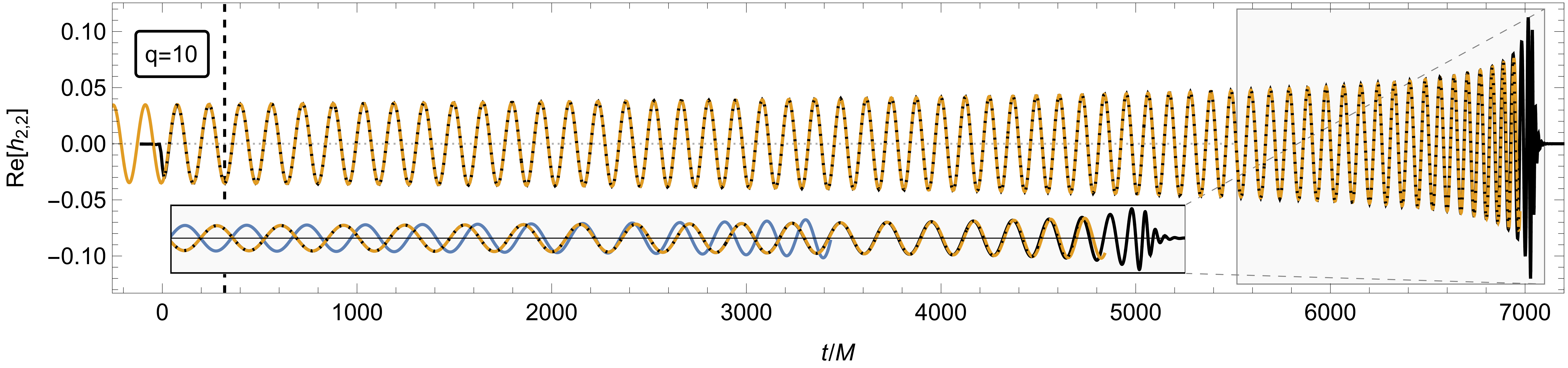



# State of the art in gravitational self-force

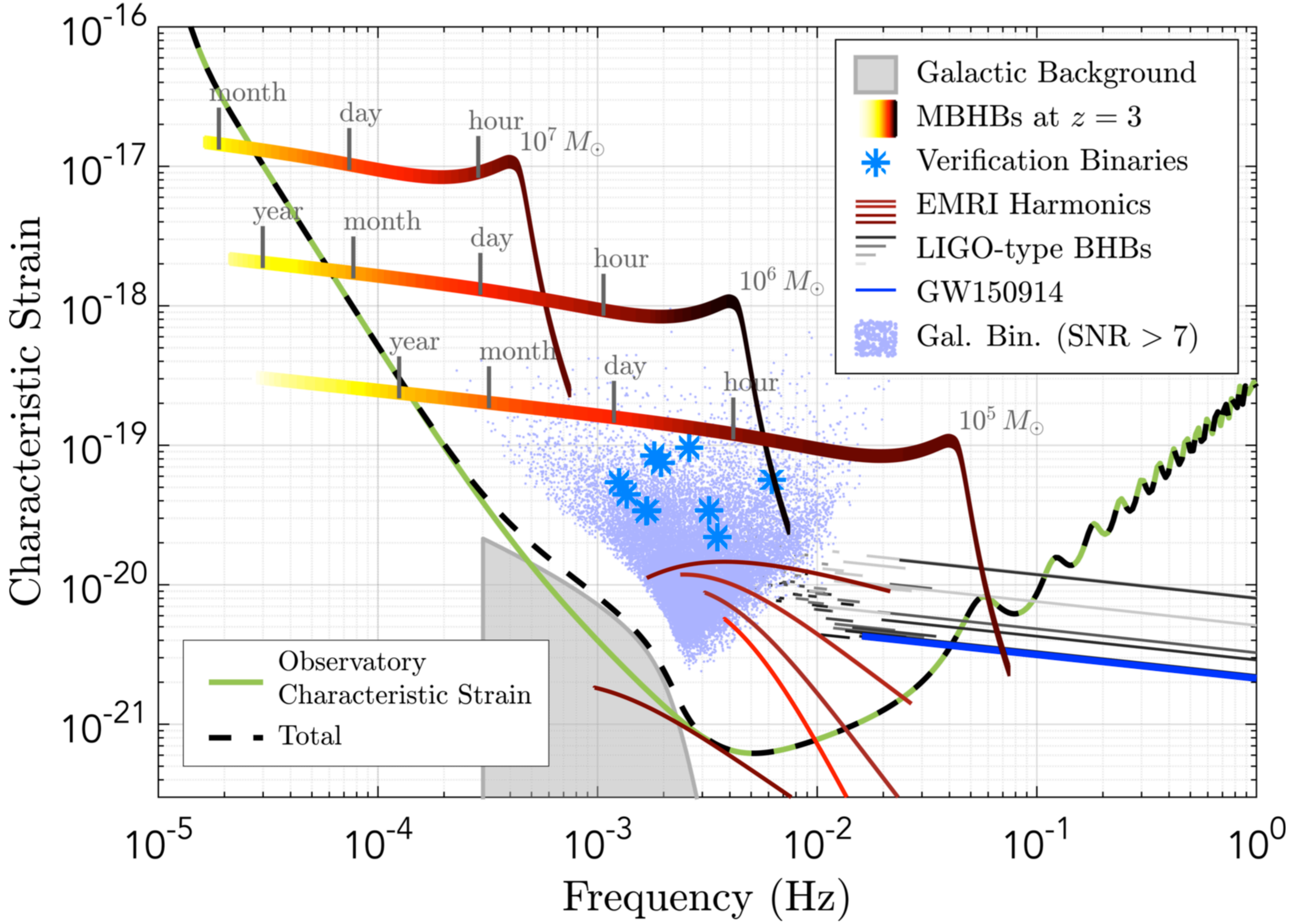

Barry Wardell
University College Dublin


**M**ultiscale Simulations with **S**elf-**F**orce Collaboration

Gravitational Self-force and Scattering Amplitudes - Edinburgh University - 19th March 2024

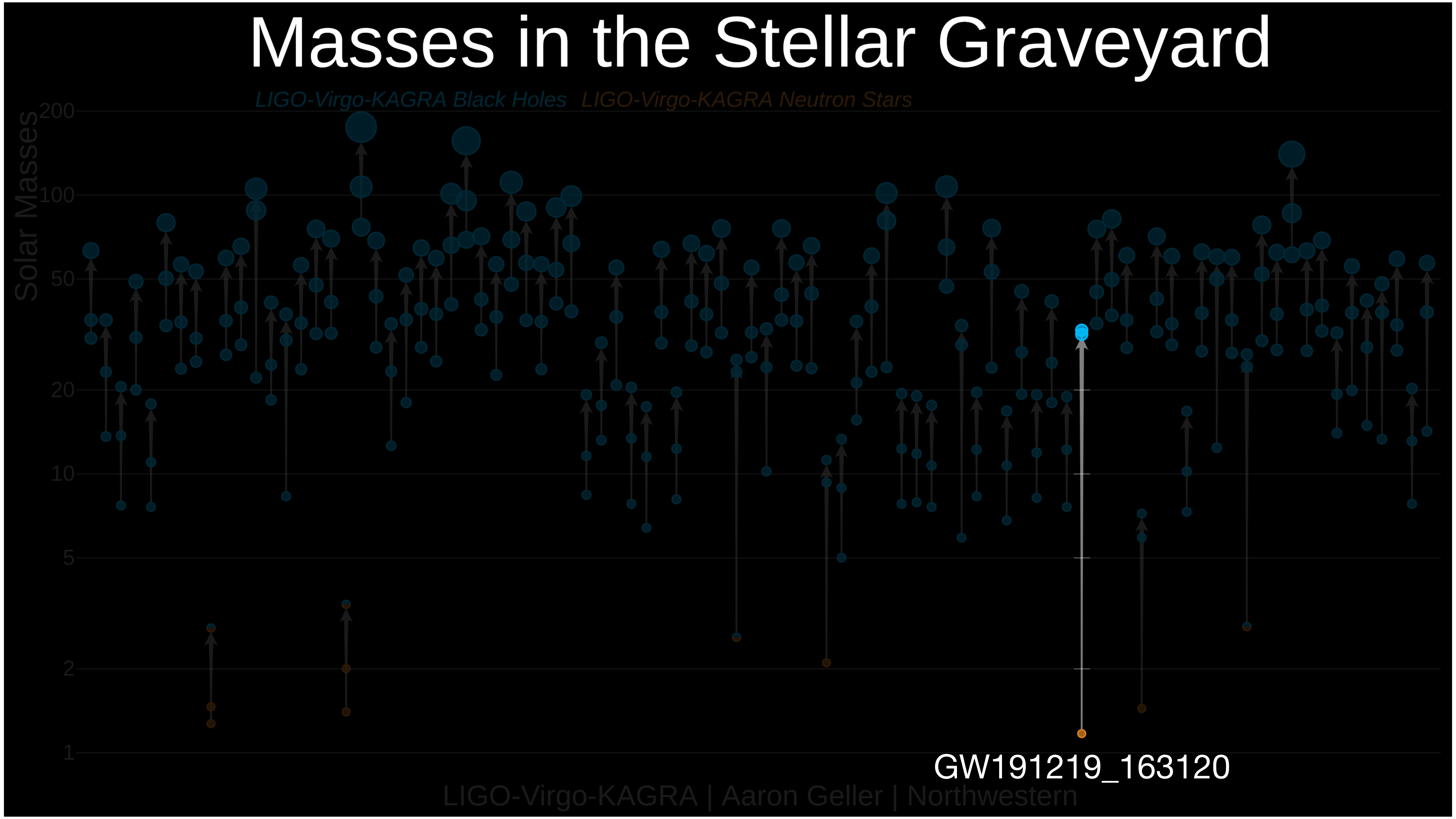
# Self-force: Why?

# Observing gravitational waves with LISA




Detect and estimate parameters for extreme mass-ratio inspirals (EMRIs) using LISA

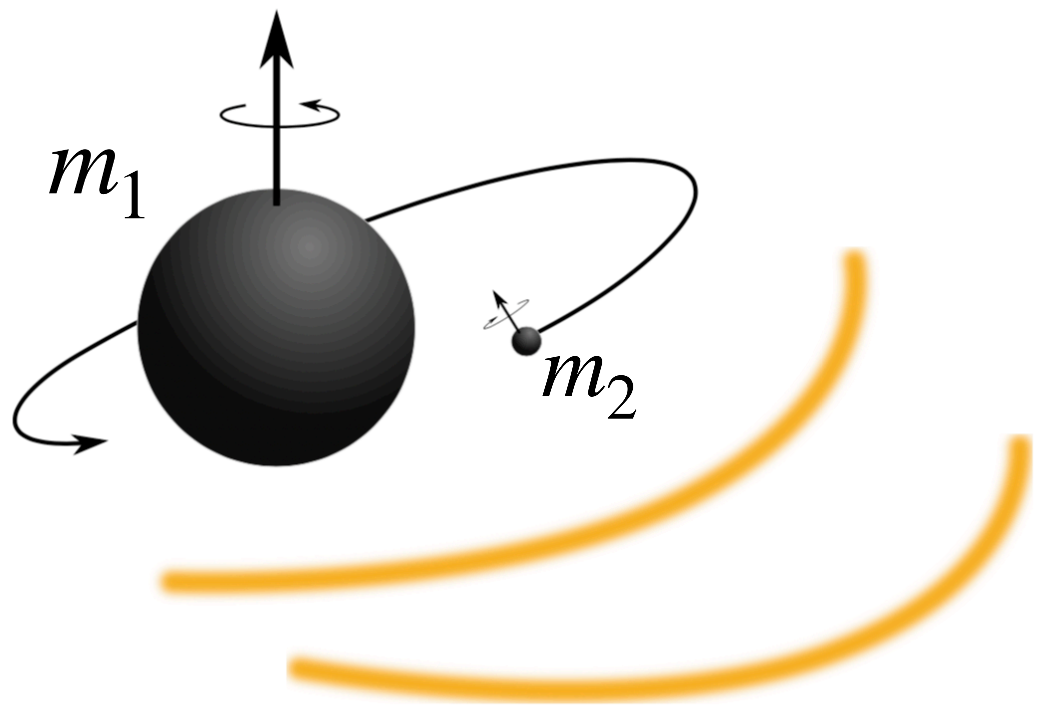
# Observing gravitational waves with LIGO




“The mass ratio of GW191219_163120's source is inferred to be $0.038^{+0.005}_{-0.004}$, which is **extremely challenging for waveform modeling**, and thus there may be **systematic uncertainties** in results for this candidate.”

# Self-force: How?

# Gravitational self-force




Expand exact binary spacetime about that of the primary

Schwarzschild/Kerr black hole

$$g^{\text{exact}}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3)$$

$$\epsilon = \frac{m_2}{m_1}$$

Substitute expansion into the Einstein equation

$$G_{\mu\nu}[\mathrm{g}] = 8\pi T_{\mu\nu}$$

Expand out in powers of $\epsilon$

$$\epsilon^0: \quad G_{\alpha\beta}[g] = 0$$

$$\epsilon^1: \quad G^1_{\alpha\beta}[h^{(1)R}] = 8\pi T_{\alpha\beta} - G^1_{\alpha\beta}[h^{(1)S}]$$

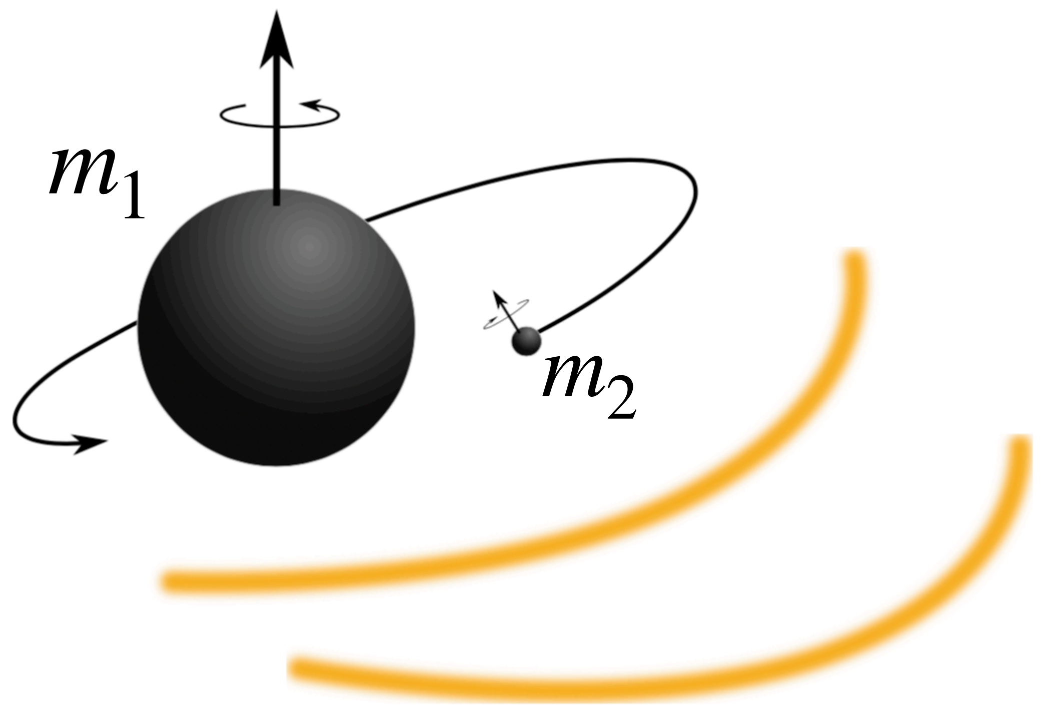
$$\epsilon^2: \quad G^1_{\alpha\beta}[h^{(2)R}] = - G^2_{\alpha\beta}[h^{(1)}, h^{(1)}] - G^1_{\alpha\beta}[h^{(2)S}] + \partial_{\tilde{t}} h^{(1)}$$

$$\partial_{\tilde{t}} h^1 = \dot{\Omega} \partial_\Omega h^1$$

This is hard.

Perform two-timescale expansion by introducing a “slow time” $\tilde{t} = \epsilon t$, use a frequency domain decomposition.

# Post-adiabatic orbit evolution




The evolution of the orbit is determined by the post-adiabatic/self-force equations of motion:

$$\frac{d\Omega}{dt} = \epsilon \left[ F_0^{\Omega}(\Omega) + \epsilon F_1^{\Omega}(\Omega) \right] \qquad \frac{d\phi_p}{dt} = \Omega$$

**Adiabatic order**

0PA: Adiabatic dissipation-driven rate of change, determined by first-order dissipative gravitational self-force/ energy flux (dissipative part of $h^{(1)}$)

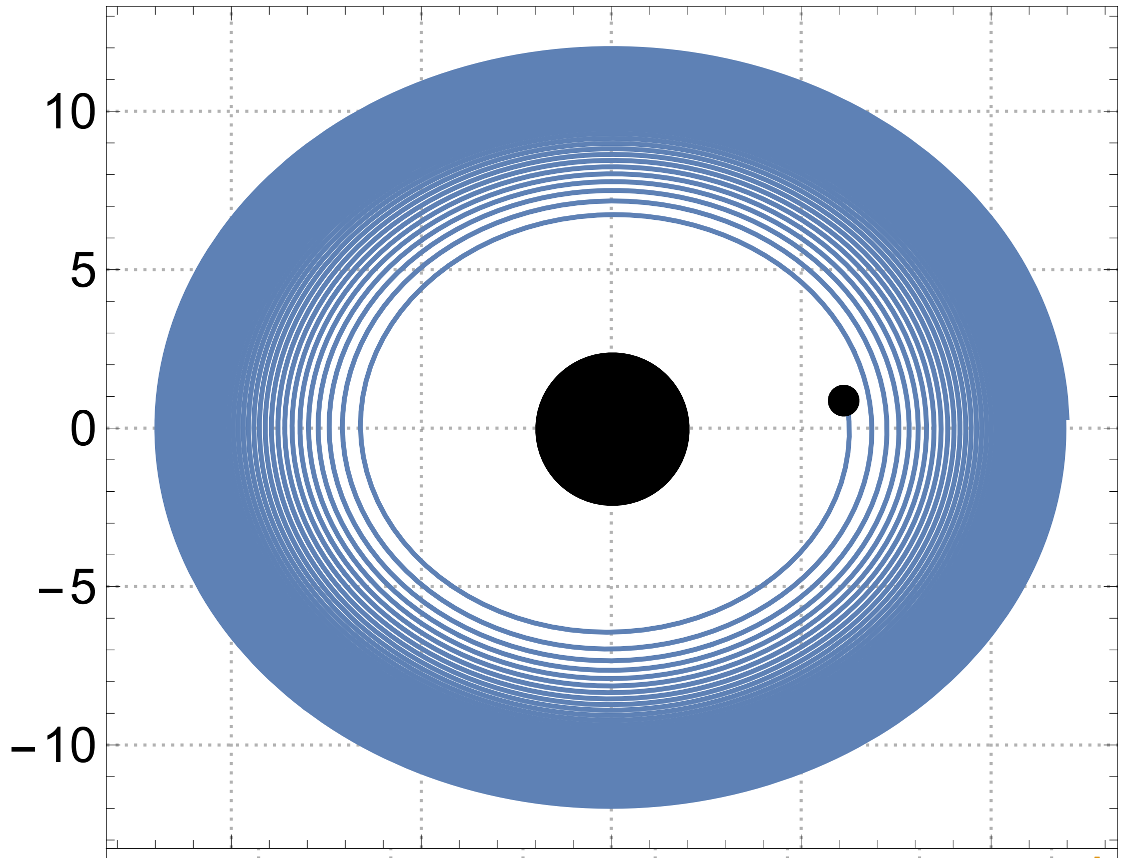
**Post-Adiabatic order**

1PA: first-post-adiabatic term determined by the full (conservative and dissipative) first-order gravitational self-force (full $h^{(1)}$) and second-order dissipation (dissipative part of $h^{(2)}$).

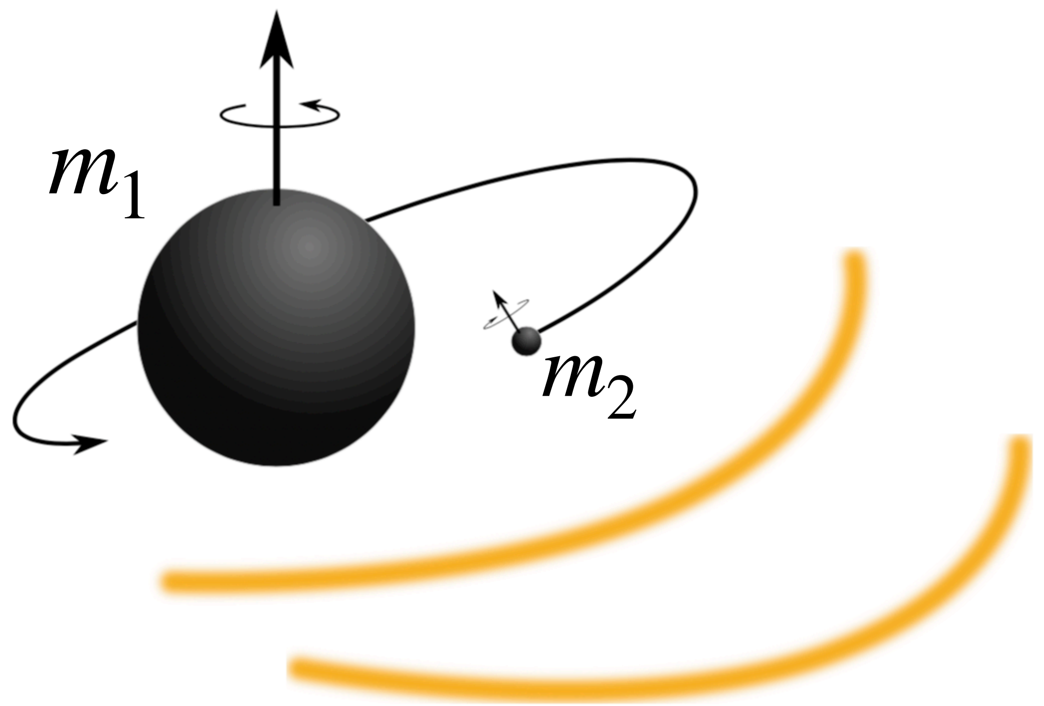
At 1PA order have to account for evolution of the mass and spin of the primary:

$$\frac{d\delta m_1}{dt} = \epsilon \mathscr{F}^{(1)}_{\mathscr{H}}(\Omega) \qquad \frac{d\delta s_1}{dt} = \epsilon\, \Omega^{-1}\, \mathscr{F}^{(1)}_{\mathscr{H}}(\Omega)$$

Specialise (for now) to quasi-circular orbits with orbital frequency $\Omega$ and spins aligned with orbital angular momentum.



# EMRI waveforms




Split the waveform into an amplitude and orbital phase:

$$h_{\ell m}(t) = \left[\epsilon h^{(1)}_{\ell m}\big(\Omega(t)\big) + \epsilon^2 h^{(2)}_{\ell m}\big(\Omega(t)\big)\right] e^{-im\phi_p(t)}$$

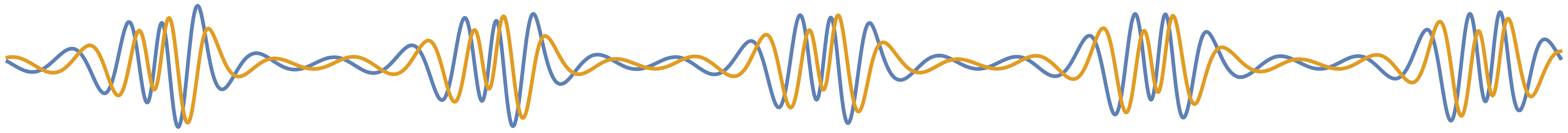
**Amplitude** is given by solving the linearised Einstein Equations.

**Frequency evolution** is given by solving the post-adiabatic equations of motion.

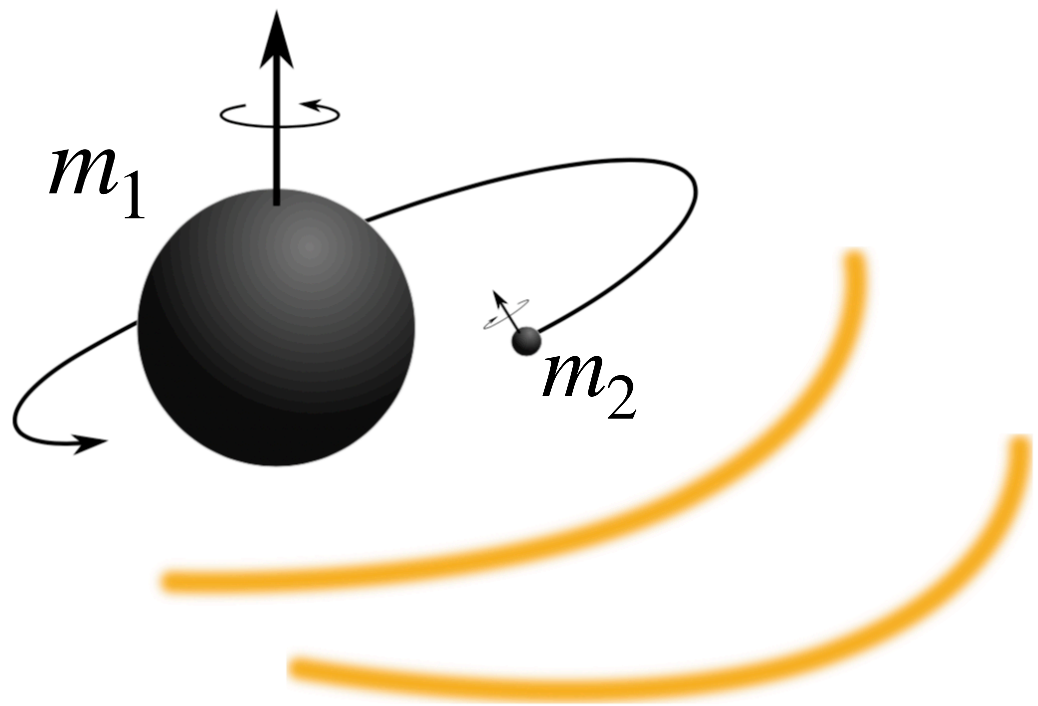
**Orbital phase** is given by integrating over the orbital frequencies.

Algorithm:

1. Precompute $h^{(1)}$ and $h^{(2)}$ on a grid of $\Omega$ values by solving Einstein's equations (hard)
2. Waveforms can be generated in **milliseconds** by solving ODEs (easy).



# EMRI waveforms




Split the waveform into an amplitude and orbital phase:

$$h_{\ell m}(t) = \left[\epsilon h^{(1)}_{\ell m}\big(\Omega(t)\big) + \epsilon^2 h^{(2)}_{\ell m}\big(\Omega(t)\big)\right] e^{-im\phi_p(t)}$$

**Amplitude** is given by solving the linearised Einstein Equations.

**Frequency evolution** is given by solving the post-adiabatic equations of motion.

**Orbital phase** is given by integrating over the orbital frequencies.

For LISA: to $\mathcal{O}\left(\epsilon^0\right)$ the phase has contributions at adiabatic and post-adiabatic orders

$$\phi_p(t) = \epsilon^{-1}\phi_0\left[\langle h^1_{\rm diss}\rangle\right] + \phi_1\left[h^1_{\rm diss,osc} + h^1_{\rm cons} + \langle h^2_{\rm diss}\rangle\right] + \mathcal{O}(\epsilon)$$

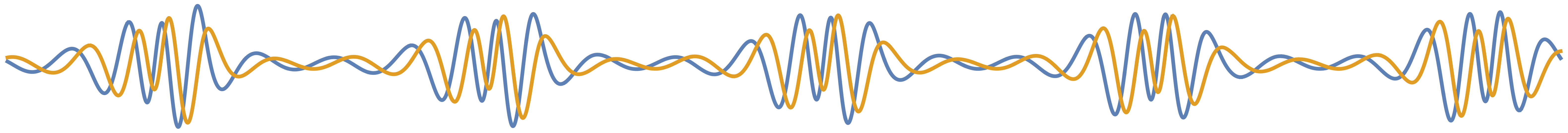
**Adiabatic order**

Can be obtained from asymptotic **fluxes**, avoiding a local calculation of the self-force

**Post-Adiabatic order**

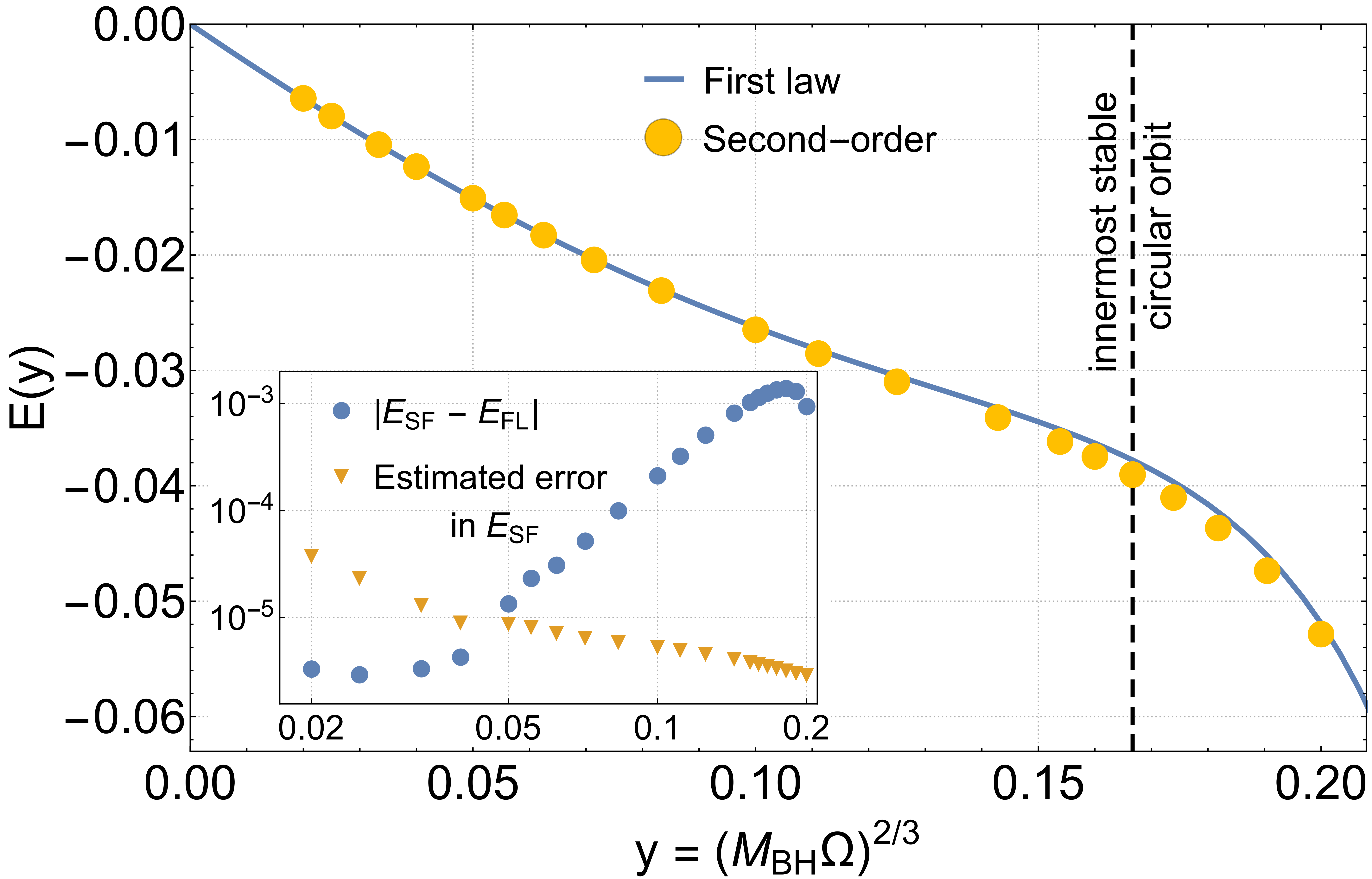
Several contributions:

- Oscillatory first order self-force
- Spin of secondary
- Second-order averaged self-force



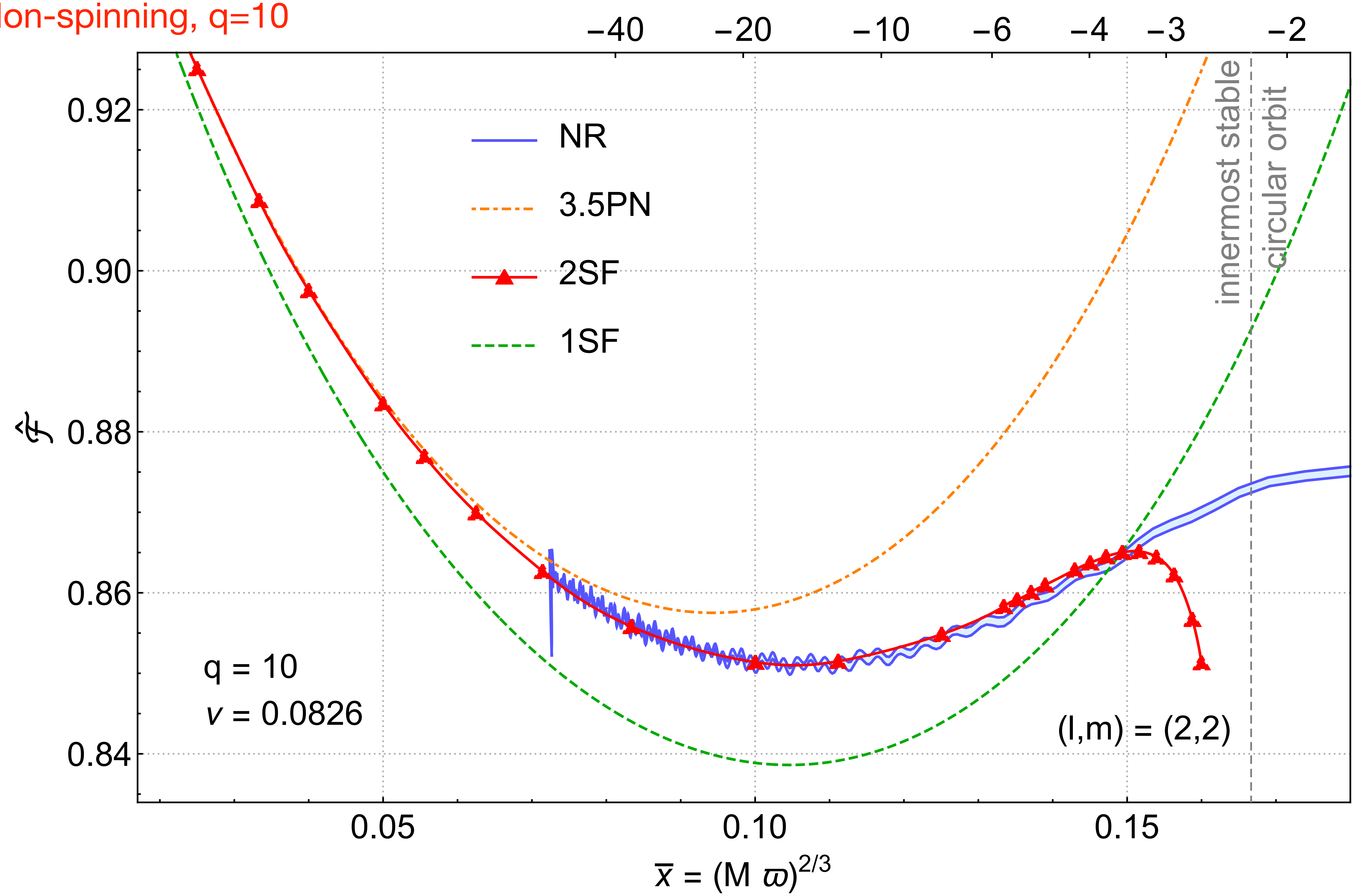
# Self-force: Latest Results

## Binding energy ($l = 0 = m$)



# Gravitational wave fluxes through $\mathscr{I}$ $(l \geq 2, m \neq 0)$

Non-spinning, q=10




$$\mathscr{F}_{lm}(t) = \frac{1}{16\pi}\left|\dot{A}_{lm}(t)\right|^2, \quad \varpi(t) = \dot{\Phi}_{22}(t)/2$$

NR waveform: SXS:BBH:1107

N. Warburton, A. Pound, B. Wardell, J. Miller and L. Durkan [Phys. Rev. Lett. 127, 151102]

# EMRI Waveforms

Factor waveform into amplitudes and **orbital** phase

$$h_{\ell m}(t) = \left[\epsilon h^{(1)}_{\ell m}\big(\Omega(t)\big) + \epsilon^2 h^{(2)}_{\ell m}\big(\Omega(t)\big)\right] e^{-im\phi_p(t)}$$

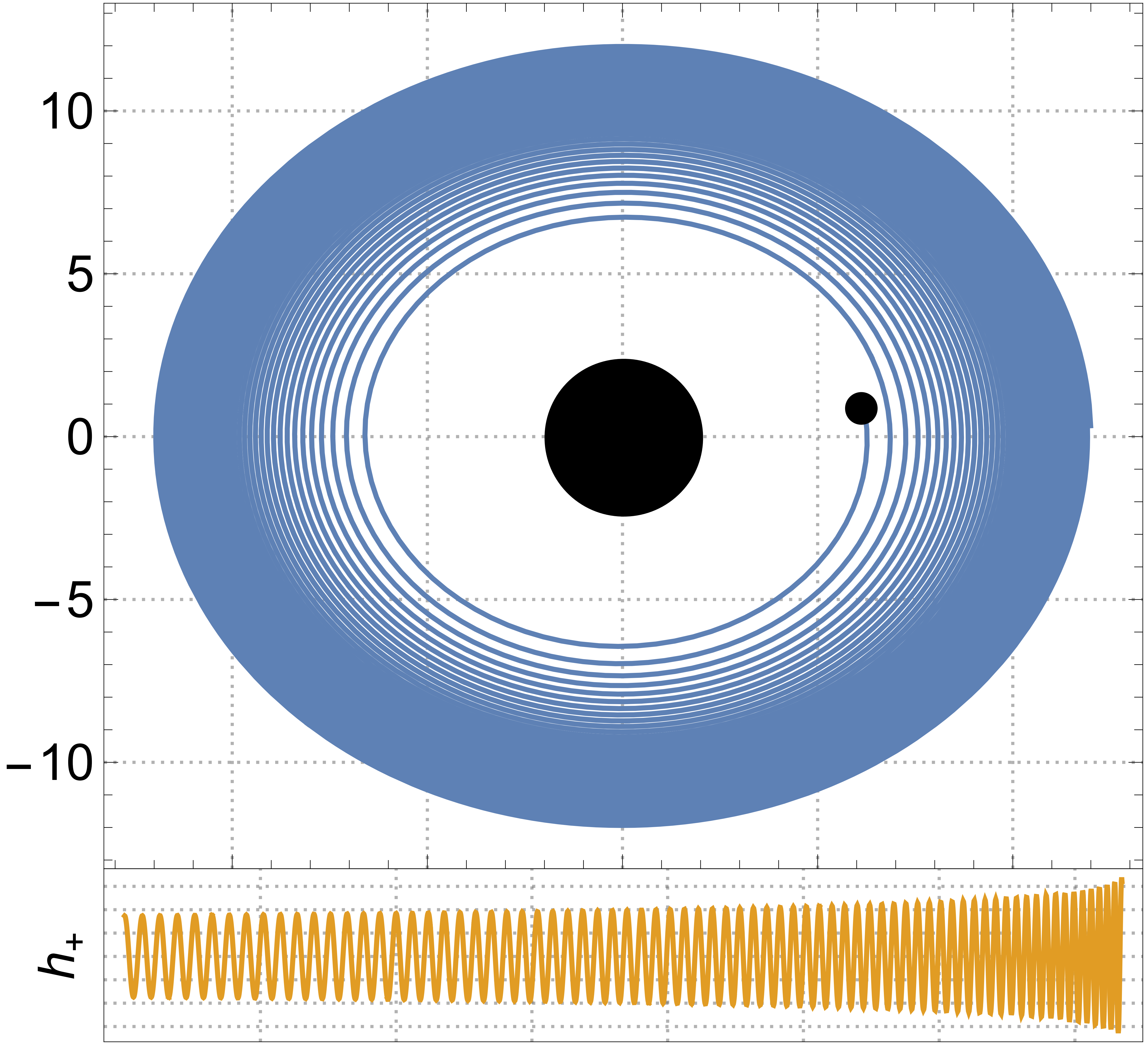
The **amplitude** is given by solving the first and second order Einstein Equations.

The **frequency evolution** is given by solving the post-adiabatic equations of motion.

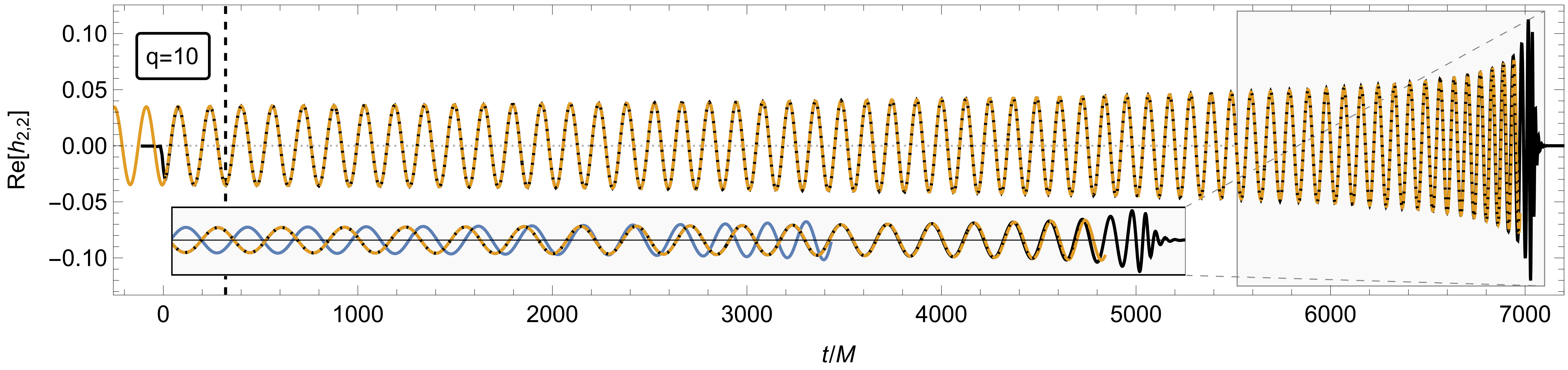
The **orbital phase** is given by integrating over the orbital frequencies.

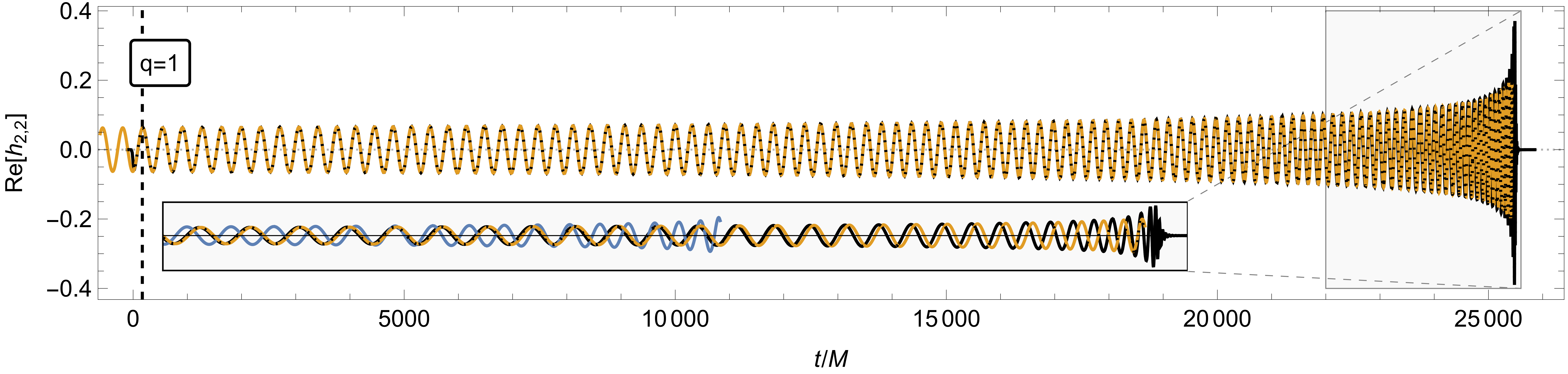
$$\frac{d\Omega}{dt} = \epsilon\left[F^{\Omega}_0(\Omega) + \epsilon F^{\Omega}_1(\Omega)\right] \qquad \frac{d\phi_p}{dt} = \Omega$$

# Mass ratio $q = 10$ quasi-circular inspiral



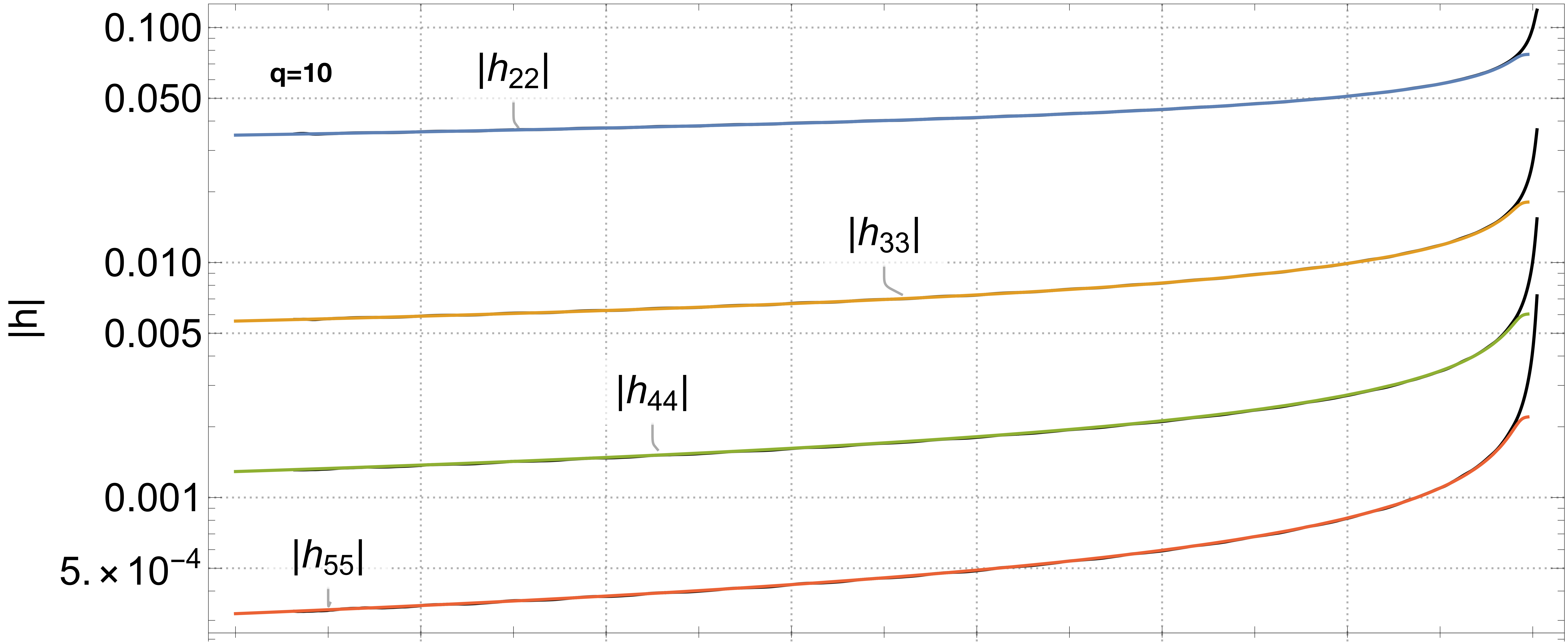
# Waveform comparison with Numerical Relativity




Barry Wardell, Adam Pound, Niels Warburton, Jeremy Miller, Leanne Durkan and Alexandre Le Tiec [Phys. Rev. Lett. 130, 241402]

# Waveform comparison: higher modes



# Including spin in waveforms

$$\chi \equiv S/\mu^2$$

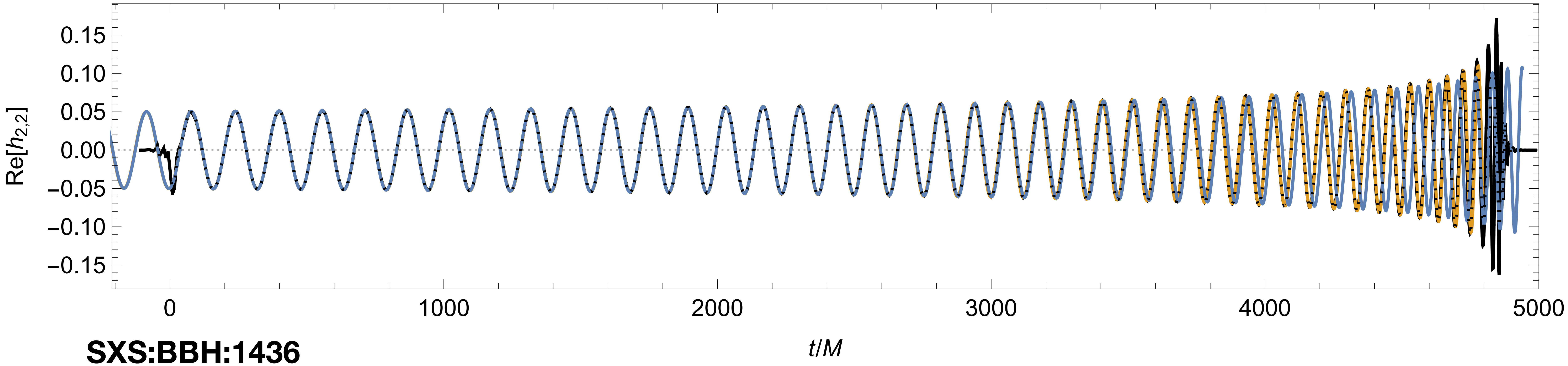
$$S^2 \equiv S^\alpha S_\alpha = \frac{1}{2} S_{\alpha\beta} S^{\alpha\beta}$$

$$\mu^2 \chi_\parallel^2 \equiv S^\theta S_\theta$$

$$\chi_\perp^2 \equiv \chi^2 - \chi_\parallel^2$$

# Aligned secondary spin

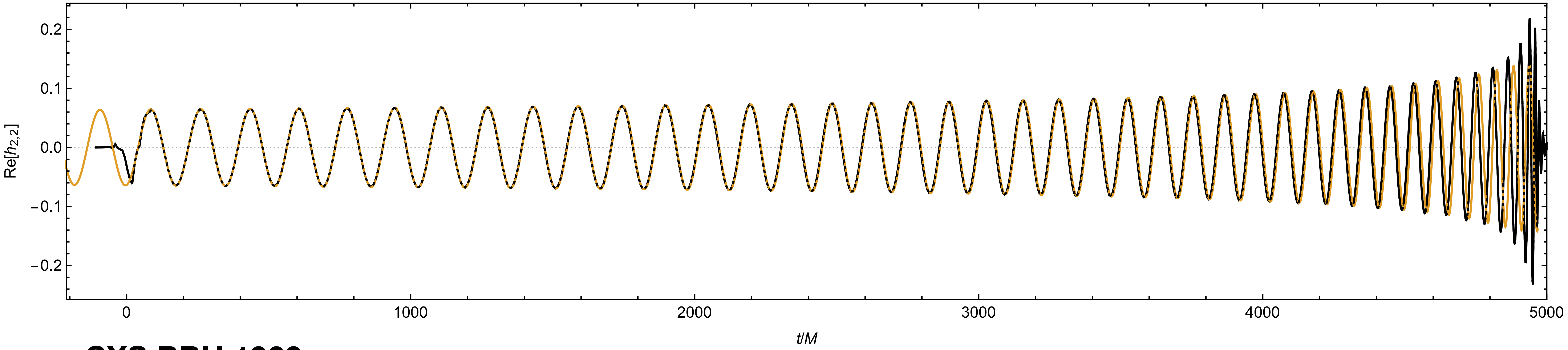
$$q = 6, \quad \chi_2 = -0.8 \quad (\chi_1 = 0)$$




**SXS:BBH:1436**

# Precessing secondary spin

$$q = 4, \quad \chi \;\approx 0.01, \quad \chi_\perp \approx 0.8, \quad (\chi_1 = 0)$$




**SXS:BBH:1909**

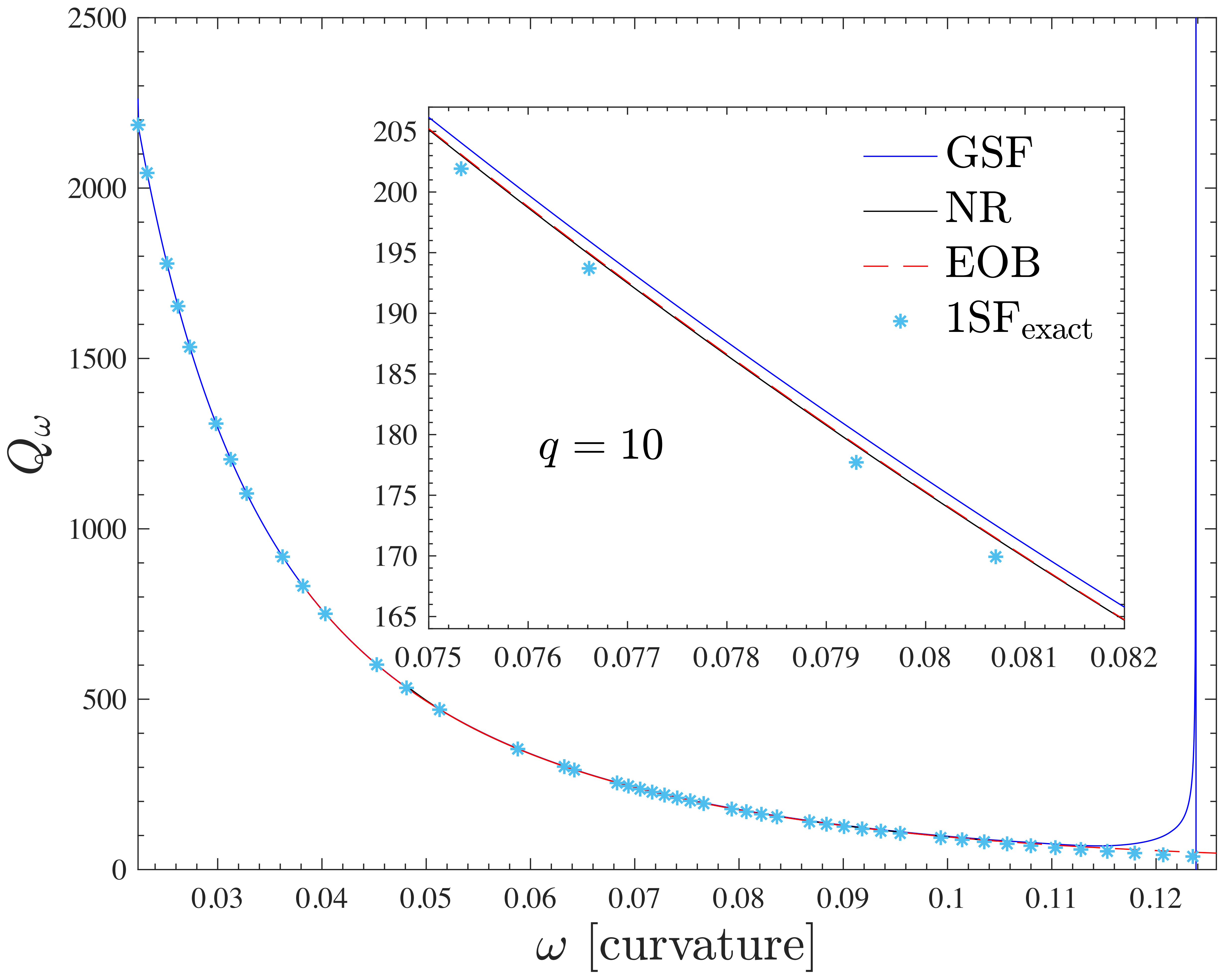
# Comparison with TEOBResumS

Detailed comparison of 1PA GSF waveforms with those from the TEOBResumS effective one body model and with numerical relativity.

1. Effects of transition to plunge significant over a large frequency interval, restricting domain of validity to orbital frequencies much smaller than ISCO frequency.

2. 1PA GSF models yield satisfactory phase errors for mass ratios $\epsilon \lesssim 1/25$.

3. Identified key areas for improvement in TEOBResumS, particularly for small mass ratios.

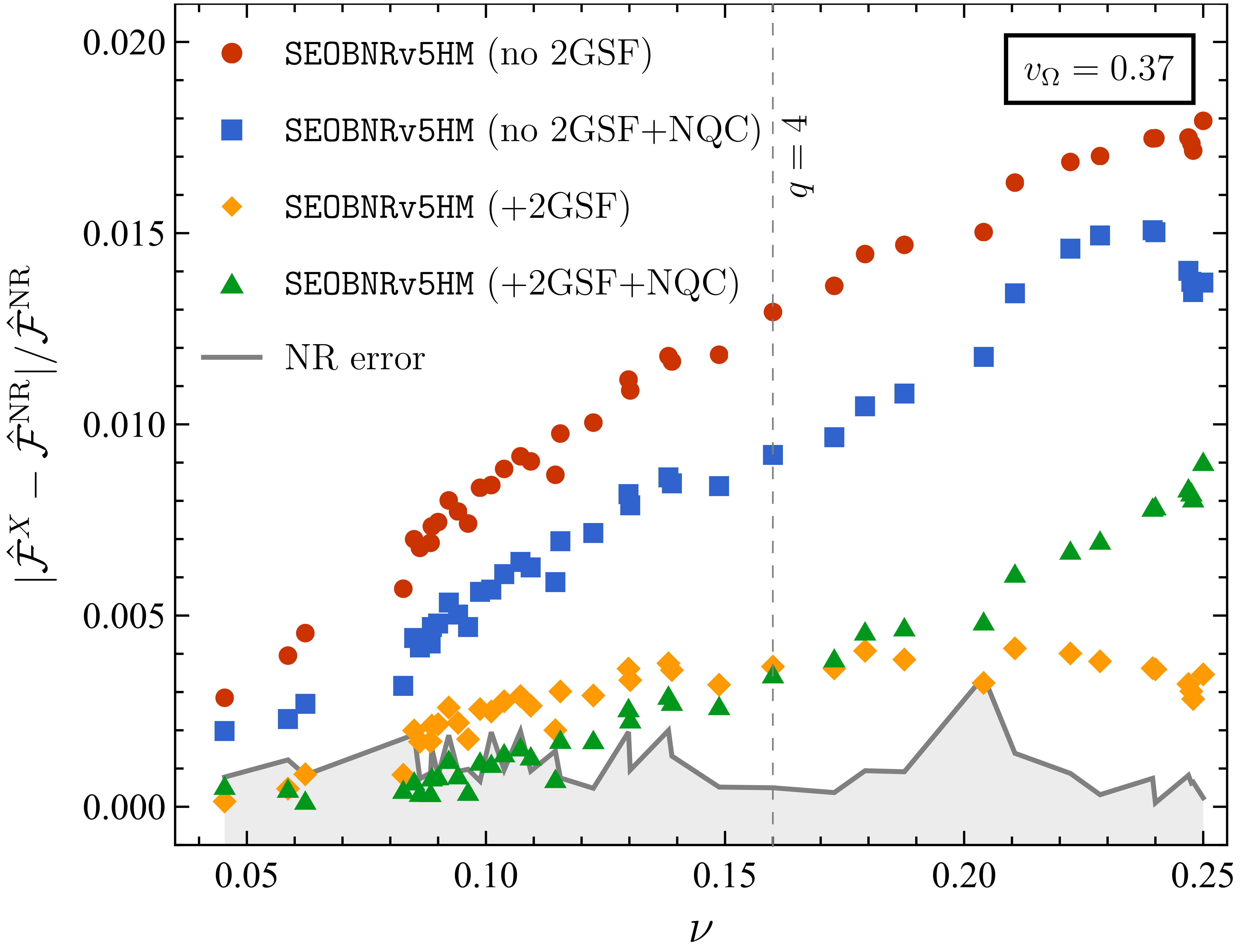


# Calibration of SEOBNRv5

Incorporated 2SF flux information into latest SEOBNR models prepared for LIGO O4 data analysis.

1. Significant improvement in agreement with reference results provided by NR.

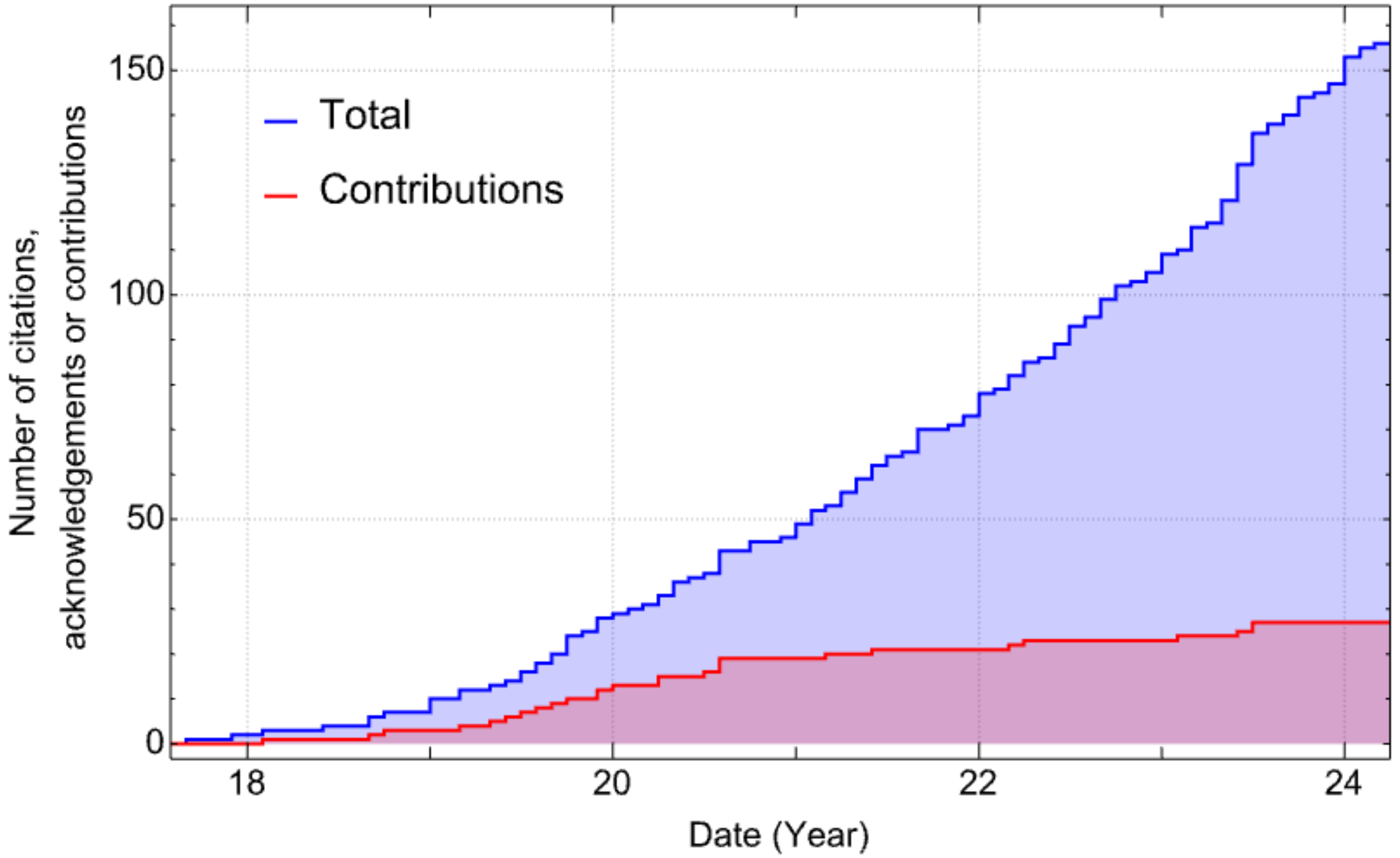
2. Reduces the need to rely on “NQC” corrections.



# Black Hole Perturbation Toolkit

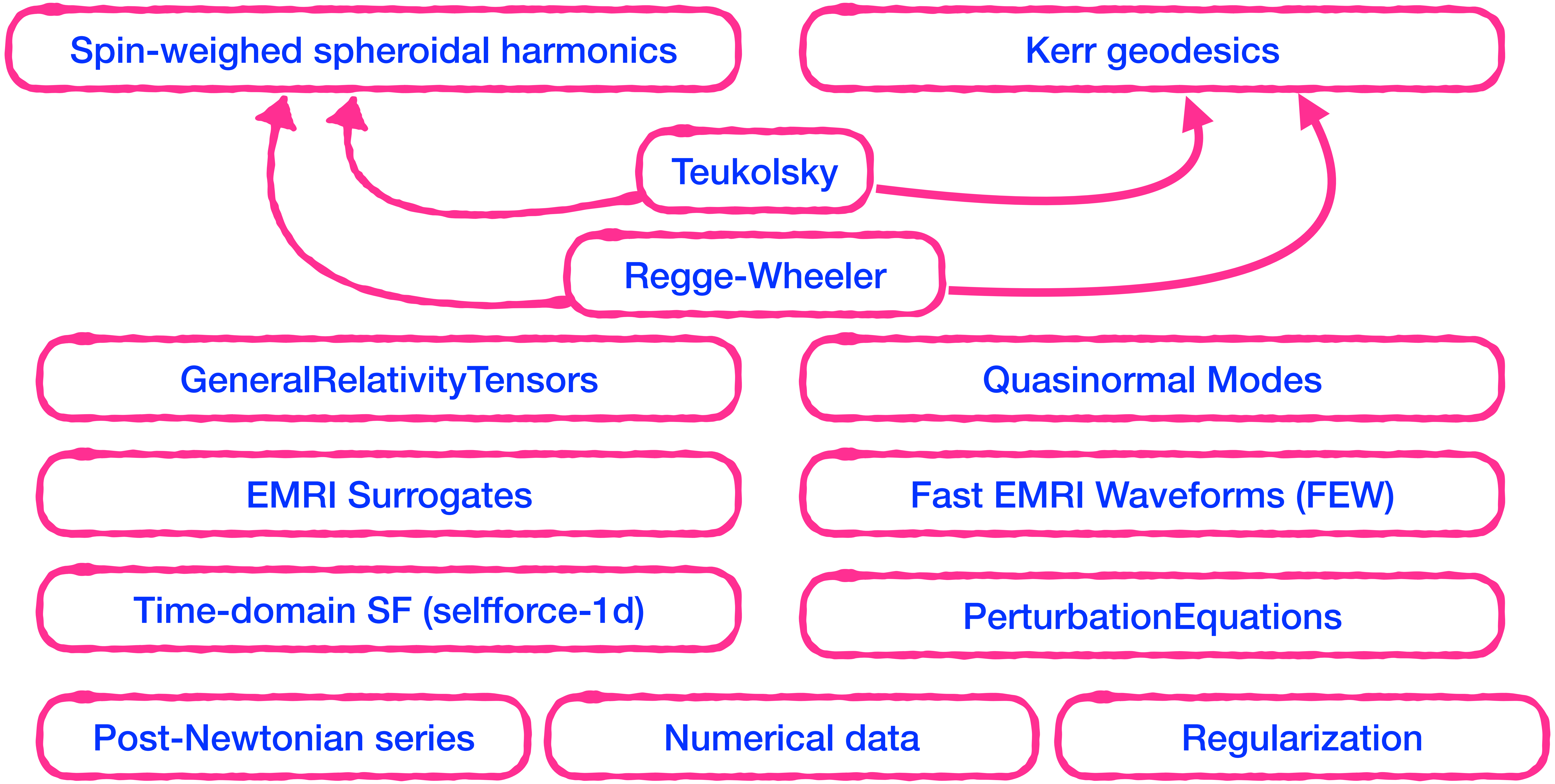
"Our goal is for less researcher time to be spent writing code and more time spent doing physics. Currently there exist multiple scattered black hole perturbation theory codes developed by a wide array of individuals or groups over a number of decades. This project aims to bring together some of the core elements of these codes into a Toolkit that can be used by all.

Additionally, we want to provide easy, open access to data from black hole perturbation codes and calculations."

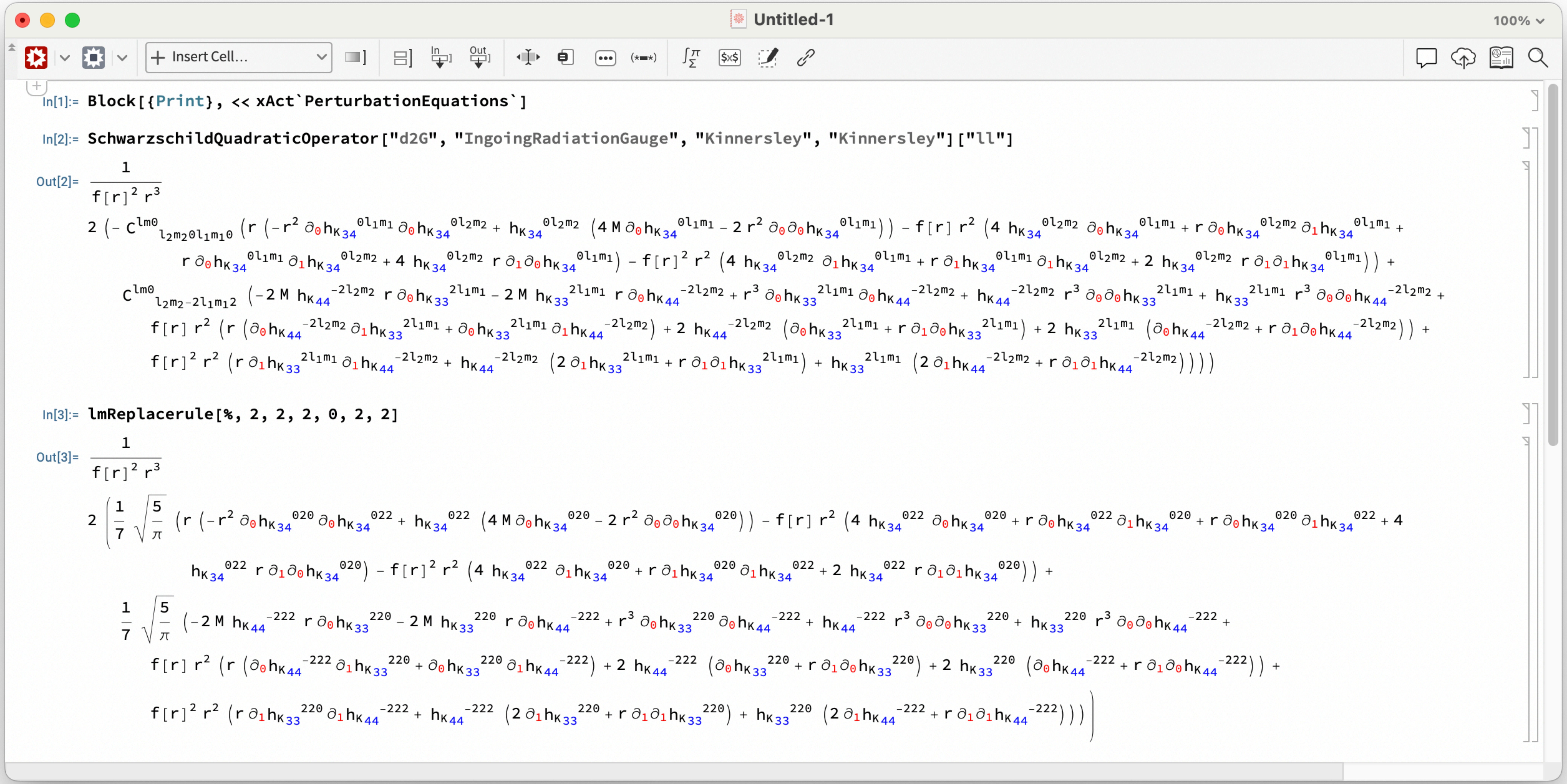



# Currently available toolkit components

The black hole perturbation toolkit has several packages for doing calculations in black hole perturbation theory, including post-adiabatic (1PA) waveforms.



# Second order Einstein equations: PERTURBATIONEQUATIONS package




Andrew Spiers, Adam Pound and Barry Wardell [arXiv:2306.17847, bhptoolkit.org/PerturbationEquations]

# Parameter estimation

Incorporated 1PA waveform into Fast EMRI Waveforms package. Fast enough to be used in LISA MCMC parameter estimation studies (~6 hours on a GPU per configuration).

Focus on three configurations:

| Config. | $\epsilon$ | $M\ [M_\odot]$ | $r_0/M$ | $D_\mathrm{S}$ [Gpc] | $T_\mathrm{obs}$ [yrs] | $\rho_{AET}$ |
|---|---|---|---|---|---|---|
| (1) | $10^{-5}$ | $10^6$ | 10.6025 | 1.0 | 2.0 | 70 |
| (2) | $10^{-4}$ | $10^6$ | 15.7905 | 2.0 | 1.5 | 65 |
| (3) | $10^{-3}$ | $5 \cdot 10^6$ | 16.8123 | 1.0 | 1.0 | 340 |

# Parameter estimation: Case (1)


Marginalized posteriors with shaded 68% credible intervals generated by injecting a true reference model cir1PA and recovering using different models

## Parameter estimation: Case (3)

# Intermediate Mass Ratio Inspiral: $q = 10^{-3}$


Marginalized posteriors with shaded 68% credible intervals generated by injecting a true reference model cir1PA and recovering using different models

# Dephasing, mismatches and degeneracy

Bias in the parameters is degenerate with mis-modelling errors

| $\epsilon$ | Model Waveform | $\Delta\Phi^{(\mathrm{inj})}$ | $\Delta\Phi^{(\mathrm{bf})}$ | $\mathcal{M}^{(\mathrm{inj})}$ | $\mathcal{M}^{(\mathrm{bf})}$ | $\rho^{(\mathrm{inj})}/\rho^{(\mathrm{opt})}$ | $\rho^{(\mathrm{bf})}/\rho^{(\mathrm{opt})}$ | $\log\mathcal{L}^{(\mathrm{inj})}$ | $\log\mathcal{L}^{(\mathrm{bf})}$ |
|---|---|---|---|---|---|---|---|---|---|
| $10^{-5}$ | Cir1PA w/o spin | 0.779 | 0.0165 | 0.143 | $4.497 \times 10^{-5}$ | 83.4% | 99.9% | $-846$ | -0.250 |
| | Cir0PA 1PA-3PN w/o spin | 0.786 | 0.00179 | 0.163 | $4.293 \times 10^{-6}$ | 81.5% | 99.8% | $-943$ | -0.0324 |
| | Cir0PA w/o spin | 3.002 | 0.00532 | 0.889 | $2.412 \times 10^{-6}$ | 6.4% | 99.8% | $-4800$ | -0.0234 |
| $10^{-4}$ | Cir1PA w/o spin | 3.994 | 0.00702 | 0.511 | $8.601 \times 10^{-6}$ | 30.3% | 99.9% | -5019 | -0.336 |
| | Cir0PA 1PA-3PN w/o spin | 4.310 | 0.0179 | 0.486 | $1.26 \times 10^{-4}$ | 34.2% | 99.9% | -4799 | -0.441 |
| | Cir0PA w/o spin | 13.093 | 0.0354 | 0.653 | $2.573 \times 10^{-5}$ | 19.0% | 99.9% | -5506 | -0.122 |
| $10^{-3}$ | Cir1PA w/o spin | 4.518 | 0.00559 | 0.922 | $3.643 \times 10^{-6}$ | 3.3% | 99.9% | -112938 | -0.226 |
| | Cir0PA 1PA-3PN w/o spin | 4.882 | 0.0218 | 0.949 | $3.443 \times 10^{-5}$ | 3.4% | 99.9% | -112827 | -2.132 |
| | Cir0PA w/o spin | 14.958 | 0.153 | 0.938 | $6.854 \times 10^{-3}$ | 4.9% | 99.1% | -122173 | -524.798 |

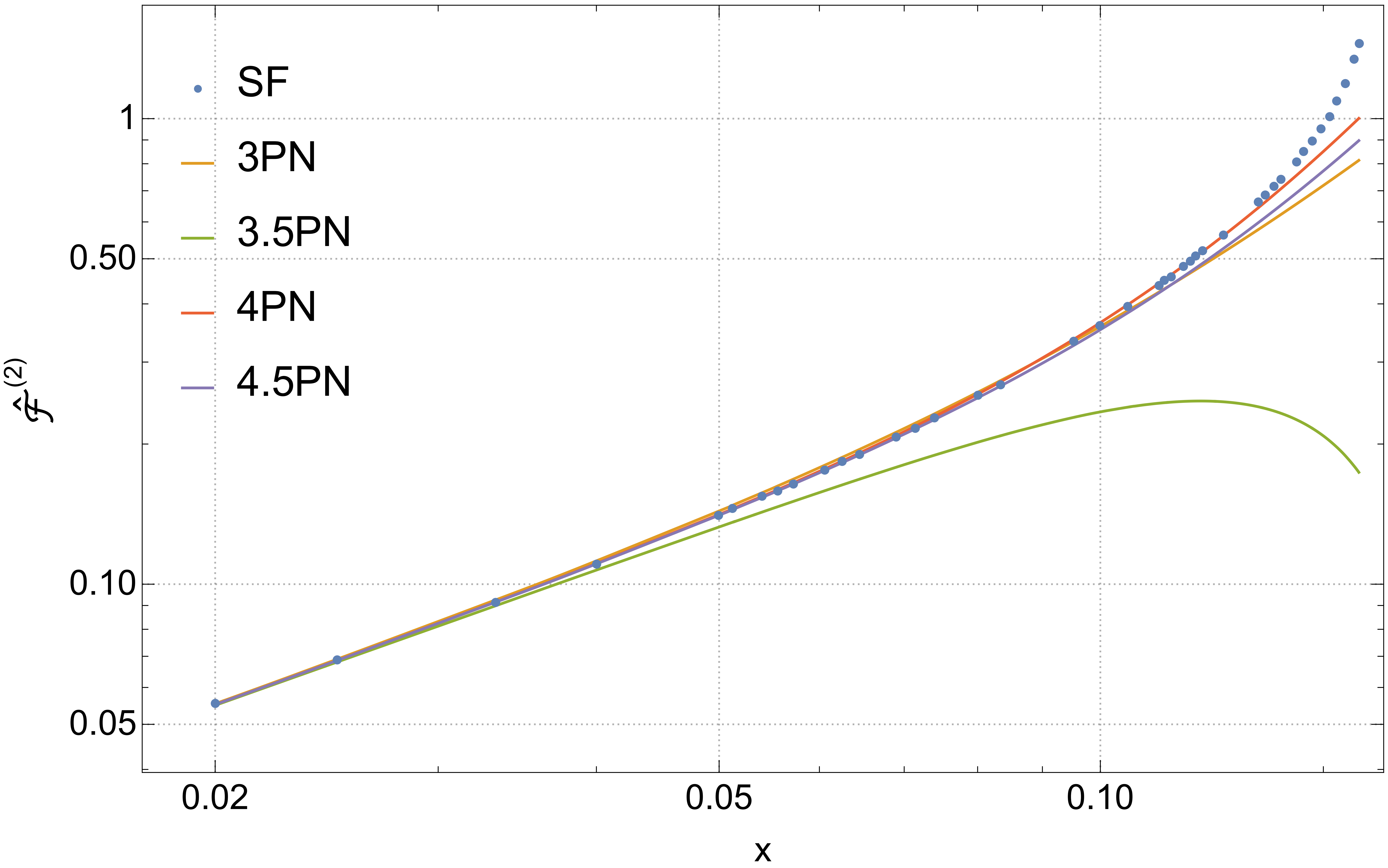
Ollie Burke, Gabriel Piovano, et al. [arXiv:2310.08927]

# 4.5PN Gravitational Wave Energy Flux for Quasicircular Binaries

Luc Blanchet, Guillaume Faye, Quentin Henry, Francois Larrouturou, David Trestini [Phys.Rev. Lett.131.121402 (2023)]

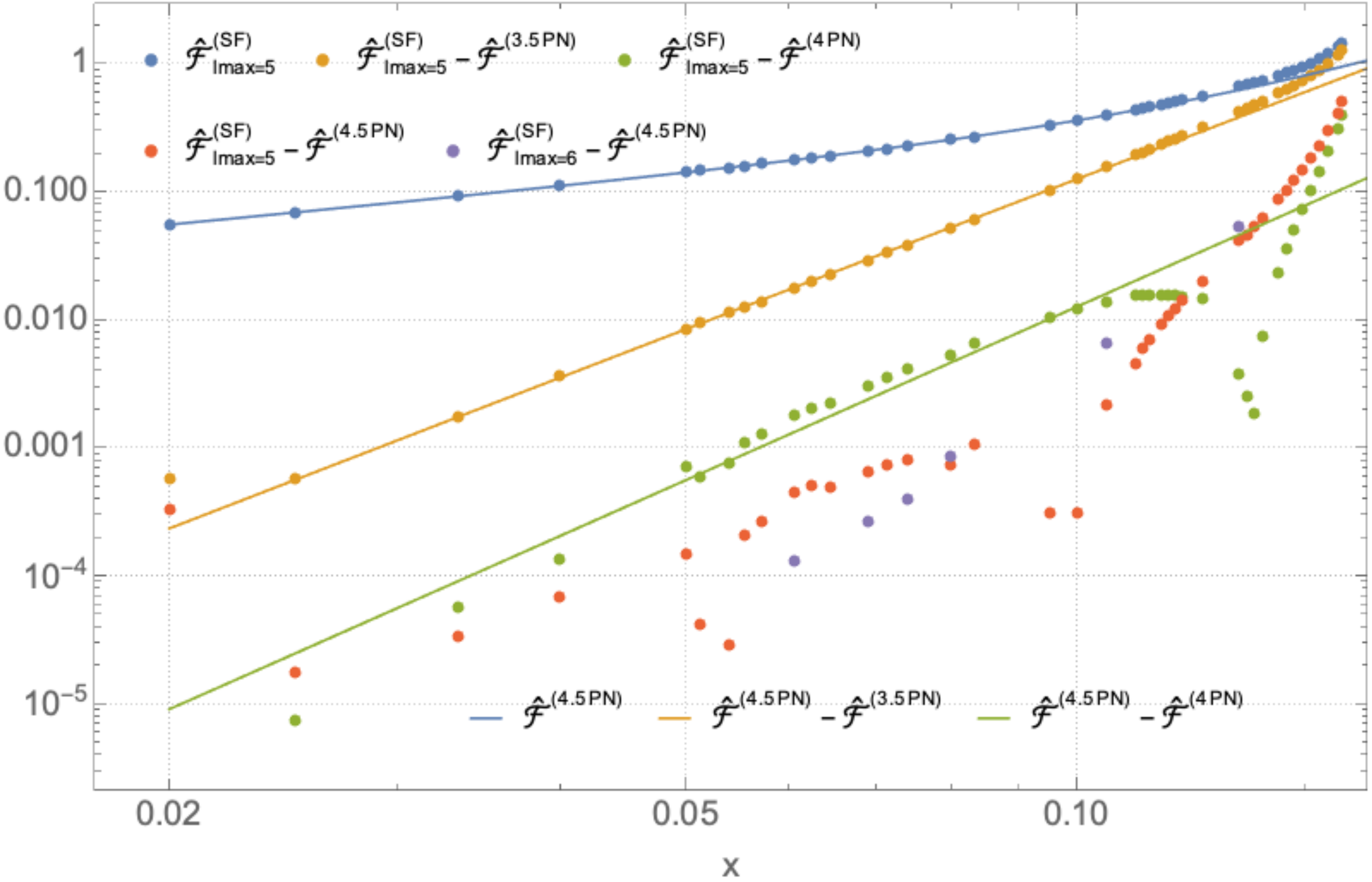
$$\begin{aligned}\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
& + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\text{E} - \frac{856}{105}\ln(16x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} \\
& + \left[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_\text{E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3 + \frac{232597}{8820}\ln x \right. \\
& + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_\text{E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3 + \frac{20739}{245}\ln x\right)\nu \\
& \left. + \left(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\right)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4\right]x^4 \\
& + \left[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\text{E} - \frac{3424}{105}\ln(16x) + \left(\frac{2062241}{22176} + \frac{41}{12}\pi^2\right)\nu \right. \\
& \left. - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\right]\pi x^{9/2} + \mathcal{O}(x^5)\Bigg\}.\end{aligned}$$

# 4.5PN comparison against second order GSF




L. Blanchet, L. Durkan, G. Faye, Q. Henry, F. Larrouturou, J. Miller, A. Pound, D. Trestini, N. Warburton, B. Wardell

# 4.5PN comparison against second order GSF

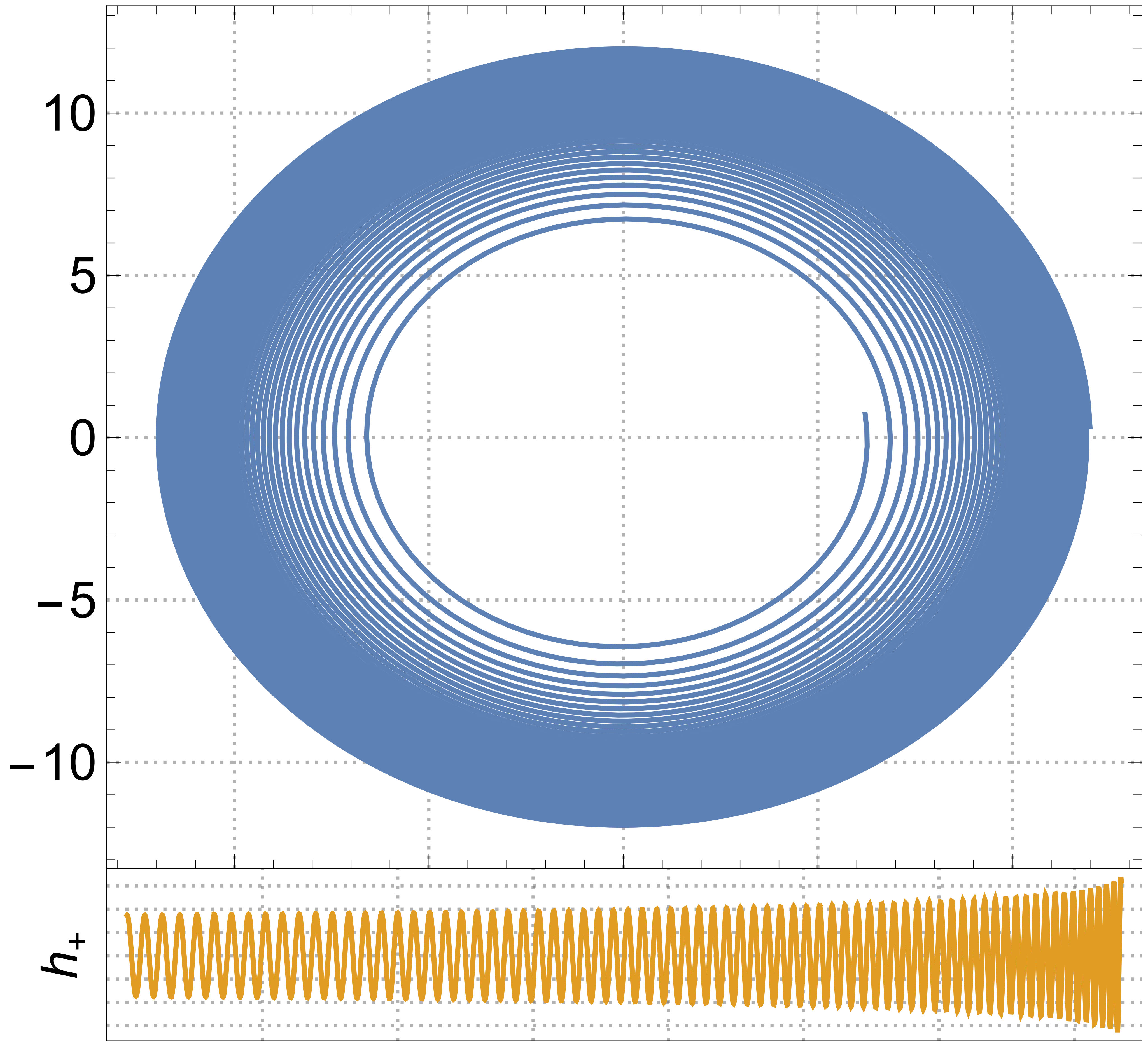



L. Blanchet, L. Durkan, G. Faye, Q. Henry, F. Larrouturou, J. Miller, A. Pound, D. Trestini, N. Warburton, B. Wardell

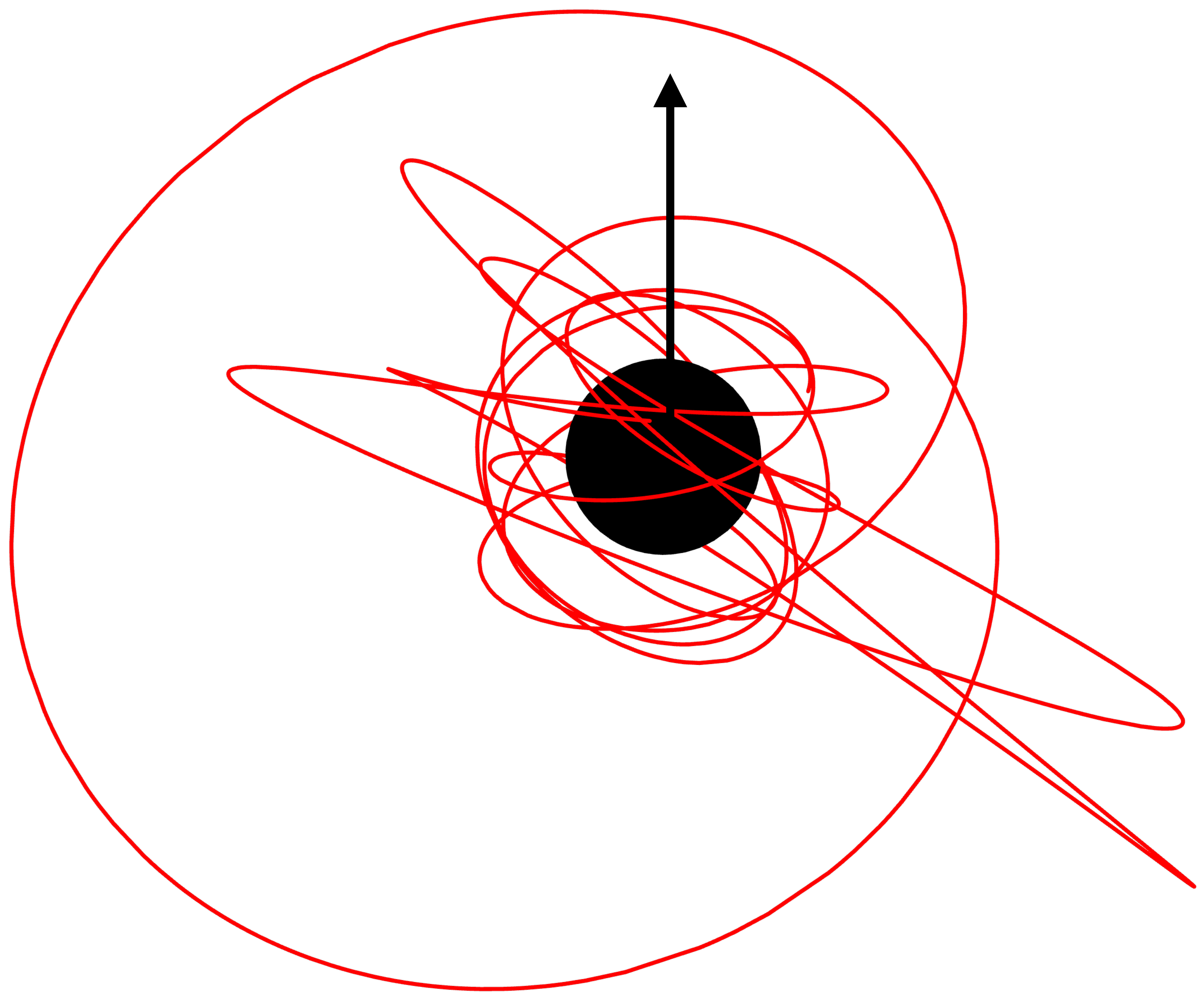
# Near future improvements

# Improvements: 1. Precession (Kerr)

Existing 2SF results limited to **quasicircular** orbits in **Schwarzschild** spacetime



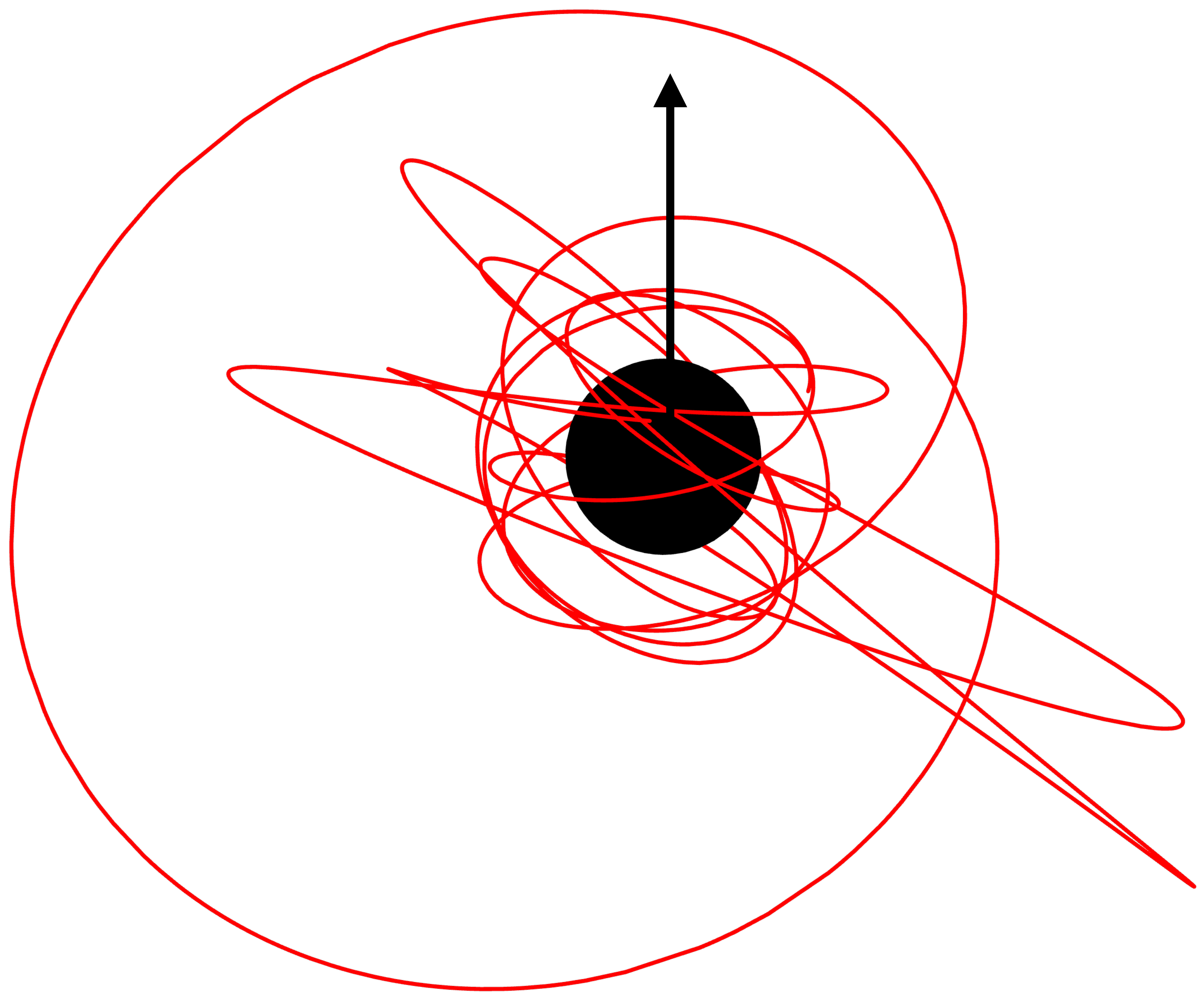

Most astrophysical EMRIs expected to have a **spinning primary**, complicated orbits with **precession** and **eccentricity**, and a **spinning secondary**.



# 4.5PN Gravitational Wave Energy Flux for Quasicircular Binaries

Challenges for incorporating **precession**

- Need to solve Einstein equations on a **Kerr** background
  - Less straightforward **separability** (spheroidal vs spherical) Recent progress on second order source construction by Spiers [arXiv:2402.00604] & Nasipak.
  - Need first order metric perturbation in a **nice gauge**
- More complicated orbits
  - Many **more modes** to compute
  - **Extended** sourced region, even at first order [Leather & Warburton, Phys. Rev. D 108, 084045]



# Lorenz gauge metric perturbation

$$\Box\,\bar{h}_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu} \qquad \nabla^{\alpha}\bar{h}_{\alpha\beta} = 0$$

- **Basic idea:**
  6 degrees of freedom in the metric perturbation captured by
  6 scalars, which are solutions of Teukolsky equations
  [S. Dolan, L. Durkan, C. Kavanagh, B. Wardell, arXiv:2306.16459 and Phys. Rev. Lett. 128, 151101]

$$s = \pm 2 \qquad s = \pm 1 \qquad s = 0$$

$$\mathcal{O}\psi_0 = 8\pi\mathcal{S}_0 T \qquad \mathcal{O}\phi_0^T = 8\pi\tilde{\mathcal{S}}_0 T \qquad \mathcal{O}h = 8\pi T$$

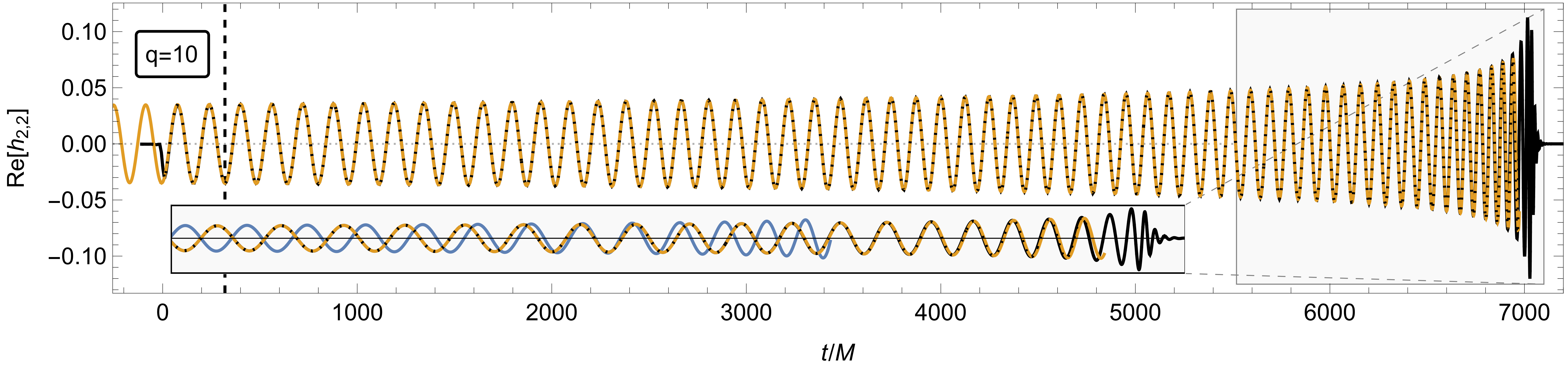
$$\mathcal{O}'\psi_4 = 8\pi\mathcal{S}_4 T \qquad \mathcal{O}'\phi_2^T = 8\pi\tilde{\mathcal{S}}_2 T \qquad \mathcal{O}\chi^T = 8\pi\mathcal{S}_\chi T$$

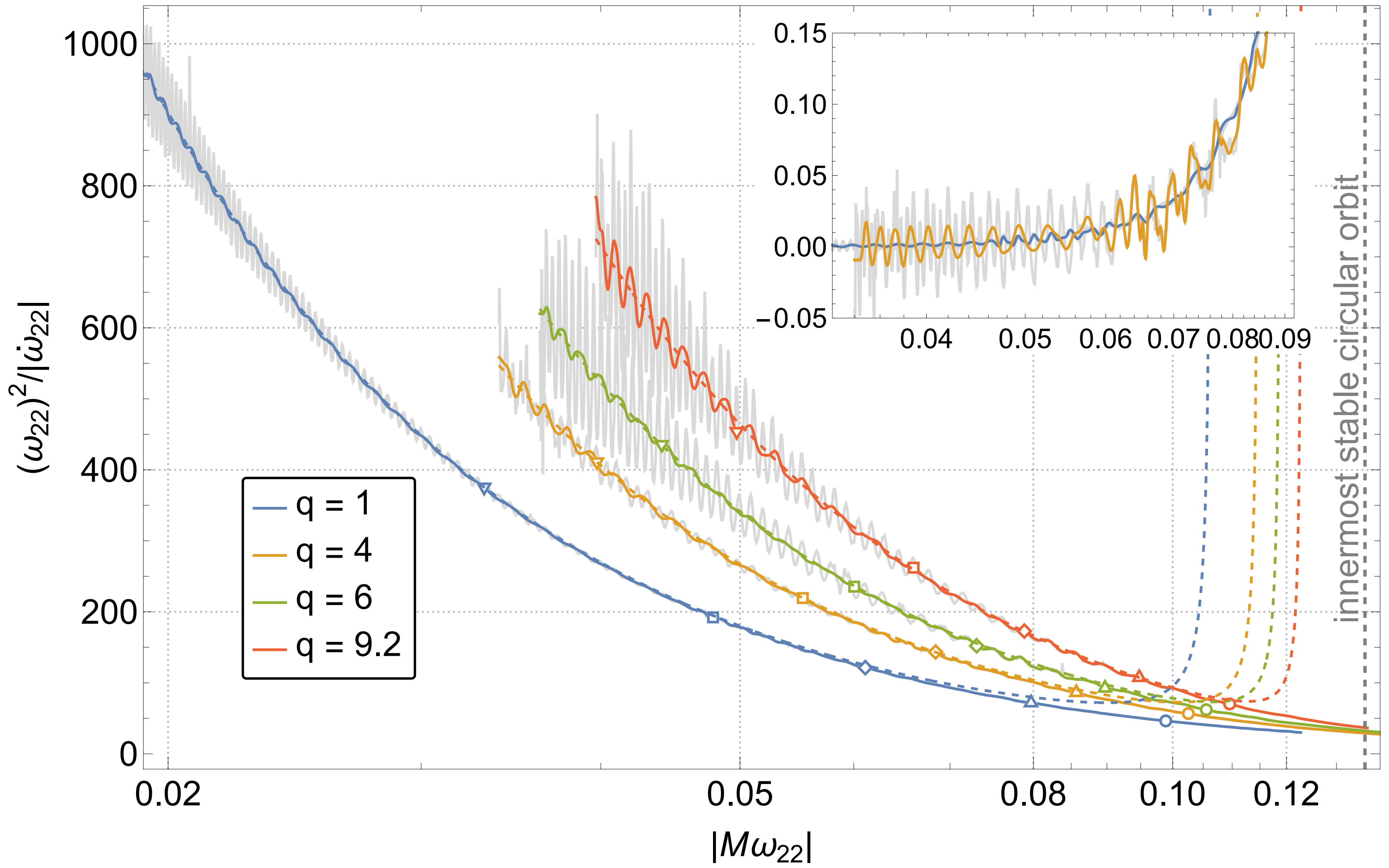
$$\mathcal{L}_T h^{\mathrm{L}}_{\alpha\beta} = h^{\mathrm{AAB}}_{\alpha\beta} - 2\xi_{(\alpha;\beta)}$$

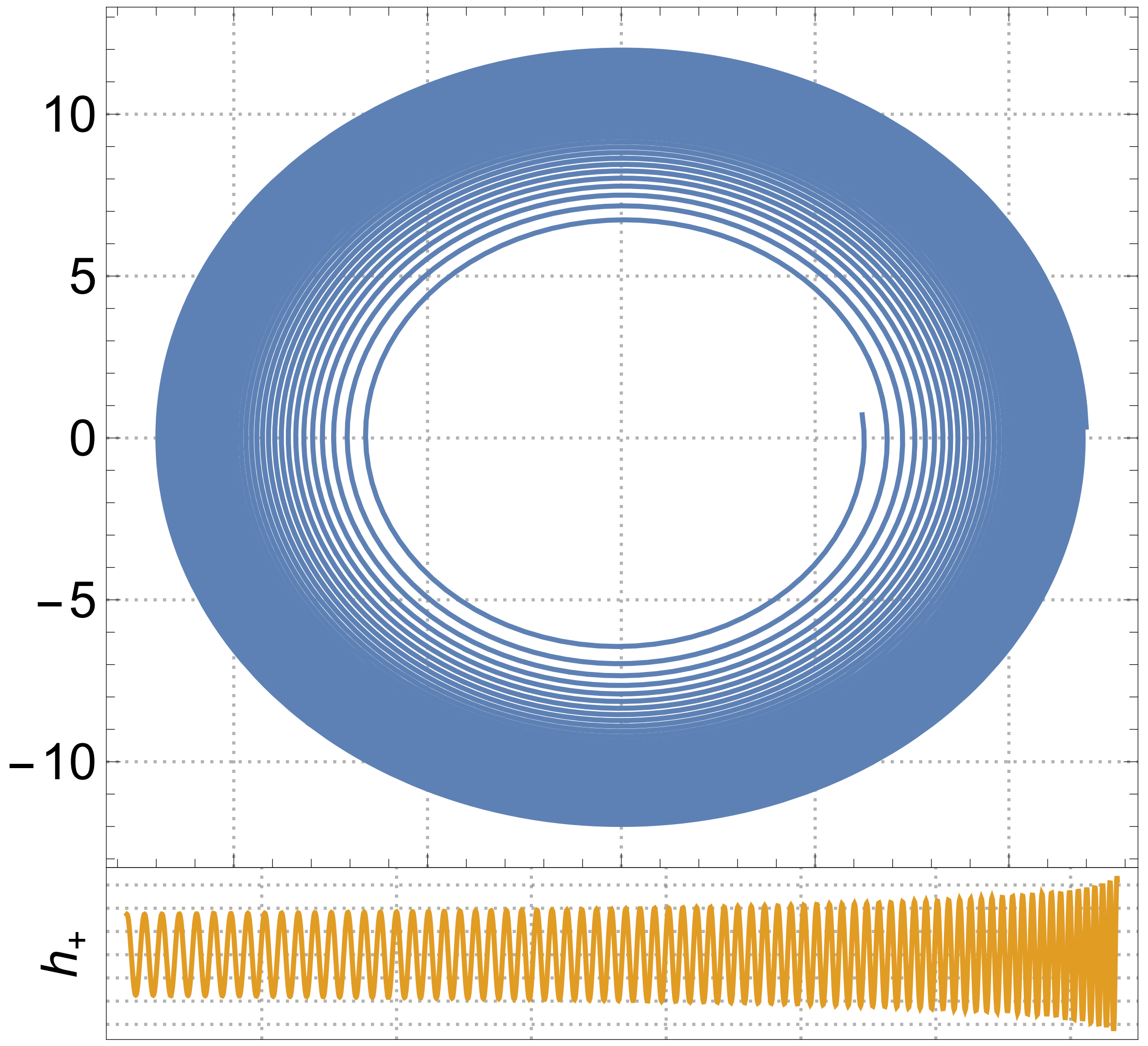
- Also "completion pieces" that capture mass and angular momentum perturbations.

- Other similar-but-different option: GHZ-Teukolsky puncture scheme [Bourg, et al.].

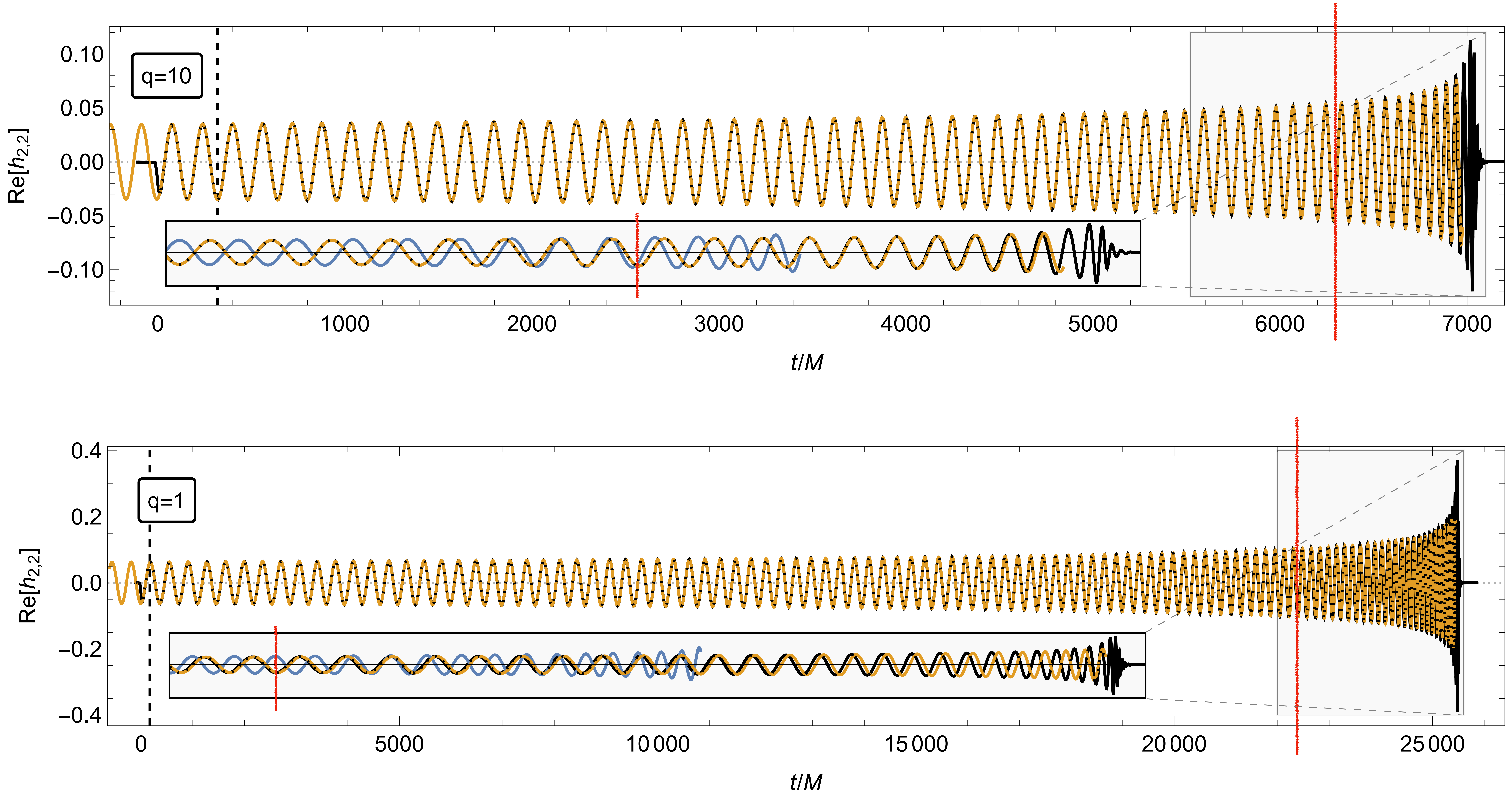
# Transition to plunge



# Transition to plunge



# Transition to plunge

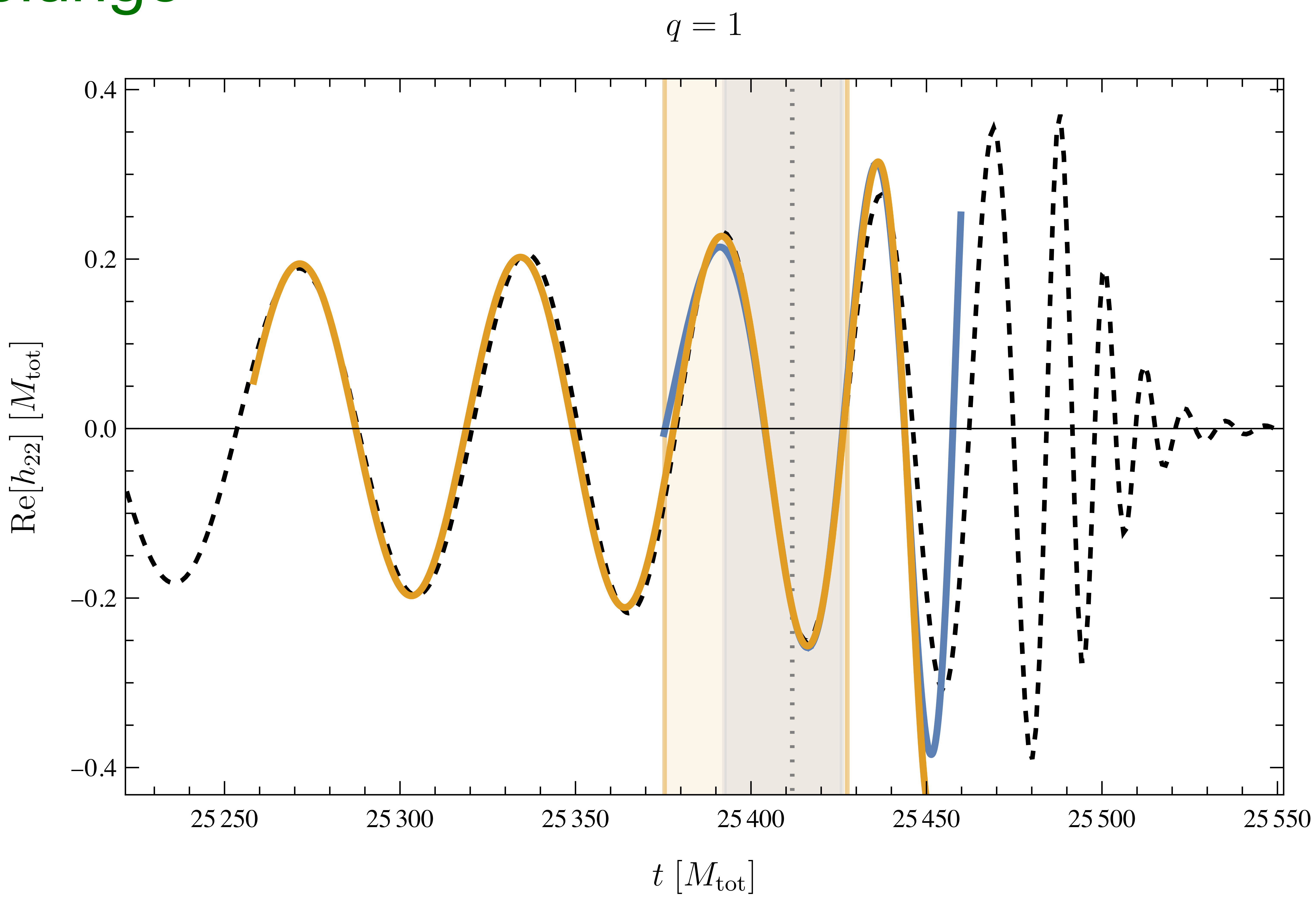



Figure credit: Leanne Durkan, Lorenzo Küchler, Geoffrey Compère, Adam Pound

# Flux balance

- Existing second-order self-force waveforms based on inspiral driven by **energy flux** calculated from **metric perturbation at $\mathscr{I}$.**
- We should also be able to drive an inspiral using the **local self-force**, computed from the **metric perturbation on the worldline**.
- At first order flux balance tells us the two are equivalent: energy dissipated through local self-force is equal to energy carried away in flux through $\mathscr{I}$.
- No **flux balance** law yet at second order.

# Second order redshift

- Connections with scattering amplitudes calculations achieved via local conservative calculations.
- Not yet any calculation of the second-order redshift, but we can compute the second-order metric perturbation.
- Challenges:
  - Difficult to accurately compute the metric perturbation **near the worldline**.
  - **Static modes** ($m = 0 = \omega$) not yet computed and potentially challenging.
  - Challenging to identify an appropriate “**conservative**” second-order spacetime.

# Outlook

## Current state-of-the-art

- We can now produce (quasi-circular) waveforms for **arbitrary mass ratios** in the time it takes to evaluate an interpolating function (milli-seconds).
- Can be used for **LISA data analysis**.
- For a complete waveform, we will need to attach a **transition to plunge** and **ringdown** at the point where our adiabatic approximation breaks down.
- Detailed comparisons with existing NR, PN and EOB show excellent agreement.
- Could be useful in the future as a test case for new EOB and PN results.
- Could be suitable for modelling IMRIs for **LIGO** once we have attached a model for the transition, plunge and ringdown.
- Used to calibrate other models (TEOBResumS and SEOBNRv5)
- It is relatively easy to add non-aligned spin on the secondary (precession), small spin on the primary, small eccentricity.

# Future directions

- We are near the end of the beginning, but there are many more important things to get EMRI waveforms ready for LISA and IMRI waveforms for LIGO:
  - Improved formulations: **Teukolsky**, **Regge-Wheeler** gauges are much easier to work with as they only require us to solve a single scalar equation, but some foundational issues still to be worked out.
  - Check that certain components of the calculation can be left out without significant effects on waveform. For example, how well justified are we to **ignore** the slow evolution of the mass and angular momentum of the big black hole?
  - Everything described here extends **in principle** to generic orbits, but significant human effort required **in practice**.
  - Need a practical method for doing things in **Kerr** spacetime.
  - Incorporate finite-size (e.g. spin effects from smaller body) into waveform.
  - Can second order be done analytically (using MST-PN expansions)?

# Thank you!