Metric Geometry and Differential Forms

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Main message

There is a strange, unfamiliar to most people construction that encodes a metric on a space into a collection of differential forms on the same space

In 3D this coincides with the more familiar to people vielbein formalism

In 4D this is known under different names, but to physics audience is probably most known as the formalism behind Ashtekar's "New Hamiltonian formulation of GR"

People have searched for generalisation to higher dimensions, but I am not aware of any convincing story

What I am presenting can be thought of as such a generalisation

The aim of the talk is to explain this construction, and also explain why a physicist may care

Main message continued

The construction that encodes a metric into differential forms has spinor origin

Works (with appropriate modifications) in any D, in any signature

Schematically

"Geometric Map"

(Metric, unit spinor) $\Longleftrightarrow$ (Collection of differential forms)

One of the aims of this talk is to explain this construction on several examples

Why interesting?

- In all known examples, gives an extremely powerful language to describe metric geometry
- Gives a very interesting perspective on 4D GR, including how to generalise spin 2 to higher spins, and on how to formulate gravity theories in higher D

  How do we describe gravity + spinor matter in higher D?

- Ideas of dimensional reduction, taken to their extreme form, show that all known bosonic and fermionic fields can be encoded into a metric + spinor in sufficiently high number of dimensions

Dynamics of such a system (PDE's satisfied) seems to be best approached by the scheme I am describing

Outline

- Motivations: Dimensional reduction - (generalised) Kaluza-Klein, spinors
- Prologue - Uniqueness of (metric) GR
- Geometric map and resulting descriptions of gravity in 3 and 4D
- Spinors and gravity in 8D
- Outlook - Geometry behind the laws of Nature?

Motivations: Kaluza-Klein

The general idea of KK-type scenarios is that gravity + YM + scalar fields all have same origin

Dynamical geometry in higher D

Dynamics of gravity in 4D | YM fields | Scalar fields

<u>Example</u>: Einstein-Hilbert gravity in 5D

Gravity in 4D | U(1) gauge field | Single scalar field

<u>Conventional KK</u>: Higher D geometry is Riemannian geometry of metrics, with dynamics dictated by the EH action

Spinors and Dimensional Reduction

Spinors behave very nicely under dimensional reduction $Spin(n) \times Spin(k) \subset Spin(n+k)$

Basically $S_{n+k} = S_n \otimes S_k$

More concretely, in even dimensions $Spin(2n) \times Spin(2k) \subset Spin(2(n+k))$

$$S^+_{n+k} = S^+_{2n} \otimes S^+_{2k} \oplus S^-_{2n} \otimes S^-_{2k}$$

$S^{\pm}$ are Weyl spinors (irreducible representations in even dimensions)

Spinors remain spinors under dimensional reduction

SM fermions and dimensional reduction

Recall that all 15 fermions (plus RH neutrino) of one generation of the SM can be put together into a single complex 16-dimensional Weyl representation of Spin(10) - this is the kinematics of SO(10) GUT

These 16 particles are also Lorentz Spin(1,3) 2-component spinors

It follows that all field components of a single fermion SM generation can be put together into a single Weyl spinor representation of a suitable Spin group in 14 dimensions

$14 = 10 + 4$

KK - all boson fields are unified by a metric in a space of sufficiently high dimension

Also the spinor fields get unified in a spinor in a space of sufficiently high dimension

This suggests we should consider metric + (Weyl) spinor in sufficiently high number of dimensions

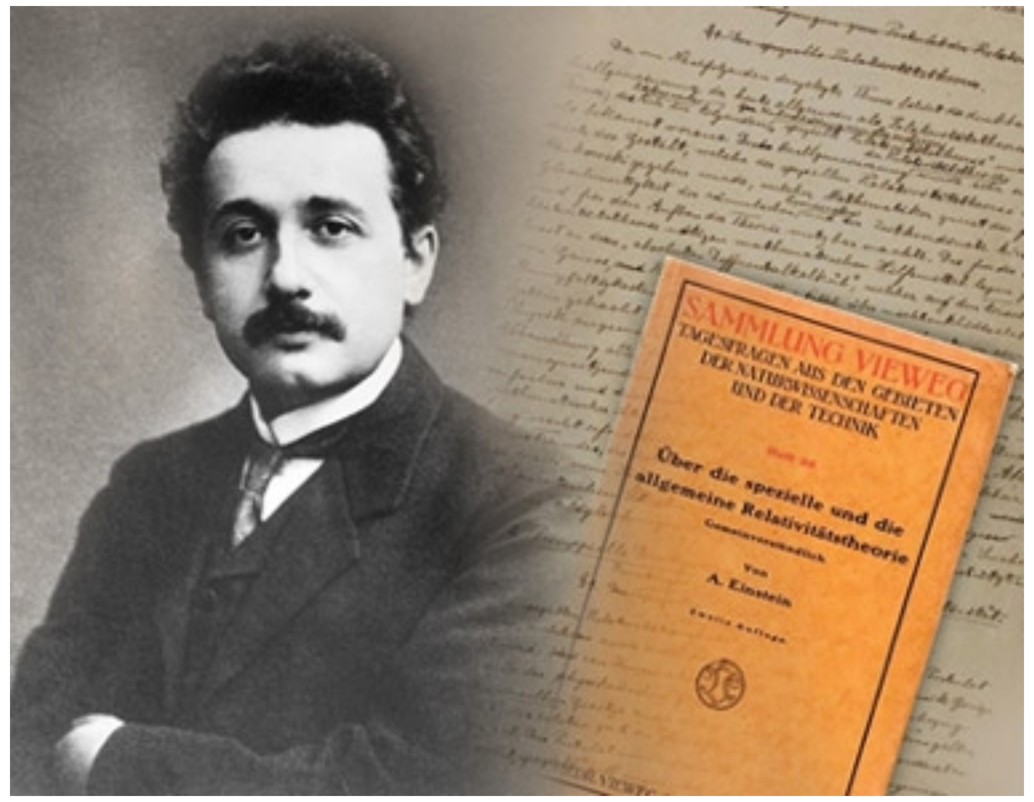
The basic question addressed by this talk is: What is the most natural dynamics for the system metric + (Weyl) spinor?

There is always the option Einstein-Hilbert + Dirac, but only in very specific dimensions works with Weyl

We will see that there are other natural possibilities

Prologue - Einstein's GR in hindsight

Einstein's "happiest thought" 1907: free fall = inertial motion



By 1912 it is clear to him that this means that gravity can be encoded into non-trivial metric geometry

Misled by an erroneous argument he abandons the covariant approach, only to return to it in 1915

He (eventually) writes what we now know as Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}$$

What we understand now (and what Einstein did not know) is that Einstein equations are the only second-order PDE's that can be written for a metric tensor

Uniqueness of GR

Many proofs, but particularly instructive is the elementary field theory proof. Let $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

What is the most general free theory Lagrangian that can be written for $h_{\mu\nu}$?

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h_{\rho\sigma})^2 + \frac{\alpha}{2}(\partial_a h)^2 - \beta h \partial^\mu \partial^\nu h_{\mu\nu} - \gamma(\partial^\mu h_{\mu\nu})^2$$

If we now assume general covariant (diffeomorphism invariance) $\delta h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}$

We are led to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h_{\rho\sigma})^2 - \frac{1}{2}(\partial_a h)^2 - h \partial^\mu \partial^\nu h_{\mu\nu} - (\partial^\mu h_{\mu\nu})^2$$

Unique diffeomorphism invariant second order in derivatives Lagrangian for the metric perturbation

Modulo an overall constant that can be absorbed into the field.

This linearised uniqueness argument works in any dimension!

Uniqueness continued

Harder to prove, but in 4D there is also the unique non-linear extension. Einstein-Hilbert action $S[g] = \int \sqrt{g} R$

In higher D, Lovelock's higher order in curvature terms are also possible, but GR is still the unique quasi-linear theory (field equations are linear in second derivatives)

This uniqueness statement is very powerful, because we now know that the only input needed to discover 4D GR is the idea that gravity can be encoded into metric geometry

Dynamics is unique and fixed by mathematics, at least in 4D, once kinematics is identified

We will follow a similar strategy, but for the system metric + (Weyl) spinor

Geometric map

One of the main points of this talk is that there exists an encoding

Given a metric and a (Weyl) spinor $\Longleftrightarrow$ Collection of differential forms that are built from the spinor

This collection depends on the "type of spinor"

An action principle can be written in terms of these differential forms.
In general, gives dynamics to both the metric and the spinor

Linearised uniqueness can be analysed similar to GR

Both the metric and the spinor (up to sign) are recovered

I will now illustrate these ideas by considering 3D, 4D and then 8D

Spinors and gravity in 3D

Spinors are "defining" representations of the Clifford algebra $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \mathbb{I}$

In $\mathbb{R}^3$ (half of) the Clifford algebra is generated by Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Spinors are 2-component columns on which σ^i act

$$\sigma^i \sigma^j + \sigma^j \sigma^i = 2\delta^{ij} \mathbb{I}$$

$$S \ni \psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C}$$

Invariant inner product on spinors $\langle \psi_1, \psi_2 \rangle = \psi_1^T \epsilon \psi_2, \qquad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

$$\langle g\psi_1, g\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle, \qquad \forall g \in SL(2, \mathbb{C})$$

Invariant conjugation on spinors $\overline{\psi} := \epsilon \psi^*$

$$\overline{g\psi} := g\overline{\psi}, \qquad \forall g \in Spin(3) = SU(2)$$

Combination of the two gives Hermitian inner product on spinors

$$\langle \overline{\psi}, \psi \rangle = \psi^\dagger \psi$$

Can construct a vector $V^i_\psi := \langle \overline{\psi}, \sigma^i \psi \rangle$

Simple computation gives $\vec{V}_\psi = (2\mathrm{Re}(\alpha^* \beta), 2\mathrm{Im}(\alpha^* \beta), |\alpha|^2 - |\beta|^2) \in \mathbb{R}^3$

Its square $(\vec{V}_\psi, \vec{V}_\psi) = (|\alpha|^2 + |\beta|^2)^2 = \langle \overline{\psi}, \psi \rangle^2$

Consider the space of unit spinors $S^3 = \{\psi : \langle \overline{\psi}, \psi \rangle = 1\}$

Then $\vec{V}_\psi$ gives a map $\vec{V}_\psi : S^3 \to S^2 \subset \mathbb{R}^3$

This map is the projection map of the Hopf fibration $S^1 \to S^3 \to S^2$

Moreover, can construct a complex vector $m^i_\psi := \langle \psi, \sigma^i \psi \rangle$

Simple computation gives $\vec{m}_\psi = (-\alpha^2 + \beta^2, -i(\alpha^2 + \beta^2), 2\alpha\beta)$

This vector satisfies $(\vec{m}_\psi, \vec{m}_\psi) = 0, \quad (\vec{m}_\psi, \vec{V}_\psi) = 0, \quad (\vec{m}_\psi, \vec{m}^*_\psi) = 2\langle \overline{\psi}, \psi \rangle^2$

This shows that for a unit spinor we get an orthonormal frame

$$(\vec{e}^{\,1}_\psi, \vec{e}^{\,2}_\psi, \vec{e}^{\,3}_\psi) := (\mathrm{Re}(\vec{m}_\psi), \mathrm{Im}(\vec{m}_\psi), \vec{V}_\psi)$$

Recall that a metric can be encoded into a choice of a frame on TM, whose vectors are declared to be orthonormal

Geometric map example → Frame = Metric + Unit spinor (mod $\mathbb{Z}_2$)

Metric + spinor encoding by differential forms

The idea of the geometric map is to produce a collection of differential forms from the metric and a spinor

The metric can be recovered from these differential forms

In this example this was the fact that the obtained triple of 1-forms was orthonormal

One can then forget about the origin of the differential forms, and continue to use the explicit formula for the metric

In this case we just take an arbitrary triple of 1-forms, and declare them orthonormal, thus encoding the metric

Field equations

Witten: 2+1 gravity as an exactly soluble system

It is well-known that in 3D, the most efficient description of a metric is provided by the frame formalism

Concretely, consider the co-frame, which is a triple of 1-forms $e^i \in \Lambda^1(M)$

Cosmological constant (scalar curvature)

We then have Cartan's structure equations $de^i + \epsilon^{ijk}\omega^j \wedge e^k = 0$

$$d\omega^i + (1/2)\epsilon^{ijk}\omega^j \wedge \omega^k = \lambda\epsilon^{ijk}e^j \wedge e^k$$

The first equation defines the components of the spin connection ω^i

The second equation is then the statement that the curvature is constant

There is a simple action principle giving these as Euler-Lagrange equations

$$S[e,\omega] = \int e^i \wedge (d\omega^i + \frac{1}{2}\epsilon^{ijk}\omega^j \wedge \omega^k) - \frac{\lambda}{3}\epsilon^{ijk}e^i \wedge e^j \wedge e^k$$

Considering spinors, and natural geometric objects that arise as spinor bi-linears, we were led to the notion of the frame, as a triple of differential 1-forms, as the best object to encode the metric. The second-order PDE's that one wants to impose on the metric (Einstein condition) become simple equations written in the language of the differential forms

Remarks on uniqueness

Uniqueness can be analysed similarly to the GR case, by considering the most general linearised Lagrangian, and then imposing gauge-invariance

The fields are perturbation of a metric + perturbation of a unit spinor = perturbation of a frame

Unique Lagrangian once diffeomoprhism invariance together with SU(2) invariance are imposed

In particular, the spinor is non-propagating in the SU(2) invariant theory

Spinors and gravity in 4D

γ-matrices are now 4x4

everything is signature dependent

we use Euclidean to simplify life

$$\gamma_4 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \qquad \gamma_i \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}$$

Dirac spinors are 4-component

Weyl spinors are 2-component

$$S = S_+ \oplus S_-, \qquad S_\pm \sim \mathbb{C}^2$$

γ-matrices are off-diagonal

$$\gamma : S_+ \to S_-$$

and vice versa

Invariant inner product on $S_\pm$

$$S_\pm \ni \psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$\langle \psi_1, \psi_2 \rangle = \psi_1^T \epsilon \psi_2$$

where $\psi_{1,2}$ are both either in S_+ or in S_-

Invariant conjugation on $S_\pm$

Spin(4) stabiliser of a spinor in S_+ is SU(2)

$$\overline{\psi} = \epsilon \psi^*$$

What are the geometric objects that can be constructed from a Weyl spinor?

We must insert an even number of γ-matrices between two copies of the spinor

Can define $\omega := \frac{i}{2}\langle\overline{\psi}, \gamma_{[\mu}\gamma_{\nu]}\psi\rangle dx^\mu \wedge dx^\nu$ real

$\Omega := \frac{i}{2}\langle\psi, \gamma_{[\mu}\gamma_{\nu]}\psi\rangle dx^\mu \wedge dx^\nu$ complex

A simple computation gives $\omega = V^i_\psi \Sigma^i, \qquad \Omega = m^i_\psi \Sigma^i$

where $\Sigma^i = dx^4 \wedge dx^i - \frac{1}{2}\epsilon^{ijk} dx^j \wedge dx^k$ is the basis of self-dual 2-forms on $\mathbb{R}^4$

and V^i_ψ, m^i_ψ are the previously encountered vectors in $\mathbb{R}^3$

Another simple computation shows that for a unit Weyl spinor $\mathrm{End}(\mathbb{R}^4) \ni J_\psi = V^i_\psi \Sigma^{i\ \nu}_{\mu}$

is a complex structure $(J_\psi)^2 = -\mathbb{I}$

We have $\Omega \wedge \Omega = m^i_\psi \Sigma^i \wedge m^j_\psi \Sigma^j \sim (\vec{m}, \vec{m}) = 0$ this means that Ω is decomposable

There exists a basis of eigenvectors $\theta_1, \theta_2 \in \Lambda^1_{\mathbb{C}}$ such that $\Omega = \theta_1 \wedge \theta_1, \qquad i\omega = \theta_1 \wedge \overline{\theta}_1 + \theta_2 \wedge \overline{\theta}_2$

The data (ω, Ω) is not arbitrary but satisfies $\Omega \wedge \Omega = 0, \quad \Omega \wedge \omega = 0, \quad 2\Omega \wedge \overline{\Omega} = \omega^2$

Theorem: A pair of 2-forms (ω, Ω) satisfying the boxed relations (on the previous slide), defines a (Euclidean signature) metric on $\mathbb{R}^4$ in which $\omega, \mathrm{Re}(\Omega), \mathrm{Im}(\Omega)$ is an orthonormal basis in the space of self-dual 2-forms

Explicitly $ig(\xi,\eta)\mathrm{vol}_g = (i_\xi \Omega \wedge i_\eta \overline{\Omega} + i_\eta \Omega \wedge i_\xi \overline{\Omega}) \wedge \omega$

Alternatively, can describe the same data by $\Sigma^1 := \mathrm{Re}(\Omega), \quad \Sigma^2 = \mathrm{Im}(\Omega), \quad \Sigma^3 := \omega$

These 2-forms satisfy $\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$

And define the metric via $g(\xi,\eta)\mathrm{vol}_g = \epsilon^{ijk} i_\xi \Sigma^i \wedge i_\eta \Sigma^j \wedge \Sigma^k$ Mysterious Urbantke formula

Another example of a geometric map

$$\dim\{\Sigma^i/\mathrm{constraints}\} = 18 - 5 = 13 = 10 + 3$$

A set of 2-forms satisfying algebraic relations = Metric + Unit spinor (mod $\mathbb{Z}_2$)

There is a close link between these ideas and Kahler geometry, where the most useful description of the geometry is known to be precisely in terms of (ω, Ω)

Like frame 1-forms in the case of 3D, a triple of 2-forms $\Sigma^i : \Sigma^i \wedge \Sigma^j \sim \delta^{ij}$

provides arguably the most efficient known description of gravity in 4D - Plebanski formalism = Ashtekar formalism

$$d\Sigma^i + \epsilon^{ijk} A^j \wedge \Sigma^k = 0$$

$$dA^i + (1/2)\epsilon^{ijk} A^j \wedge A^k = (\Psi^{ij} - \frac{\Lambda}{3}\delta^{ij})\Sigma^j$$

Cosmological constant

Encodes Weyl curvature

Theorem: When fields Σ^i, A^i satisfy the above equations, the metric defined by the 2-forms Σ^i is Einstein

Importantly, we see that as dimension increases, the object that spinors tell us we should use to describe the metric are not frames - rather they are a collection of differential forms of higher degree

There is also a beautiful action principle giving the above as its Euler-Lagrange equations

$$S[\Sigma, A, \Psi] = \int \Sigma^i \wedge (dA^i + \frac{1}{2}\epsilon^{ijk} A^j \wedge A^k) - \frac{1}{2}(\Psi^{ij} - \frac{\Lambda}{3}\delta^{ij})\Sigma^i \wedge \Sigma^j$$

Remarks on uniqueness

Like in the 3D case, linearised uniqueness is easy to analyse

The relevant linearised fields are metric perturbation + unit spinor perturbation

Again, one finds a unique linearised Lagrangian once diffeomorphism and SU(2) gauge invariance are imposed

Like in 3D, the spinor is non-propagating in the SU(2) invariant theory

G-structures

Very convenient to formalise the examples we have encountered as so called G-structures

Metric geometry is not the only possible one

After the approach to geometry pioneered by Cartan, we now know many more types of geometry

Definition: G-structure is a reduction of the principal GL(n,R) bundle of frames on an n-manifold, to a G-subbundle

Examples:

- Volume form - reduction to SL(n,R)
- Metric - reduction to O(n,R)
- Almost complex structure - reduction of GL(2m,R) to GL(m,C)
- Hermitian metric - further reduction to U(m)

Can think of them as a collection of tensors invariant under G

The two examples we have encountered

- {0}-structure in 3D
- SU(2)-structure in 4D

G-structures and Spinors

Given a metric - rank 2 symmetric, non-degenerate tensor in S^2T^*M - can form the Clifford algebra

$$uv + vu = 2g(u,v), \qquad u, v \in TM$$

Its fundamental representation is known as the spinor representation

Remark: In order for spinors to exist globally on M, it must be spin (second Whitney class vanishes)

Now, given a spinor ψ there exists the stabiliser subgroup (possibly trivial) $G_\psi \subset \mathrm{Spin}(n)$

Many types of G-structures arise as those stabilising a spinor - they can be referred to as "spinorial"

Examples: Calabi-Yau SU(m) structure - pure spinor of Spin(2m,R)

Stabiliser of a non-zero spinor in 3D is {0} - get the already encountered {0} structures in 3D

$$\dim(\mathrm{GL}(3,\mathbb{R})/\{0\}) = 9 = 6 + 3$$

Stabiliser of a non-zero Weyl spinor in 4D is SU(2) - get the already encountered SU(2) structures in 4D

$$\dim(\mathrm{GL}(4,\mathbb{R})/\mathrm{SU}(2)) = 13 = 10 + 3$$

Geometry from spinors

The point of the examples considered is that the same phenomena continue and extend to arbitrary dimension and arbitrary signature. Thus, given a spinor of Spin(r,s)

- Geometric map - Spinor bi-linears define geometric objects - in general differential forms on M $S \otimes S = \oplus_{k=0}^{n} \Lambda^k(M)$
- The collection of geometric objects (differential forms) one obtains from a spinor can be used to encode the metric.
- In all known examples, this encoding is the best possible (most useful) description of the metric+spinor system!
- This construction explains why the metric can be encoded into objects of a different type - differential forms

Spinors and gravity in 8D

As one increases the dimension the story is analogous. More and more exotic ways to encode the metric arise

In 8D, the Majorana (real) spinor of Spin(8)

$$\psi \in S_+ \sim \mathbb{R}^8$$

$$\langle \psi, \psi \rangle = 1$$ unit spinor

The only independent geometrical object one can construct is

$$\Phi = \frac{1}{24} \langle \psi, \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \psi \rangle dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

This 4-form satisfies 27 independent algebraic relations. Known as the Cayley form. Encodes the octonionic product

Explicit formula for the metric defined by the Cayley form is available, but will not be needed

The stabiliser of a unit real spinor in 8D is Spin(7). Also the GL(8,R) stabiliser of a Cayley form

$$\dim(\mathrm{GL}(8,\mathbb{R})/\mathrm{Spin}(7)) = 64 - 21 = 43$$

$$\dim(\text{metrics}) = 36 \qquad \dim(\text{unit spinors}) = 7$$

$$\dim(\Lambda^4)/\text{constraints} = 70 - 27 = 43$$

This explains why the geometric map can work in this case

Linearised dynamics

Most general diffeomorphism invariant free theory Lagrangian for perturbations of the Cayley form?

Proposition: A computation shows that there is a one-parameter family of such Lagrangians

$$\frac{1}{2}(1+\frac{\kappa}{6})(\partial_a h_{bc})^2 - \frac{1}{6}(1-\frac{\kappa}{2})(\partial_a h)^2 - \frac{1}{3}(1-\frac{\kappa}{2})h\partial^a\partial^b h_{ab} - \frac{2}{3}(\partial^a h_{ab})^2$$
$$+\frac{1}{24}(1+\frac{\kappa}{2})(\partial_a \xi_{bc})^2 - \frac{2}{3}(1+\frac{\kappa}{2})\partial_b h^{ba}\partial^c \xi_{ca}$$

Here κ is a parameter and field ξ_{ab} belongs to the 7-dimensional representation of Spin(7),

here encoded as a 2-form satisfying some algebraic conditions

Remark: $\kappa = -2$ gives the same linearised Lagrangian as Einstein-Hilbert

For all other values we have 7 additional propagating degrees of freedom

Non-linear completion

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There is a one-parameter family of actions, analogous to the Plebanski action in 4D

$$S_\kappa[\Phi, C] = \int_M \Phi \wedge (dC - 6C \wedge_\Phi C) + \frac{\kappa}{6}(C)^2 v_\Phi + \frac{\lambda}{6} v_\Phi + \text{constraint terms.}$$

Here C is a 3-form and $v_\Phi = \frac{1}{14}\Phi \wedge \Phi$

8D version of the Plebanski formalism

$$(C \wedge_\Phi C)_{\mu\nu\rho\sigma} := C_{\mu\nu\alpha} C_{\rho\sigma\beta} g^{\alpha\beta}$$

Its linearisation gives the Lagrangian described on the previous slide

This describes (Euclidean) gravity in 8D, coupled to spinorial matter, which, in general, carries propagating DOF

Summary

- There is the geometric map $\text{spinor} \otimes \text{spinor} \to$ geometric objects (differential forms)
- Differential forms arising via this map can be used to encode the metric (in a way that is dimension and signature specific)
- In all known examples, the way to encode the metric as suggested by spinors is the most efficient and useful way to describe the metric + spinor geometry, and also impose various differential equations on it
- Many interesting geometries arise in the way described
- Many known examples fall into the same pattern. In particular the known (but still exotic to many people) descriptions of 3D and 4D gravity are covered

Do not use the metric to describe geometry. Use differential forms that originate in spinors

Speculative outlook

Coming back to the Kaluza-Klein dimensional reduction ideas

My hope is that in the search for a geometry that is relevant to the description of the real world

some version of this construction (with a Majorana-Weyl spinor) of Spin(12,4) is useful,

in the sense of providing the right geometric setup for unification by dimensional reduction ideas.

The basic idea is to look for a spinor orbit in 16D that can break Spin(16) in a realistic fashion

Symmetry breaking effected by the spinor matter itself

The metric is then described by a collection of differential forms, as relevant for that particular spinor orbit

Machian - no (spinor) matter, no metric

Thank you!