Black hole dynamics in large number of space-time dimensions

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Motivation - Black Holes

- Black holes are singular solutions of Einstein equation
- The space-time singularity is causally shielded behind a null hypersurface called event horizon.
- Like most natural objects, black hole geometries also evolve with time, i.e., their event horizon becomes dynamical.
- Also they radiate a lot of energy as gravitational waves in the process.
- According to General Theory of Relativity all these dynamics of Black Holes are governed by Einstein's equation.

Motivation (contd)

1. Einstein's equation is a deceptively simple looking equation. It relates curvature of the space-time with the energy distribution.
2. Curvature in encoded in the Ricci Tensor $R_{\mu\nu}$ and Ricci scalar R and energy distribution is given by the stress energy tensor $T_{\mu\nu}$.
3. However, in reality this is a set of very complicated nonlinear partial differential equations where the basic variables are the different components of the space-time metric.
4. Exact analytic solutions are almost impossible to construct unless we have special symmetries and stationarity.
5. Even numerical solutions are not easy to find in a completely dynamical situation

Motivation

1. In such a scenario perturbative techniques often prove very useful.
2. One standard way to construct perturbative solutions around an exact one is to slightly break any of the symmetry of the exact solutions.
3. For example, we could have a static Black hole solution perturbed by a time dependent component of small amplitude.
4. In this talk, our goal would be to develop another new perturbation technique, particularly suited to handle the dynamics of black holes.
5. We shall use the dimension of the space-time (D) as new perturbation parameter.

Why we could use D as perturbation parameter

- It turns out that Black hole solution (even when there is dynamics) simplifies a lot in the limit of large D.
- In fact in strict $D \to \infty$ limit, we can construct a simple and exact dynamical black hole solution
- This leads to a natural extension of this exact solution in powers of $\left(\frac{1}{D}\right)$
- Further we could also track the non-perturbative effects falling off like e^{-D}.

Goal of this talk

In this talk we shall see that

- New dynamical Black hole solutions with any asymptotic geometry could be constructed as a power series in $\frac{1}{D}$
- Boundary data for these solutions are encoded on the horizon - viewed as a massive codimension-1 membrane embedded in the asymptotic geometry.
- There is a one to one duality between the dynamics of this membrane and the a dynamical black hole solution
- The gravitational radiation out of this massive membrane maps to the radiation out of the dynamical black hole. This turns out to be the nonperturbative effect in the context of the $\left(\frac{1}{D}\right)$ expansion.

Plan of the Talk

- First we shall give the intuition behind the large D simplification of the black hole type solution.
- Next using this intuition we shall propose an ansatz for the solution at strict $D \to \infty$ limit.
- We shall show how this leading solution could be corrected order by order in a series in $\left(\frac{1}{D}\right)$ and how some components of the Einstein's equation lead to the dual membrane dynamics.
- Next we shall sketch the method to track the gravitational radiation as the nonperturbative effect.
- Finally we conclude with some open questions and possible applications of this technique for astrophysical phenomena.

Simplification at large D

- Gravitation potential around a localized spherical massive source will fall off with distance r as $\left(\frac{M}{r^{D-3}}\right)$ in D dimensions. here M is the energy scale associated with the problem.
- Note $\lim_{D\to\infty}\left(\frac{M}{r^D}\right) = 0$ whenever $\left(r - M^{\frac{1}{D}}\right)$ is finite and positive.
- Now consider the case when r could be written as $r = M^{\frac{1}{D}}\left(1 + \frac{x}{D}\right)$. We could take $D \to \infty$ limit keeping x fixed.

$$\lim_{D\to\infty}\left(1 + \frac{x}{D}\right)^{-D} = e^{-X}$$

- So the first observation:
  as D increases, the gravitational force of any localized massive body confines within a thin shell of thickness $(1/D)$ around the outer boundary of the mass distribution.

Simplification at large D (contd)

- Now suppose the gravitational potential has the schematic form

$$\text{Potential} \sim \left(\frac{M(y^\mu)}{r^D}\right), \quad \{y^\mu\} \equiv \text{ all other transverse coordinates}$$

- Note

$$\partial_r \left(\frac{M}{r^D}\right)_{r=M^{\frac{1}{D}}} = D\left(\frac{M}{M^{\frac{D-1}{D}}}\right), \qquad \partial_{y^\mu}\left(\frac{M}{r^D}\right)_{r=M^{\frac{1}{D}}} = \left(\frac{\partial_{y^\mu} M}{M}\right)$$

- Clearly the variation of the potential along the radial direction is D times faster than the other $\{y^\mu\}$ directions.

So the second observation:

- In the strict $D \to \infty$ limit, it is the derivative along the radial direction that contributes to the equations ;
- As $D \to \infty$, the metric acquires effective translational invariance along all the non-radial directions.

Simplification at Large D - Summary

As $D \to \infty$

- The gravitational force of any localized massive body confines within an infinitesimally thin shell (of thickness of the order $\mathcal{O}(1/D)$.

In strict $D = \infty$ limit

- It is the derivative along the radial direction that contributes to the equations ;
- The metric acquires effective translational invariance along all the non-radial directions.

Deviation from the strict $D \to \infty$ limit amounts to a breaking of the translational invariance along the $\{y^\mu\}$ directions within this infinitesimally thin strip.

Every derivative along y^μ directions will have a factor of $\frac{1}{D}$ compared to the radial derivatives, which could be handled perturbatively in a series in $\left(\frac{1}{D}\right)$.

The leading solution

- As it is true for any perturbative technique, the starting point is the exact leading solution.
- We shall use the exact stationary black hole solutions to determine this starting ansatz.
- Schwarzschild black holes in Kerr Schild (chk) coordinates:

$$ds^2 = \underbrace{-dt^2 + dr^2 + r^{D-2} d\Omega_{d-2}^2}_{\text{Flat space}} + M\ r^{-(D-3)}(dt + dr)^2$$

- We evaluate each component of Einstein's equation on this metric
- Inspecting their large D limit we distill out the those structures of the above metric that are essential for solving the leading large D equation and therefore must be there in the leading ansatz.
- We should emphasise that the we do not have any systematic derivation for the leading ansatz. In the end it is an educated guess that could be used as the starting point for the whole technique.

Our final ansatz for the leading solution

Our final ansatz (in asymptotically flat space)

$$ds^2 = G_{AB}\ dx^A\ dx^B = ds^2_{flat} + \psi^{-D}(O_A\ dx^A)^2$$

$$\text{where } ds^2_{flat} = \text{Flat space metric} = \eta_{AB}\ dx^A\ dx^B$$

- Here ψ is a smooth function where $\psi = 1$ defines the event horizon.
  Hence $d\psi$ is null on $\psi = 1$ hypersurface.
- Define
  1. norm of $d\psi$ w.r.t the flat metric $= N \equiv \sqrt{(\partial_A\psi)\,(\partial_B\psi)\,\eta^{AB}}$
  2. $n_A\ dx^A = \frac{d\psi}{N}$

     unit normal to the $\psi = const$ surface, viewed as embedded in the flat space.
- $O \equiv O_A\ dx^A$ is null oneform satisfying
  1. $O_A\ n_B\ G^{AB} = 1 \qquad O_A\ O_B\ G^{AB} = 0$ - everywhere.

Metric at leading order

$ds^2 = \left[\eta_{AB} + \psi^{-D} O_A O_B\right] \; dx^A \; dx^B,$

Event Horizon: $\psi = 1$, O is a null oneform, $N \equiv$ norm of $(d\psi)$ w.r.t η_{AB}, $n \equiv \frac{d\psi}{N}$

- There are two sources of $\mathcal{O}(D)$ terms.
  1. when a derivative act on ψ^{-D}
  2. When we trace over the $(D-1)$ transverse derivatives.

  A single derivative along a transverse direction is of $\mathcal{O}(1)$. But when we sum up the derivatives in all transverse directions it will be of $\mathcal{O}(D)$.

  i.e., $\partial_A O^B \sim \mathcal{O}(1)$ but $\sum_A \partial_A O^A = \nabla \cdot O =$ sum of $(D-1)$ terms, each of order $\mathcal{O}(1) \sim \mathcal{O}(D)$

- We could easily see that the leading term in Einstein tensor , once evaluated on the above metric will be a product of the above two types of terms and therefore is of $\mathcal{O}(D^2)$.
- The final form of the Einstein tensor on $\psi = 1$ hypersurface;

$$E_{AB}|_{\psi=1} = \left(\frac{DN}{2}\right) \Big[(\nabla \cdot n + \nabla \cdot O - 2DN) \; O_A O_B + (\nabla \cdot O - DN) \; (O_A n_B + O_B n_A)\Big] + \mathcal{O}(D)$$

here ∇ denotes derivative w.r.t the flat space metric and all index contractions are also w.r.t η_{AB}

Metric at leading order (contd)

$ds^2 = \left[\eta_{AB} + \psi^{-D} O_A O_B\right] \, dx^A \, dx^B,$

Event Horizon: $\psi = 1$, O is a null oneform, $N \equiv$ norm of $(d\psi)$ w.r.t η_{AB}, $n \equiv \frac{d\psi}{N}$

So we conclude

- Any metric of the above form will satisfy the Einstein equation at leading order provided on the horizon $\psi = 1$

$$\nabla \cdot n = \nabla \cdot O = DN \tag{1}$$

- Note it is enough to satisfy the Einstein tensor only at $\psi = 1$ because
  - The metric and therefore the Einstein tensor is nontrivial only within a thin shell of width $(1/D)$ around $\psi = 1$ hypersurface.
  - Within this shell, any deviation from $\psi = 1$ hypersurface will therefore be multiplied by extra factors of $\left(\frac{1}{D}\right)$ and hence suppressed compared to the leading term.
- Further all terms on the constraint equation (1) is defined w.r.t flat space metric.
  So this equation could be viewed as constraints on some membrane embedded in flat space defined by the equation $\psi = 1$.

The leading solution compared to exact Black hole solution

- Our leading ansatz: $ds^2 = ds^2_{flat} + \psi^{-D} O_A O_B \ dx^A \ dx^B$
- Sch Black Hole solution:

$$ds^2 = ds^2_{flat} + M \ r^{-(D-3)}(dt+dr)^2 \approx ds^2_{flat} + M \ r^{-D}(dt+dr)^2$$

- Comparing we could see that on Sch BH

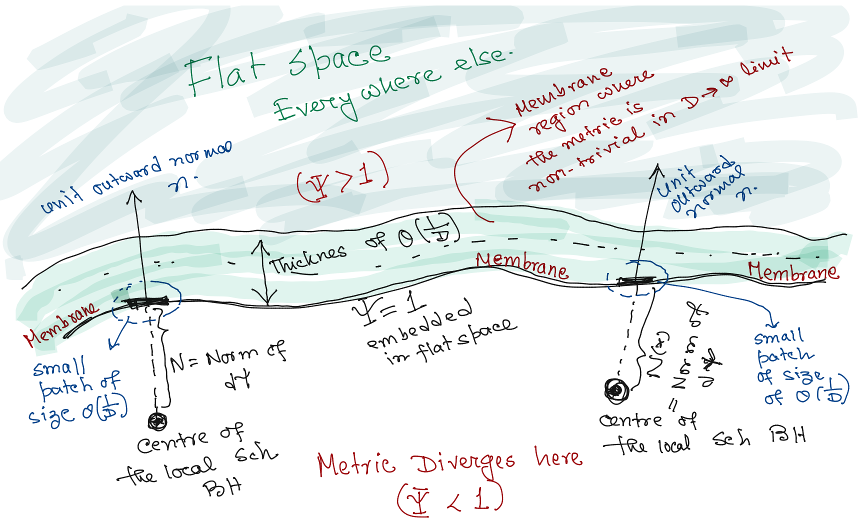
$$\psi_{(Sch)} = M^{-\left(\frac{1}{D}\right)} r, \quad O_A^{(Sch)} \ dx^A = dr + dt, \quad n_A^{(Sch)} \ dx^A = dr$$

- It is easy to check that $\psi_{(Sch)}, \ \ O^{(Sch)}, \ \ n^{(Sch)}$ also satisfy the two constraints

$$\nabla \cdot n^{(Sch)} = \nabla \cdot O^{(Sch)} = DN^{(Sch)} + \mathcal{O}(1) \sim D \ \underbrace{M^{\frac{1}{D}}}_{\text{BH radius}} \ + \mathcal{O}(1)$$

- It turns out that locally we could always cast it in the form of Sch BH provided we identify
  1. N (local norm of $d\psi$) $\longrightarrow$ radius of the BH
  2. $u_A \ dx^A \equiv (O_A - n_A) \ dx^A = dt \ \longrightarrow$ differential of local time

Our leading ansatz - schematic diagram



Subleading corrections

Once we have the leading ansatz, it is straightforward (though a bit tedious at higher order) to compute the subleading corrections in the metric.

- Metric is corrected as

$$ds^2_{flat} + \psi^{-D}\left[O_A\ O_B + \left(\frac{1}{D}\right) g^{(1)}_{AB} + \left(\frac{1}{D}\right)^2 g^{(2)}_{AB} + \cdots\right] dx^A\, dx^B$$

- To determine $g^{(1)}_{AB}$ we evaluate Einstein equation on the above metric upto order $\mathcal{O}(D)$. Recall at leading order Einstein equation was of order $\mathcal{O}(D^2)$
- This will give a linear inhomogeneous ODE on the components of $g^{(1)}_{AB}$, which could be solved
  It is an ODE as opposed to PDE because variation of g^{AB} along all transverse directions are further suppressed by factors of $(1/D)$.

  ODE is inhomogeneous because the leading ansatz itself will generate some $\mathcal{O}(D)$ terms involving derivatives of ψ and O.
- We could show that the same pattern will be repeated at all orders.

The data of the solution and the constraints on it

- The data that generates these class of perturbative BH solutions
  1. The horizon $\psi = 1$ viewed as a dynamic membrane embedded in flat space-time
  2. The null oneform O_A or more precisely the timelike oneform $u_A \equiv O_A - n_A$ to be identified locally as the time coordinate.

  Recall $n_A \propto d\psi$ and unit-normalized w.r.t flat metric.

- It turns out that we could consistently solve higher order corrections $g^{(n)}_{AB}$ provided the $\psi = 1$ hypersurface and the velocity-like field u_A satisfies a set of coupled constraint equations.
- These constraint equations have the structure of the 'equations of motion' of a dynamical massive membrane, embedded in flat space-time coupled with a velocity.
- For every solution of these membrane equations we could generate a unique BH type solution, thus establishing the membrane-BH duality in large D expansion.

Field redifinition ambiguity

- Note we shall express our geometry in terms of the field ψ. But this function is constrained to have value 1 only on the horizon but other than that we have the freedom of redefining it.
- If one geometry is expressed in terms of the function ψ and its derivatives, we could also choose it to express in terms of a different function $\tilde{\psi}$ defined as

  $$\tilde{\psi} = \psi + \left(\frac{1}{D}\right) \Delta\psi$$

  so that $\Delta\psi$ vanishes when $\psi = 1$
- Such a redefinition will affect the expression of the subleading corrections (though not the actual geometry).
- it turns out that all of the first subleading correction to the metric could be absorbed into some appropriate redefinition of the ψ field.

Metric correction at first subleading order

$$ds^2 = ds^2_{flat} + \psi^{-D} \left[O_A \; O_B + \left(\frac{1}{D}\right) g^{(1)}_{AB} + \left(\frac{1}{D}\right)^2 g^{(2)}_{AB} + \cdots \right] dx^A \; dx^B$$

- If we fix the field redefinition ambiguity in ψ and O_A, by imposing the conditions

$$\nabla^2 \psi^{-D} = 0, \quad (O \cdot \nabla) O_A \propto O_A$$

the first metric correction $g^{(1)}_{AB}$ vanishes.

$$ds^2 = ds^2_{flat} + \psi^{-D} \left[O_A \; O_B \; dx^A \; dx^B + \mathcal{O}\left(\frac{1}{D^2}\right) \right]$$

- But the constraint equation on the membrane is invariant under these type of field redefinitions and is nontrivial at first subleading order.

Constraint equation at first subleading order

- The constraint equations constrain the data on the dual membrane dynamics.
- It is a set of equations involving extrinsic curvature K_{AB} of the dual membrane at $\psi = 1$, embedded in flat space and the velocity field u_A on it.
- The equation takes the following form

$$P_A^C \left[\frac{\hat{\nabla}^2 u_C - \nabla_C K}{K} + u^D K_{DC} - (u \cdot \nabla) u_C \right] = 0$$

$$\hat{\nabla} \cdot u = 0$$

Here K_{AB} is the extrinsic curvature of the membrane, K is the trace of the extrinsic curvature and P_{AB} is the projector perpendicular to both n and u.
$\hat{\nabla}$ is derivative projected along the hypersurface.

Extensions

- At the second subleading order both the metric and the membrane equations receive nontrivial corrections.
- Though it has been explicitly computed, the answer looks cumbersome and so we are not presenting it here.
- This technique, almost without any modification, could be extended to any other asymptotic geometry, in particular AdS or even more singular situations where the background itself contains other black holes.
- It has also been extended to
    1. Einstein Maxwell equations, leading to a dual charged membrane
    2. Higher derivative gravity theories like Einstein-Gauss Bonnet gravity.

Radiation

- It turns out that the constraint equation could be recast as an equation of the form of stress tensor conservation.

$$T_{\mu\nu} = \left(\frac{1}{8\pi}\right)\left[\left(\frac{K}{2}\right) u_\mu u_\nu + K_{\mu\nu} - \left(\frac{\hat{\nabla}_\mu u_\nu + \hat{\nabla}_\nu u_\mu}{2}\right) - \left(\frac{u_\mu \hat{\nabla}^2 u_\nu + u_\nu \hat{\nabla}^2 u_\mu}{K}\right) - \left(\frac{1}{2}\right)\left[u^\alpha u^\beta K_{\alpha\beta} + \left(\frac{K}{D}\right)\right] g^{ind}_{\mu\nu}\right]$$

  Here $g^{ind}_{\mu\nu}$ is the induced metric on the membrane, which is used to define the covariant derivative $\hat{\nabla}_\mu$.
  $K_{\alpha\beta}$ = the extrinsic curvature of the membrane embedded in the flat space-time.

- This stress tensor is completely intrinsic to the membrane and conserved as a consequence of the constraint equations.

Radiation (contd)

- So in these geometries the horizon $\psi = 1$ could also be viewed as a massive dynamical membrane embedded in flat space-time and associated with a conserved stress tensor.
- Therefore this membrane will act as a source of gravitational radiation which could be computed simply by convoluting this stress tensor with retarded Green's function in flat space.
- The radiation thus constructed would be dual to the gravitational radiation emitted from a dynamical black hole.
- The fact that the radiation is nonperturbative (goes as D^{-D}) is a consequence of the properties of Green's function in large number of space-time dimension.
- However, it has been precisely computed in the first few orders.

Application

- As mentioned before, it is very difficult to construct dynamical BH solutions though we know that there is a rich physics hidden behind such solutions.
- Therefore any technique generating dynamical BH solutions also provides a nice toolkit to analyse formal theoretical questions that necessarily involve time-evolution.
  For example
  - We could use such solutions to address the question entropy production in higher derivative theories of gravity.
    (already has been addressed in the context of Einstein Gauss Bonnet theory)
  - Instabilities and their end points (explored in detail in the context of black string to black hole transition)
- But we hope that such solutions could be used to model real astrophysical phenomena in some limit and therefore provide a quantitative matching point for numerics.

Application and future direction

- Here the BH dynamics has been mapped to the dynamics of membrane or soap bubbles which are well studied in some completely different context.
- In the soap bubble picture, the violent astrophysical phenomenon of two BHs merging mapped to the merging two soap bubbles and this analogy has already been explored in a quantitative.
- It might be possible to adapt the radiation calculation to the dynamics of such merging and thus we could analytically compute the radiation coming out of such merging process (though in the unrealistic corner of large D, but still might be useful)

Future direction

- Finally it is a perturbative technique to solve complicated differential equation based on a simple property of the background solution that
  it depends exponentially on the number of dimensions, but such dependence is there only along one coordinate.
- Therefore it might be possible to use this method to any other system of equations where the background has this property.

Thank you