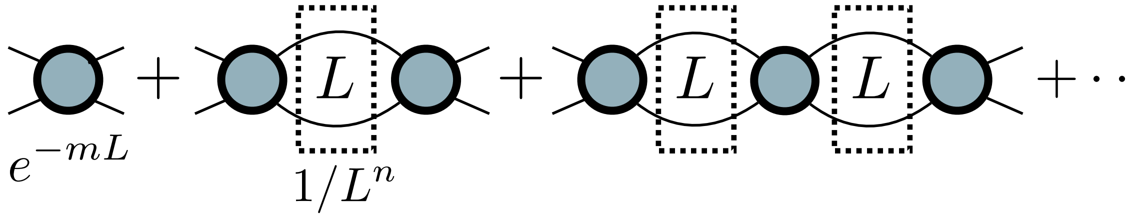
$\mathcal{M}_L(P) =$ 

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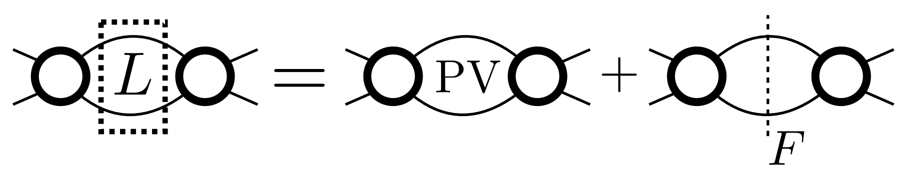

F = matrix of known geometric functions

• Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

Derivation (all orders diagrammatic)

$\mathcal{M}(s)$ probability amplitude | $\mathcal{M}_L(P)$ poles give f.v. spectrum

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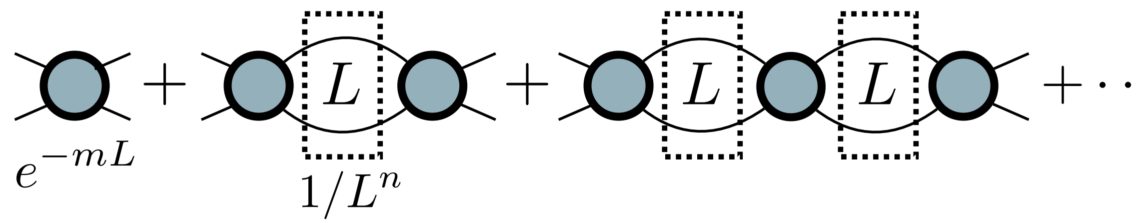
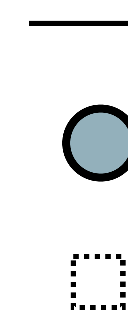
$\mathcal{M}_L(P) =$ 

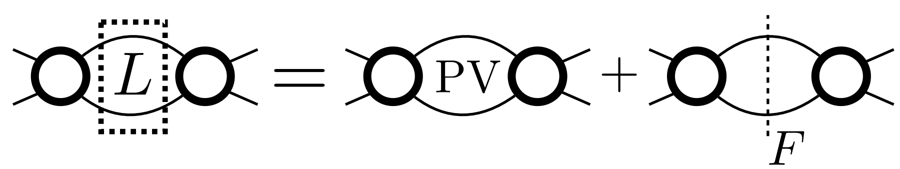


propagating pion

Bethe-Salpeter kernel

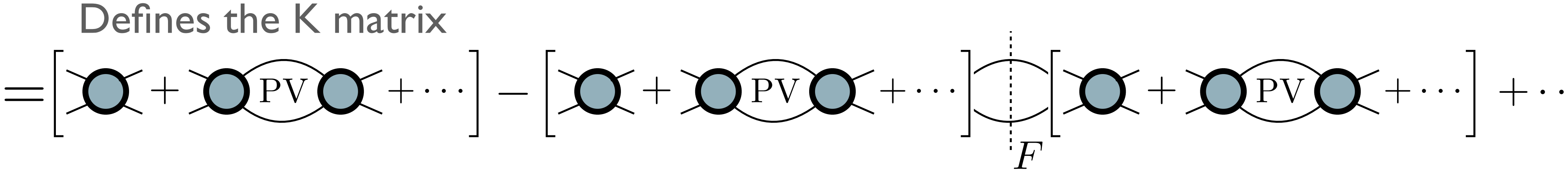
$= \sum_{\mathbf{k}}$

For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?

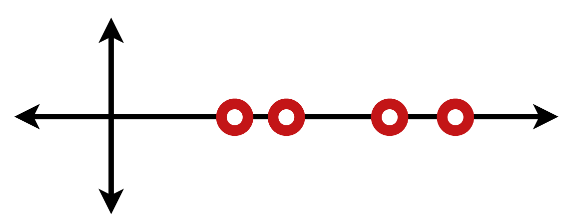


F = matrix of known geometric functions

Defines the K matrix



$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$



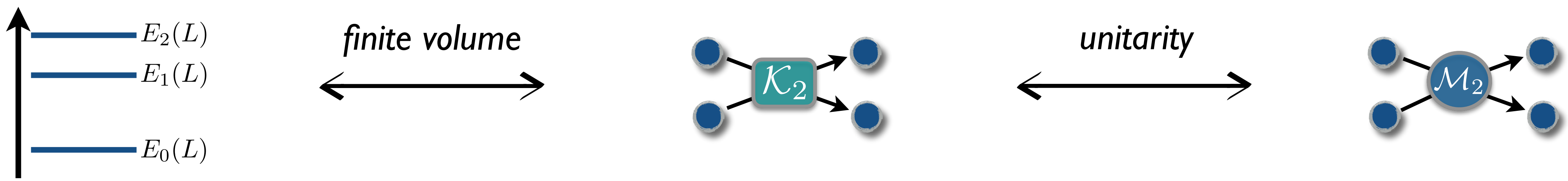
$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

• Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

General relation

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$$F(P,L) \equiv$$ Matrix of known geometric functions



Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to ***non-degenerate masses, multiple channels, spinning particles***

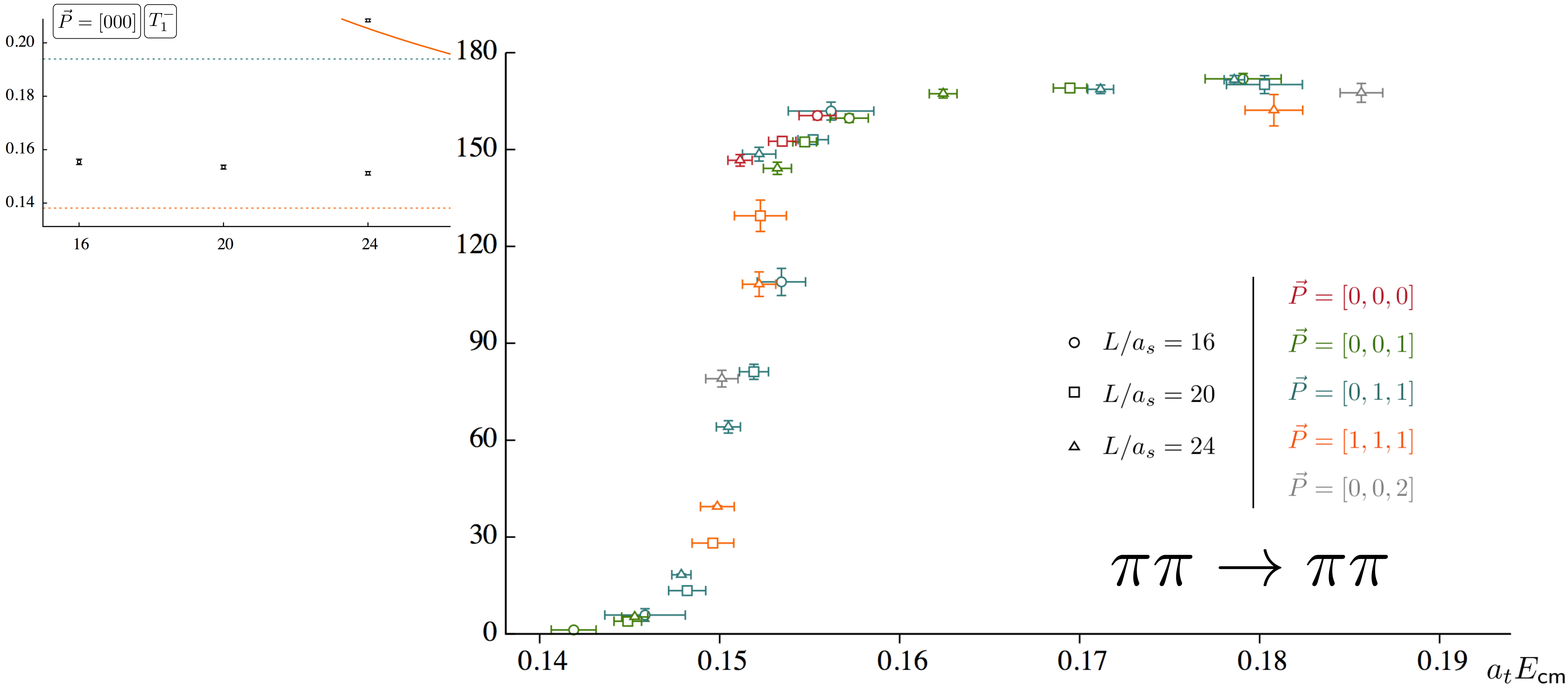
Encodes angular momentum mixing

• Lüscher (1989) • *many others* •

Using the result

- Single-channel case *(pions in a p-wave)*

$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$



• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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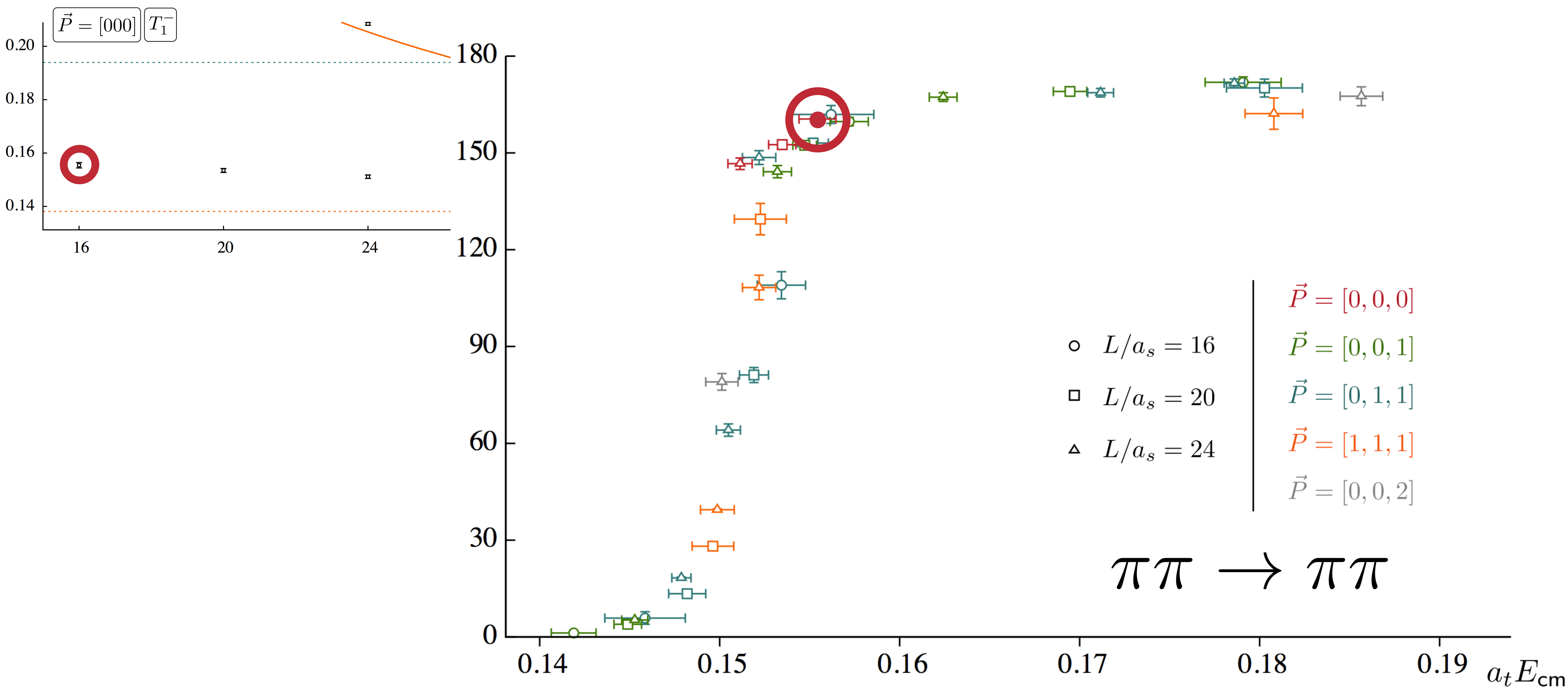

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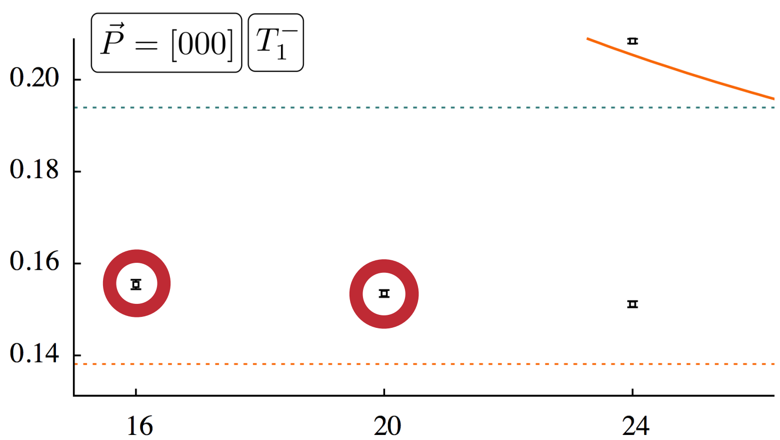

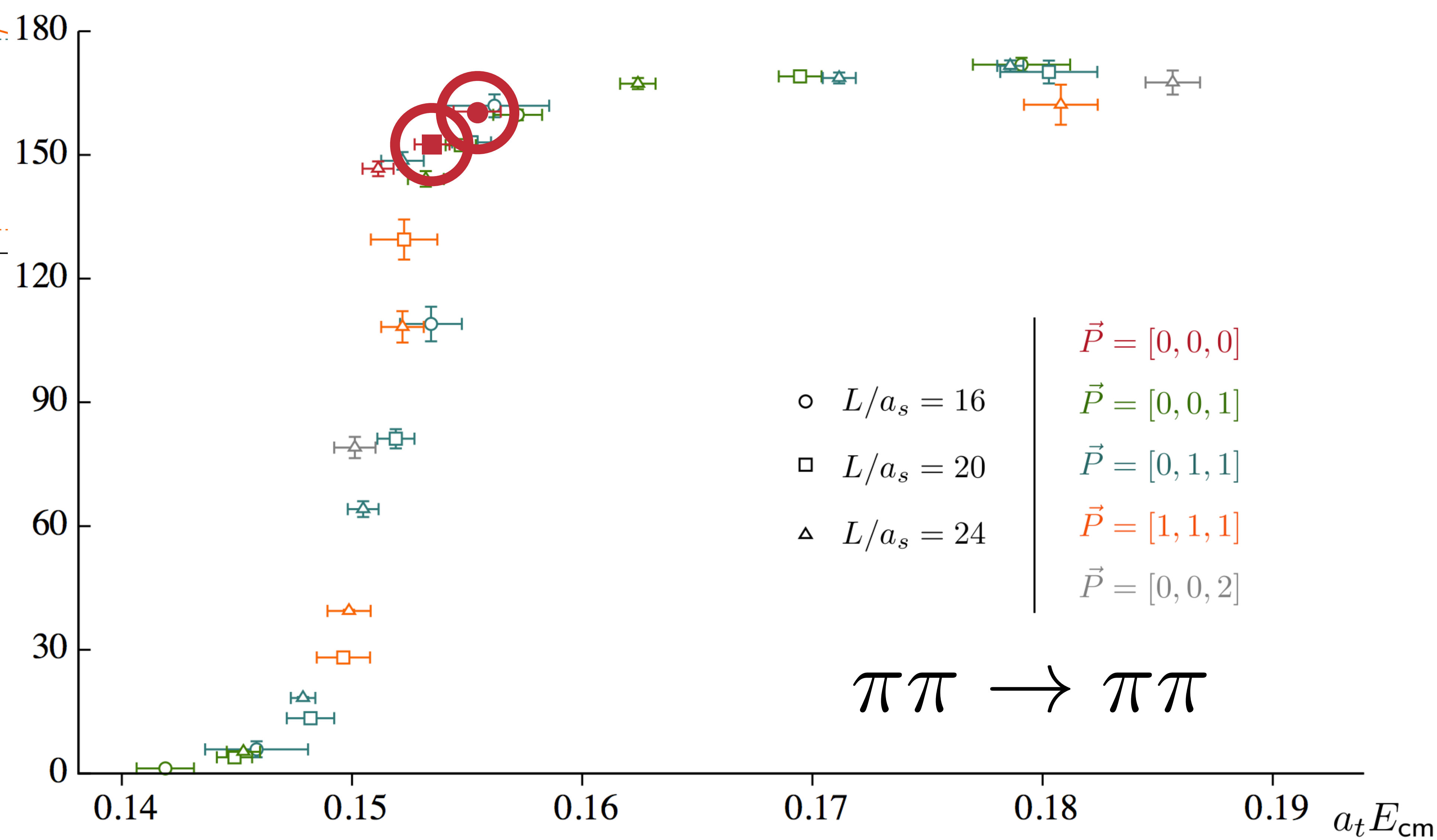

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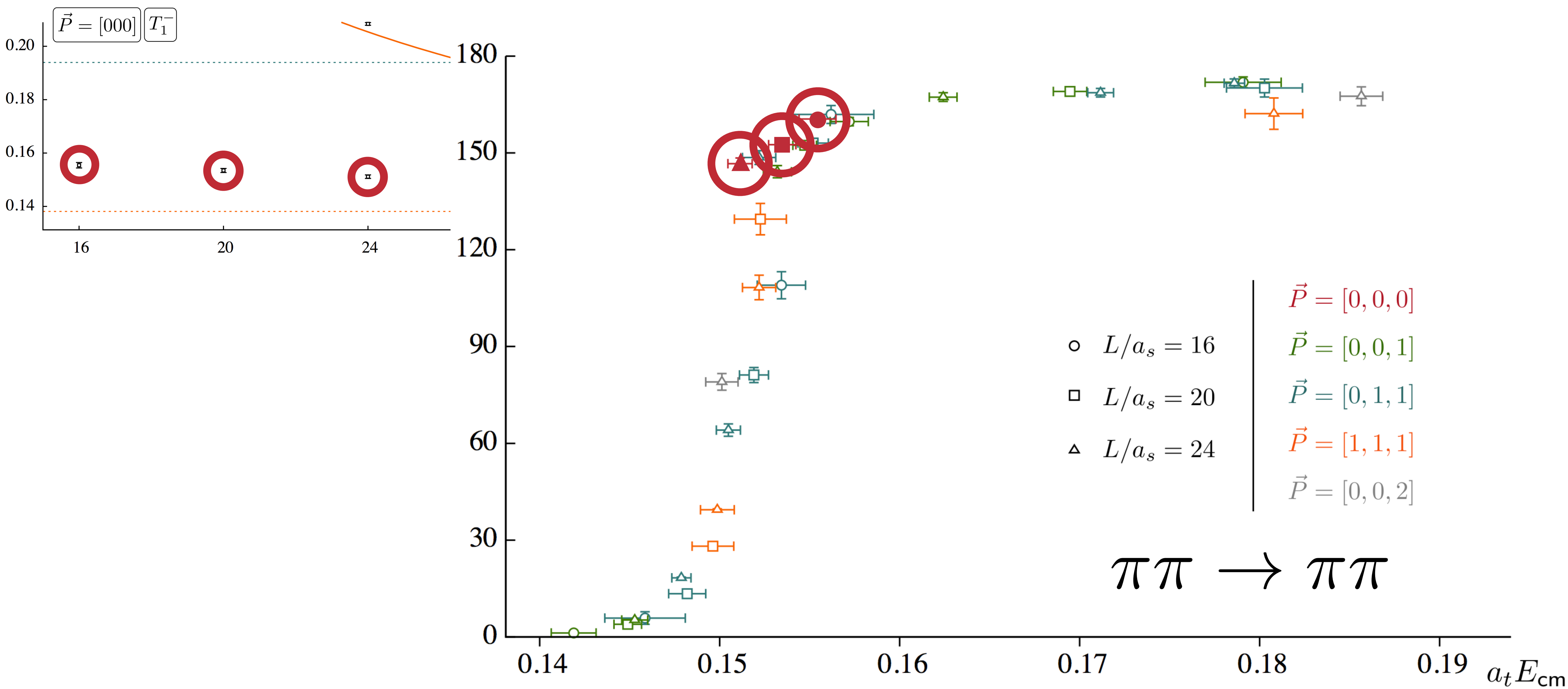

• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

Using the result

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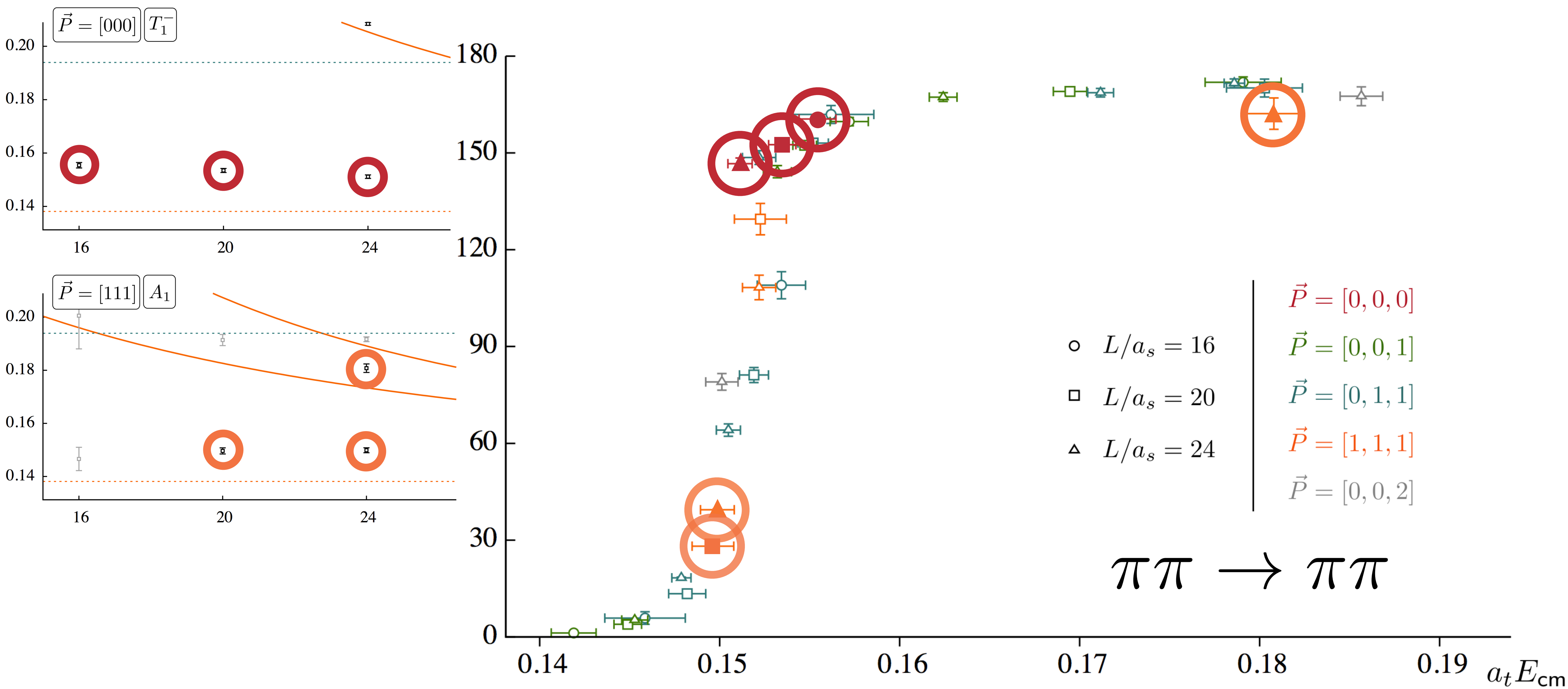
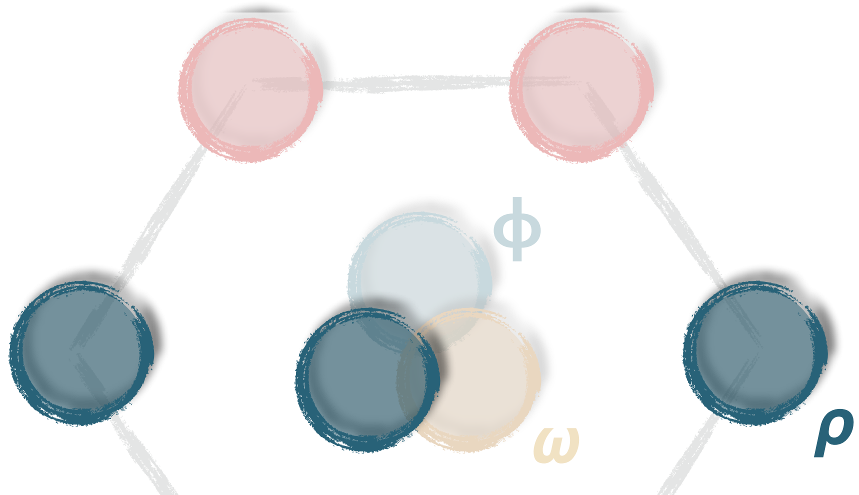
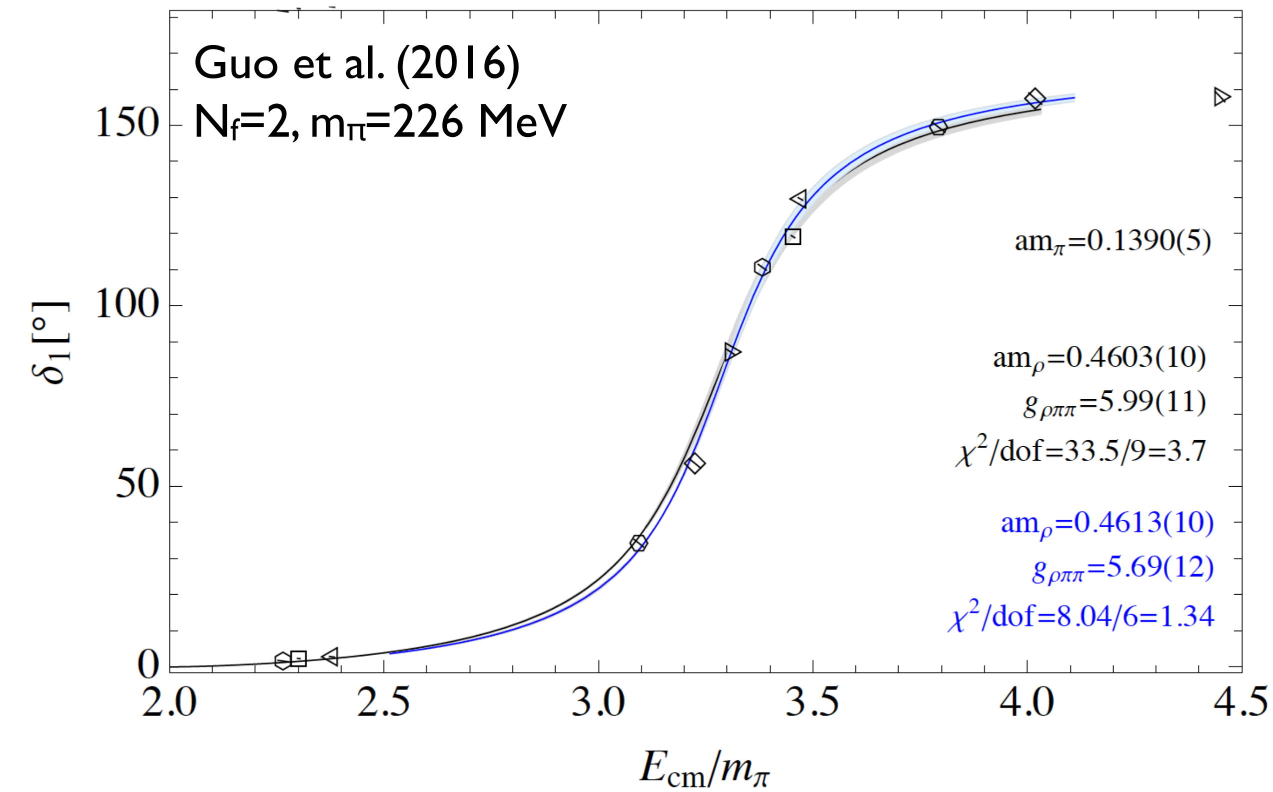


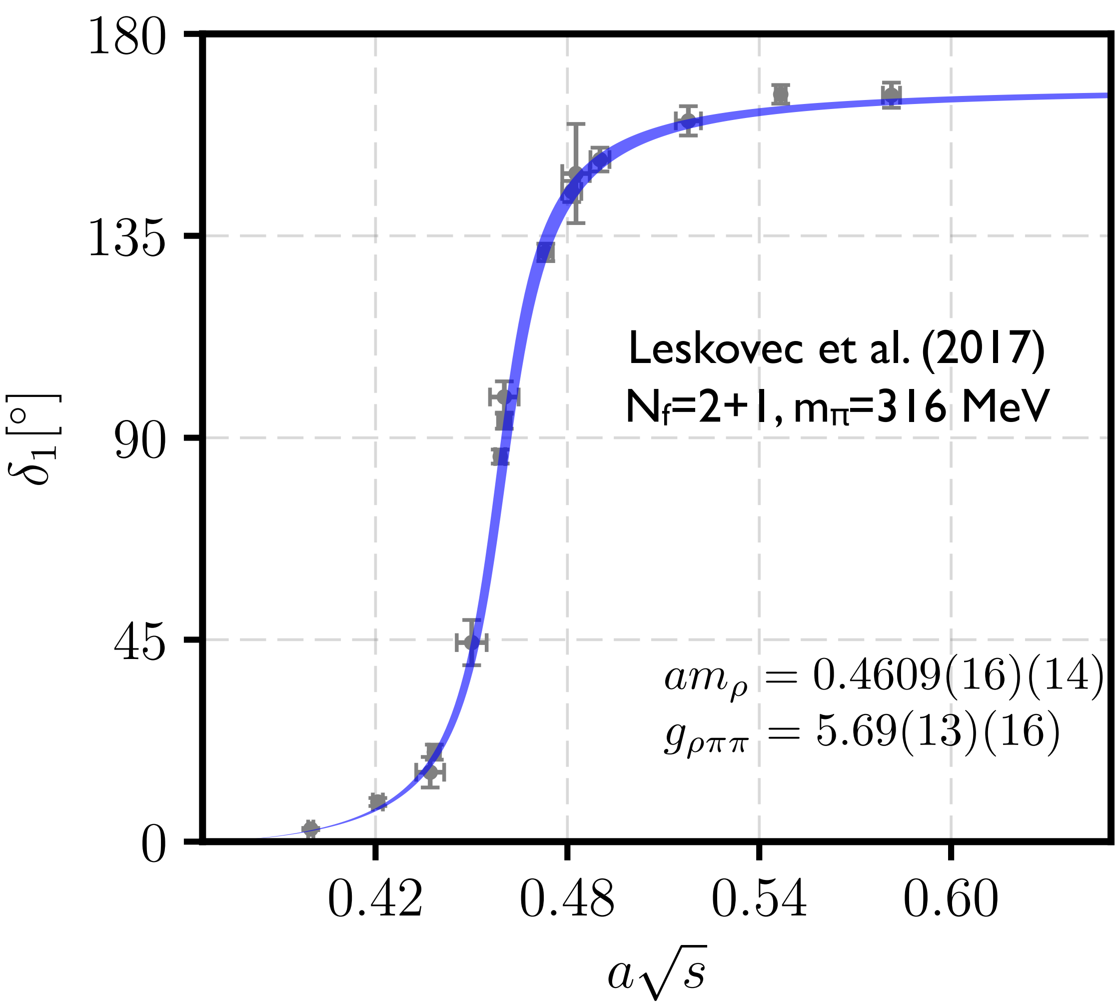
• Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505 •

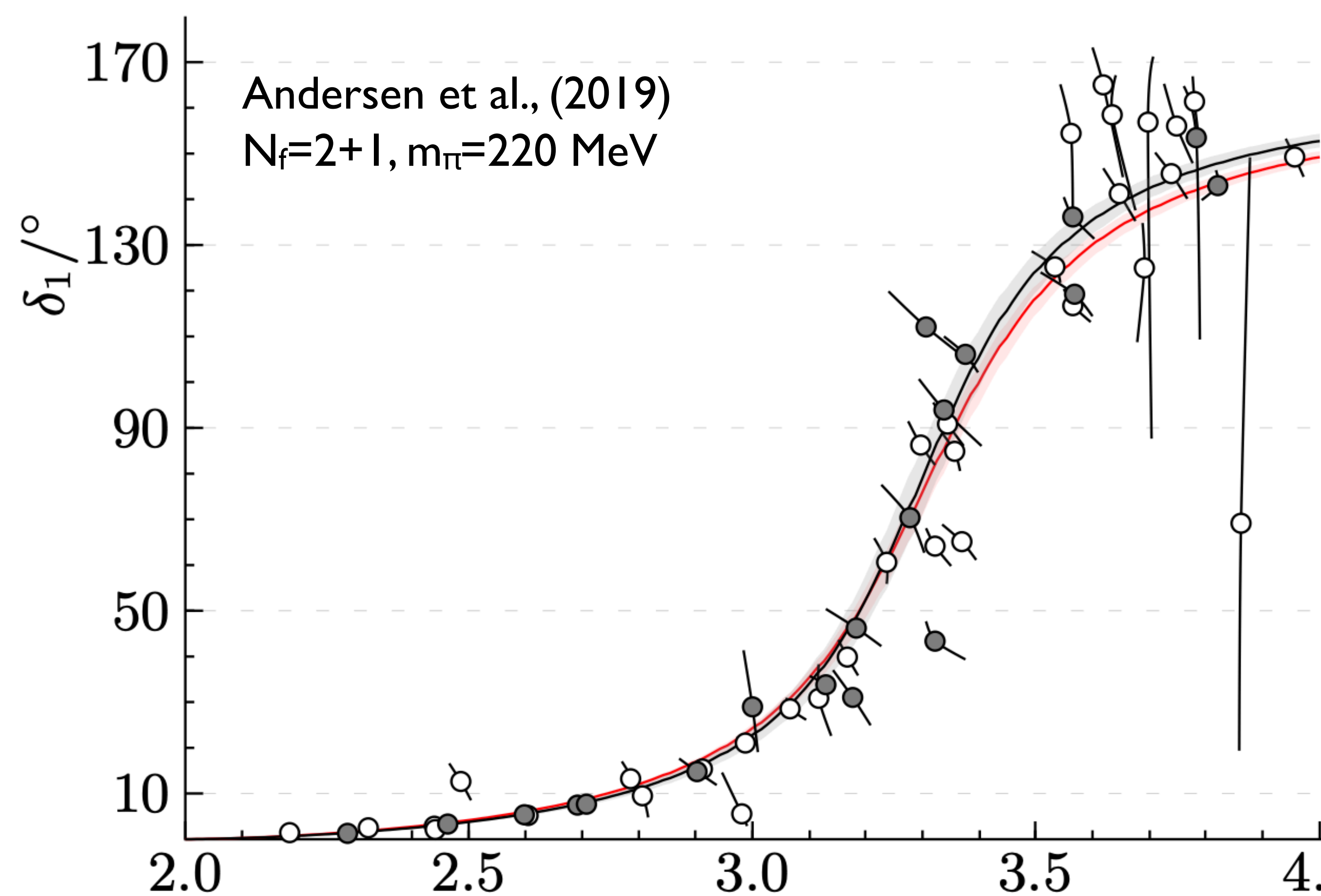
$$\rho \rightarrow \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$

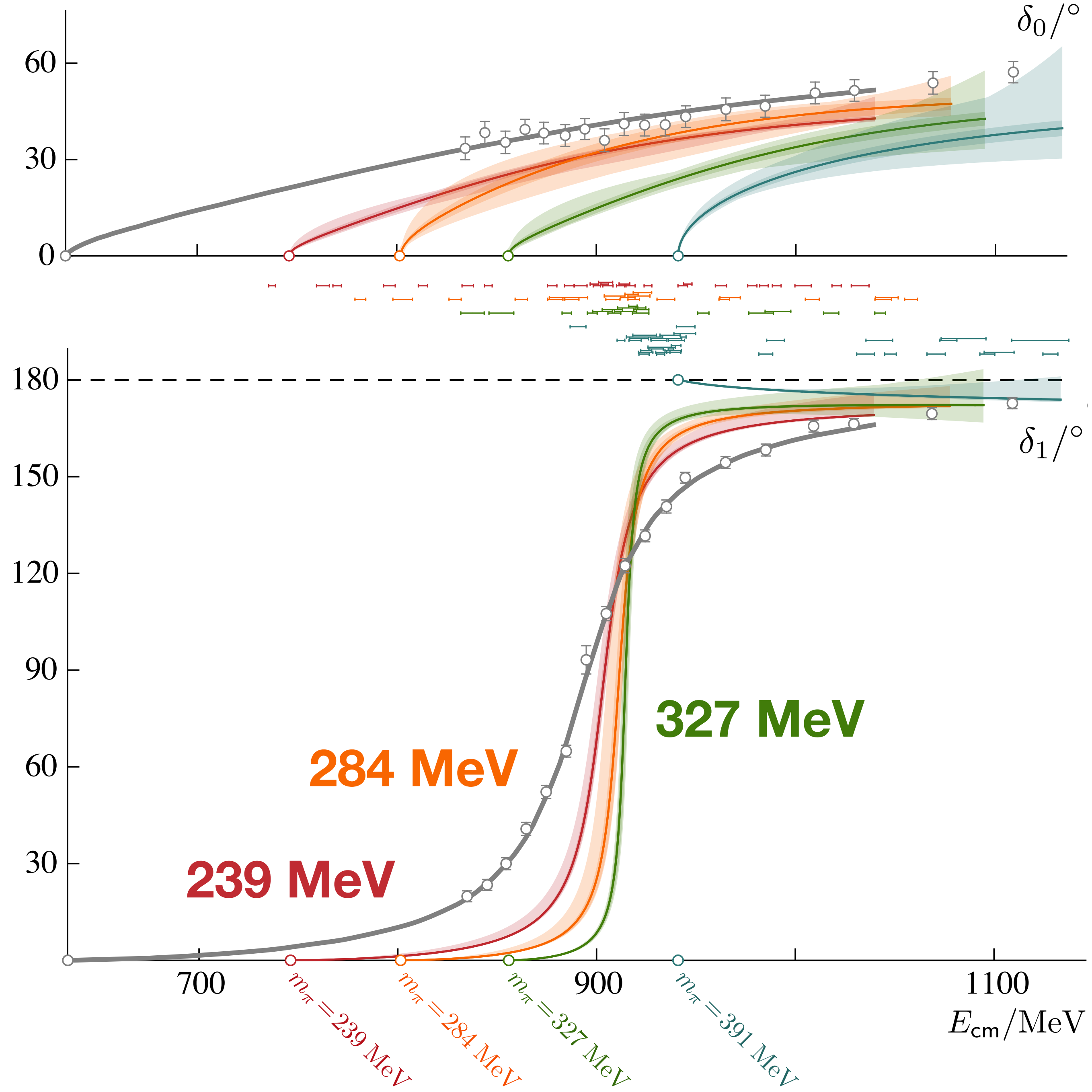


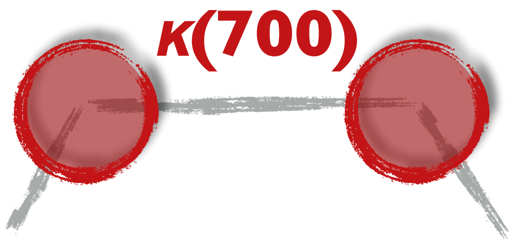




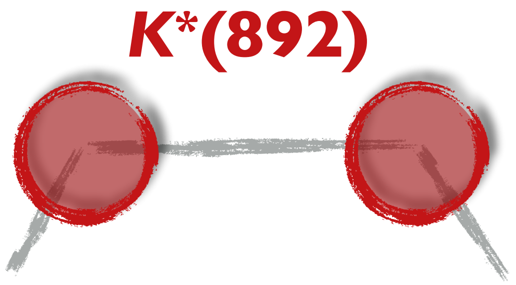


$\kappa, K^* \to K\pi$





$$I(J^P) = 1/2(0^+)$$



$$I(J^P) = 1/2(1^-)$$

- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002* •

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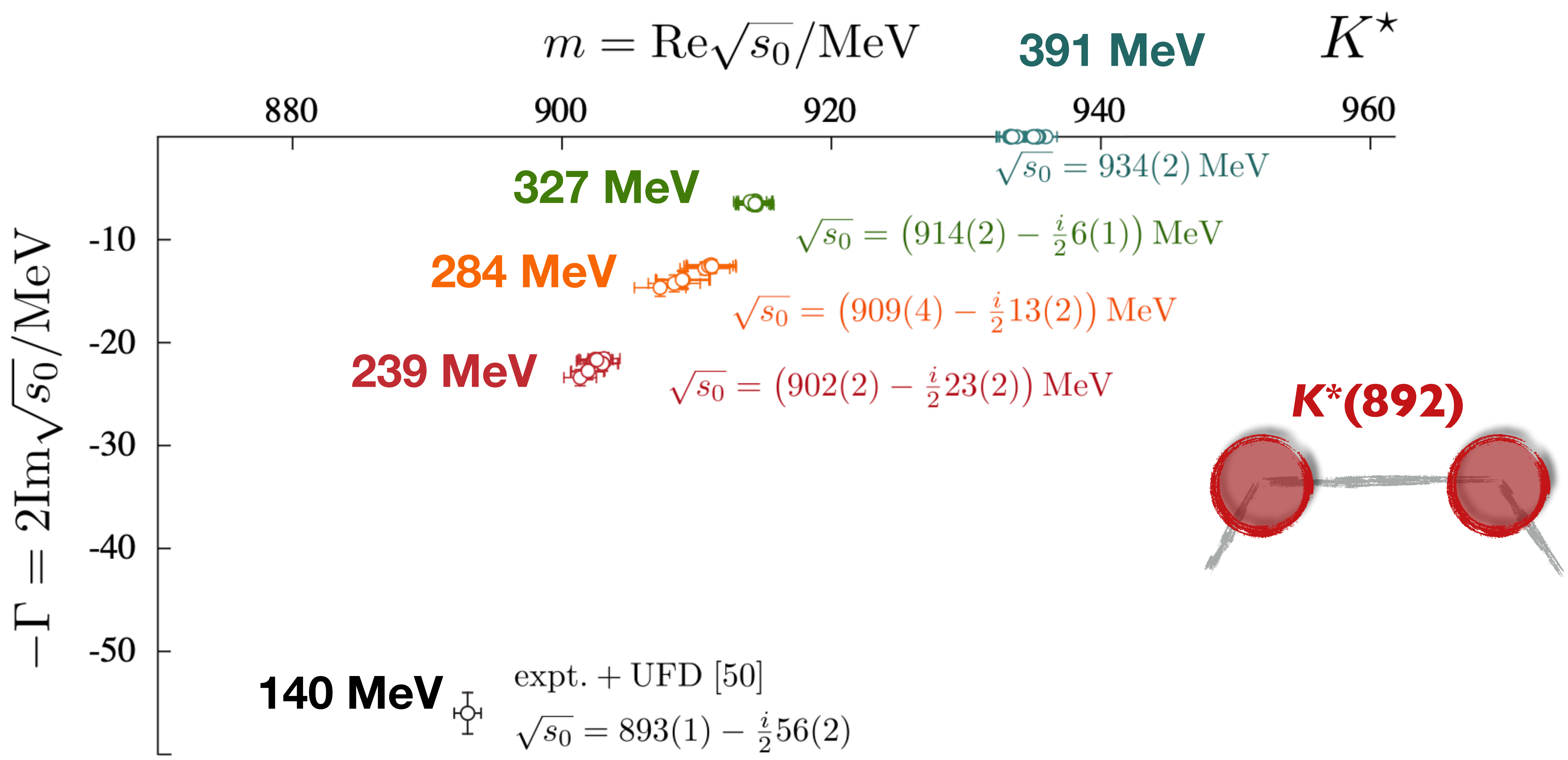
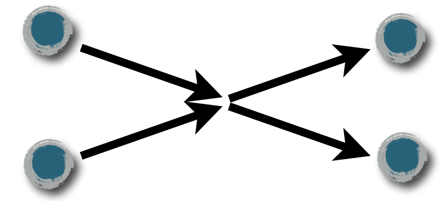


- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002*

Landscape of amplitudes

- [x] Two-to-two scattering: $2 \to 2$



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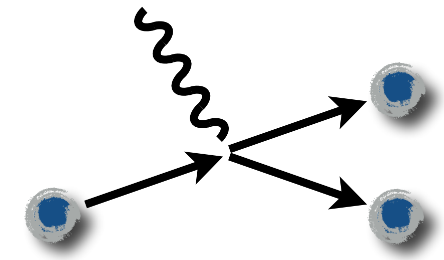

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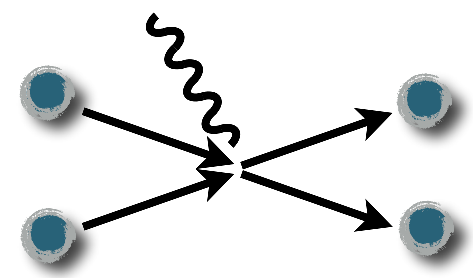

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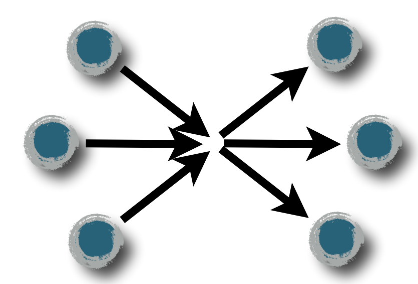
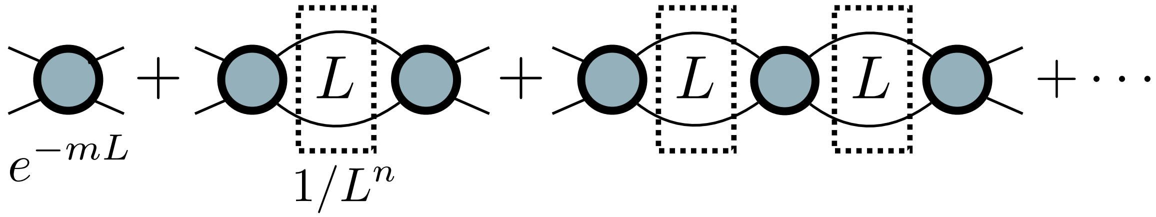


Slightly modified version ($i\epsilon$)

- Consider the finite-volume correlator:

$\mathcal{M}_L(P) =$ 

For two-particle energies $(2m)^2 < s < (4m)^2$, what is the *L* dependence?

$\mathcal{M}(s)$	$\mathcal{M}_L(P)$
probability amplitude	poles give f.v. spectrum

— propagating pion

● Bethe-Salpeter kernel

⬚ $= \sum_{\mathbf{k}}$

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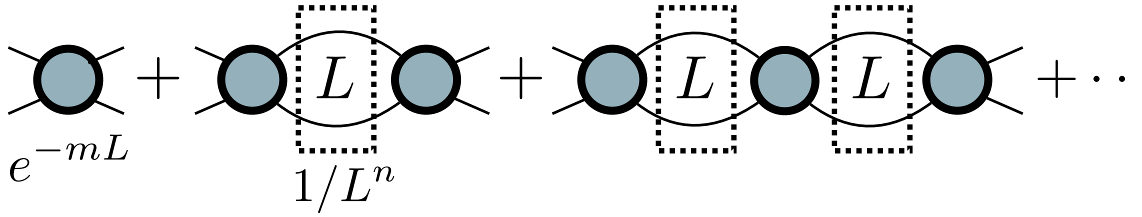

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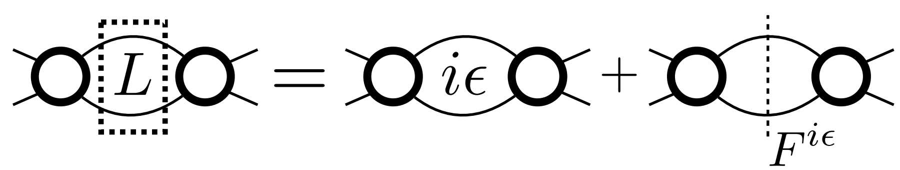

Cut projects loop to ***on-shell energies***

$F^{i\epsilon}$= matrix of known geometric functions

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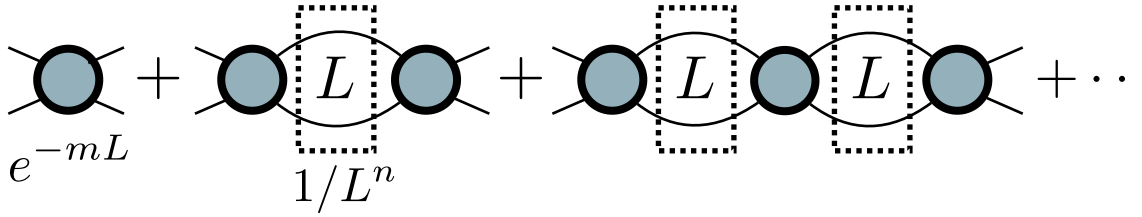
$\mathcal{M}_L(P) =$ 

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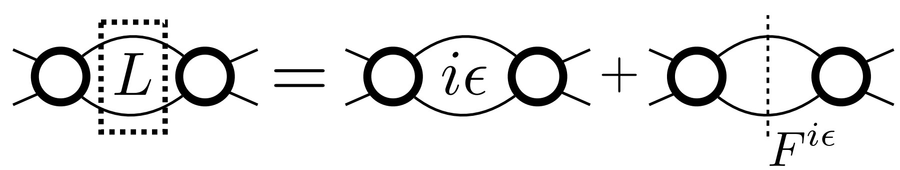

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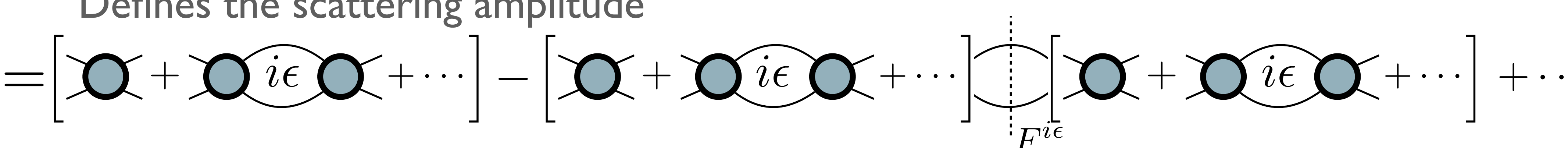
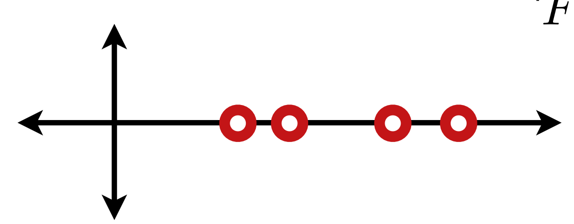
Defines the scattering amplitude



$$= \frac{1}{\mathcal{M}(s)^{-1} + F^{i\epsilon}(P, L)}$$

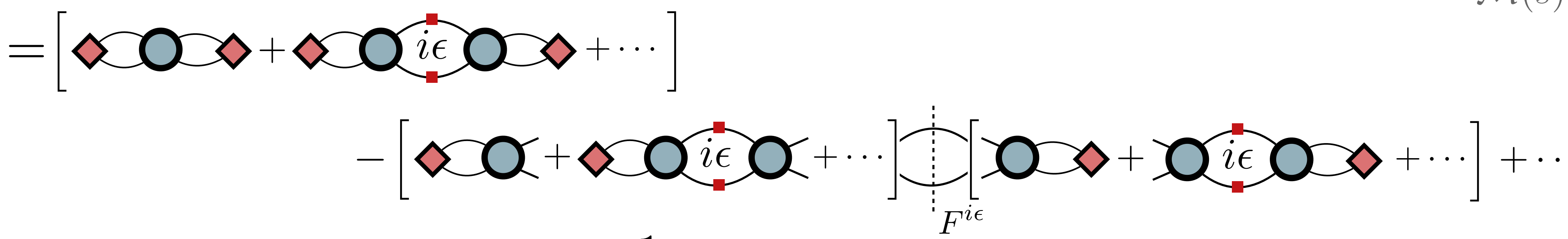


$1 + \mathscr{J} \to 2$

$$C_L(P) = \langle \pi | \mathcal{J}\mathcal{J} | \pi \rangle_L$$

Instead of this…

$$\mathcal{M}_L(P) = \mathcal{M}(s) - \mathcal{M}(s)\frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}}\mathcal{M}(s)$$



$$C_L(P) = C_\infty^{i\epsilon}(s) - A_{\text{in}}^{i\epsilon}(s)\frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}}A_{\text{out}}^{i\epsilon}(s)$$

The "endcaps" define the matrix element… $A_{\text{out}}^{i\epsilon}(s) = \langle \pi\pi, \text{out} | \mathcal{J} | \pi \rangle$

MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016) •

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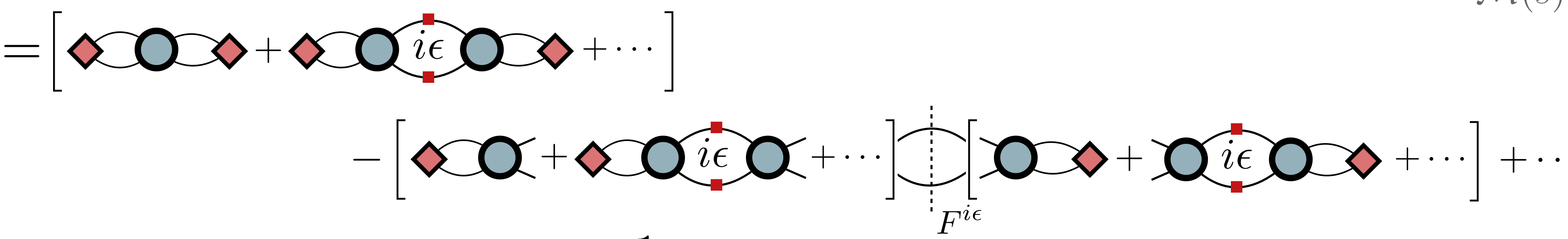

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Crucial information = residue at the pole

$$\mathcal{R}(P,L) = -\lim_{E \to E_n(L)} \frac{E - E_n(L)}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} = \frac{1}{\mu'(E)}\mathbf{v}^T\mathbf{v}$$

MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016) •

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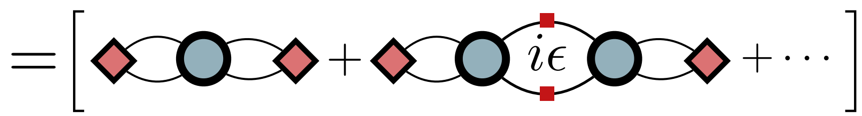

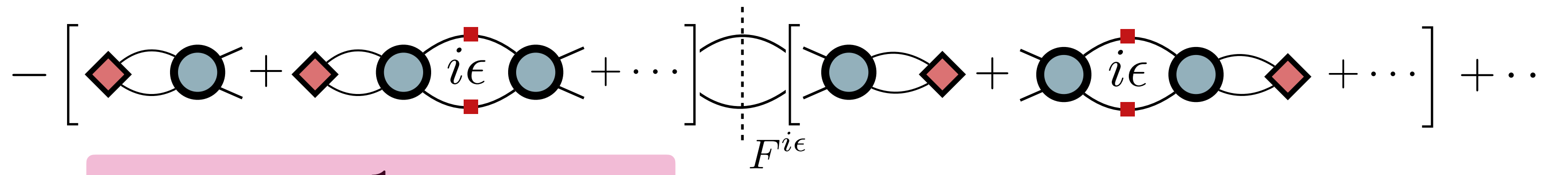

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Transition amplitudes

- An effective relation on states

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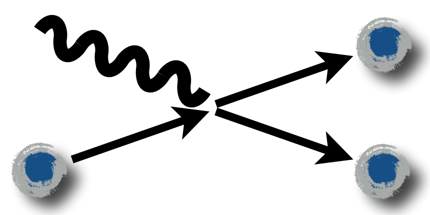
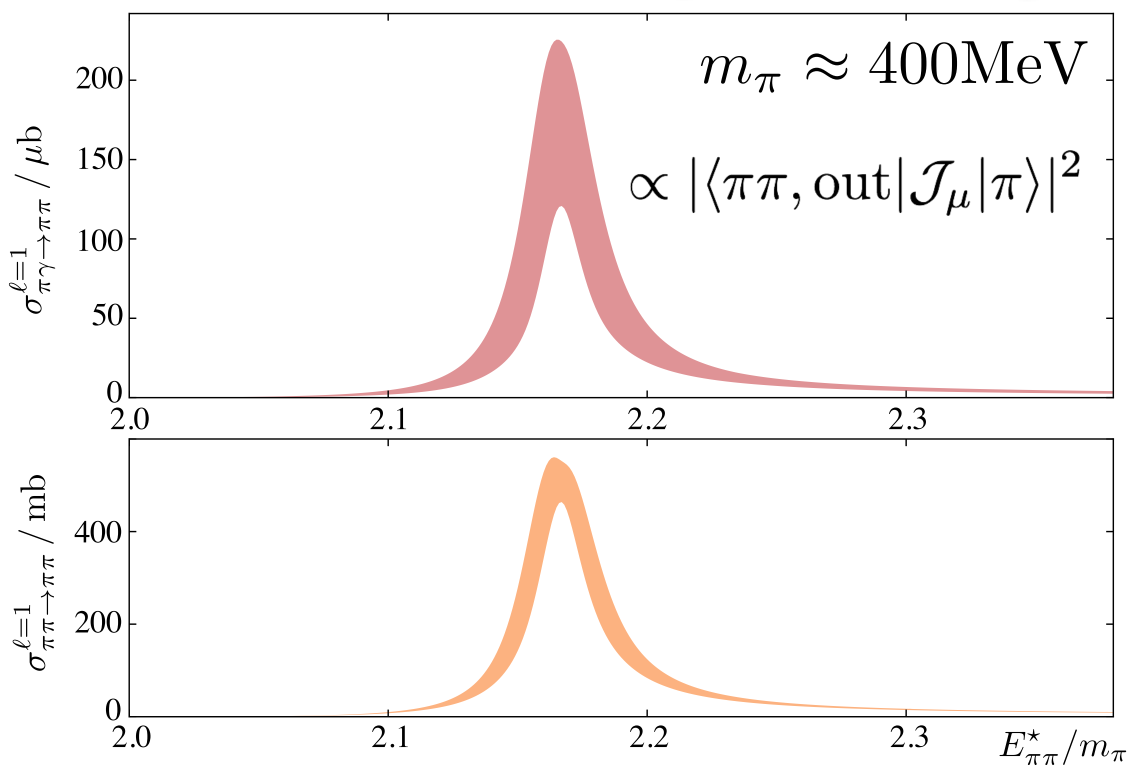
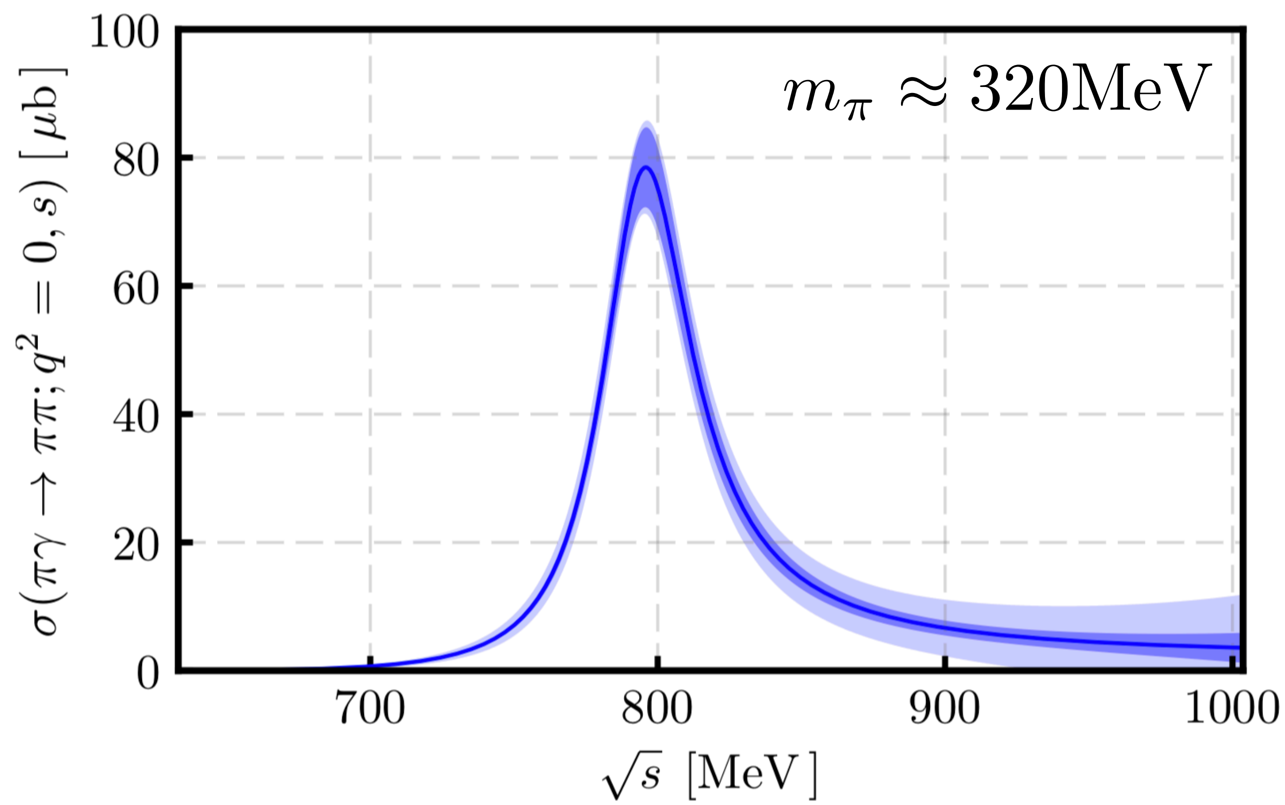




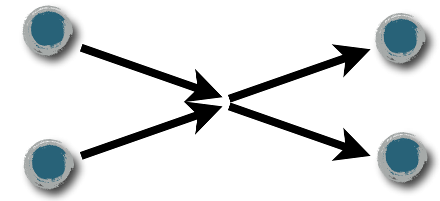
Briceño et. al., Phys. Rev. D93, 114508 (2016)



Alexandrou et. al., Phys. Rev. D98, 074502 (2018)

Landscape of amplitudes

- ☑ Two-to-two scattering: $2 \to 2$



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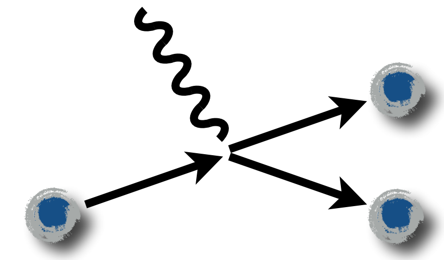

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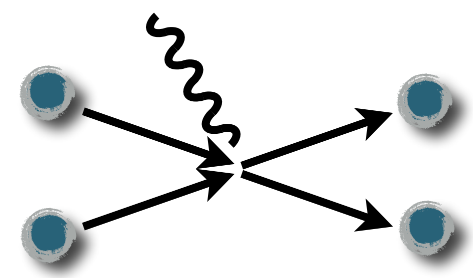

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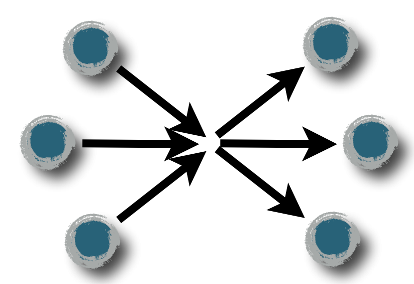
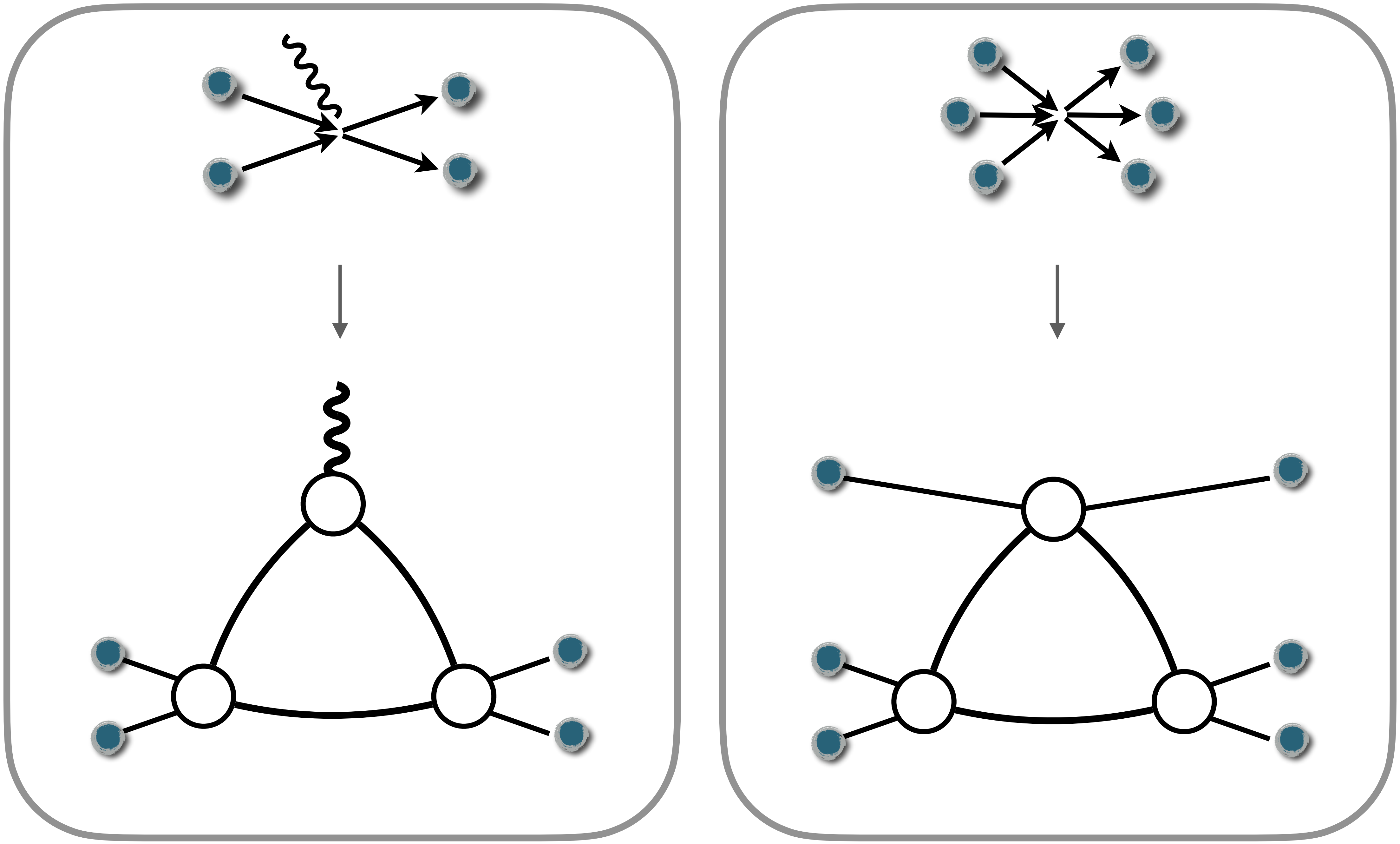


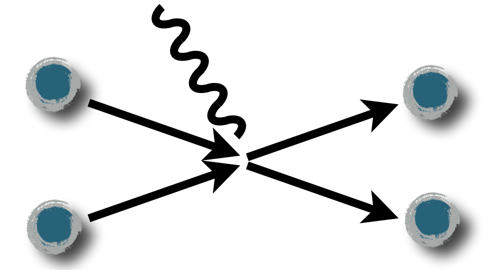
Physically observable subprocesses



$2 + \mathscr{J} \to 2$

- Formalism for multi-hadron form factors

$$\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi\pi, \mathrm{in} \rangle \equiv$$



- Continuation to the pole → ***resonance form factors***

Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

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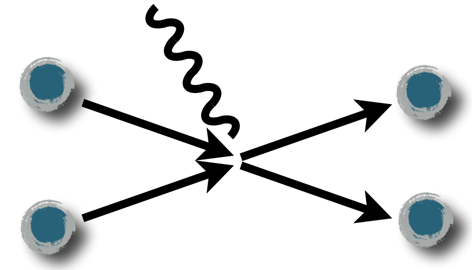
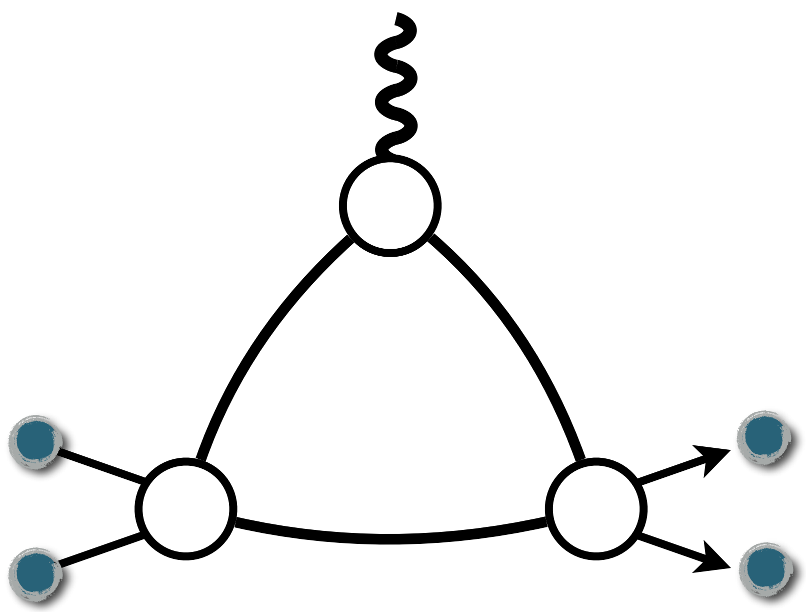
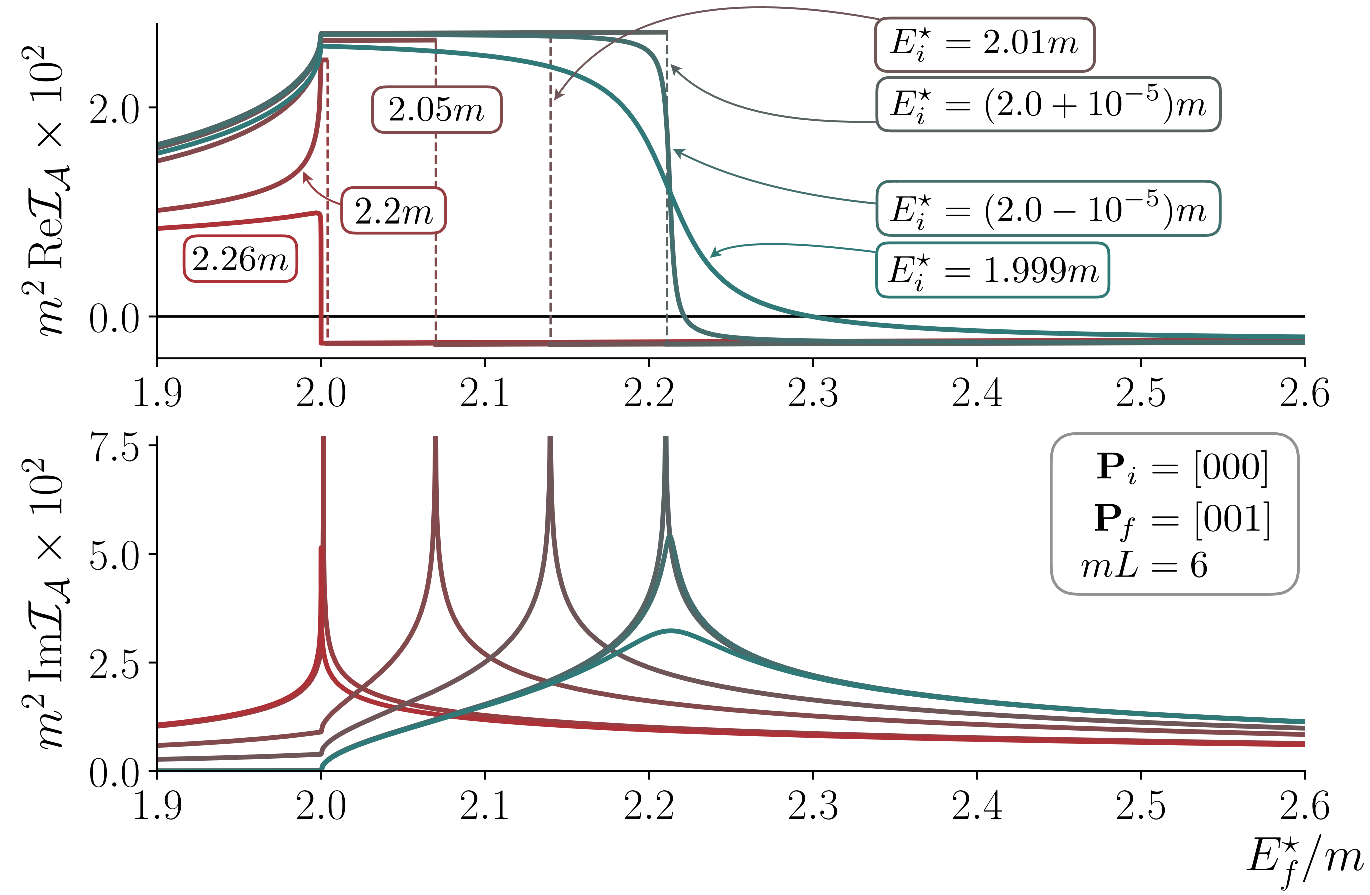


- Continuation to the pole → ***resonance form factors***
- Must carefully treat ***triangle*** singularities



$\sim$

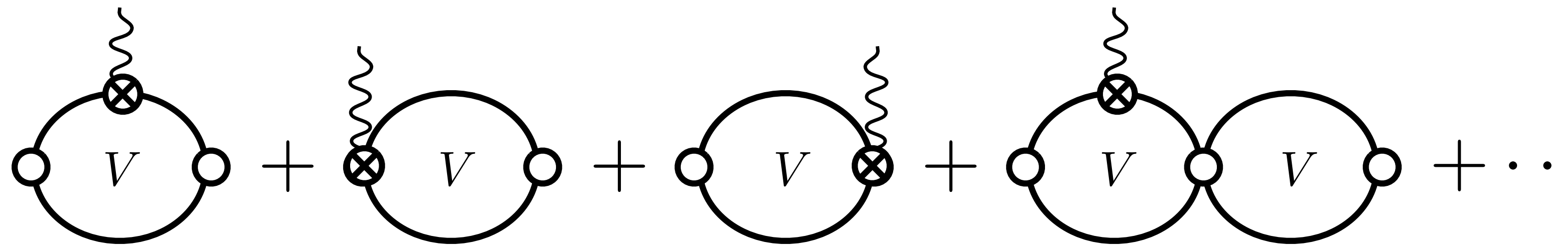


Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

In a nutshell

- By analysing an all orders skeleton expansion…

$$C_L^{2\to 2}(P_f, P_i) =$$



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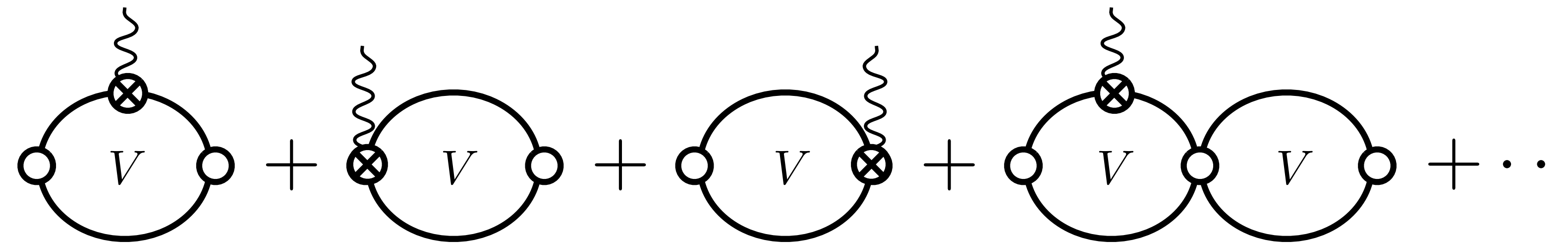
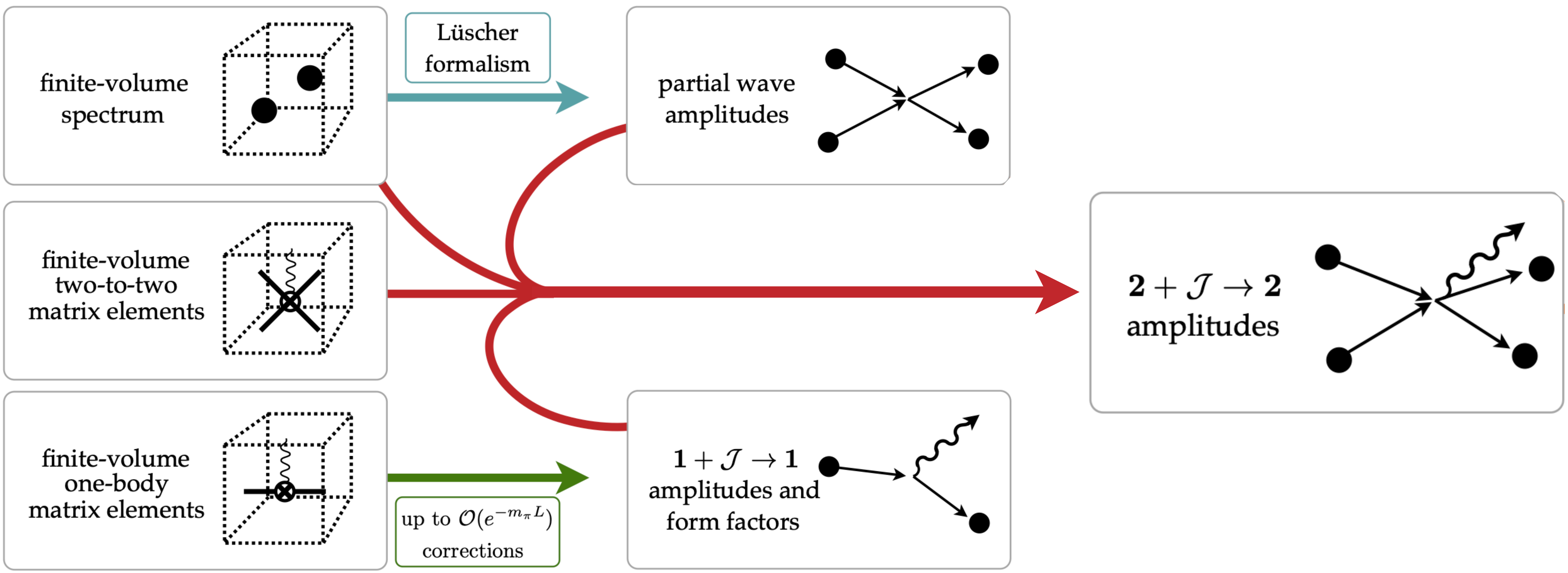
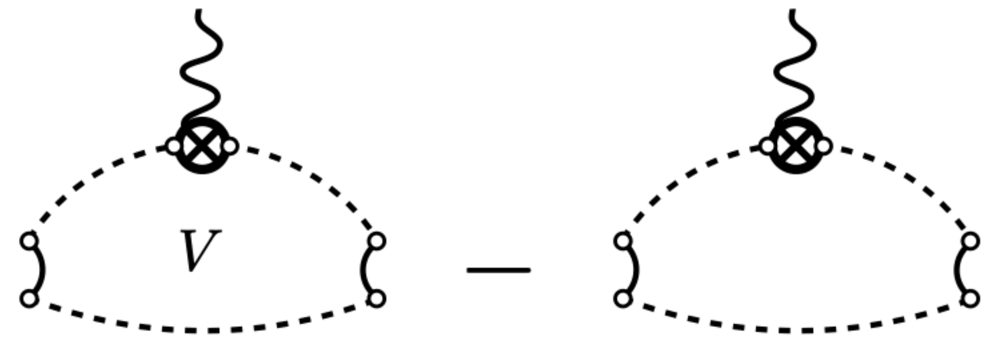


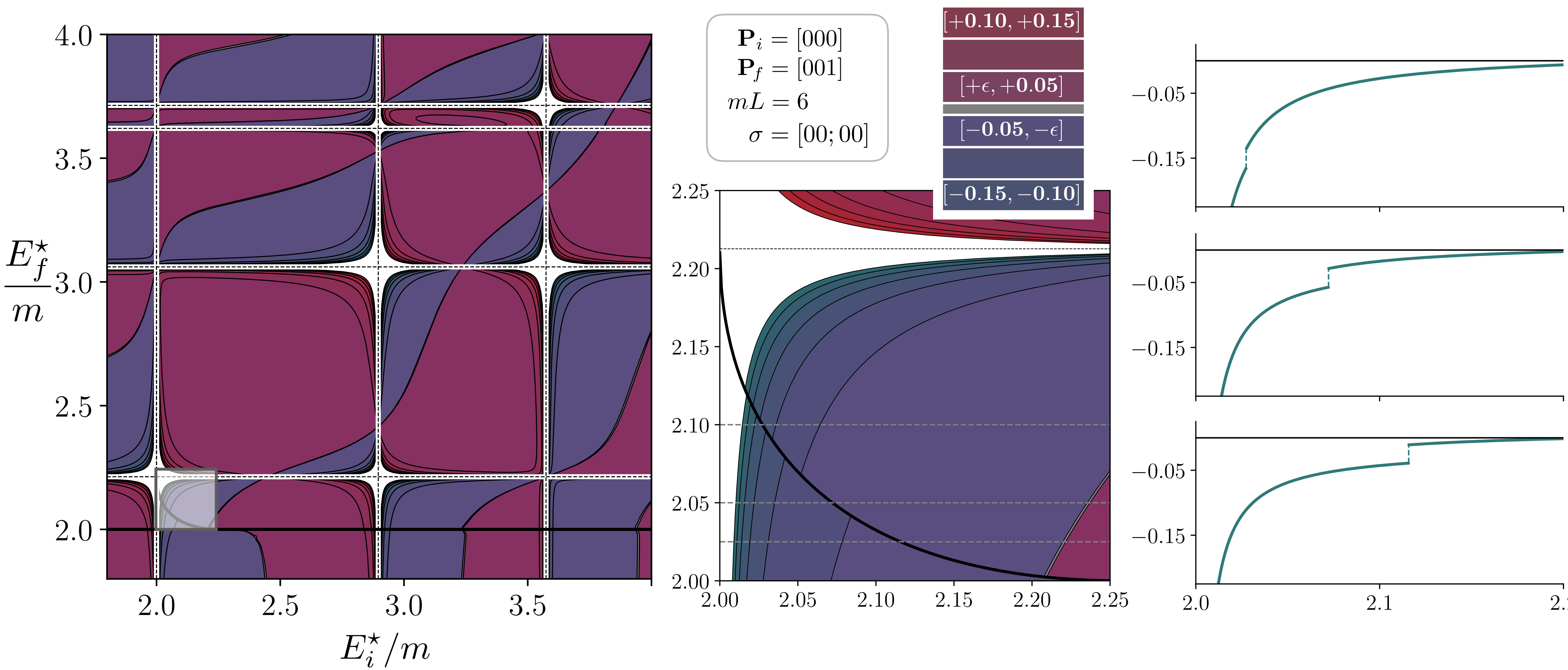
- … we derived a framework to calculate the $\mathbf{2} + \mathcal{J} \to \mathbf{2}$ amplitude



Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

New finite-volume function

$$G_{\mu_1 \cdots \mu_n}(P_f, P_i, L) =$$ 



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Landscape of amplitudes

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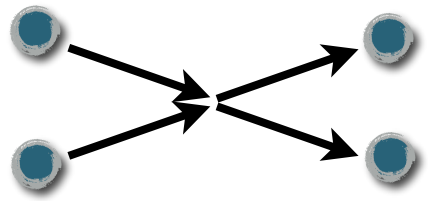

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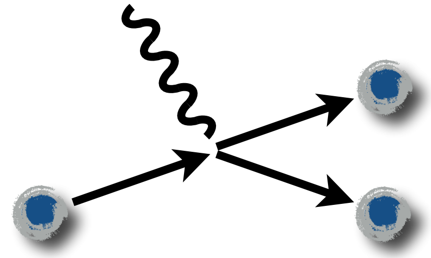

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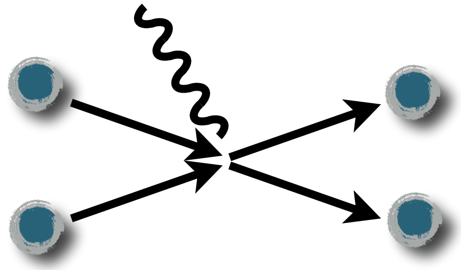

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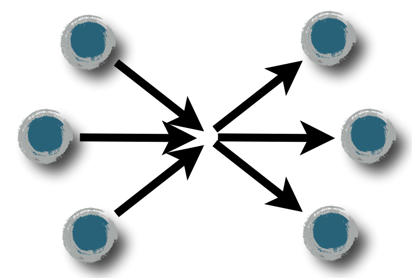


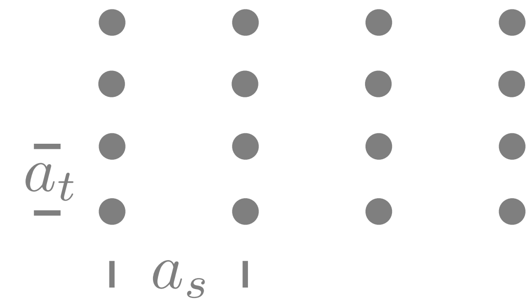
… skipping many details (in the backup slides)…

$$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$$

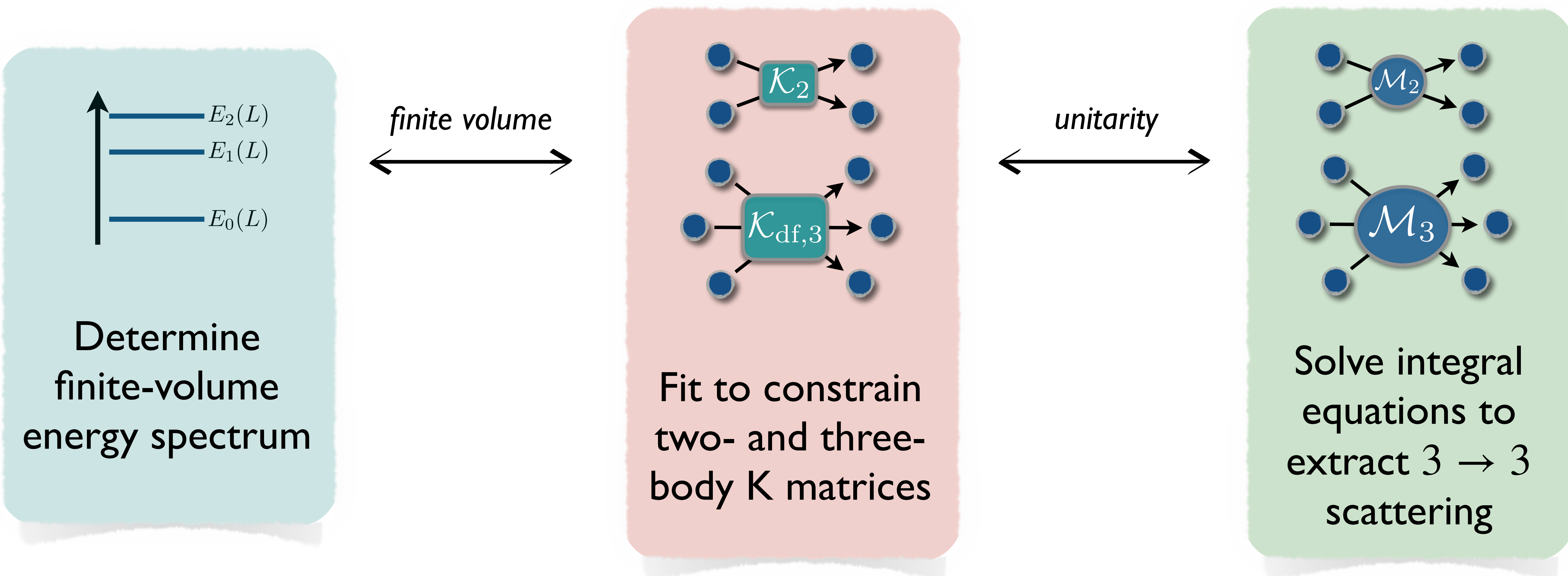
lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$



- Workflow outline

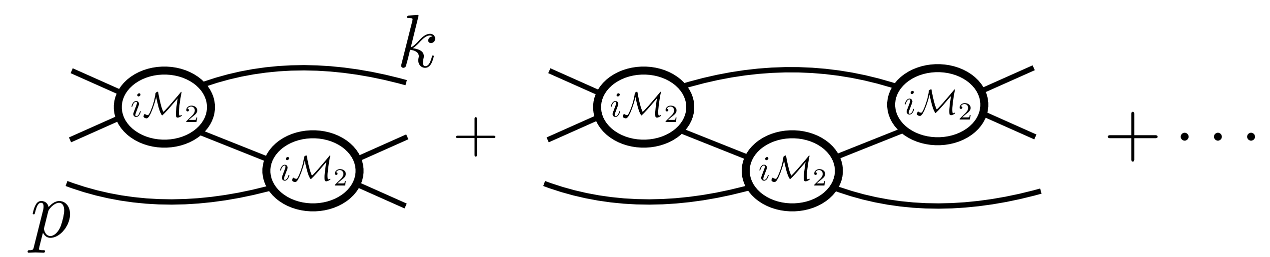


MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

… jump to the final step, just to give an idea of the integral equations predicting the amplitude…

Integral equation

$$\mathcal{M}_3^{\mathsf{un}}(E_3^\star,\boldsymbol{p},\boldsymbol{k}) = \mathcal{D}^{\mathsf{un}}(E_3^\star,\boldsymbol{p},\boldsymbol{k}) + \mathcal{E}^{\mathsf{un}}(E_3^\star,\boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\mathsf{un}}(E_3^\star,\boldsymbol{k})$$



Vanishes for $K_{\mathrm{df},3} = 0$

$$\boldsymbol{D}(N,\epsilon) = -\boldsymbol{\mathcal{M}}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{P}\cdot\boldsymbol{D}(N,\epsilon)$$

$$\mathcal{D}^{\mathsf{un}}(E_3^\star,\boldsymbol{p},\boldsymbol{k}) = \lim_{\epsilon\to0}\lim_{N\to\infty}\boldsymbol{D}_{pk}(N,\epsilon)$$

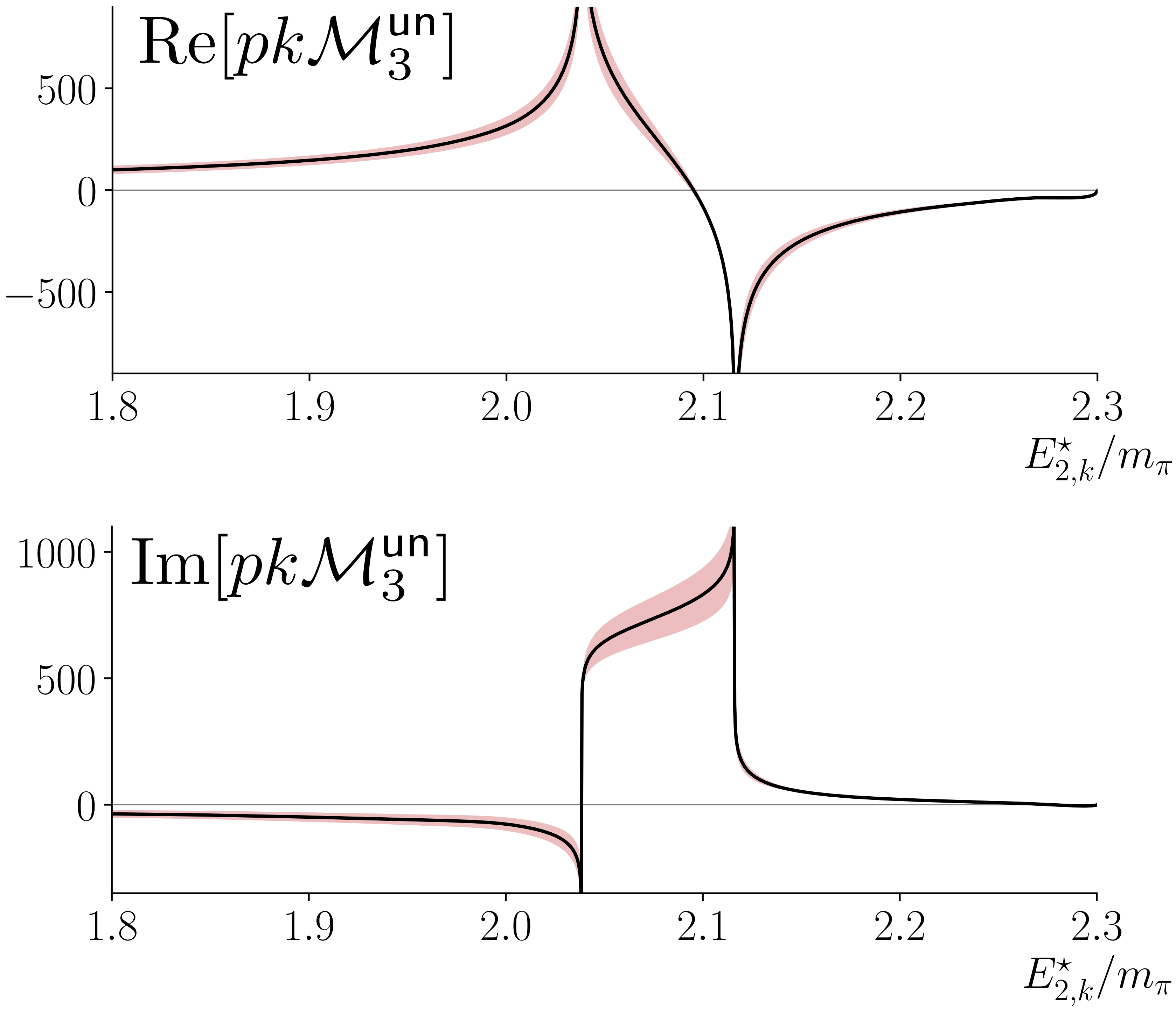
- See also…

Solving relativistic three-body integral equations in the presence of bound states

Andrew W. Jackura,[1,2,*] Raúl A. Briceño,[1,2,†] Sebastian M. Dawid,[3,4,‡] Md Habib E Islam,[2,§] and Connor McCarty[5,¶]

arXiv: 2010.09820

Integral equation



Total angular momentum = 0

Two-particle sub-system angular momentum = 0

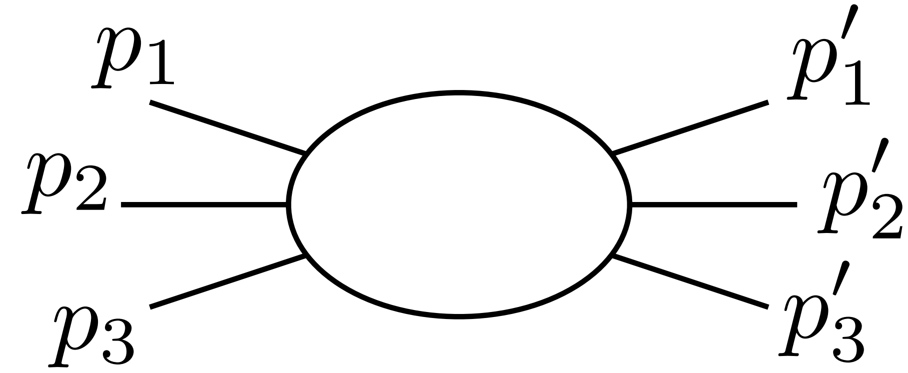
Plot at fixed $E_3^\star$ and p

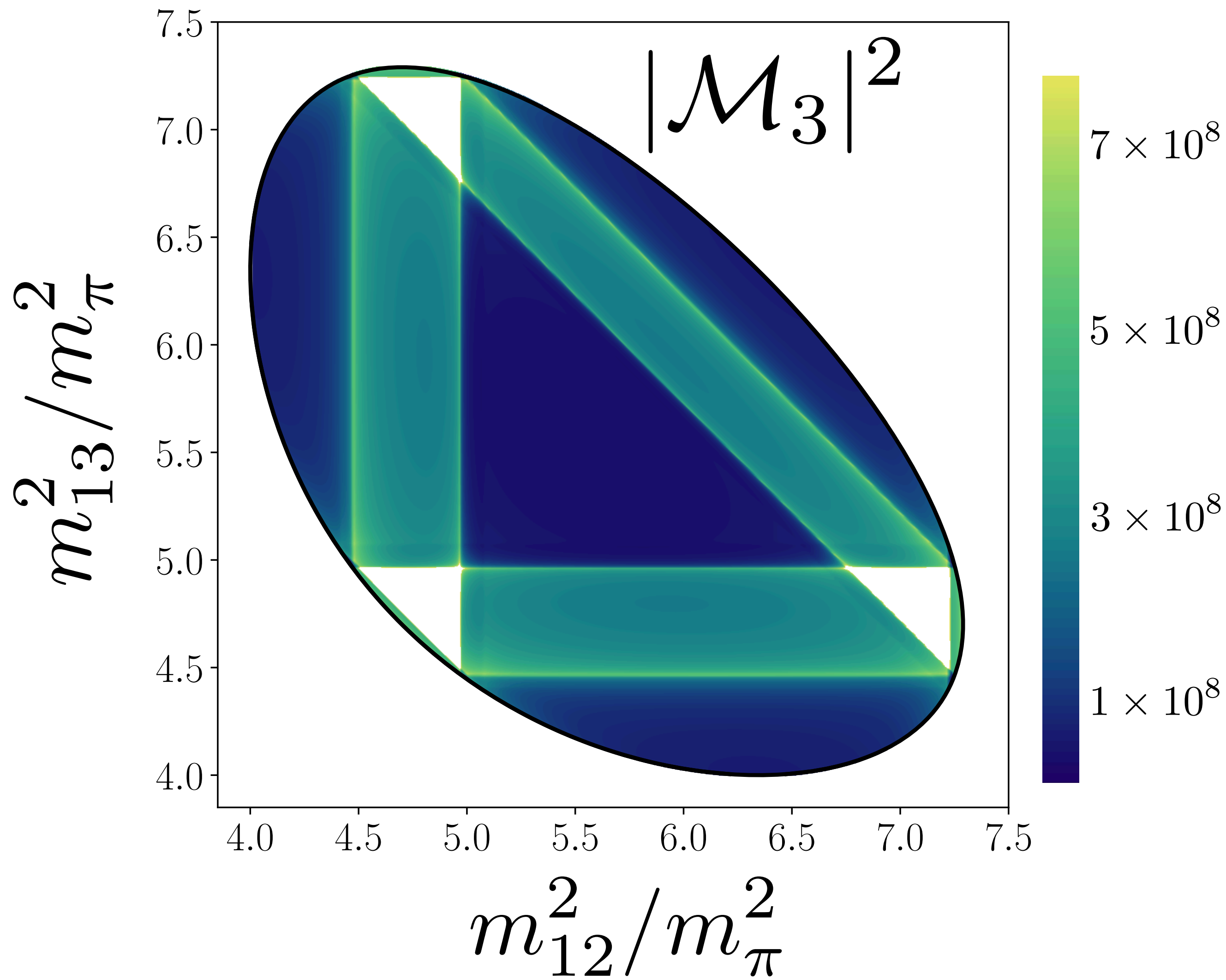
Both two- and three-body uncertainties estimated

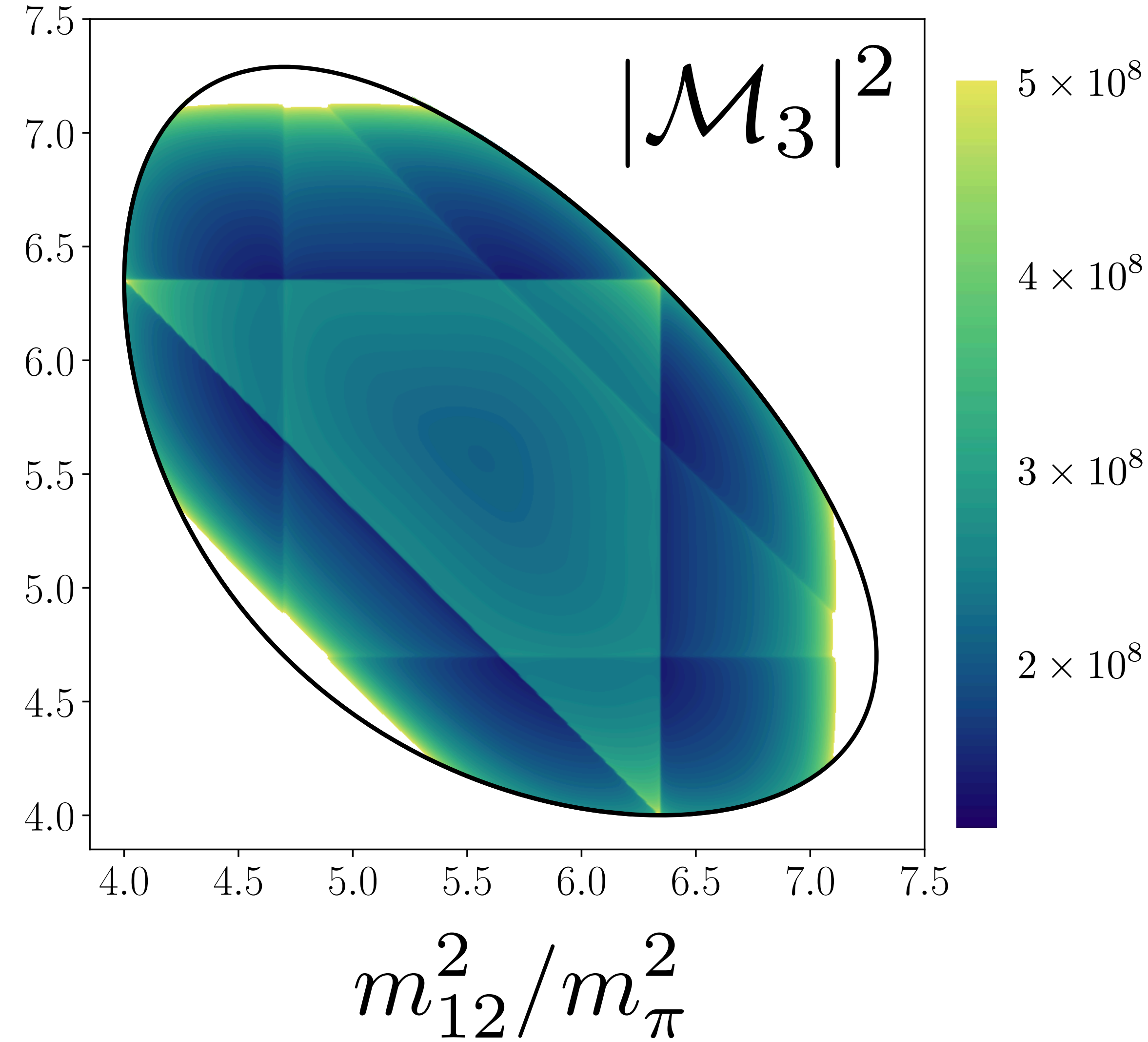
Still need to symmetrize

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\mathcal{M}_3 = \sum_{i,j\in\{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p_i', p_j)$$







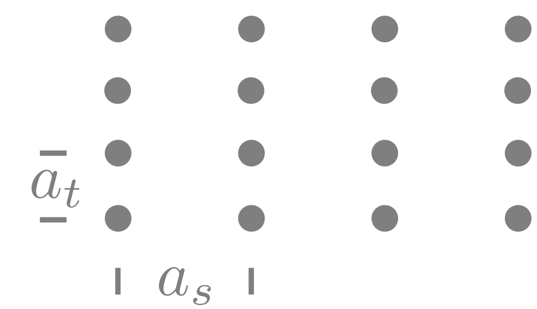
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$$

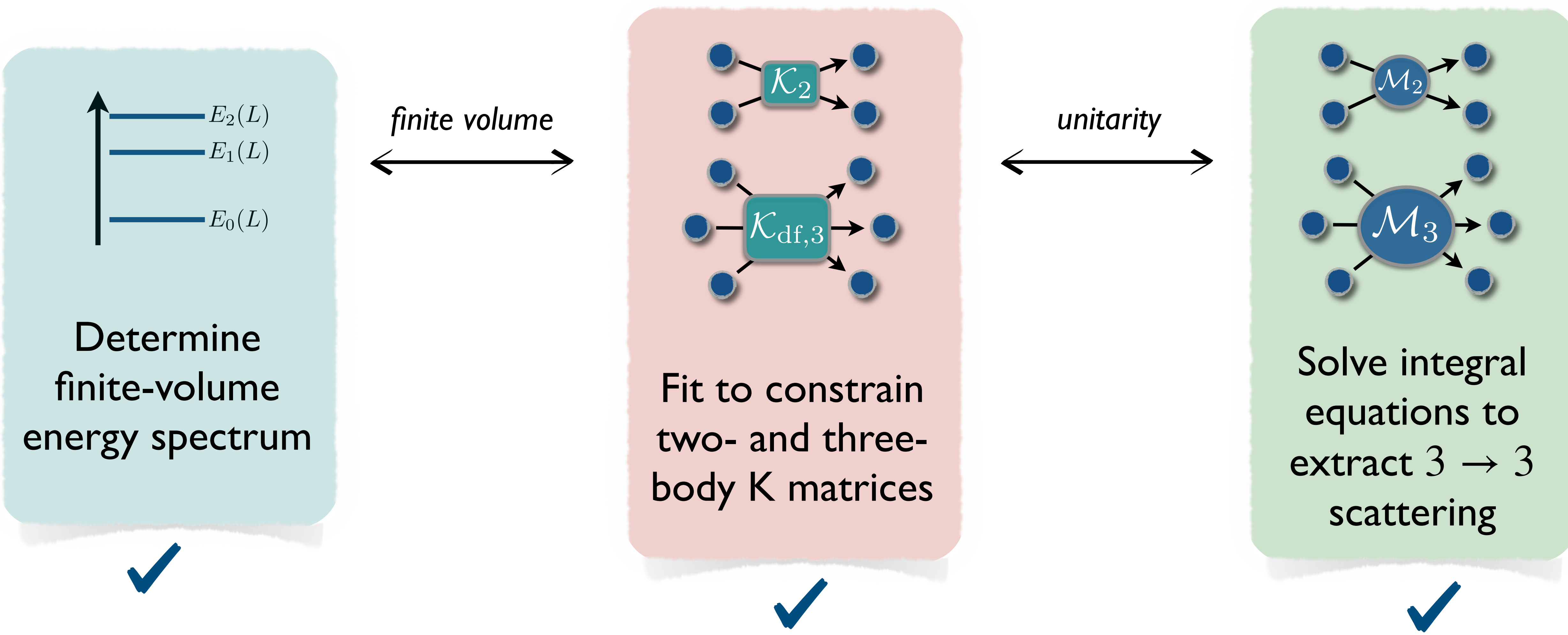
lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$



❑ Workflow outline



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

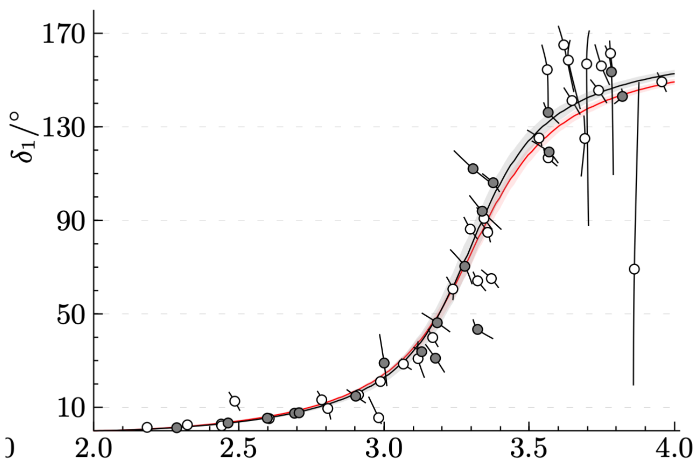
Conclusions

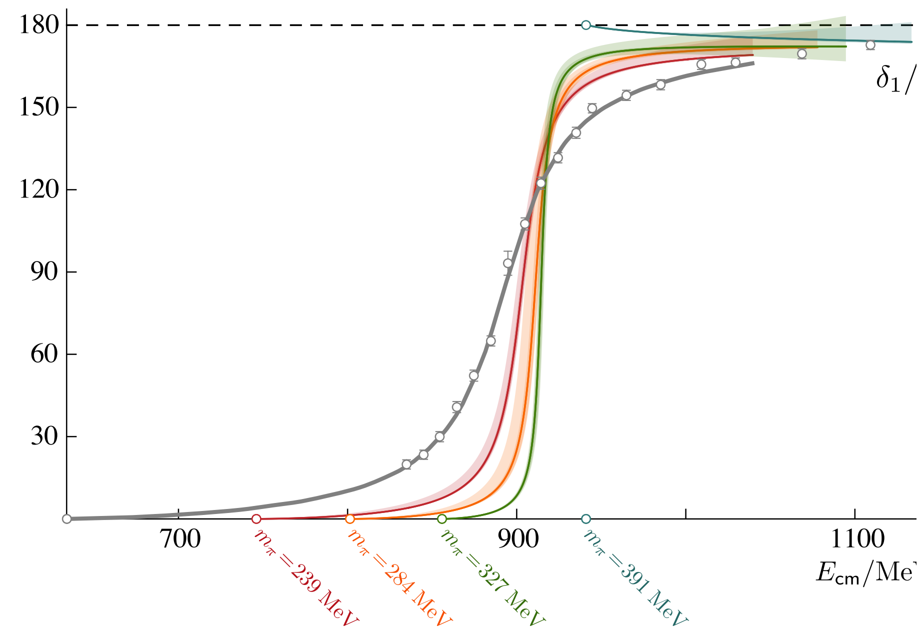
- LQCD is in the era of ‘rigorous resonance spectroscopy’
- The finite-volume = ***a useful tool***
- Relations between finite-volume data and amplitudes necessarily restore anomalous thresholds
- Next steps…

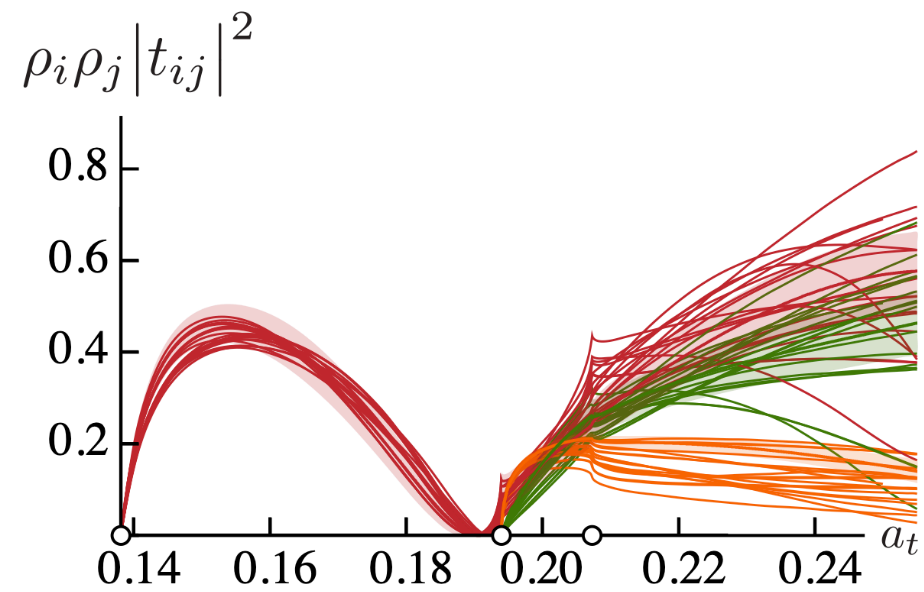
  complete 3-particle formalism → *extend to N-particle formalism*

  extend studies involving an external current

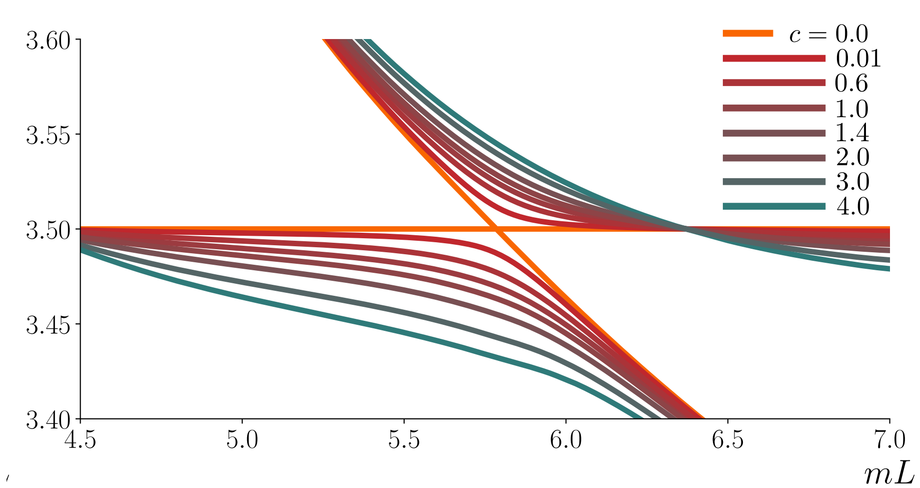
  precision

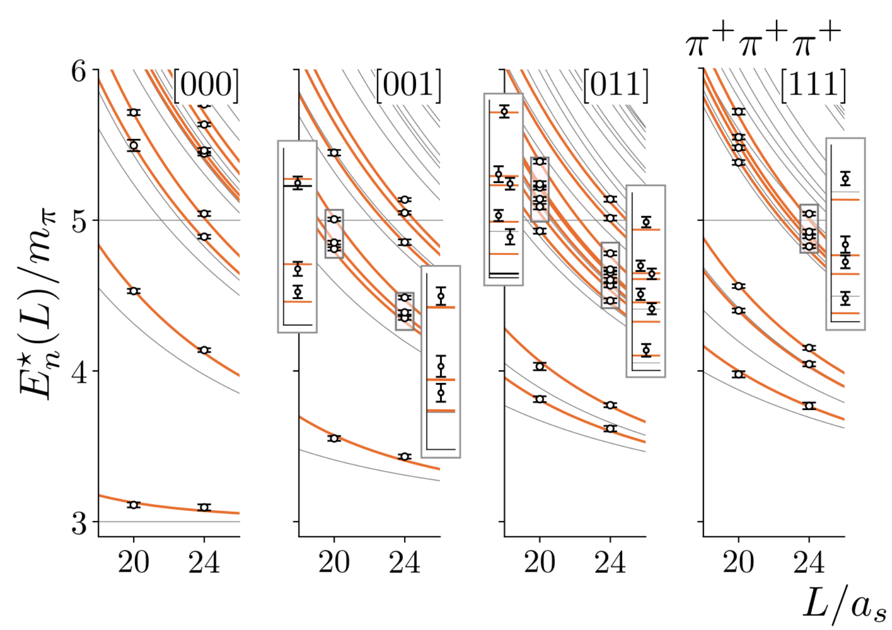


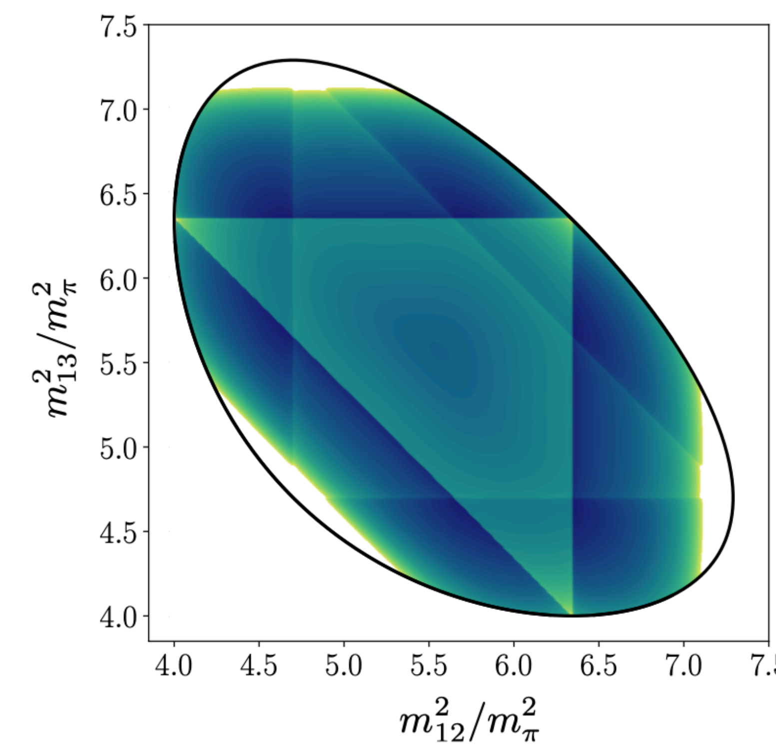




Thanks for listening!







Back-up slides

Two strategies…

- ☑ Finite-volume as a tool
  - LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

  - Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***
  - Applicable only in limited energy range for two- and three-hadron states

- ☐ Spectral function method
  - Formally applies for any number of particles / any energy range
  - An answer to the question… "Can't you just analytically continue?"
  - Still important challenges and limitations to consider

Two strategies…

- [x] Finite-volume as a tool
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Correlation functions → observables

- Lattice QCD gives finite-volume Euclidean correlators

$$\langle 0|\, \mathcal{O}_1(0)\, e^{-\hat{H}\tau}\, \mathcal{O}_2(0)\, |0\rangle_L$$

have

- Complete physical information is contained in…

$$\langle 0|\, \mathcal{O}_1(0)\, f(\hat{H})\, \mathcal{O}_2(0)\, |0\rangle_\infty$$

want

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$$\langle 0|\, \mathcal{O}_1(0)\, f(\hat{H})\, \mathcal{O}_2(0)\, |0\rangle_\infty$$

want

- Detailed choice of *f(E)* and operators determines the observable

R-ratio

$$\langle 0|\, j_\mu(0)\, \delta(\hat{H} - \omega)\, j_\mu(0)\, |0\rangle_\infty$$

Meyer • Bailas, Hashimoto, Ishikawa (2020)
Alexandrou et al. (2022)

D-meson total lifetime

$$\langle D|\, \mathcal{H}_W(0)\, \delta(M_D - \hat{H})\, \mathcal{H}_W(0)\, |D\rangle_\infty$$

MTH, Meyer, Robaina (2017)

π π → π π amplitude

$$\langle \pi|\, \pi(0)\, \frac{1}{E - \hat{H} + i\epsilon}\, \pi(0)\, |\pi\rangle_\infty$$

j → π π amplitude

$$\langle \pi|\, \pi(0)\, \frac{1}{E - \hat{H} + i\epsilon}\, j_\mu(0)\, |0\rangle_\infty$$

Bulava, MTH (2019)

Linear reconstruction

$$\langle \mathcal{O}(0) e^{-\hat{H}\tau} \mathcal{O}(0) \rangle = \int d\omega\, e^{-\omega\tau} \, \langle \mathcal{O}(0) \delta(\omega - \hat{H}) \mathcal{O}(0) \rangle$$

have — want

$$G(\tau) = \int d\omega\, e^{-\omega\tau} \, \rho(\omega)$$

have — want

- ***Linear, model-independent reconstruction*** (e.g. Backus-Gilbert-like, Chebyshev)

$$\sum_\tau \mathcal{K}(\bar{\omega}, \tau) G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega}, \tau) \int d\omega\, e^{-\omega\tau} \, \rho(\omega)$$

Linear reconstruction

$$\langle \mathcal{O}(0) e^{-\hat{H}\tau} \mathcal{O}(0) \rangle = \int d\omega \, e^{-\omega\tau} \, \langle \mathcal{O}(0) \delta(\omega - \hat{H}) \mathcal{O}(0) \rangle$$

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$$\sum_\tau \mathcal{K}(\bar{\omega}, \tau) G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega}, \tau) \int d\omega \, e^{-\omega\tau} \, \rho(\omega) = \int d\omega \, \Big[\sum_\tau \mathcal{K}(\bar{\omega}, \tau) \, e^{-\omega\tau} \Big] \, \rho(\omega)$$

$$= \int d\omega \, \widehat{\delta}_\Delta(\bar{\omega}, \omega) \, \rho(\omega)$$

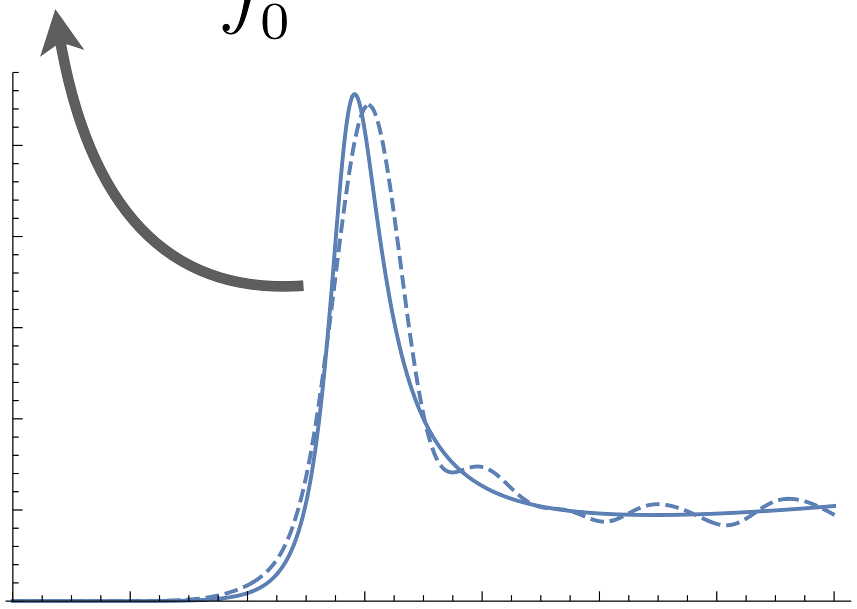
δ is exactly known

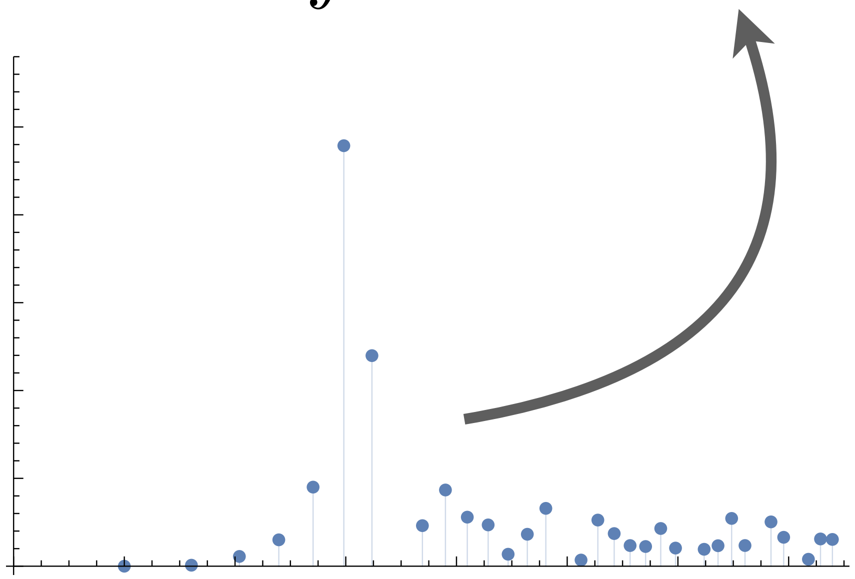
- Non-linear (***not discussed here…***)
  - Maximum Entropy Method (MEM)
  - Direct fits
  - Neural networks

See multiple ECT and CERN workshops, work by Aarts, Allton, Amato, Brandt, Burnier, Del Debbio, Francis, Giudice, Hands, Harris, Hashimoto, Jäger, Karpie, Liu, Meyer, Monahan, Orginos, Robaina, Rothkopf, Ryan, …*

Role of the finite volume

$$\widehat{\rho}_L(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega) \qquad G_L(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho_L(\omega)$$





- Any reconstructed spectral function that ≠ forest of deltas…
  contains implicit smearing (or else $L \to \infty$)

We require…

$$1/L \ll \Delta \ll \mu_{\text{physical}}$$

smearing function covers many delta peaks

smearing does not overly distort observable

MTH, Meyer, Robaina (2017)

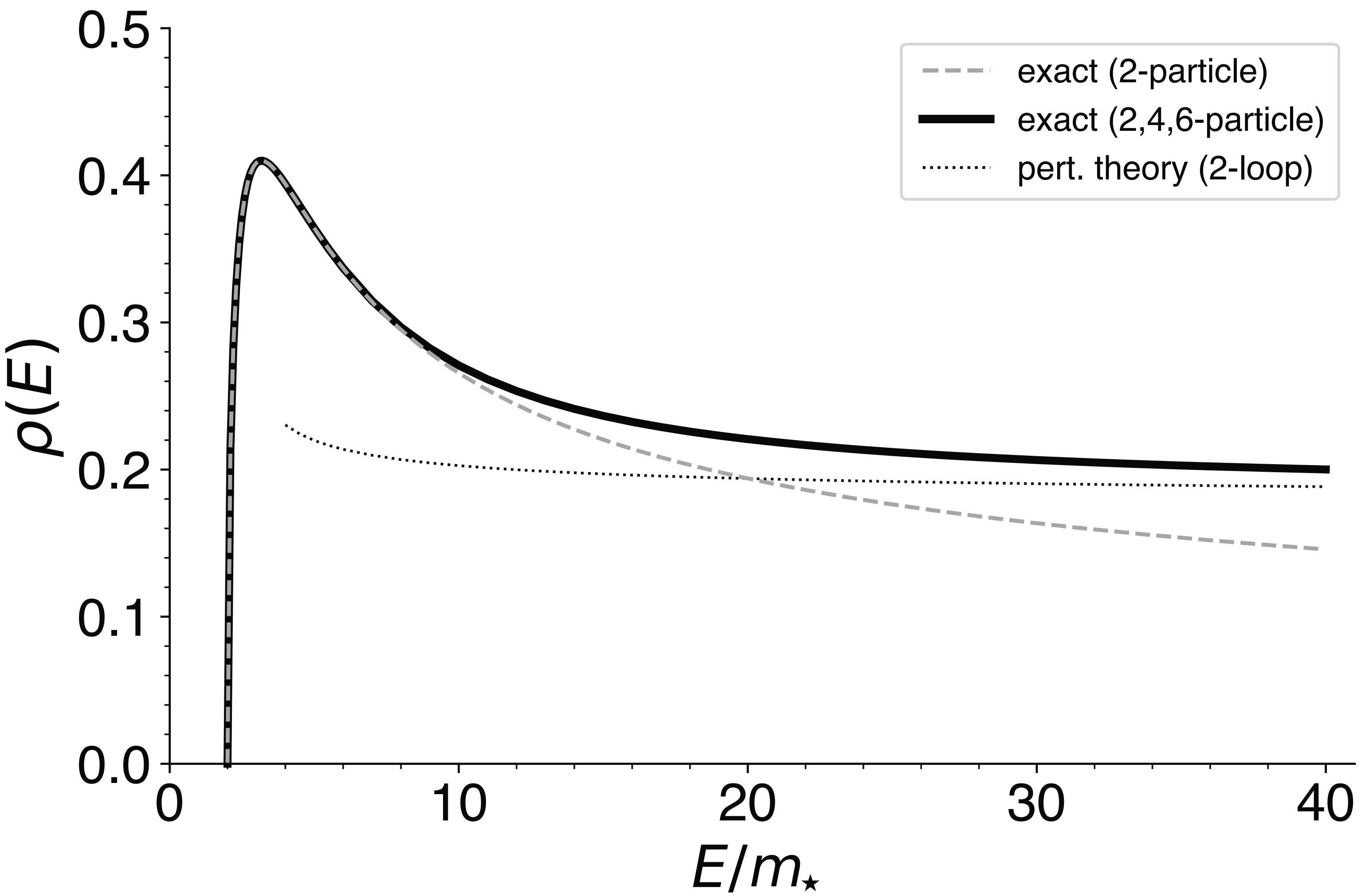
1+1 O(3) Model

- *Integrable* theory with some nice similarities to QCD
  - Asymptotically free
  - Dynamically generated mass gap
  - Iso-spin like symmetry
  - Conserved iso-vector vector current

$$S[\sigma] = \frac{1}{2g^2} \int d^2x\, \partial_\mu \sigma(x) \cdot \partial_\mu \sigma(x)$$

$$j^c_\mu(x) = \frac{1}{g^2} \epsilon^{abc} \sigma^a(x) \partial_\mu \sigma^b(x)$$

conserved current



$$\rho(E) = 2\pi \langle \Omega | \hat{j}^a_1(0)\, \delta^2(\hat{P} - p)\, \hat{j}^a_1(0) | \Omega \rangle$$

spectral function

$$\rho^{(2)}(E) = \frac{3\pi^3}{8\theta^2} \frac{\theta^2 + \pi^2}{\theta^2 + 4\pi^2} \tanh^3 \frac{\theta}{2}$$

$$\theta = 2 \cosh^{-1} \frac{E}{2m}$$

Smeared spectral function vs analytic result

- Construct different smearings of $\rho(\omega)$

$$\rho_\epsilon^\lambda(E) = \int_0^\infty d\omega\, \delta_\epsilon^\lambda(E,\omega)\, \rho(\omega)$$

$$\delta_\epsilon^{\mathrm{g}}(x) = \frac{1}{\sqrt{2\pi}\epsilon} \exp\left[-\frac{x^2}{2\epsilon^2}\right], \qquad \delta_\epsilon^{\mathrm{c0}}(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2},$$

$$\delta_\epsilon^{\mathrm{c1}}(x) = \frac{2}{\pi}\frac{\epsilon^3}{(x^2+\epsilon^2)^2}, \qquad \delta_\epsilon^{\mathrm{c2}}(x) = \frac{8}{3\pi}\frac{\epsilon^5}{(x^2+\epsilon^2)^3}.$$

Bulava, MTH, Hansen, Patella, Tantalo (2021)

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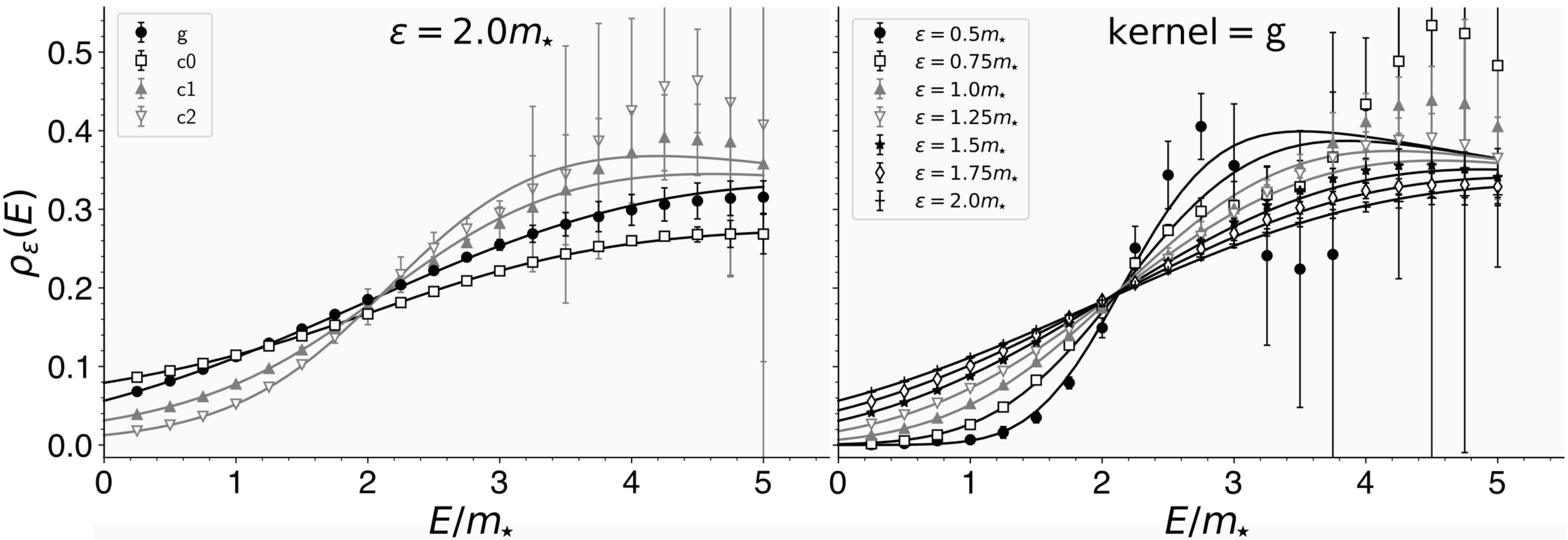
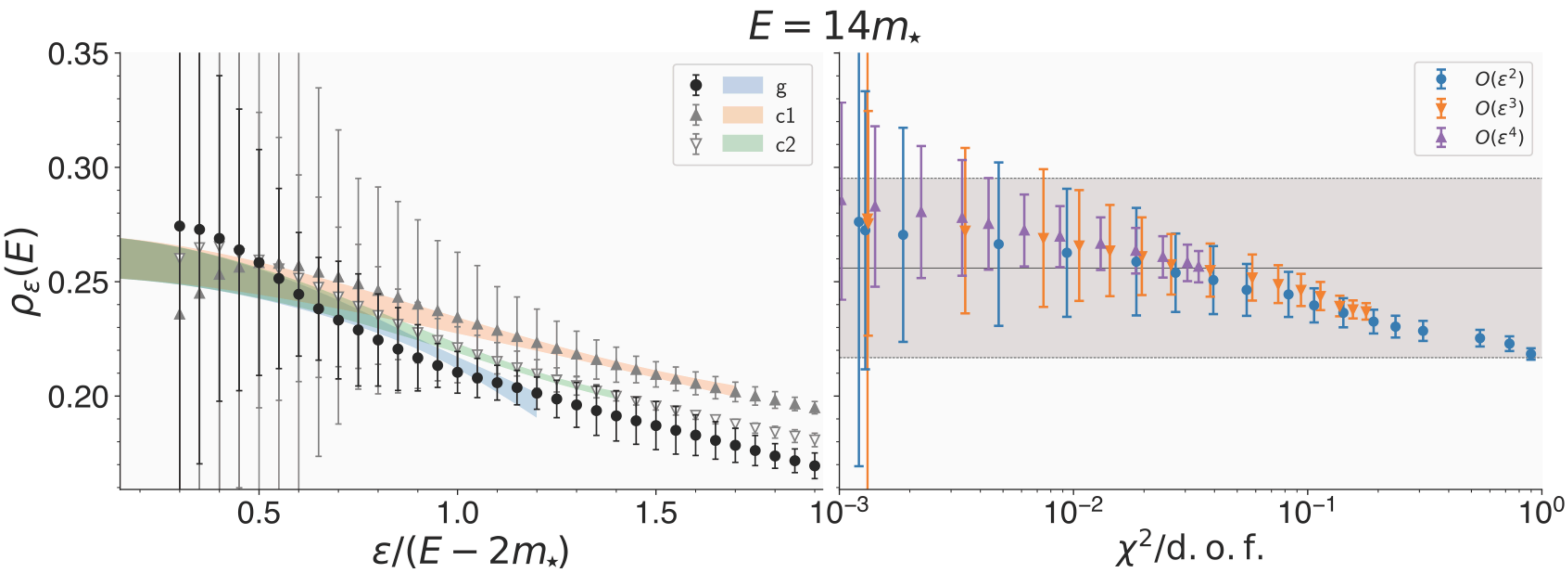


Bulava, MTH, Hansen, Patella, Tantalo (2021)

Extrapolation

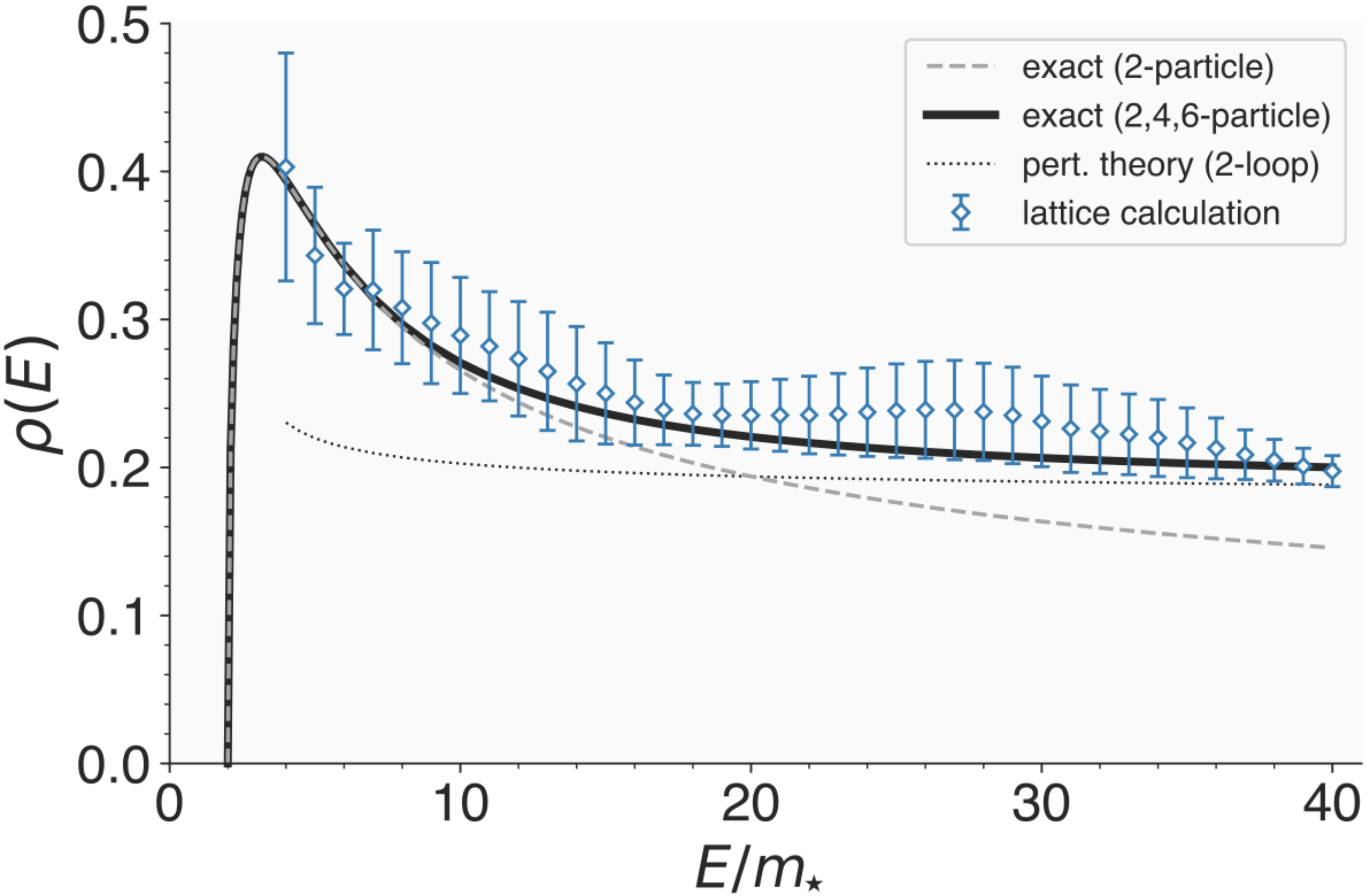
- Targeting $\rho(E)$ for $E = 14m$ here



- Use known relations between different smearing kernels

Bulava, MTH, Hansen, Patella, Tantalo (2021)

Result

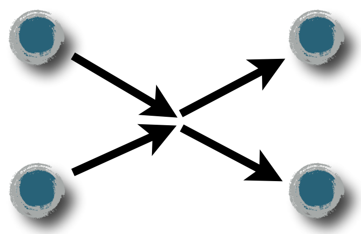


Bulava, MTH, Hansen, Patella, Tantalo (2021)

Many QCD applications already published… see work by A. Barone, S. Hashimoto, A. Jüttner, T. Kaneko, R. Kellermann, R. Frezzotti, G. Gagliardi, V. Lubicz, F. Sanfilippo, S. Simula …

Three-particle scattering

Complication: degrees of freedom



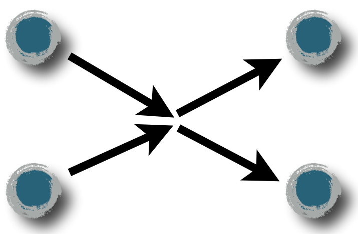
12 momentum
components

-10 Poincaré generators

2 degrees of freedom

$$\vec{p}_1 + \vec{p}_2 \ \rightarrow \ \vec{p}_3 + \vec{p}_4 \quad \longrightarrow \quad \text{Mandelstam } s, t$$

Complication: degrees of freedom



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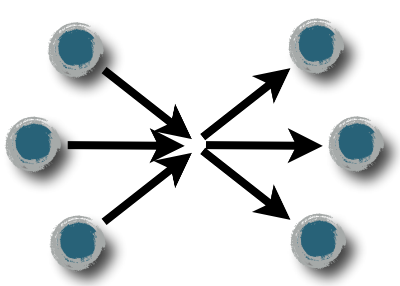
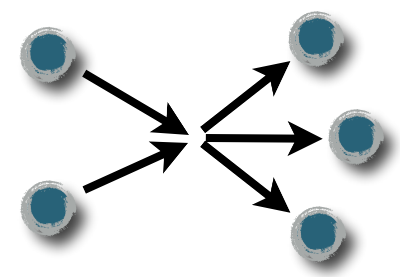


18 momentum components

-10 Poincaré generators

8 degrees of freedom



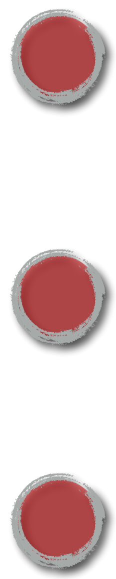
15 momentum components

-10 Poincaré generators

5 degrees of freedom

Complication: on-shell states

- Classical pairwise scattering



Complication: on-shell states

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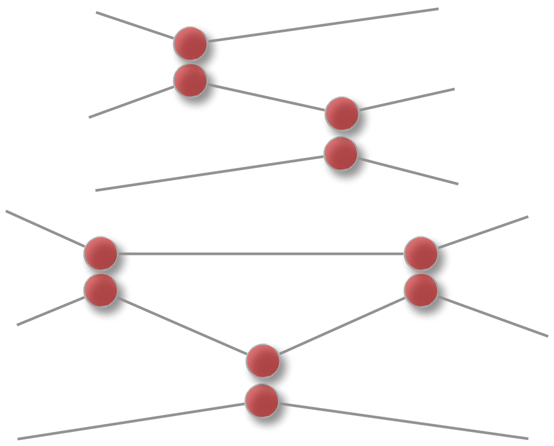
- Classical pairwise scattering

for $m_1 = m_2 = m_3$ up to ***3***
binary collisions are possible



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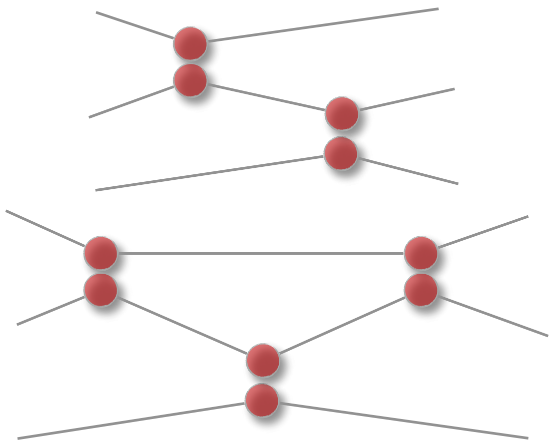

Dispersion Relations for Three-Particle Scattering Amplitudes. I*

MORTON RUBIN
Physics Department, University of Wisconsin, Madison, Wisconsin

AND

ROBERT SUGAR
Physics Department, Columbia University, New York, New York

AND

GEORGE TIKTOPOULOS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received 31 January 1966)

$$b=\frac{(m_1+m_3)(m_2+m_3)}{m_1m_2}$$

It follows that if

$$b^{n-2}(b-1)>1\,, \qquad \text{(IV.18)}$$

then $2n+1$ successive binary collisions are kinematically impossible.

Complication: on-shell states

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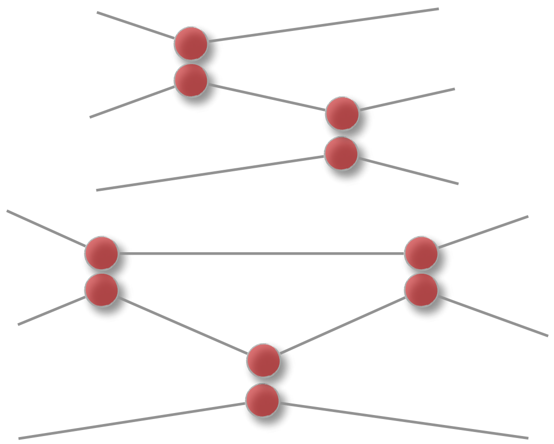

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$m_1 = m_2 = m_3 - \varepsilon$:
4 collisions possible

$\pi\pi K$

$b < 2$
5 collisions possible

$\pi K K$

Complication: on-shell states

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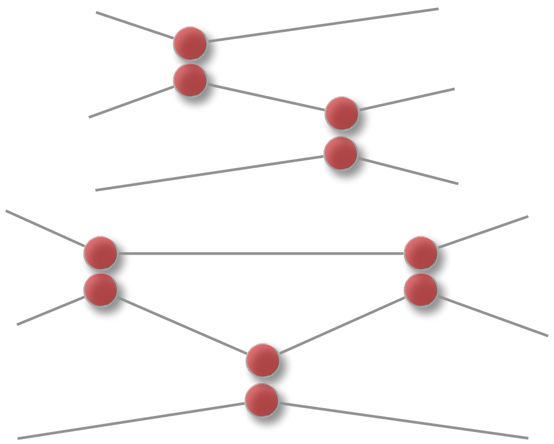

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$\pi\pi K$

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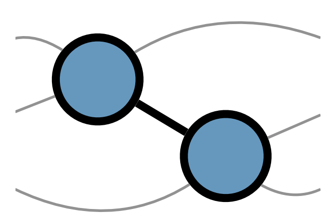
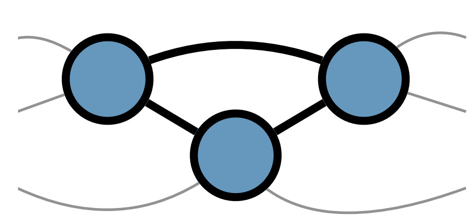
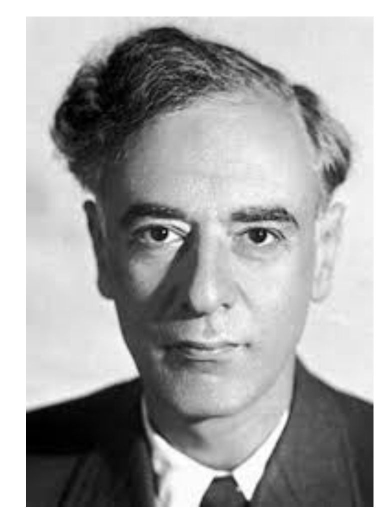
$\pi K K$

- Correspond to Landau singularities

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator $=$  $+$  $+\cdots$

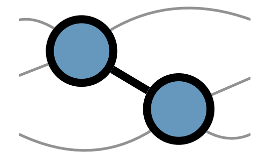
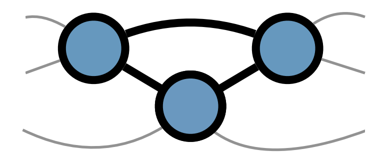


complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles

Two key observations

- Intermediate $K_{\mathrm{df},3}$ removes singularities

$\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$  $+$  $+ \cdots$

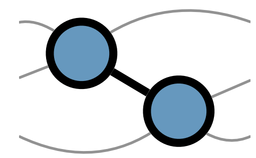
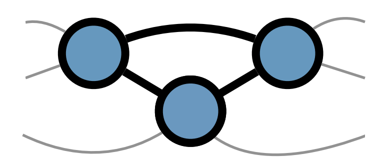
same degrees of freedom as M_3

smooth real function

relation to M_3 = known

Two key observations

- Intermediate $K_{\text{df},3}$ removes singularities

$\mathcal{K}_{\text{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$  $+$  $+\cdots$

same degrees of freedom as M_3 | smooth real function | relation to M_3 = known

- $K_{\text{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\text{df},3}(p_3,p_2,p_1;k_3,k_2,k_1) = \mathcal{K}_{\text{df},3}^{\text{iso},0} + \mathcal{K}_{\text{df},3}^{\text{iso},1}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

smooth real function

analogous to effective range expansion

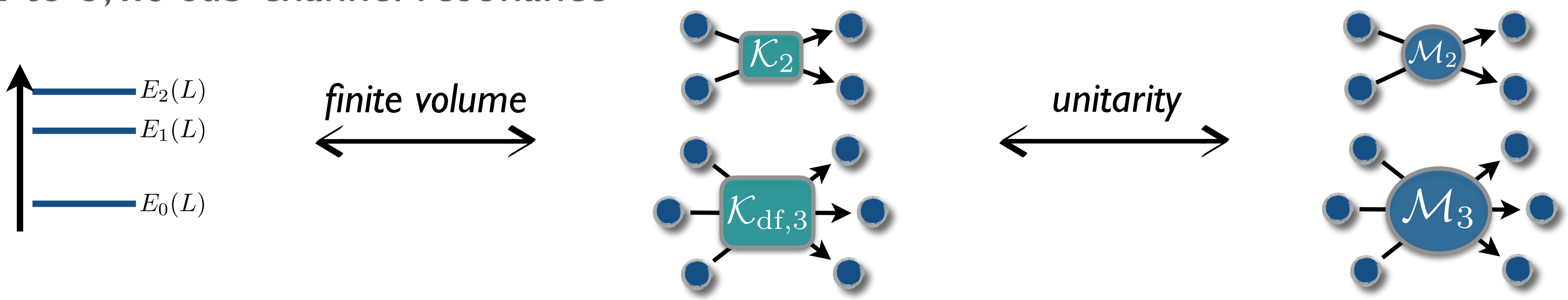
$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\,r\,p^2 + \mathcal{O}(p^4)$$

gives handle on many *degrees of freedom*
(DOFs enter order by order)

Status…

- General relation between *energies* and *two-and-three scalar scattering*

No 2-to-3, *no* sub-channel resonance

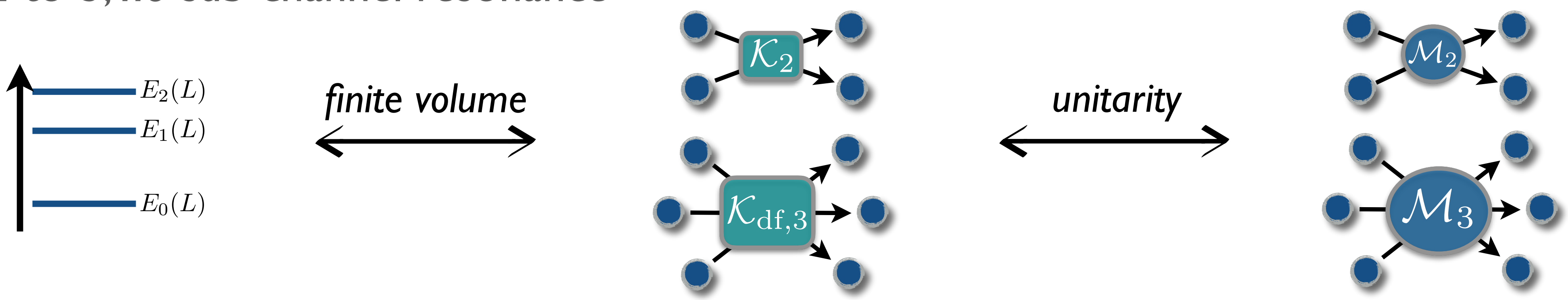


• MTH, Sharpe (2014, 2015) •

Status…

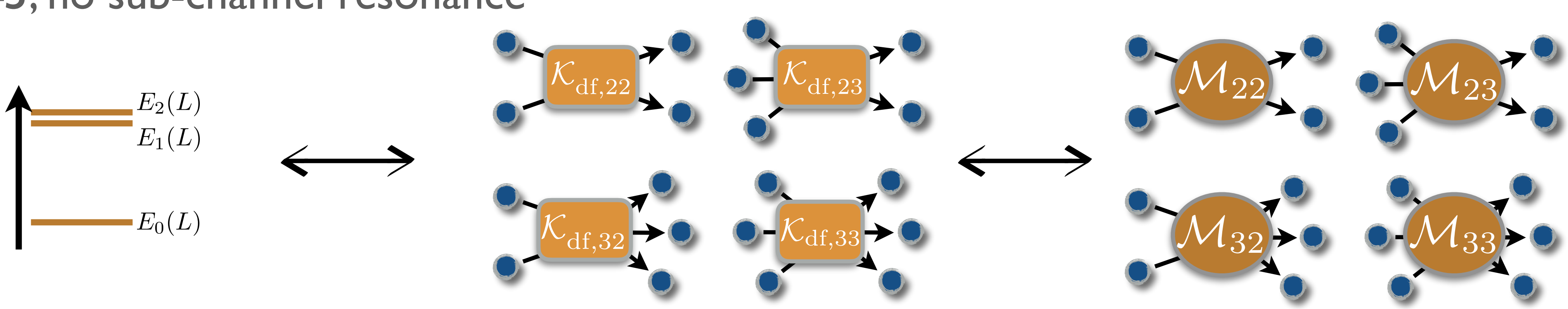
- ☐ General relation between ***energies*** and ***two-and-three scalar scattering***

No 2-to-3, ***no*** sub-channel resonance



• MTH, Sharpe (2014, 2015) •

2-to-3, no sub-channel resonance



• Briceño, MTH, Sharpe (2017) •

Including sub-channel resonances + ***different isospins*** + ***non-degenerate***

$$\pi\pi\pi \longrightarrow \rho\pi \longrightarrow \omega \longrightarrow \rho\pi \longrightarrow \pi\pi\pi$$

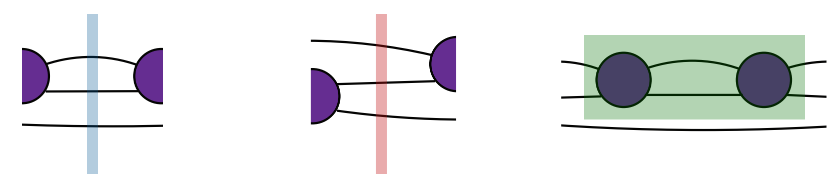
• Briceño, MTH, Sharpe (2018) • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

General relation

$$\det\left[\mathcal{K}_{\mathrm{df},3}^{-1}(s) + F_3(P, L|\mathcal{K}_2)\right] = 0$$

$F_3(P, L|\mathcal{K}_2) \equiv$ Matrix of functions depending on kinematics + two-particle dynamics

$$F_3 \equiv \frac{1}{3}F + F\,\mathcal{K}_2\,\frac{1}{1-(F+G)\,\mathcal{K}_2}\,F$$



Holds only for three-particle energies

Neglects e^{-mL}

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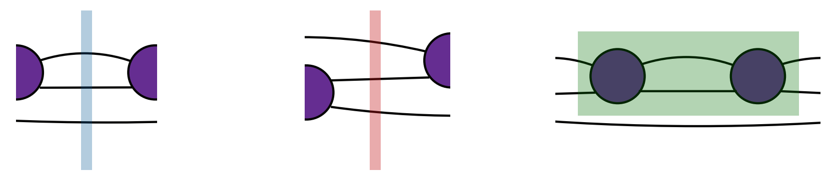


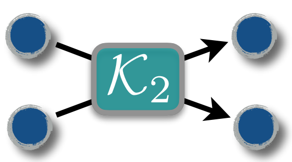
Holds only for three-particle energies

Neglects e^{-mL}

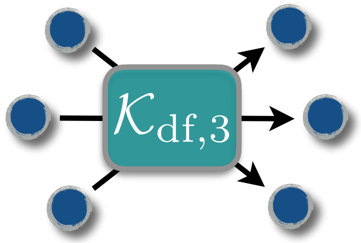
• MTH, Sharpe (2014-2016) • *See also Döring, Mai, Hammer, Pang, Rusetsky* •

Review: *Lattice QCD and Three-particle Decays of Resonances*
MTH and Sharpe, *1901.00483*

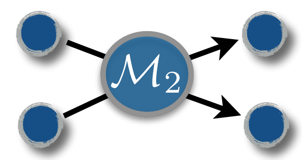
Two-particle interactions



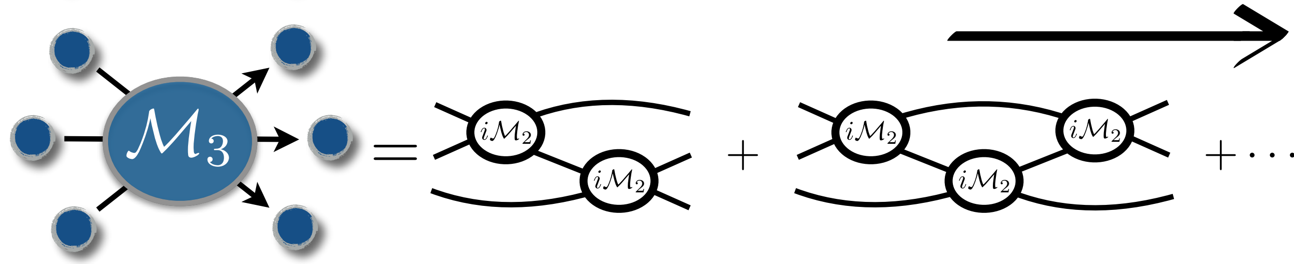
$= -16\pi\sqrt{s}\, a$

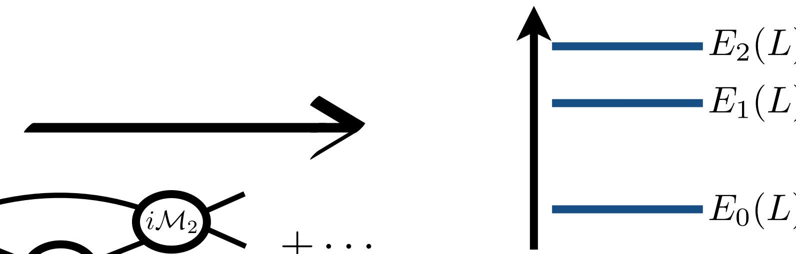


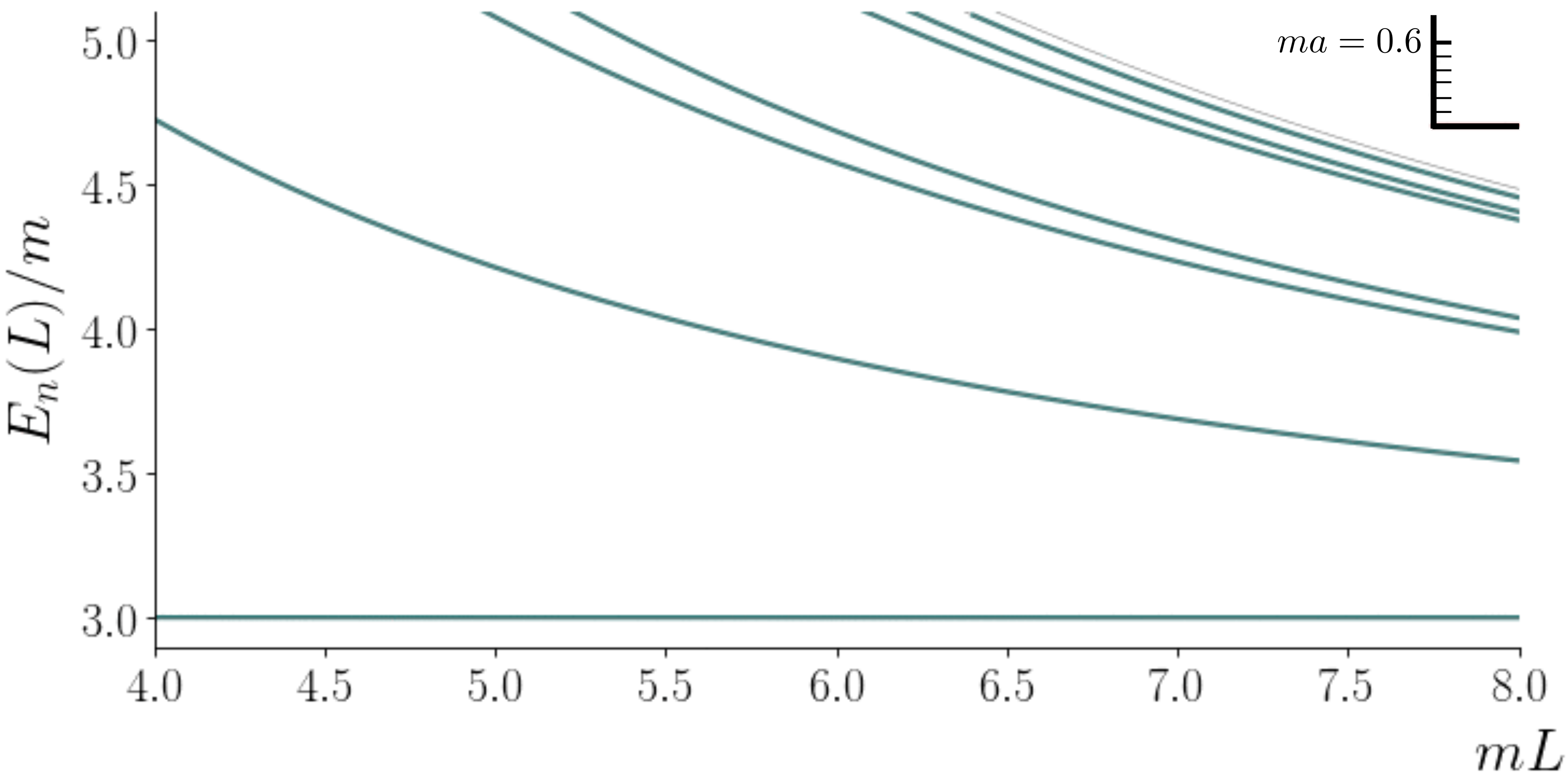
$= 0$



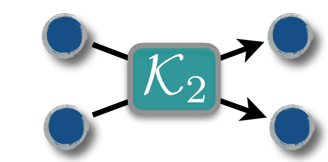
$$= \frac{16\pi\sqrt{s}}{-1/a - ip}$$



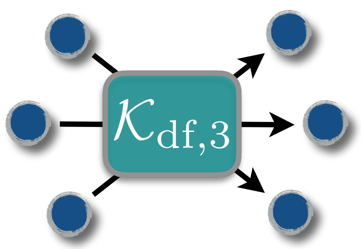




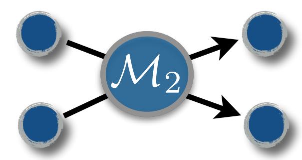
Two-particle interactions



$$= -16\pi\sqrt{s}\,a$$



$$= 0$$



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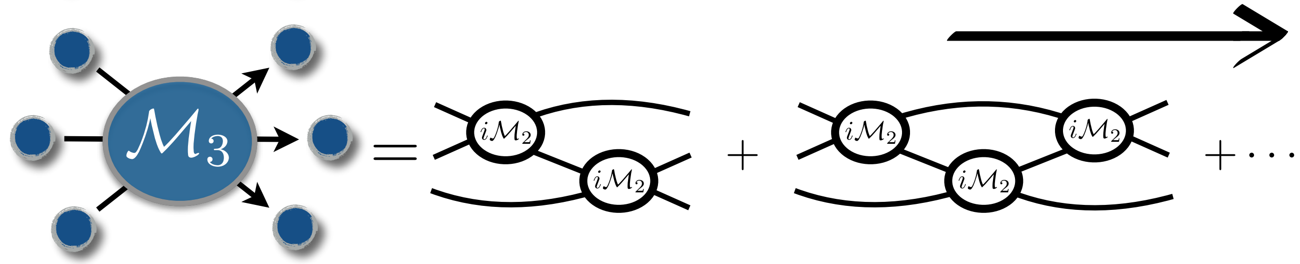
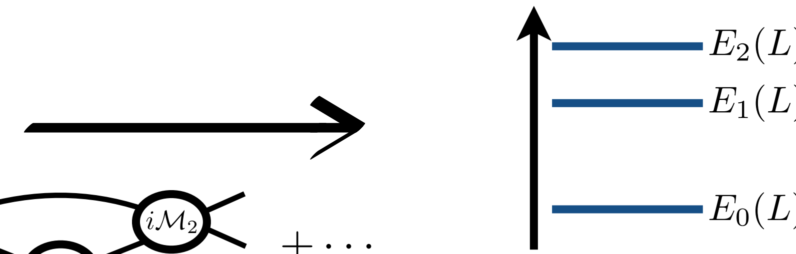
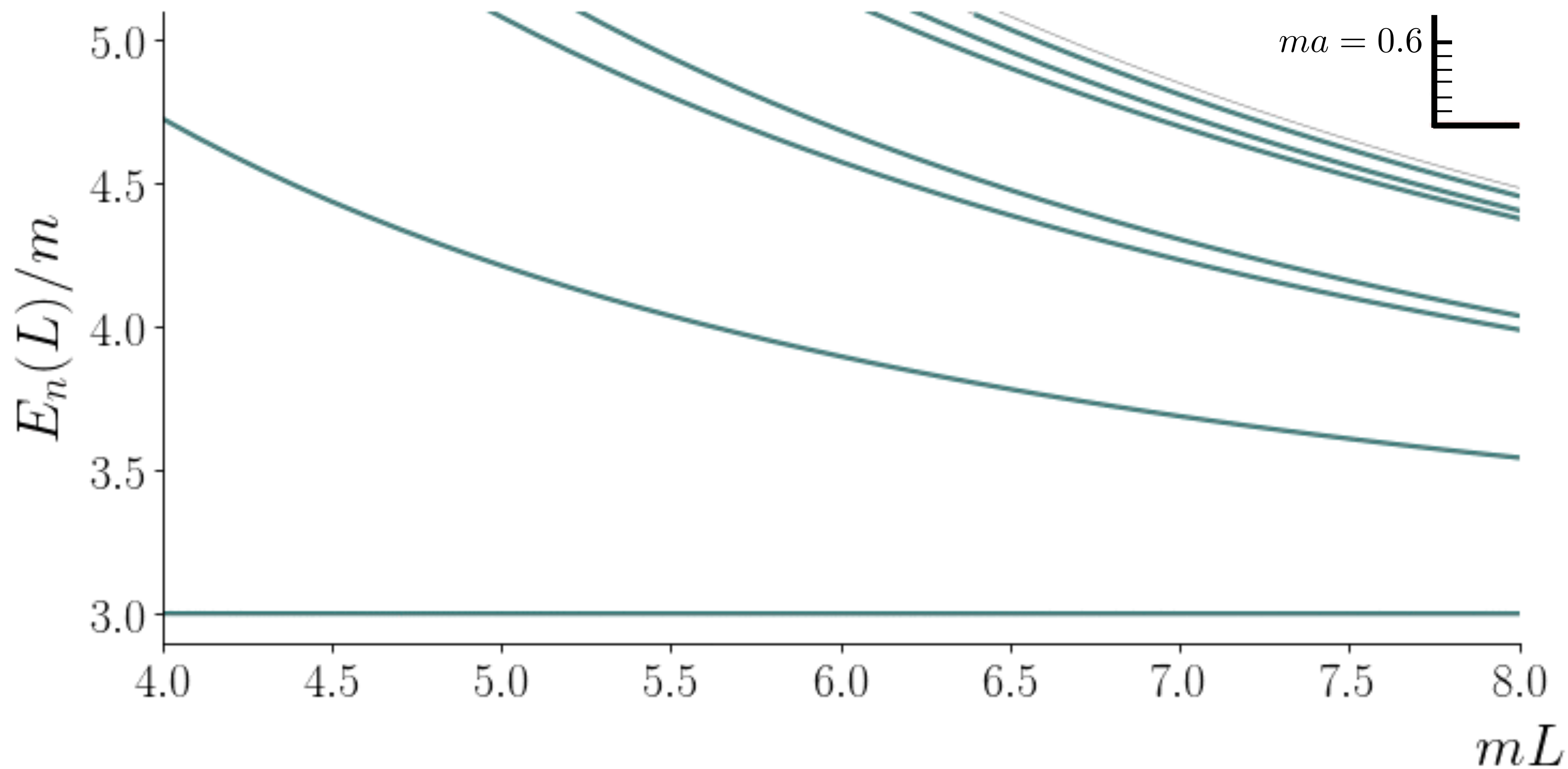
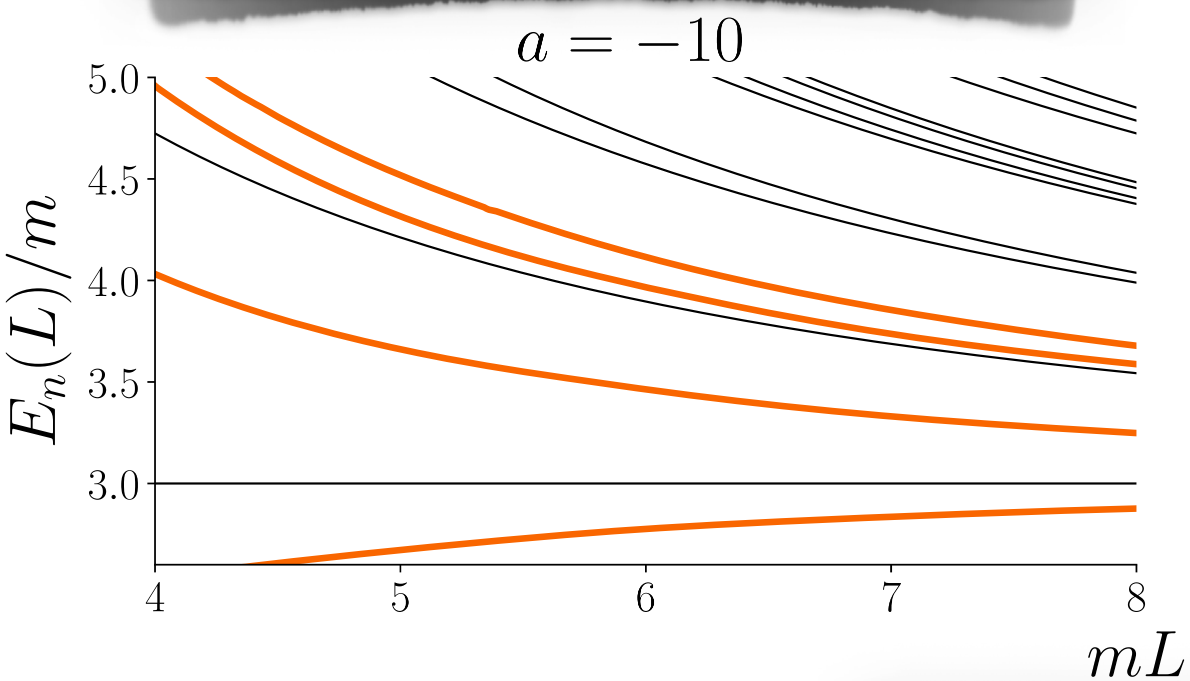




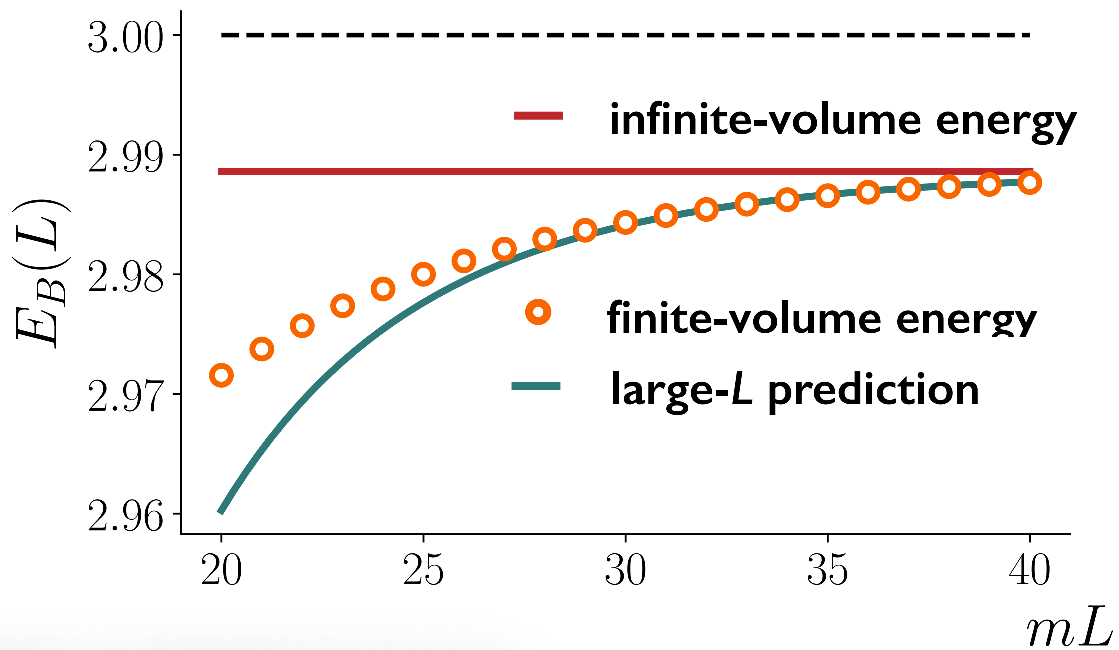


Many toy results

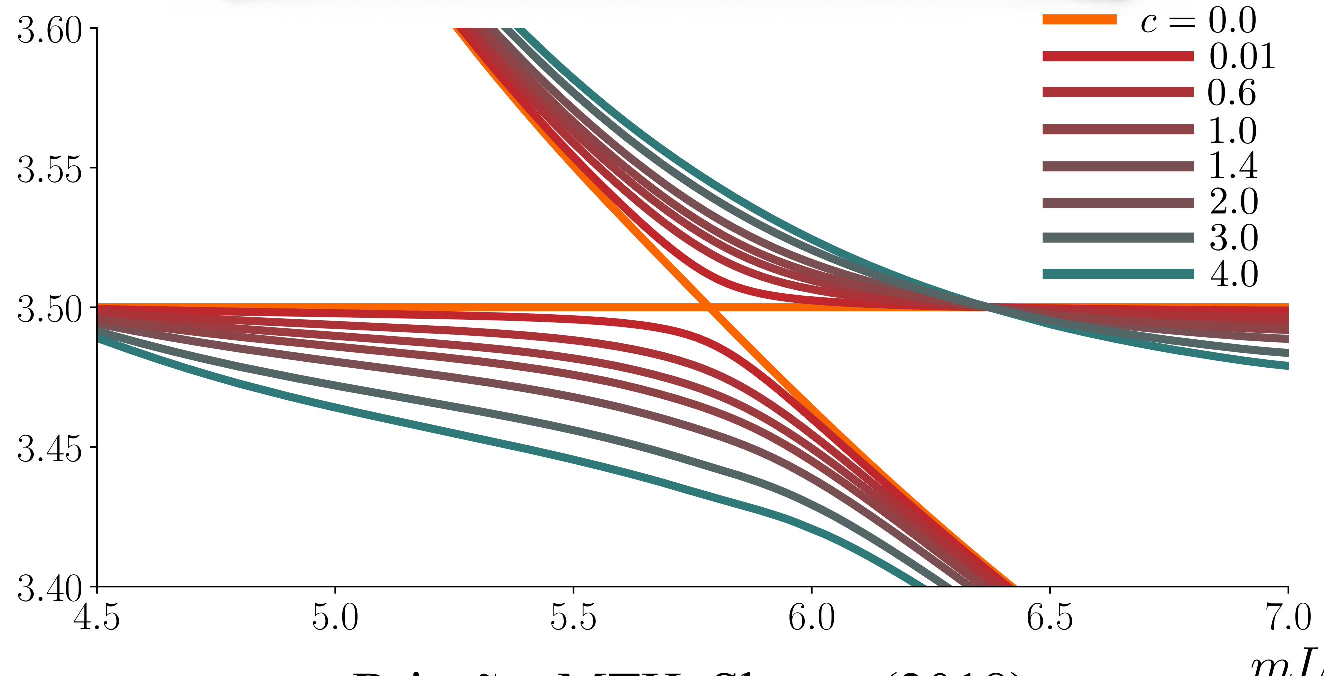
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



Model of a 3-particle resonance



• Briceño, MTH, Sharpe (2018) •

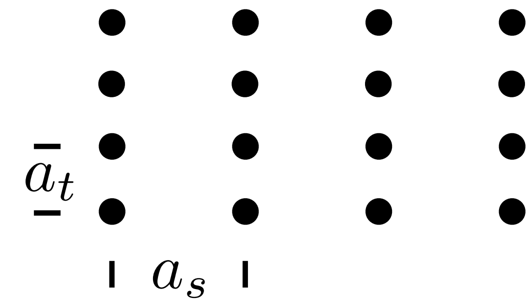
Lattice QCD calculation

$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$ in lattice QCD

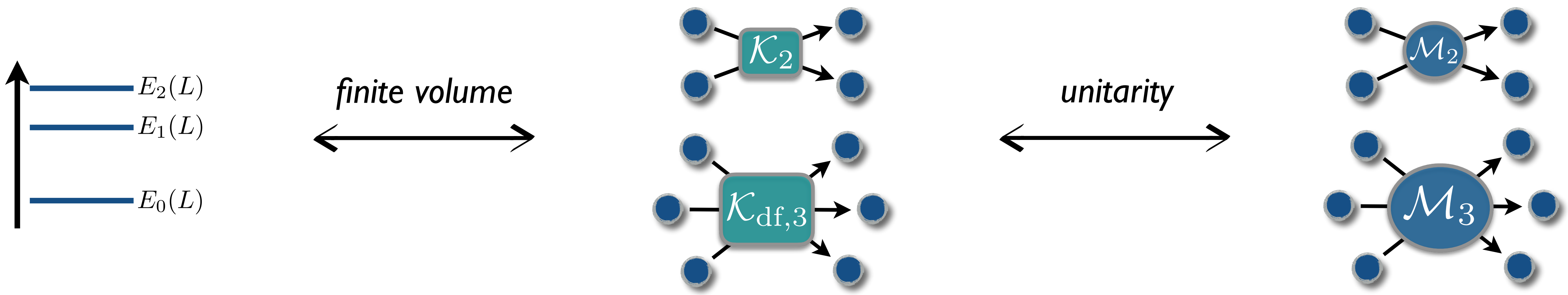
lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$



❑ Workflow outline



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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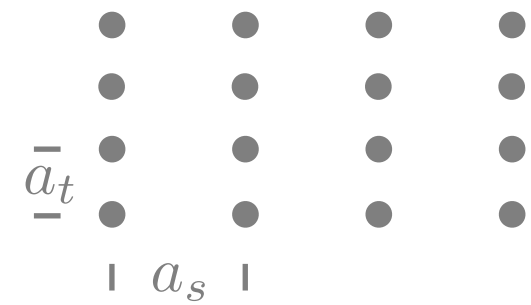
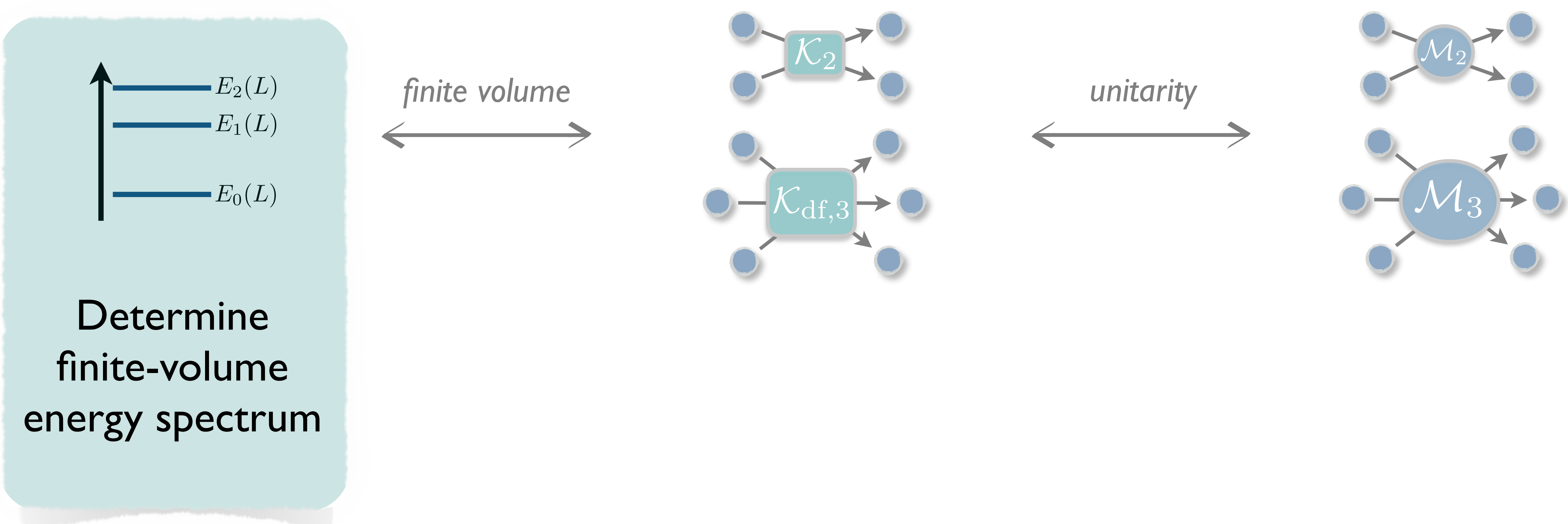
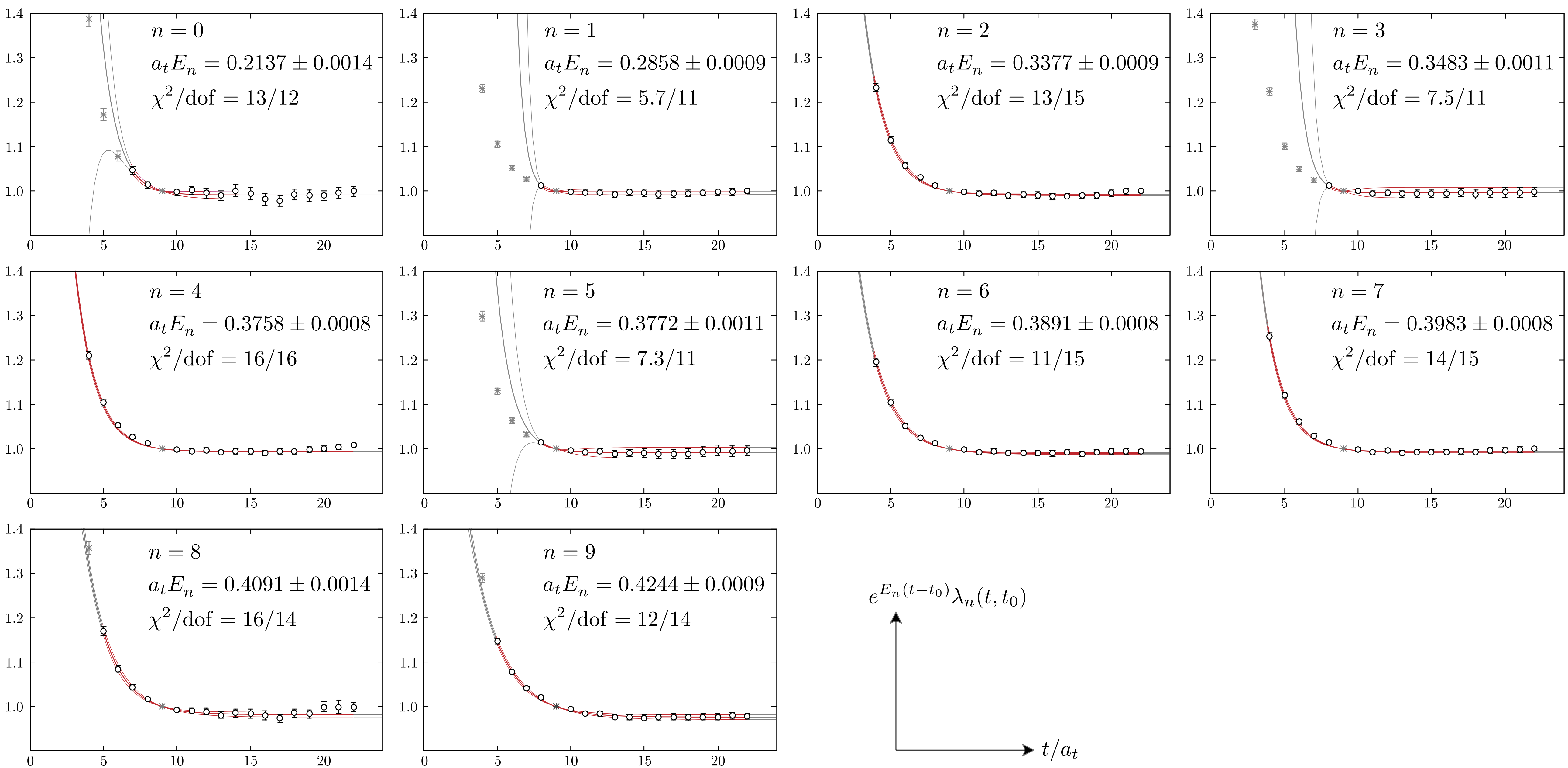


- [ ] Workflow outline



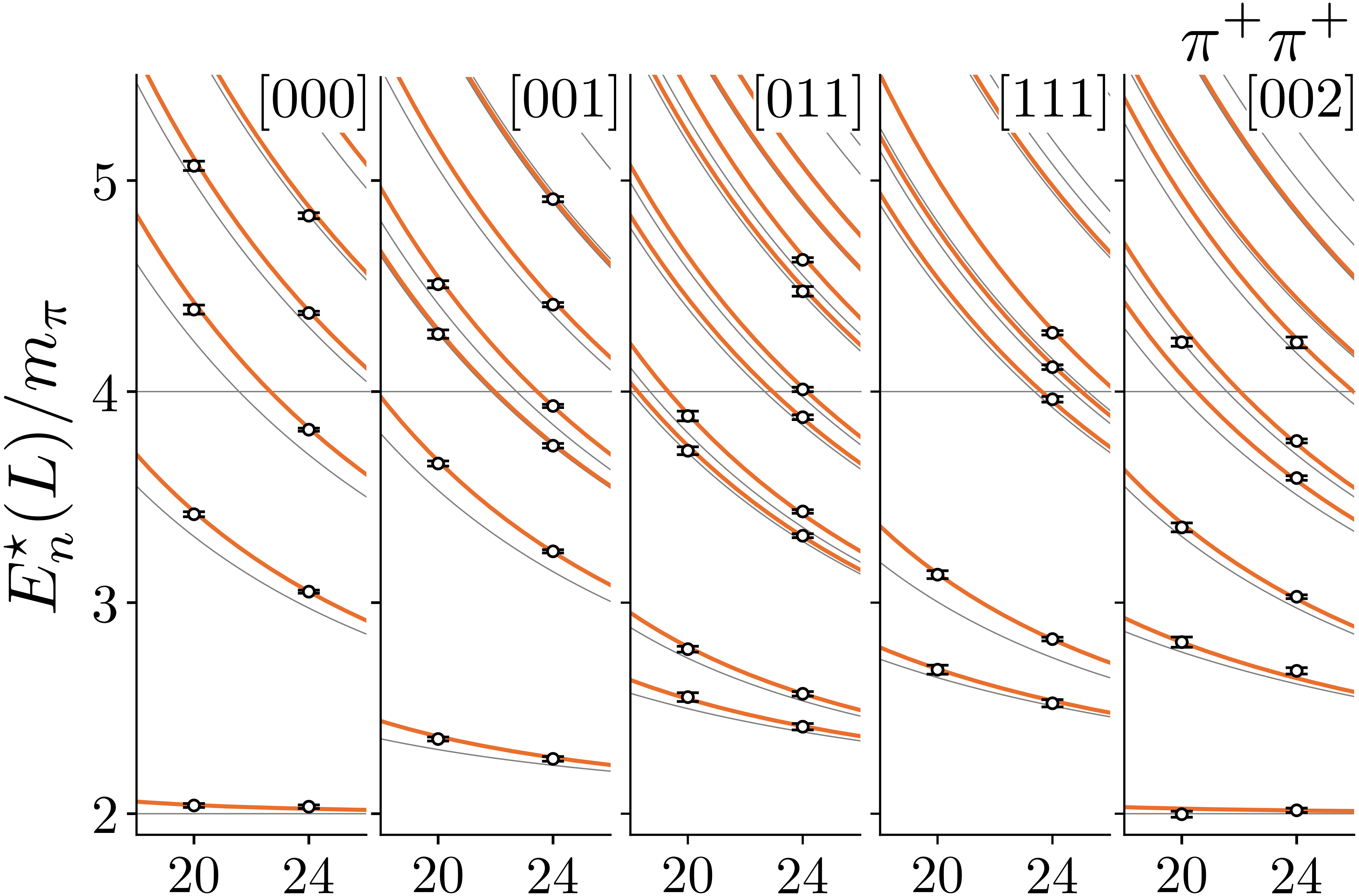
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$I = 3\,(\pi^+\pi^+\pi^+),\ \boldsymbol{P} = [000],\ \Lambda = A_1^-,\ L/a_s = 24$



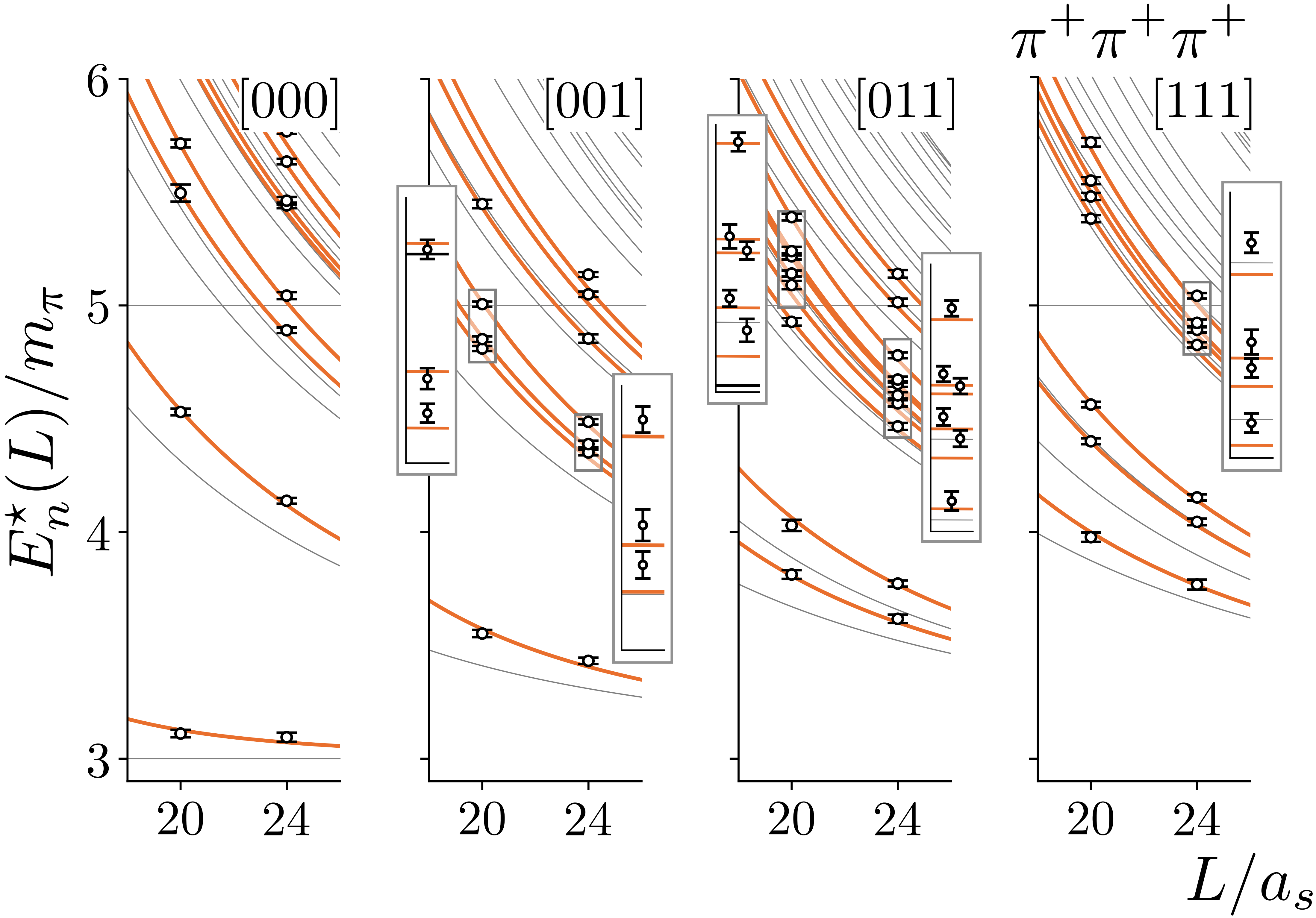
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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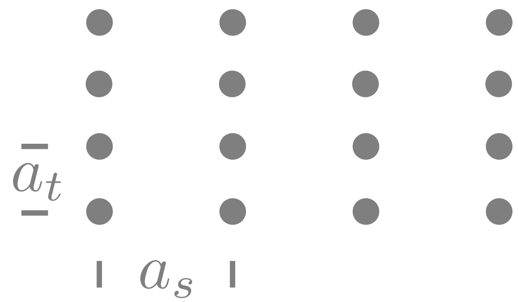

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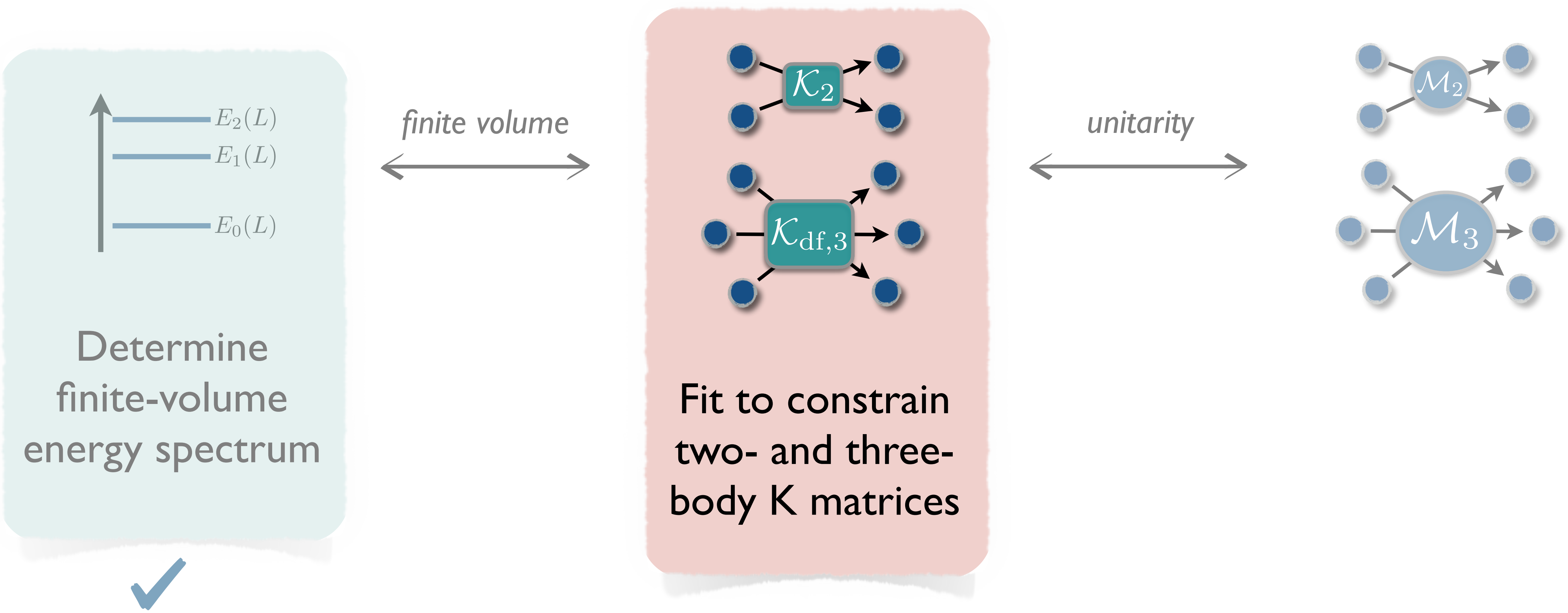
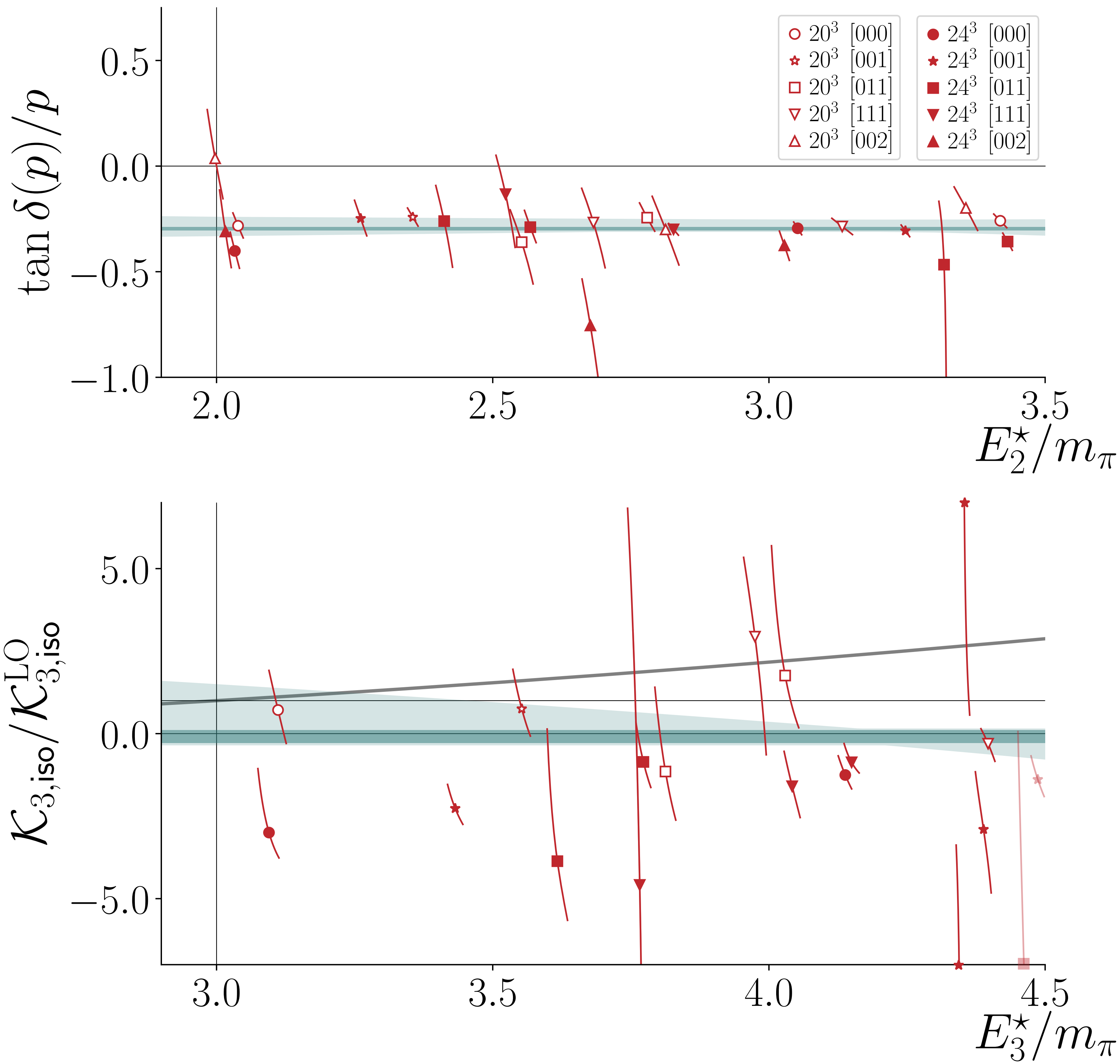


MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

K matrix fits



Finite-volume formalism relates energies to K matrices

One-to-one for $K_{\text{df},3}$ depending only on $E_{\text{cm}} = E^\star$

Fit both two and three-body K to various polynomials

Cut on the CM energy in the fits

$K_{\text{df},3}$ is scheme dependent (removed upon converting to $\mathcal{M}_3$)

$$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$$

lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$

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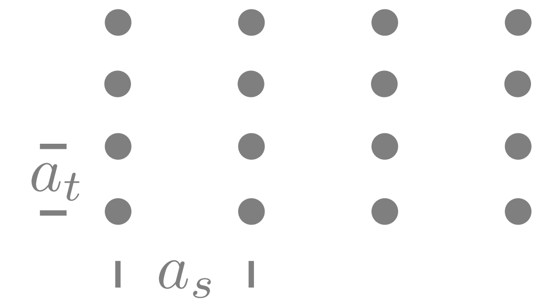
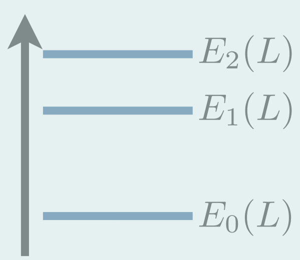


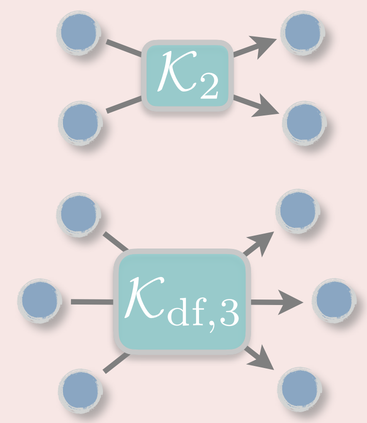
☐ Workflow outline



Determine finite-volume energy spectrum

✓

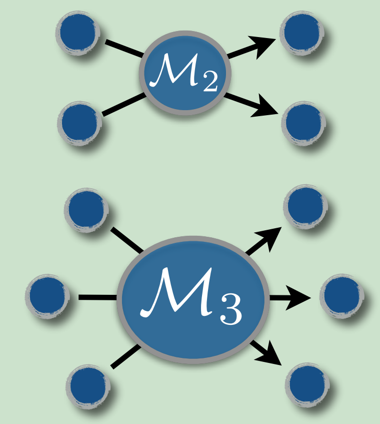
finite volume ⟷



Fit to constrain two- and three-body K matrices

✓

unitarity ⟷

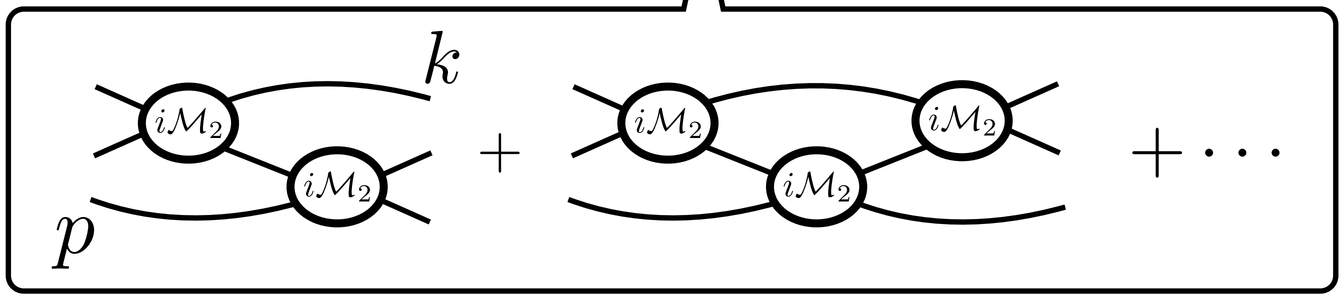


Solve integral equations to extract $3 \to 3$ scattering

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

Integral equation

$$\mathcal{M}_3^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) + \mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{k})$$



Vanishes for $K_{\text{df},3} = 0$

$$\boldsymbol{D}(N,\epsilon) = -\boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{P} \cdot \boldsymbol{D}(N,\epsilon)$$

$$\mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \lim_{\epsilon \to 0} \lim_{N \to \infty} \boldsymbol{D}_{pk}(N,\epsilon)$$

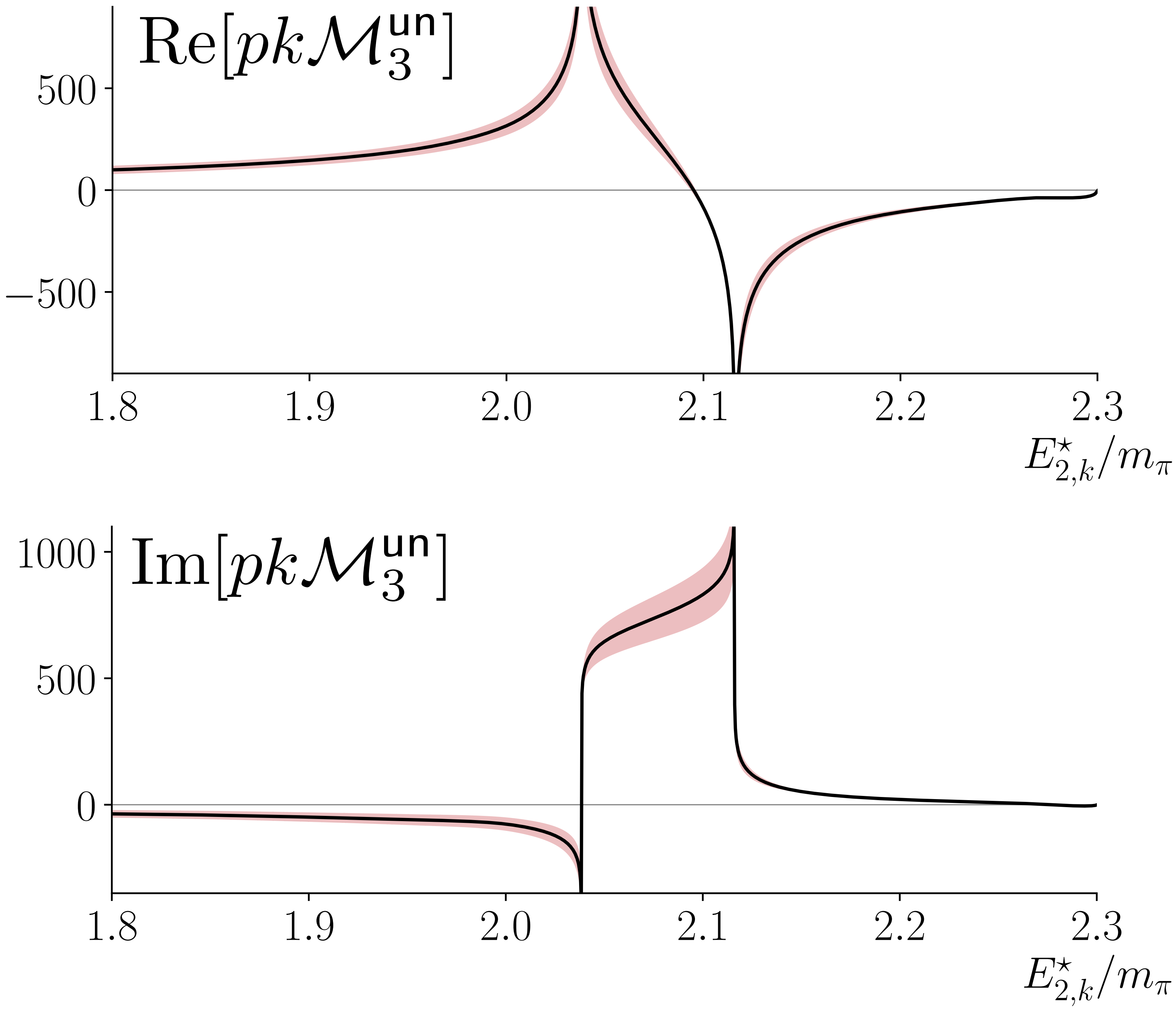
- See also…

Solving relativistic three-body integral equations in the presence of bound states

Andrew W. Jackura,[1,2,*] Raúl A. Briceño,[1,2,†] Sebastian M. Dawid,[3,4,‡] Md Habib E Islam,[2,§] and Connor McCarty[5,¶]

arXiv: 2010.09820

Integral equation



Total angular momentum = 0

Two-particle sub-system angular momentum = 0

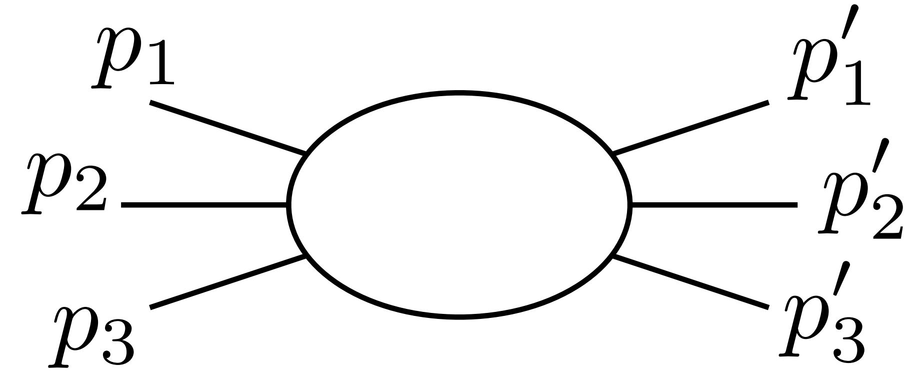
Plot at fixed $E_3^\star$ and p

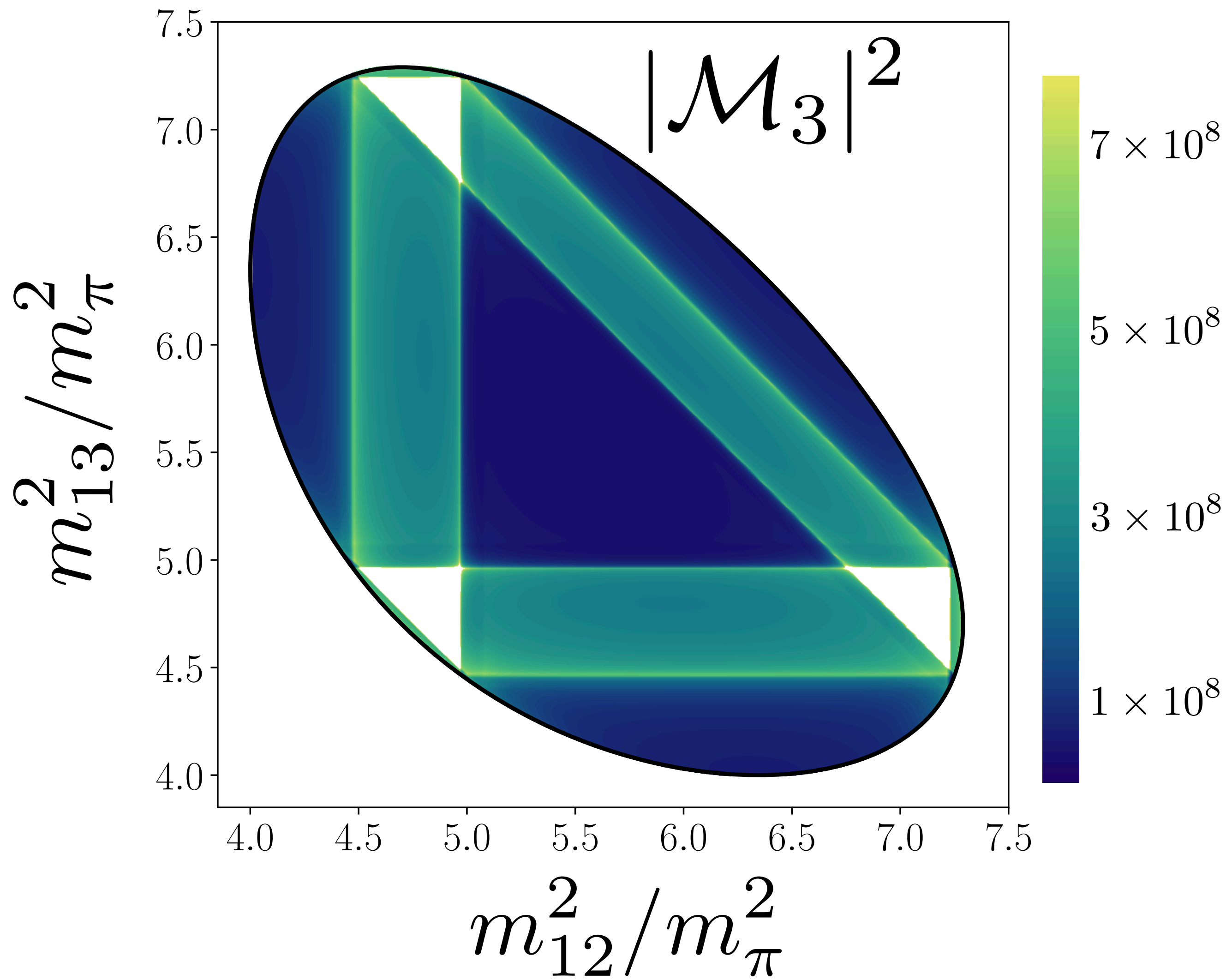
Both two- and three-body uncertainties estimated

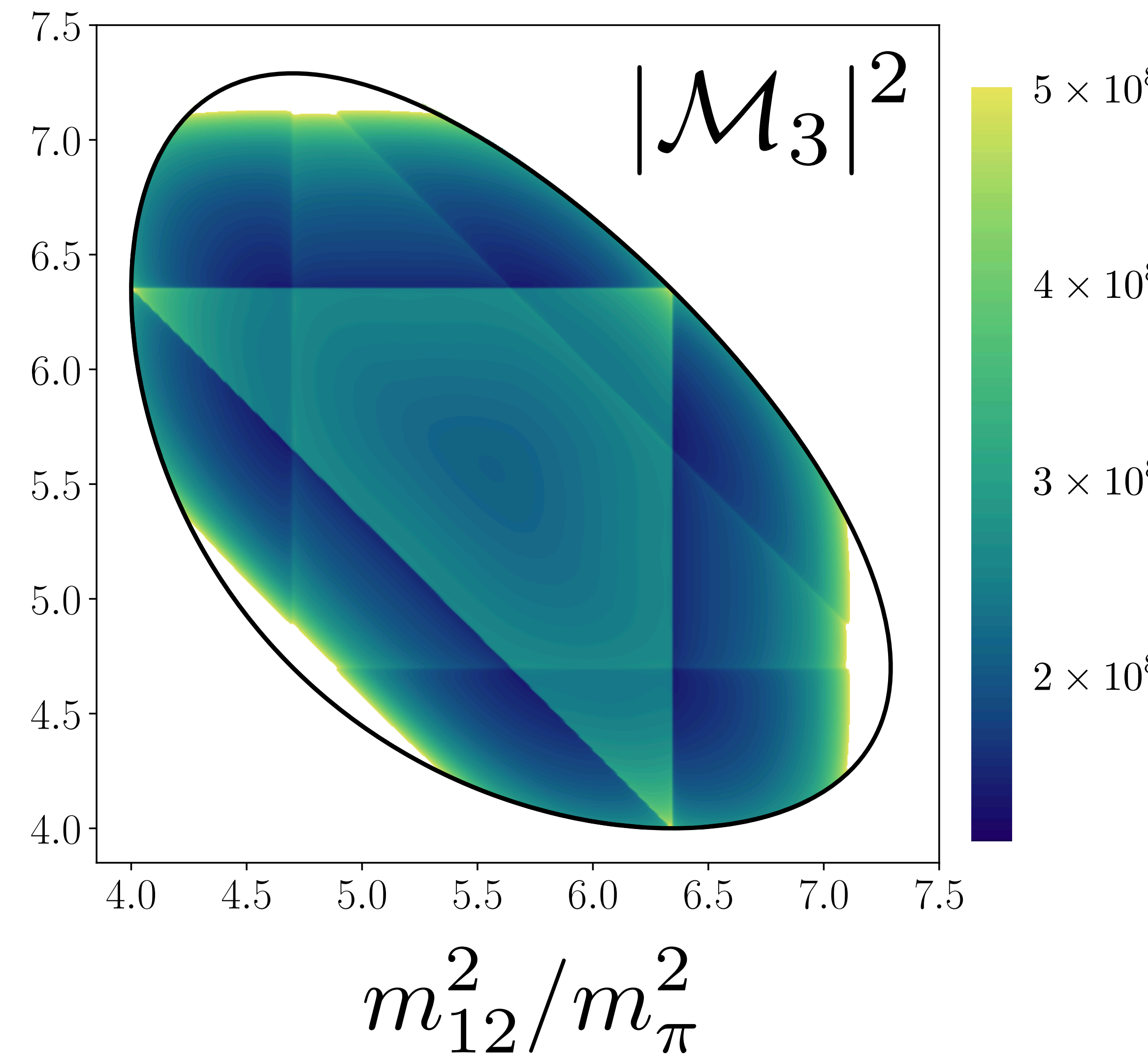
Still need to symmetrize

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\mathcal{M}_3 = \sum_{i,j\in\{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p_i', p_j)$$







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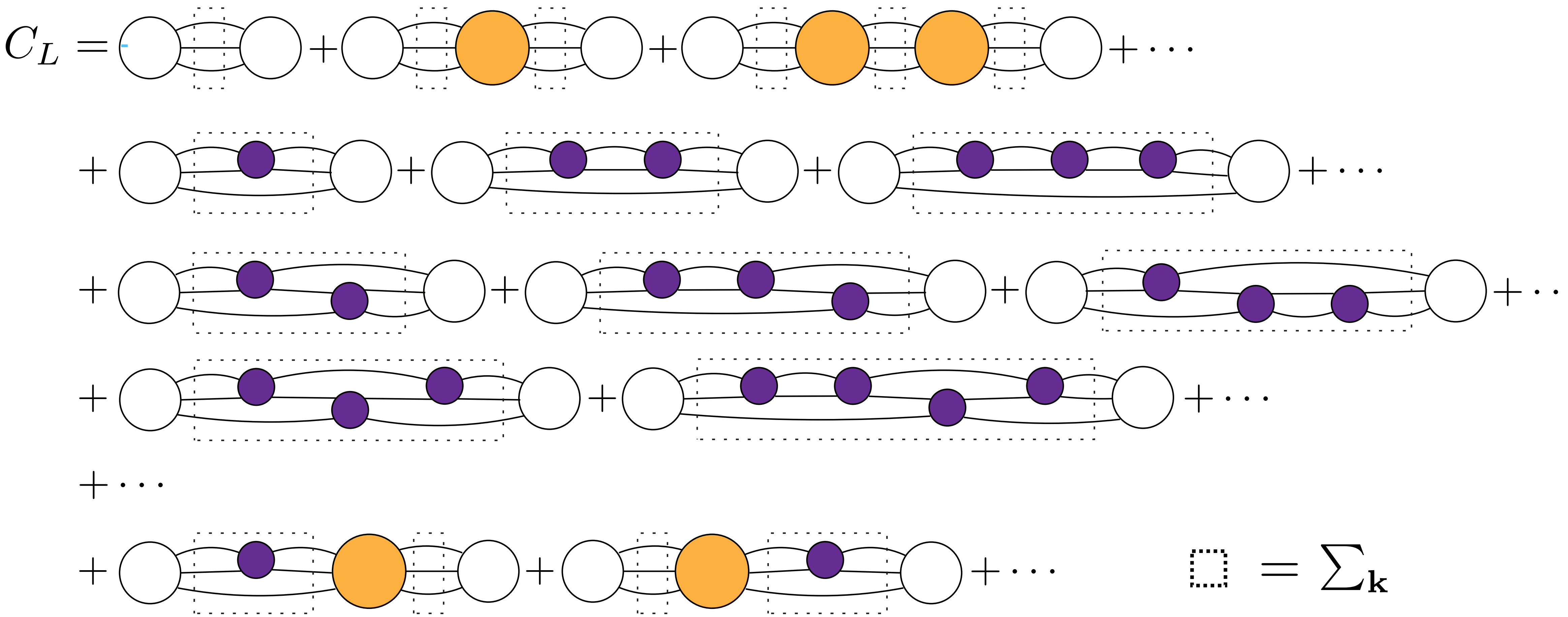
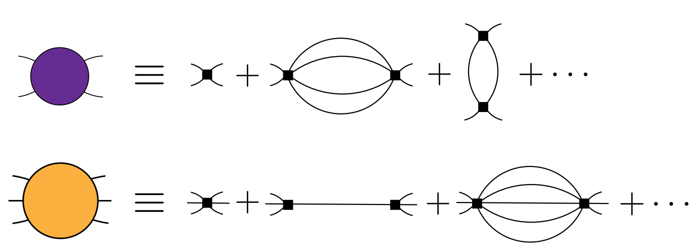
Details on the derivation

3-particle derivation

- Study 3-body correlator in an ***all-orders skeleton expansion***



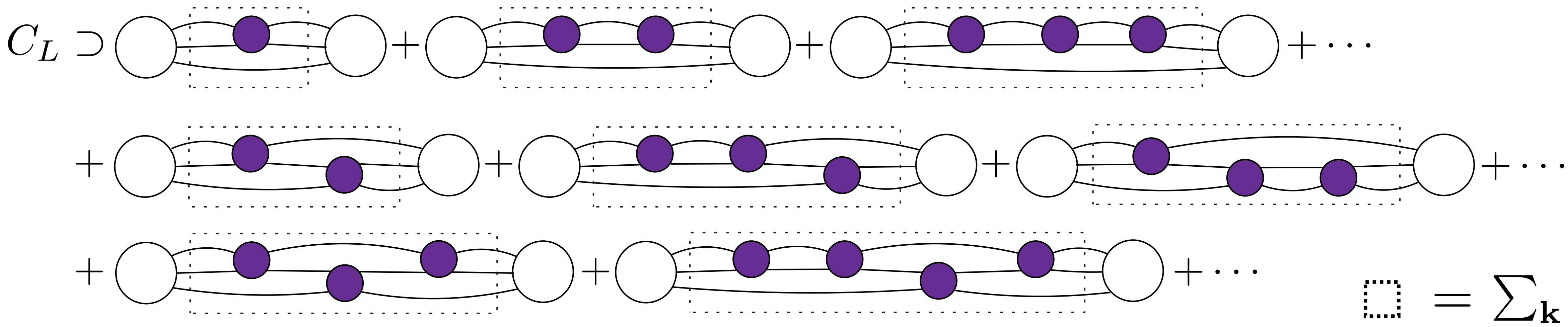


kernels have suppressed *L* dependence

lines = fully dressed hadrons

• MTH, Sharpe (2014) •

Two types of cuts



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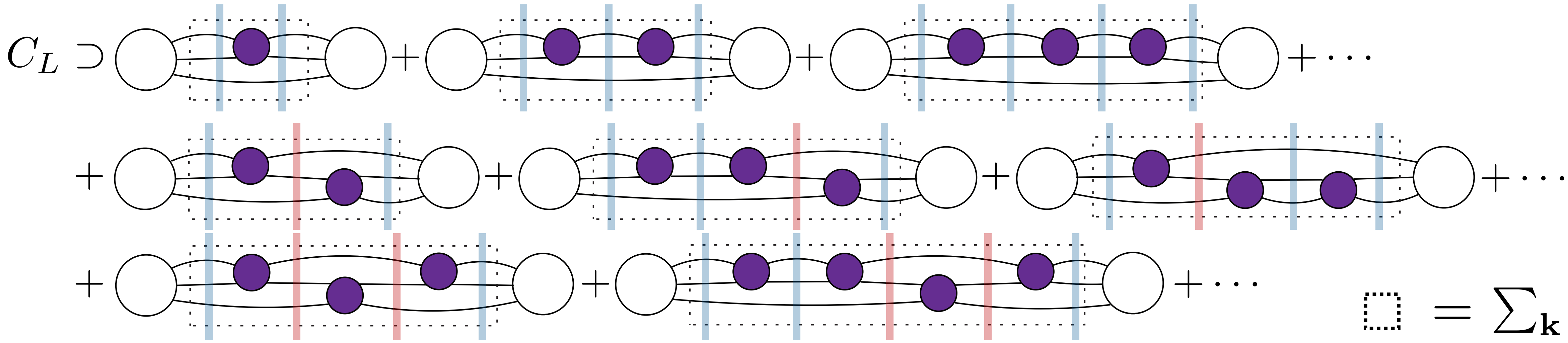


$$\mathbf{A}_3' \, \mathbf{F} \, \mathbf{K}_2 \, \mathbf{F} \, \mathbf{A}_3 + \mathbf{A}_3' \, \mathbf{F} \, [\mathbf{K}_2 \, \mathbf{F}]^2 \, \mathbf{A}_3 + \mathbf{A}_3' \, \mathbf{F} \, [\mathbf{K}_2 \, \mathbf{F}]^3 \, \mathbf{A}_3 + \cdots = \mathbf{A}_3' \, \mathbf{F} \frac{1}{1 - \mathbf{K}_2 \mathbf{F}} \mathbf{K}_2 \mathbf{F} \, \mathbf{A}_3$$

$$\mathbf{A}_3' \, \mathbf{F} \, \mathbf{K}_2 \, \mathbf{G} \, \mathbf{K}_2 \, \mathbf{F} \, \mathbf{A}_3 + \cdots$$

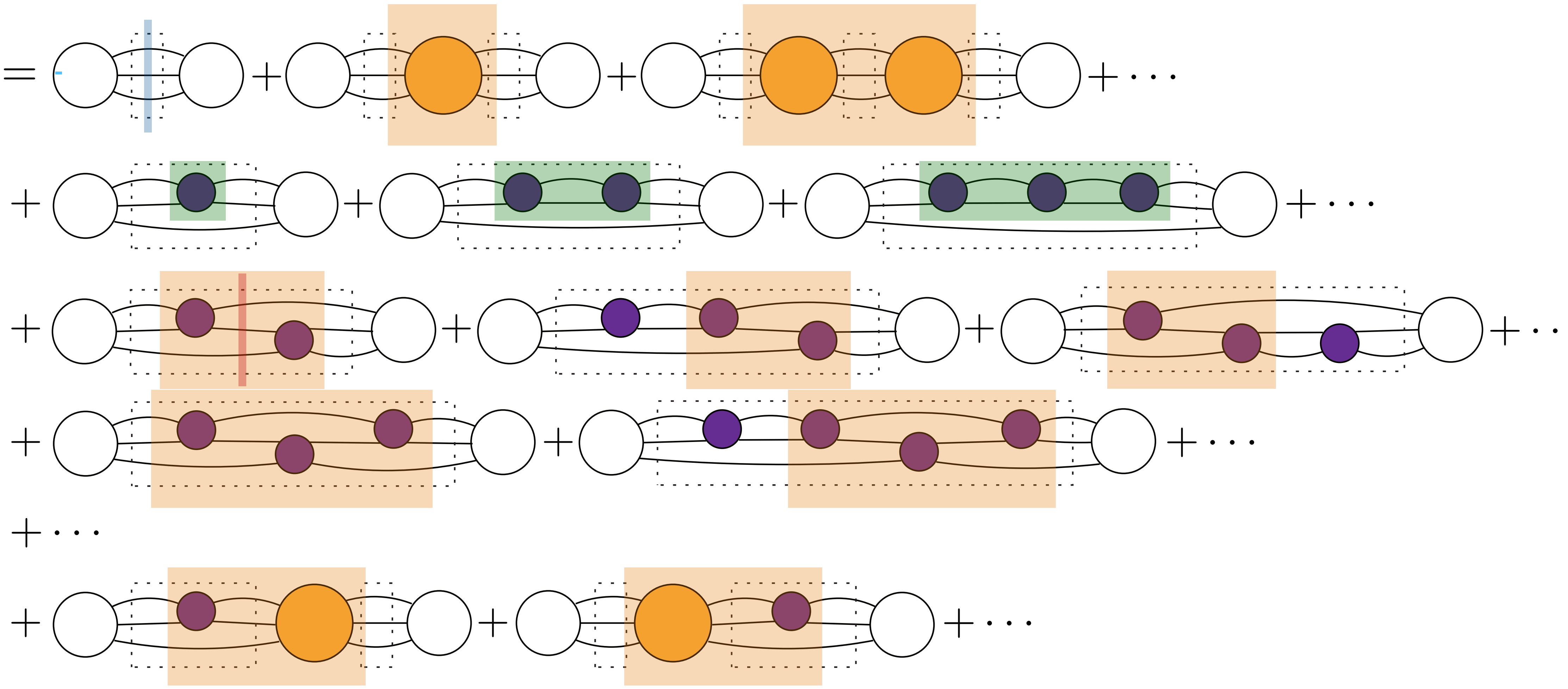
have not yet considered entire diagram contributions

missing contributions from ***off-shellness***

missing smooth terms (short-distance parts)

Short-distance parts & summation

$C_L =$



$$C_L - C_\infty = \mathbf{A}'_3\,\mathbf{F}_{33}\,\mathbf{A}_3 + \mathbf{A}'_3\,\mathbf{F}_{33}\,\mathbf{K}_{\mathrm{df},3}\,\mathbf{F}_{33}\,\mathbf{A}_3 + \cdots$$

$$= \mathbf{A}'_3\,\frac{1}{\mathbf{F}_{33}^{-1} + \mathbf{K}_{\mathrm{df},3}}\,\mathbf{A}_3$$

$$\mathbf{F}_{33} \equiv \frac{1}{3}\mathbf{F} + \mathbf{F}\,\mathbf{K}_2\,\frac{1}{1 - (\mathbf{F} + \mathbf{G})\,\mathbf{K}_2}\,\mathbf{F}$$

no term left behind