Comparing numeric and analytic methods for black hole scattering in unequal mass systems

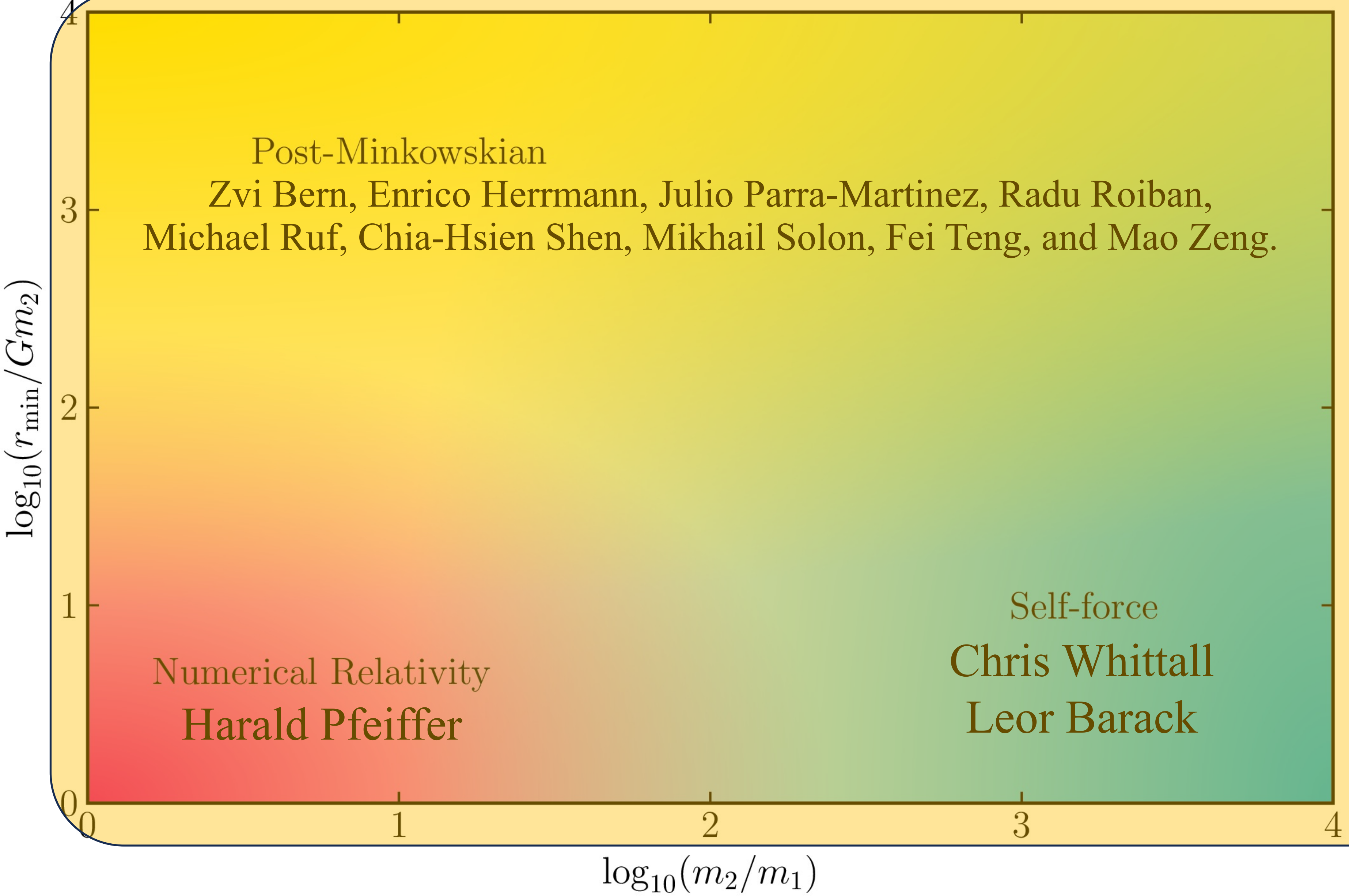
Oliver Long

Gravitational Self-Force and Scattering Amplitudes Workshop

20th March 2024

MAX-PLANCK-GESELLSCHAFT

Contributors and collaborators



Effective One Body

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- Self-force numerical methods:
  - Time-domain model.
  - Hybrid frequency/time-domain model.
- Comparison to post-Minkowskian in the scalar field toy model:
  - Energy and angular momentum dissipation.
  - Self-force correction to the scattering angle.
- Extension to gravitational self-force.
- Scattering in Numerical Relativity:
  - Current status.
  - Comparison to Effective One Body.
  - Challenges for unequal masses.

Self-force vs post-Minkowskian expansions

Two independent expansions of the same system:

	1PM	2PM	3PM	4PM	$\dots$	
0SF	G	G^2	G^3	G^4	$\dots$	$\to$ analytical
1SF	qG	qG^2	qG^3	qG^4	$\dots$	$\to$ numerical
2SF	q^2G	q^2G^2	q^2G^3	q^2G^4	$\dots$	$\to$ impossible (for now)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	

nSF contains all orders in G

nPM contains all orders in q

$$q := \frac{m_1}{m_2} \ll 1$$

Scalar field toy model

Schwarzschild spacetime: Endow particle with a spin-0 scalar charge Q.

Keep inertial mass but ignore gravitational mass.

Expansion in small parameter at fixed energy and angular momentum:

m_1 : Small mass

m_2 : Large mass

$$\chi = \chi^{(0)} + q_s\,\delta\chi + \ldots$$

Geodesic

$$q_s := \frac{Q^2}{m_1 m_2}$$

Scalar field Φ obeys the Klein-Gordon equation:

$$\Box\Phi = Q\int_{-\infty}^{\infty} \delta^4\left(x^\mu - x_p^\mu(\tau)\right)\, d\tau$$

Decompose into (time-domain) modes:

$$\Phi = \frac{Q}{r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \psi_{\ell m}(t,r) Y_{\ell m}(\theta,\varphi)$$

$$\boxed{\psi_{,uv} + V(\ell;r)\psi = S_\psi(\ell;x_p^\mu)\,\delta\left(r - R\right)}$$

u, v : Eddington-Finkelstein coordinates

1+1D evolution scheme [Barack & OL '22]

Orbit parameterised by either:

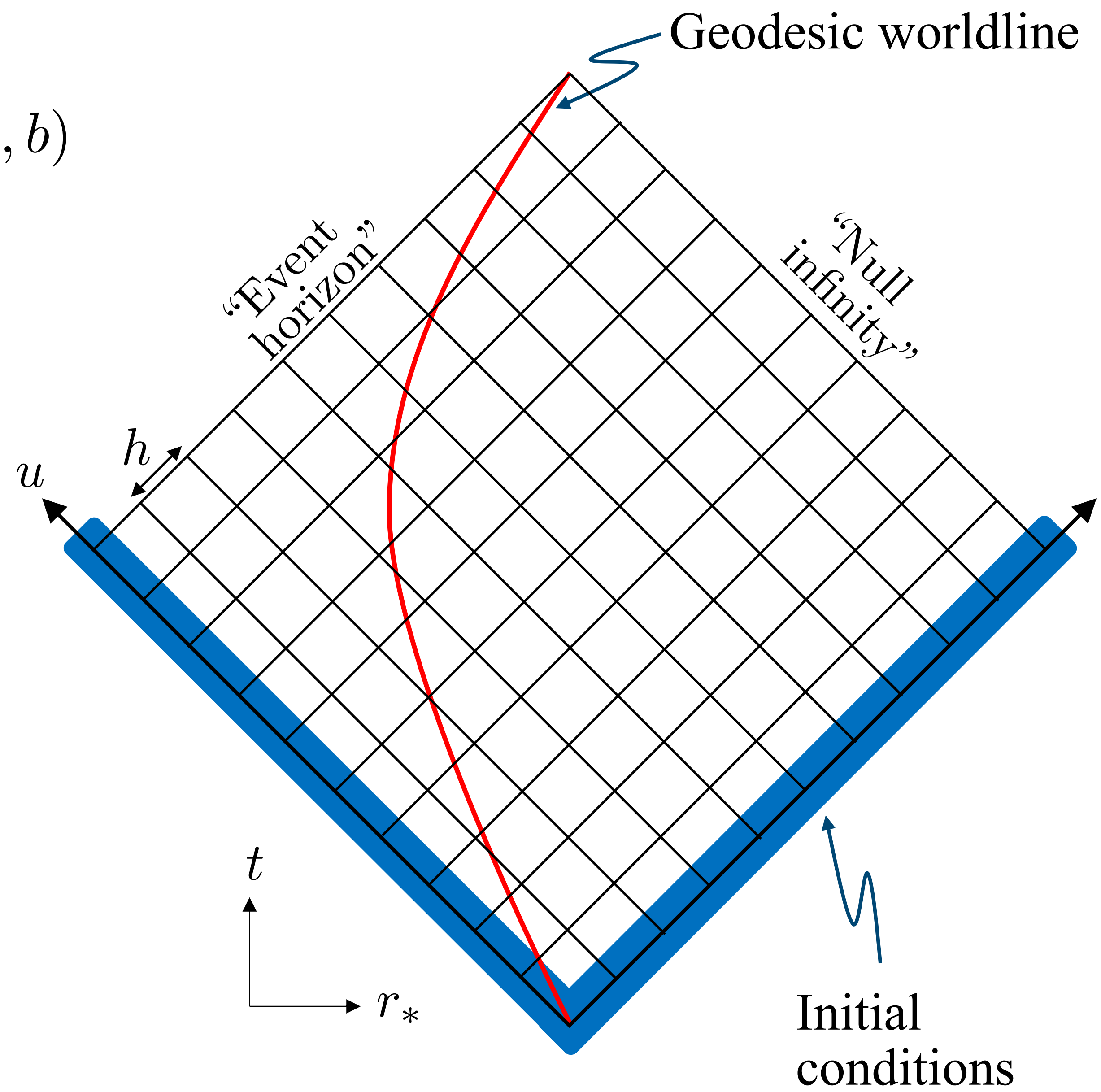
- Velocity at infinity and impact parameter (v, b)
- Energy and angular momentum (E, L)

Numerically solve sourced field equation:

$$\psi_{,uv} + V(\ell; r)\psi = S_\psi(\ell; x_p^\mu)\,\delta(r - R)$$

Sum over modes:

$$\Phi = \frac{Q}{r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\psi_{\ell m}(t,r)Y_{\ell m}(\theta,\varphi)$$



Alternative hybrid frequency/time-domain method

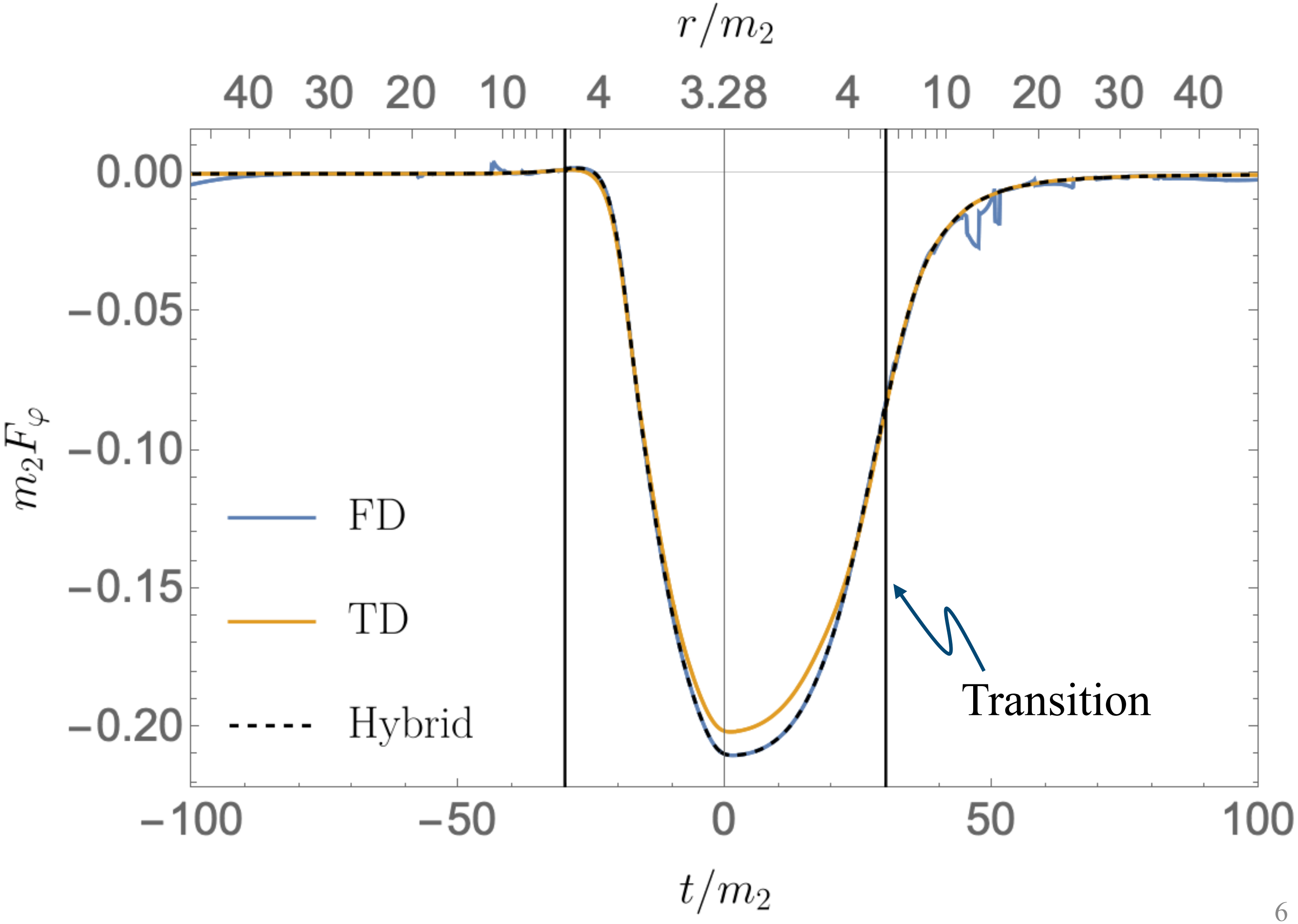
Combine frequency and time domain methods [Whittall & Barack ‘23, Barack & OL ‘22].

Use frequency domain (FD) data near periastron where we need a larger number of modes.

FD method automatically truncates number of modes based on accuracy.

When accuracy of FD drops below time domain (TD), use TD data.

Important for strong-field orbits at large velocities.



Self-force energy and angular momentum dissipation

Self-force definition:

$$m_1 \frac{du^\alpha}{d\tau} = Q(\delta^\alpha_\beta + u^\alpha u_\beta)\nabla^\beta \Phi^R =: m_1 q_s F^\alpha$$

τ : Proper time

u^α : 4-velocity

Φ^R : Regular field

Can split SF into conservative and dissipative pieces using symmetries about periastron:

$$F^{\text{cons}}_\alpha(r,\dot{r}) = -F^{\text{cons}}_\alpha(r,-\dot{r}) \qquad F^{\text{diss}}_\alpha(r,\dot{r}) = F^{\text{diss}}_\alpha(r,-\dot{r}) \qquad \alpha = t,\varphi$$

Dissipated energy and angular momentum as integrals over the orbit:

$$\Delta E_{1\text{SF}} = -\int_{-\infty}^{\infty} F^{\text{diss}}_t d\tau \qquad \Delta L_{1\text{SF}} = \int_{-\infty}^{\infty} F^{\text{diss}}_\varphi d\tau$$

Includes flux radiated to infinity and the horizon.

PM expansion of dissipation [Barack et al. '23, Jones & Ruf '24]

Expansion around flat space in powers of G.

2PM:

$$\Delta E_{\rm 2PM} = 0 \qquad \Delta L_{\rm 2PM} = \frac{2(1+v^2)}{3(1-v^2)}\frac{Gm_2}{b}$$

v : Velocity at infinity

b : Impact parameter

3PM:

$$\Delta E_{\rm 3PM} = \left(r_1 + r_2\,\mathrm{arctanh}\,(v) + r_3 \log\left[\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right] + \frac{\pi}{2}\frac{v}{\sqrt{1-v^2}} \right)\left(\frac{Gm_2}{b}\right)^3$$

$$\Delta L_{\rm 3PM} = \Delta L_{\rm 2PM} + \left(r_3 + r_4\,\mathrm{arctanh}\,(v) + r_6 \log\left[\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right]\right)\left(\frac{Gm_2}{b}\right)^2$$

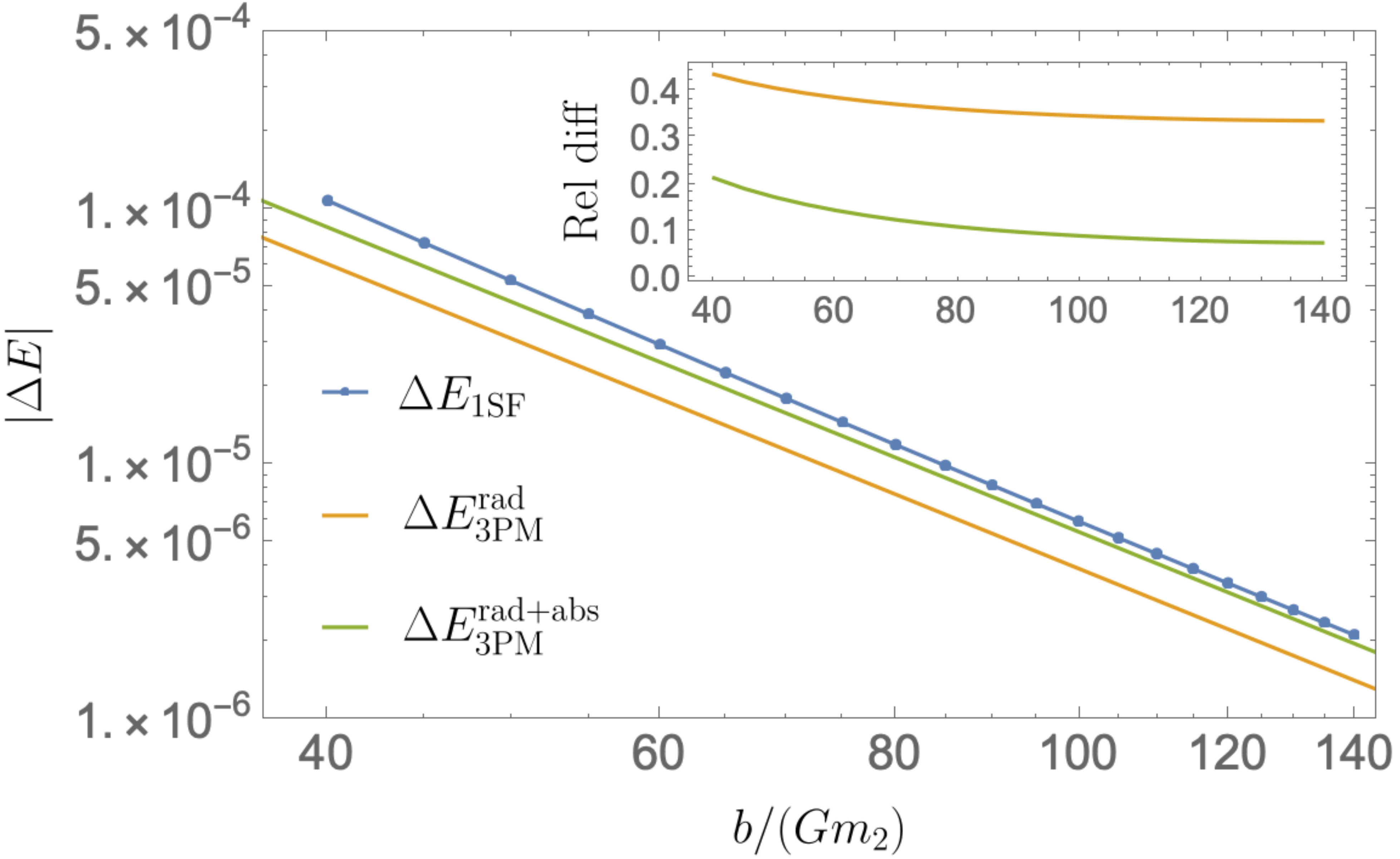
$$r_i = \text{rational coefficients}$$

Flux to infinity

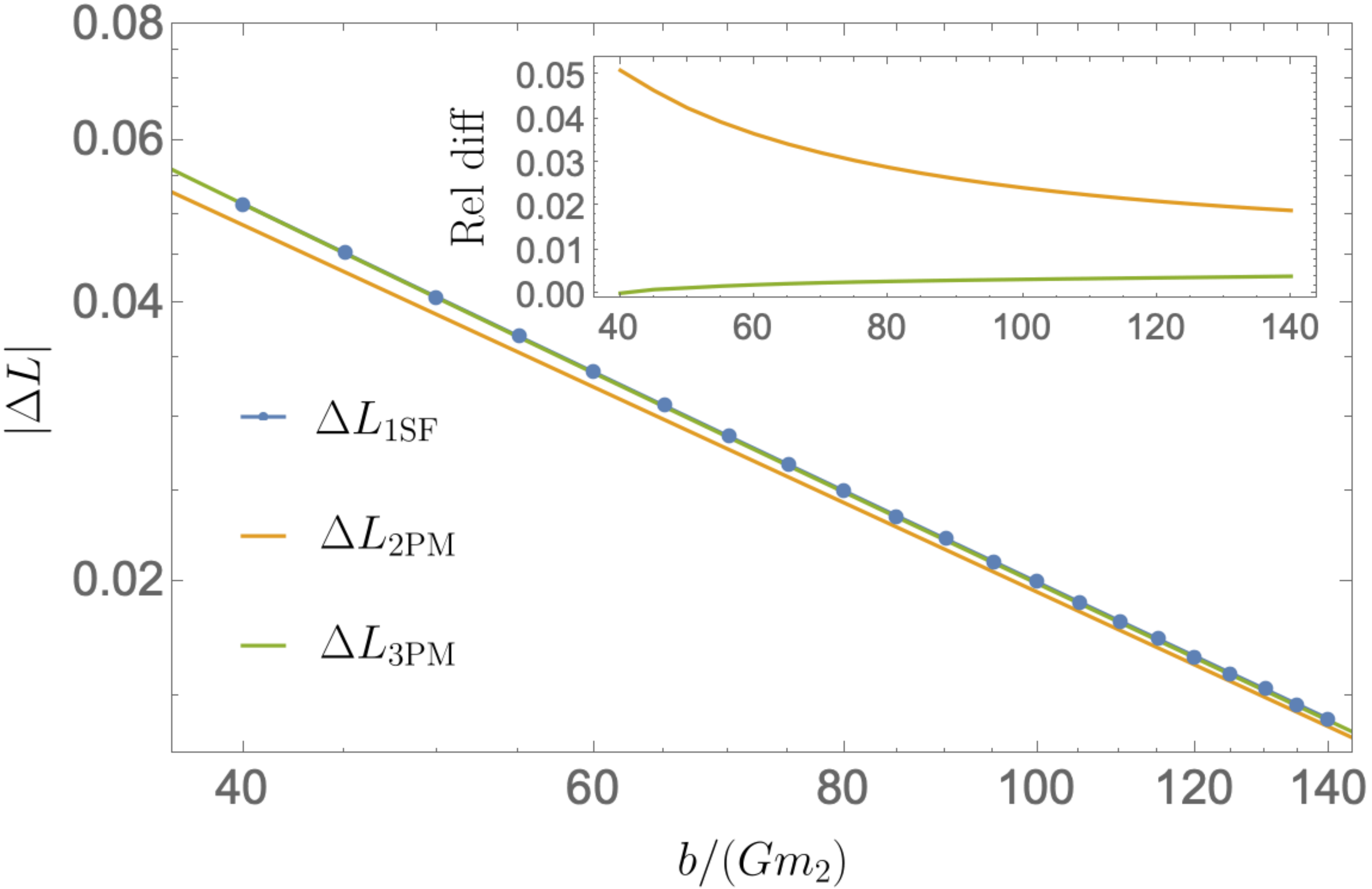
Flux to horizon

Flux radiated to the horizon appears due to monopole term.

Energy flux comparison



Angular momentum flux comparison

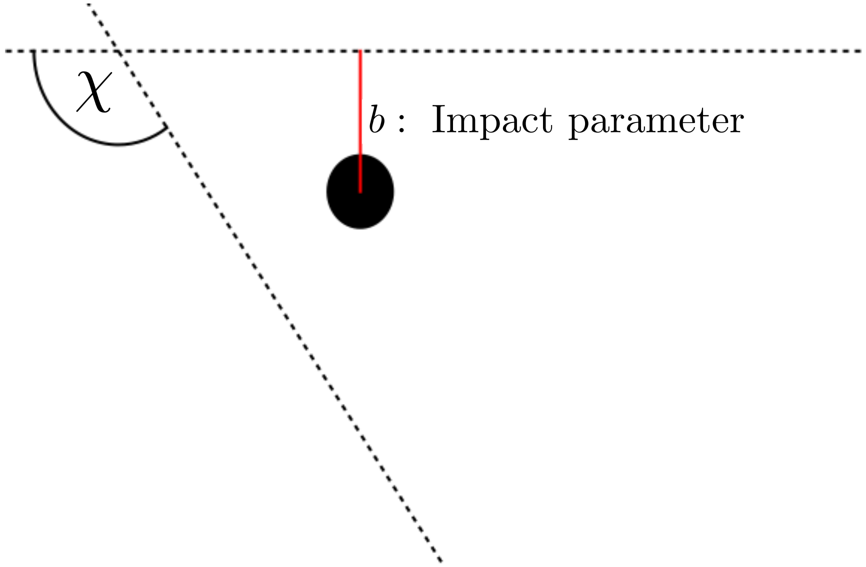


Self-force correction to the scattering angle [Barack & OL '22]

Scattering angle:

$$\chi = \chi^{(0)} + q_s\,\delta\chi + \ldots$$

Geodesic



$$\delta\chi = \sum_{\pm}\int_{r_{\min}}^{\infty}\left[\mathcal{G}_E^{\pm}(r)F_t^{\pm} - \mathcal{G}_L^{\pm}(r)F_\varphi^{\pm}\right]dr$$

Incoming/outgoing leg

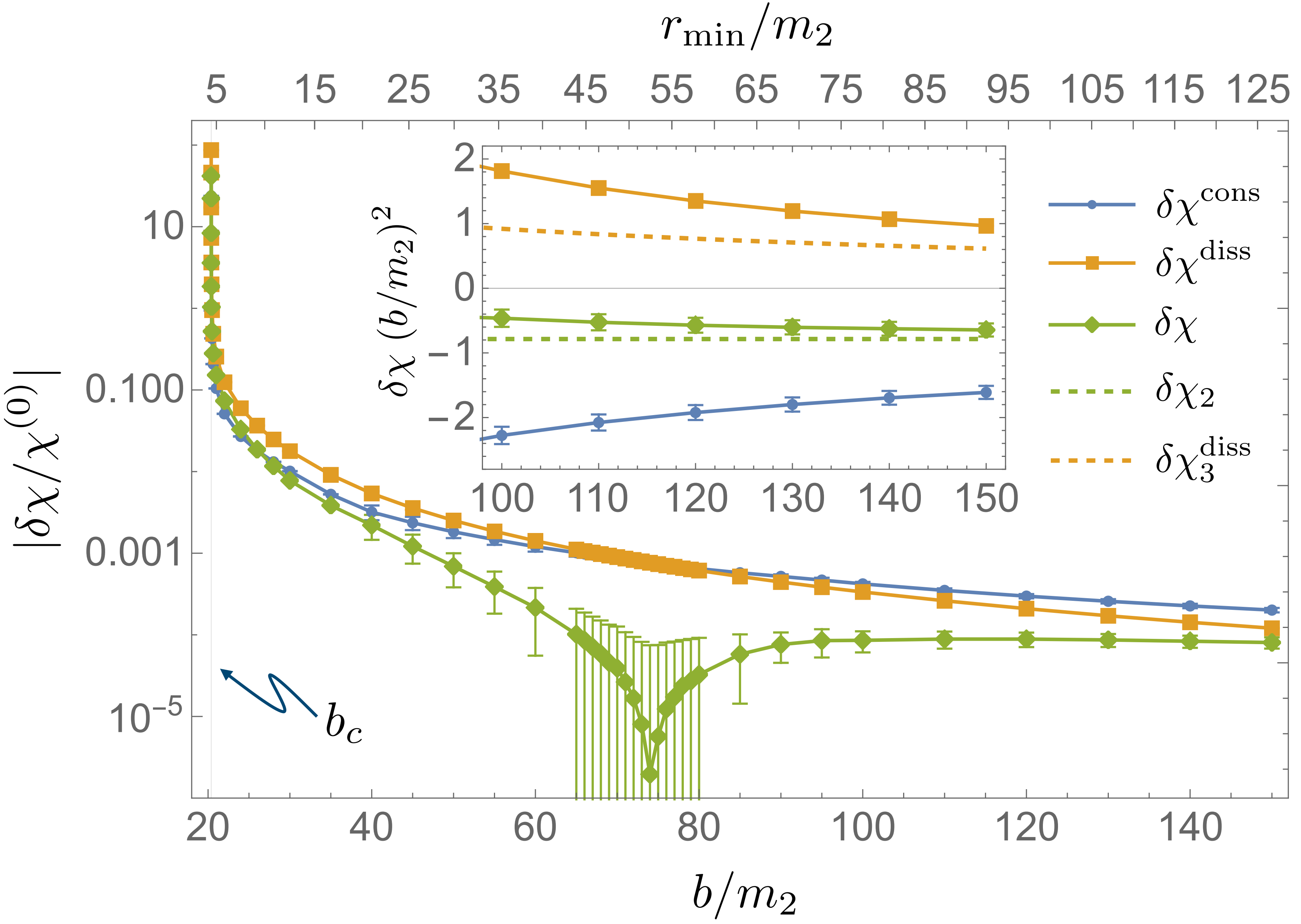
Functions of geodesics

Can get dissipative angle directly from radiated fluxes:

$$\delta\chi^{\text{diss}} = \frac{1}{2}\left(\alpha_E E\,\Delta E_{1\text{SF}} + \alpha_L L\,\Delta L_{1\text{SF}}\right)$$

Functions of geodesics

Scattering angle results: $v = 0.2$ [Barack & OL '22]



Scattering angle correction: PM expansion [Barack et al. '23]

Expansion around flat space:

$$\delta\chi^{\mathrm{PM}} = \sum_{i=0}^{\infty} \delta\chi_i \left(\frac{Gm_2}{b}\right)^i$$

2PM [Gralla & Lobo '22]:

$$\delta\chi_2^{\mathrm{cons}} = -\frac{\pi}{4}\left(\frac{m_2}{b}\right)^2 \qquad \delta\chi_2^{\mathrm{diss}} = 0$$

3PM:

$$\delta\chi_3^{\mathrm{cons}} = -\frac{4\left(3-v^2\right)}{3v^2\sqrt{1-v^2}}\left(\frac{m_2}{b}\right)^3 \qquad \delta\chi_3^{\mathrm{diss}} = \frac{2\left(v^2+1\right)^2}{3v^3\sqrt{1-v^2}}\left(\frac{m_2}{b}\right)^3$$

LO

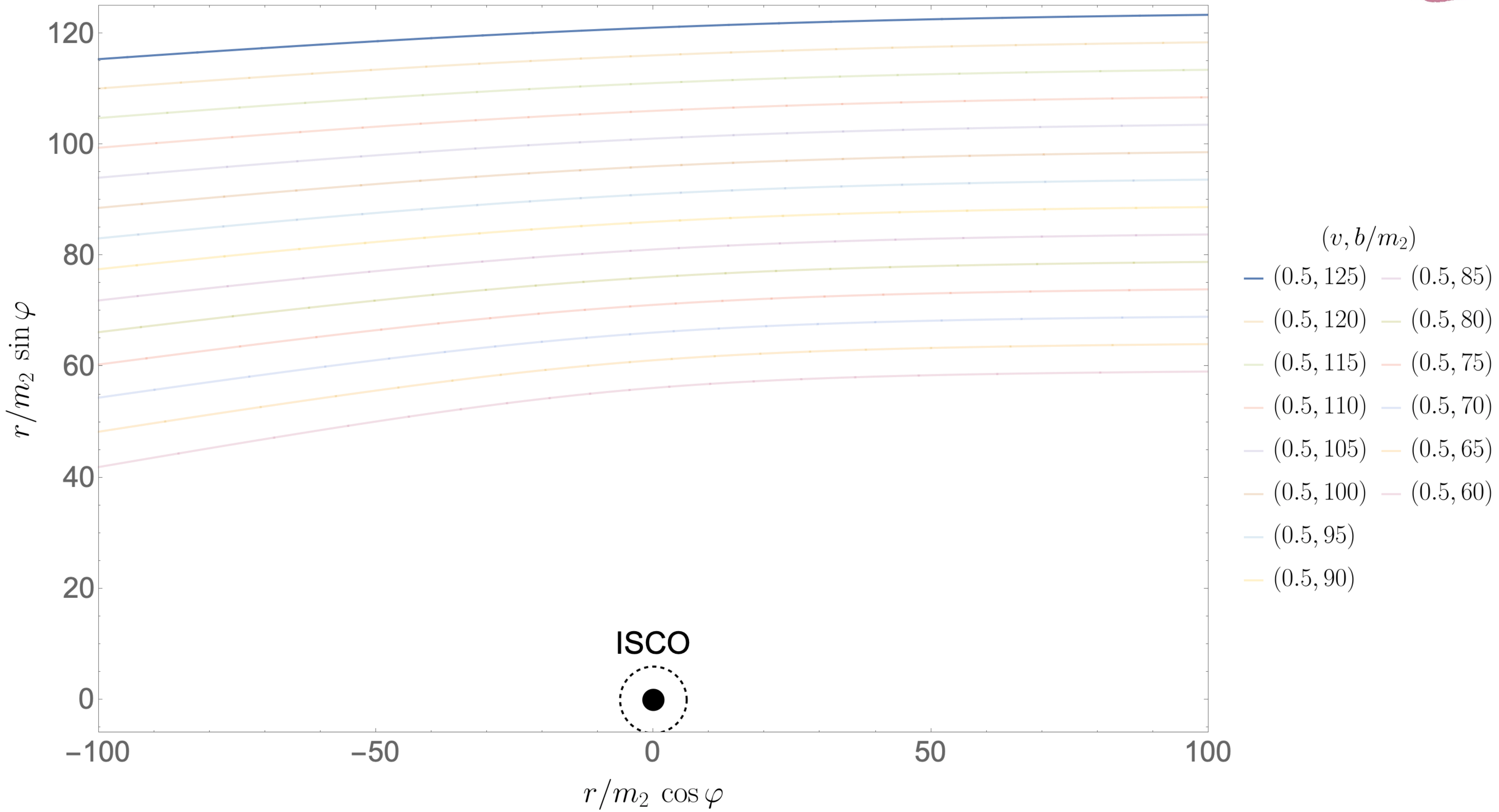
NLO

4PM dissipative:

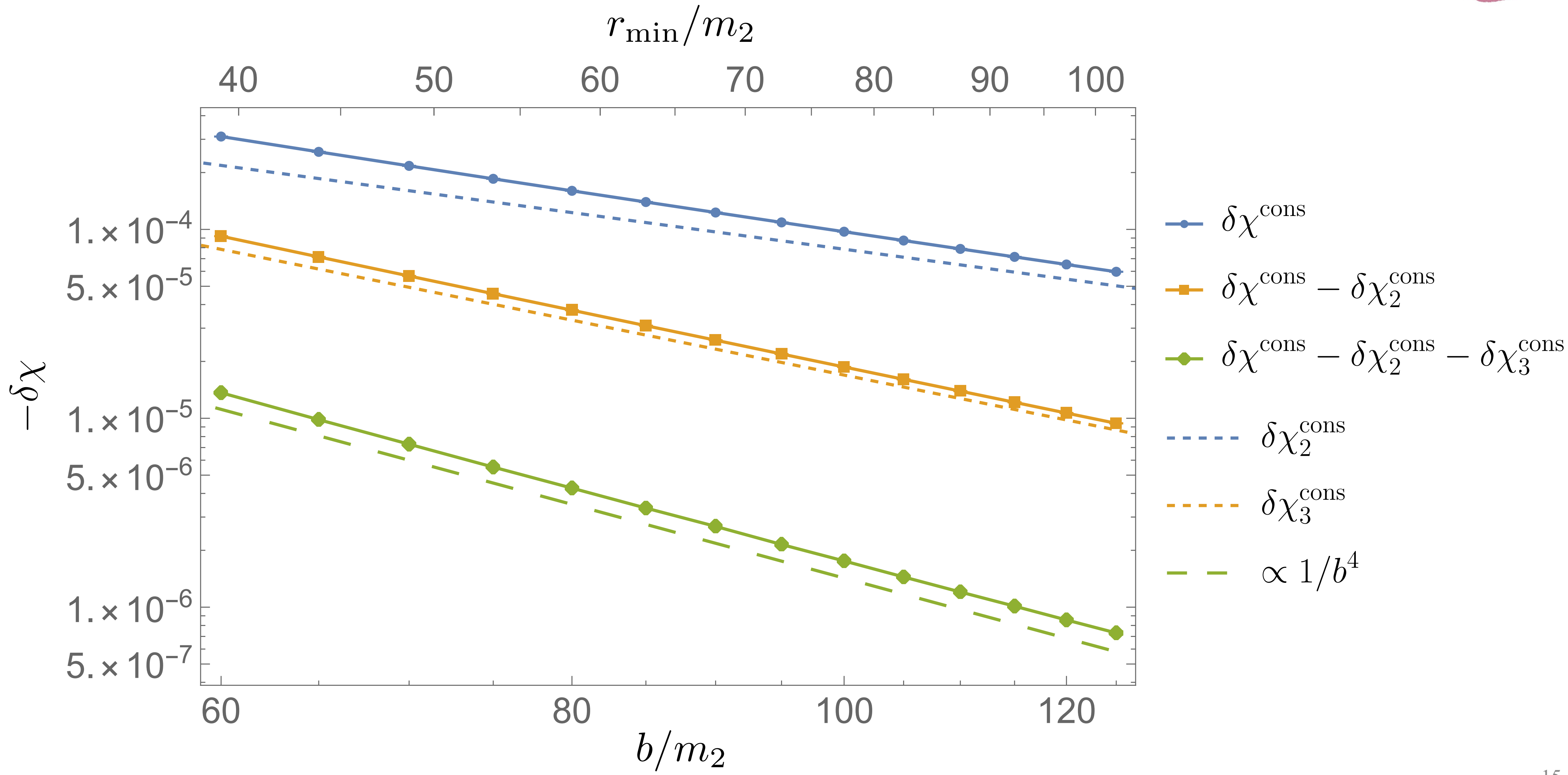
$$\delta\chi_4^{\mathrm{diss}} = \left(r_1 + r_2\,\mathrm{arcsech}\left(\sqrt{1-v^2}\right) + r_3 \log\left[\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right]\right)\left(\frac{m_2}{b}\right)^4$$

r_i = rational coefficients

Weak-field orbits



Conservative: $v = 0.5$ [Barack et al. '23]



Extraction of high-order conservative PM results [Barack et al. '23]

PM expansion with free parameters:

$$\delta\chi^{\text{cons}} = \frac{a_2}{b^2} + \frac{a_3}{b^3} + \frac{a_4}{b^4} + \frac{a_5}{b^5} + \dots$$

Up to 3PM can fit value or use **analytic value**.

a_2	a_3	a_4	a_5
-1.0886	–	–	–
-0.7535	-21.77	–	–
-0.7899	-16.17	-206.5	–
-0.7803	-18.49	-25.0	-4620
$\mathbf{-0.785398}$	-19.18	–	–
$\mathbf{-0.785398}$	-16.93	-176.2	–
$\mathbf{-0.785398}$	-17.20	-131.1	-1793
$\mathbf{-0.785398}$	$\mathbf{-16.9356}$	-175.9	–
$\mathbf{-0.785398}$	$\mathbf{-16.9356}$	-174.4	-107
$< 1\%$	$\sim 1\%$	~ -175	$< 0(?)$

Scattering angle: 4PM conservative [Barack et al. '23]

$$\delta\chi_4^{\text{cons}} = \Bigg(r_1 + r_2 \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right) + r_3 \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right)^2 + \boxed{r_4 \mathsf{E}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)^2 + r_5 \mathsf{K}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)\mathsf{E}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right) + r_6 \mathsf{K}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)^2}$$

$$+ r_7 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) + r_8 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right)$$

$$+ r_9 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) \log\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) + r_{10} \log\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right)$$

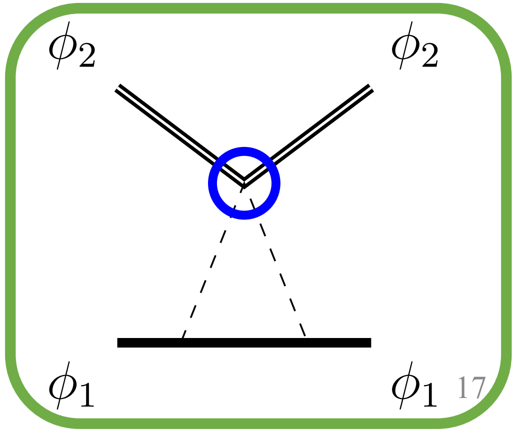
$$+ r_{11} \log^2\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) + r_{12}\,\boxed{\alpha} + r_{13}\frac{\boxed{\beta}}{v^2} + r_4 \log(b)\Bigg)\left(\frac{m_2}{b}\right)^4$$

Elliptic integrals

Free coefficients

Log term

r_i = rational coefficients



Extraction of high-order conservative PM results [Barack et al. '23]

Subtract known analytic parts of conservative 4PM:

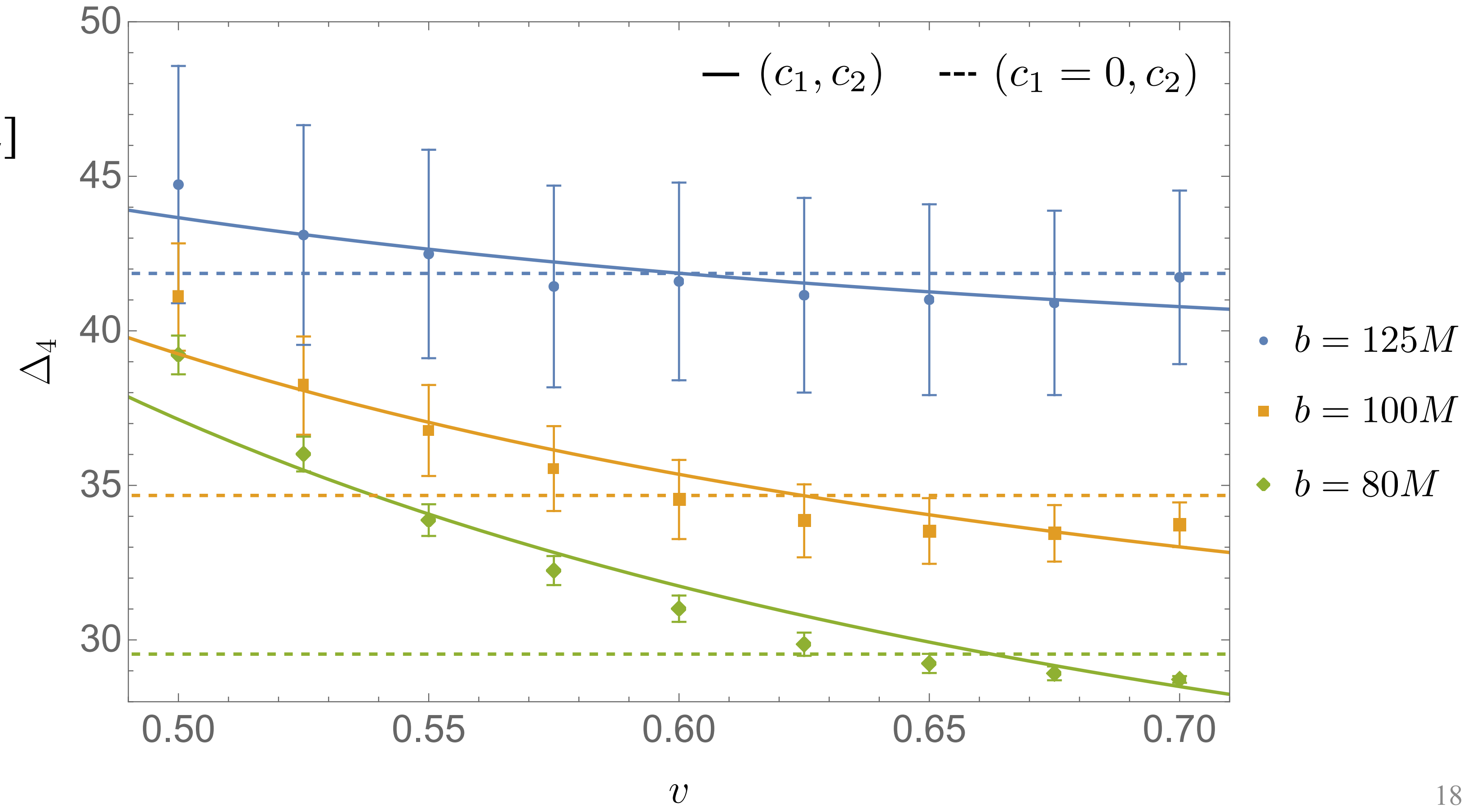
$$\Delta_4(v) := (\delta\chi_4^{\text{cons}} - \delta\chi_4^{\text{known}})b^4 = \boxed{\frac{3}{8}\pi m_2^4 \left[c_2 + c_1(5 - 4/v^2)\right]} + \mathcal{O}(1/b)$$

c_1 & c_2 are Wilson coefficients.

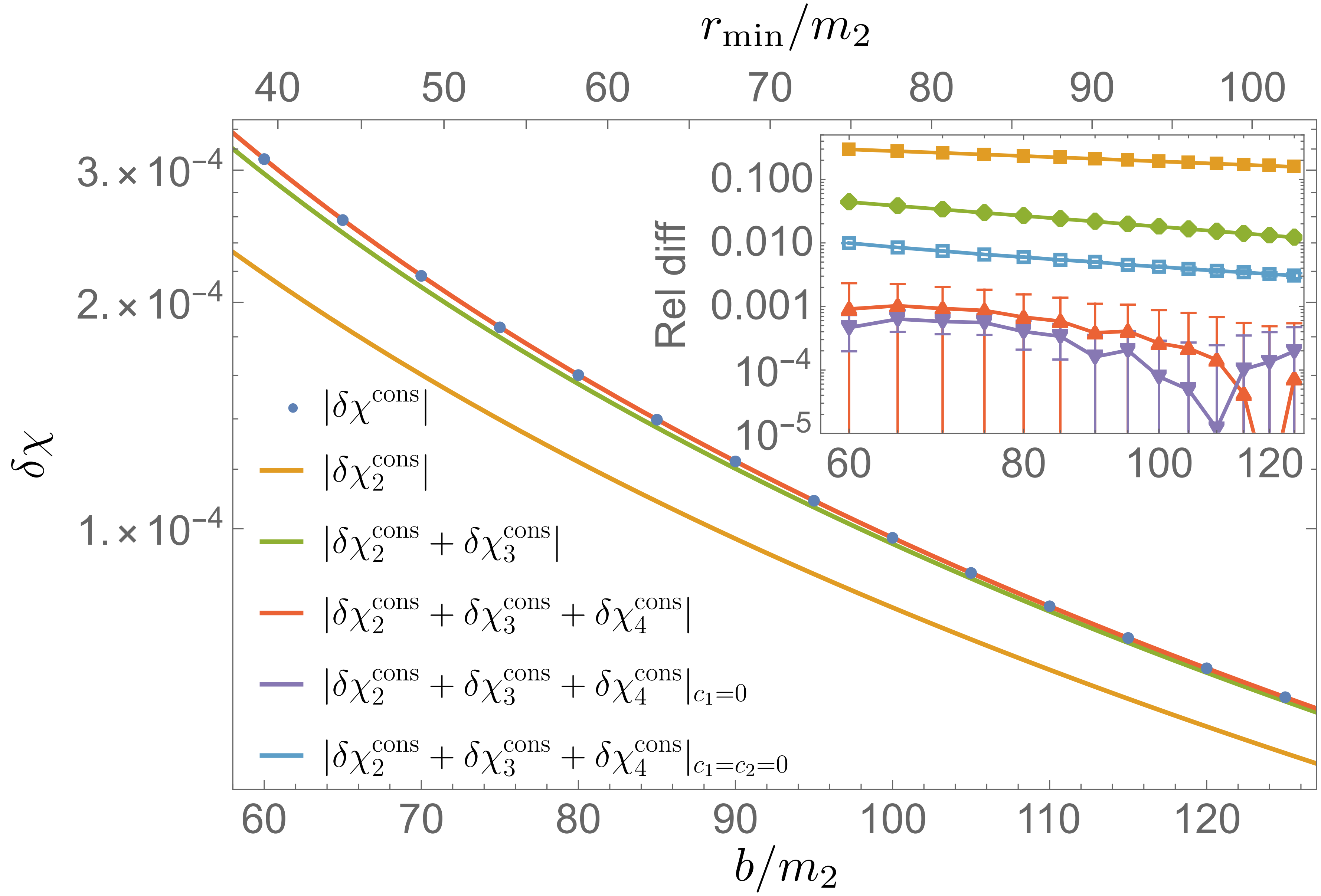
Expect $c_1 = 0$ [Ivanov & Zhou '22]

b/M	c_1	c_2
80	0.94	−21.2
100	0.68	−25.9
125	0.31(±0.38)	−33.6
80	**0**	−25.1
100	**0**	−29.4
125	**0**	−35.5

Fixed (Fitting error)



PM comparison: Conservative $v = 0.5$ [Barack et al. ’23]



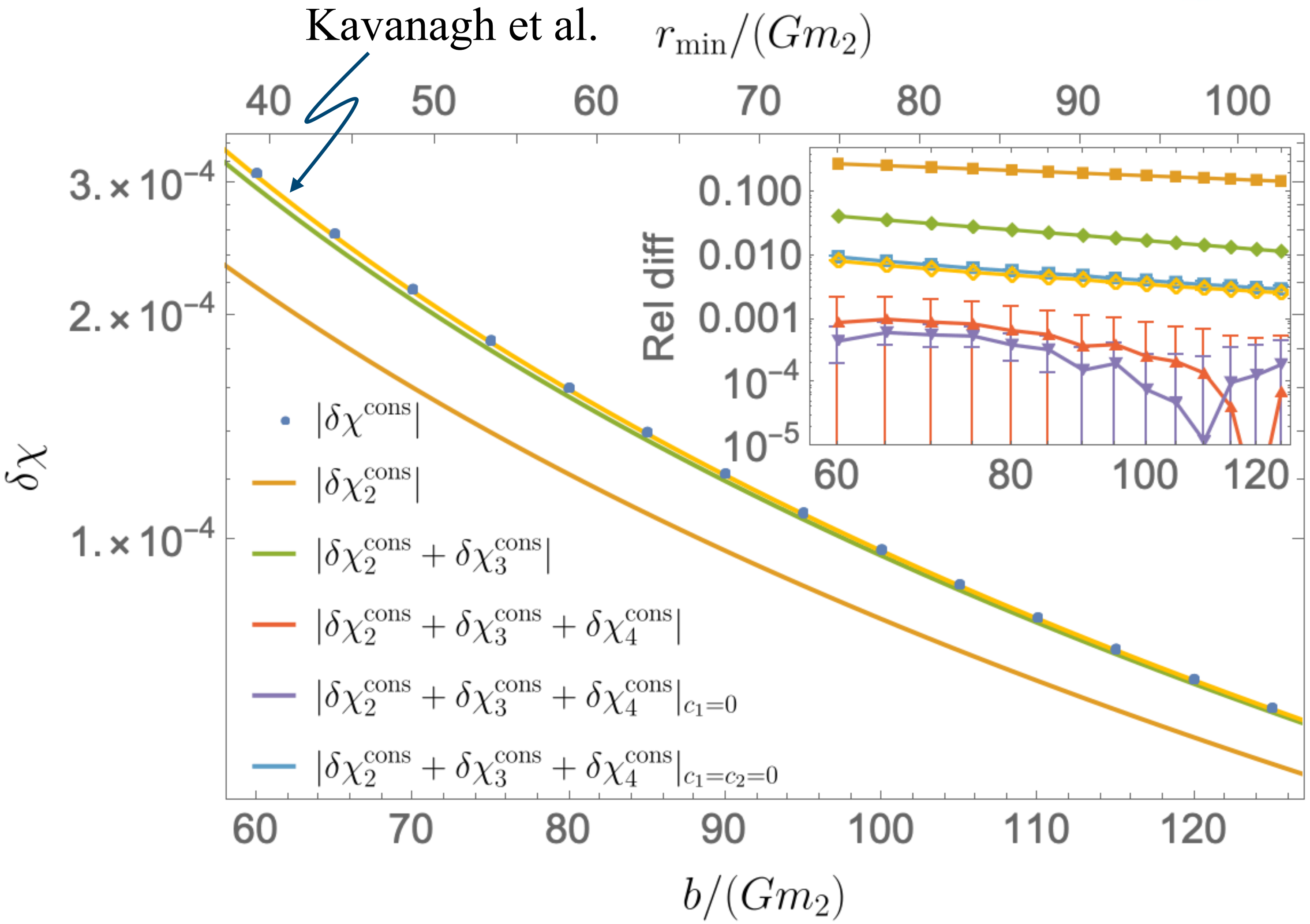
PM comparison: Conservative $v = 0.5$

Kavanagh, Usseglio, Pound, Bini, Geralico:

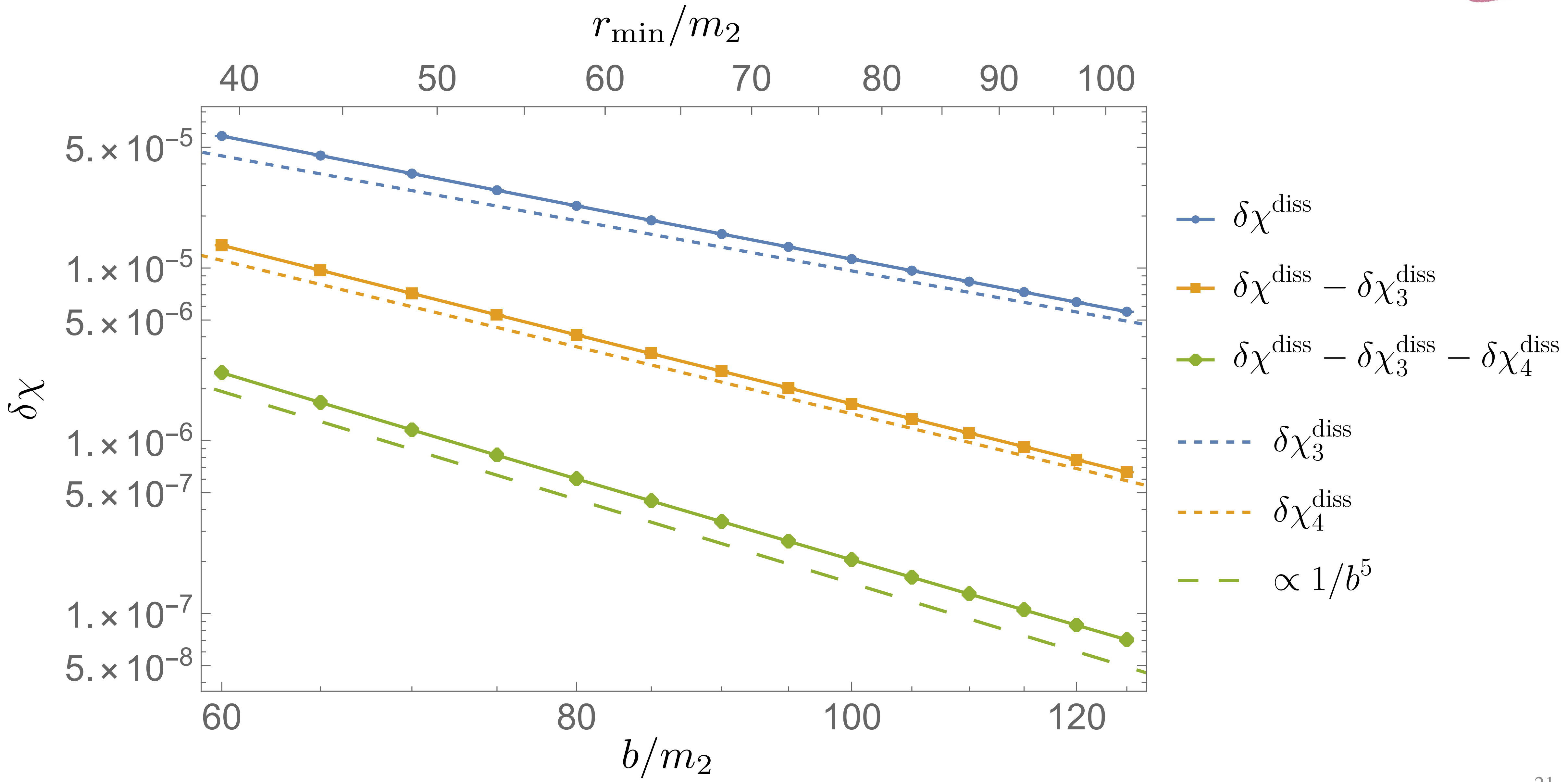
$$c_1 = \frac{1}{6}$$
$$c_2 = -2.76$$

Barack et al '23 (with **fitting** errors):

b/m_2	c_1	c_2
80	0.94 ± 0.04	-21.2 ± 0.2
100	0.68 ± 0.14	-25.9 ± 0.8
125	0.31 ± 0.38	-33.6 ± 2.5
80	**0**	-25.1 ± 0.1
100	**0**	-29.4 ± 0.3
125	**0**	-35.5 ± 0.9



Dissipative: $v = 0.5$ [Barack et al. '23]



Extraction of high-order dissipative PM results [Barack et al. '23]

PM expansion with free parameters:

$$\delta\chi^{\text{diss}} = \frac{\alpha_3}{b^3} + \frac{\alpha_4}{b^4} + \frac{\alpha_5}{b^5} + \frac{\alpha_6}{b^6} + \dots$$

Up to 4PM can fit value or use **analytic value**.

α_3	α_4	α_5	α_6
11.19	–	–	–
9.44	188	–	–
9.64	142	1900	–
9.61	154	920	26615
9.6225	169	–	–
9.6225	147	1720	–
9.6225	149	1321	15859
9.6225	**143.344**	1965	–
9.6225	**143.344**	2248	−20216
$< 1\%$	$\sim 1\%$	$\sim 2000(?)$	???

Gravitational self-force

Formulation:

- Have several methods to calculate GSF (Teukolsky equation, Lorenz gauge).
- Correction to the scatter angle formula valid for GSF in large BH rest frame.
- Need additional corrections to move to centre-of-mass (CoM) frame.

Calculation:

- Current methods not applicable to hyperbolic GSF calculation:
    - Most bound GSF codes cannot be easily extended to the unbound case.
    - Hyperbolic time-domain code has growing modes for spin ±2 Teukolsky equation.
    - Hyperbolic frequency-domain code currently only formulated for the scalar case.
- Hyperbolic time-domain code with hyperboloidal slicing [with R. Panosso Macedo]:
    - Currently only scalar case.
    - Relatively easy extension to spin ±2 Teukolsky.

Scattering in Numerical Relativity

The basics of Numerical Relativity

Numerical solving of Einstein's equations:

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Spacetime split into 3-hypersurfaces.

Two types of equations:

- 4 elliptic constraint equations.
- 12 first-order hyperbolic evolution equations.
- Plus 4 coordinate gauge degrees of freedom.

Scattering with SpEC uses ~10^4 core hours:

- ~1-2 weeks on a cluster.

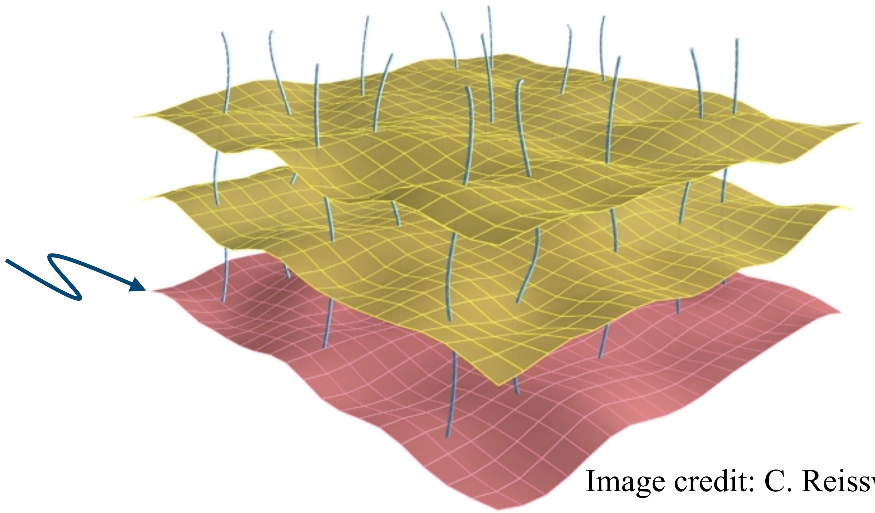
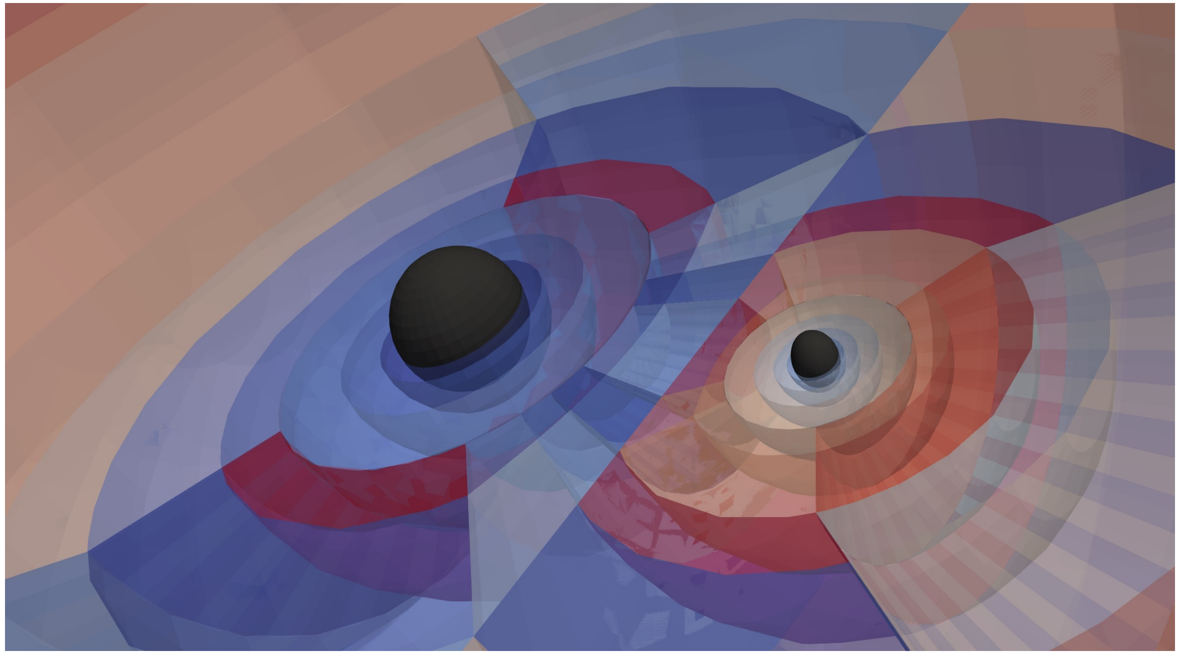


Image credit: C. Reisswig

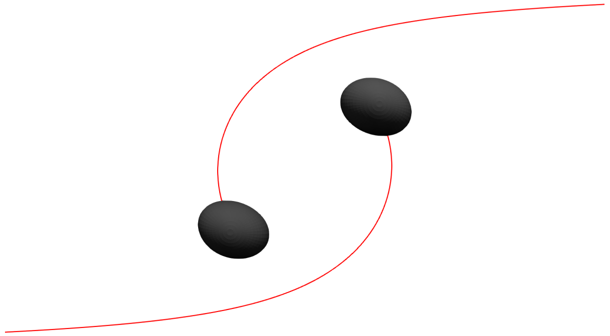


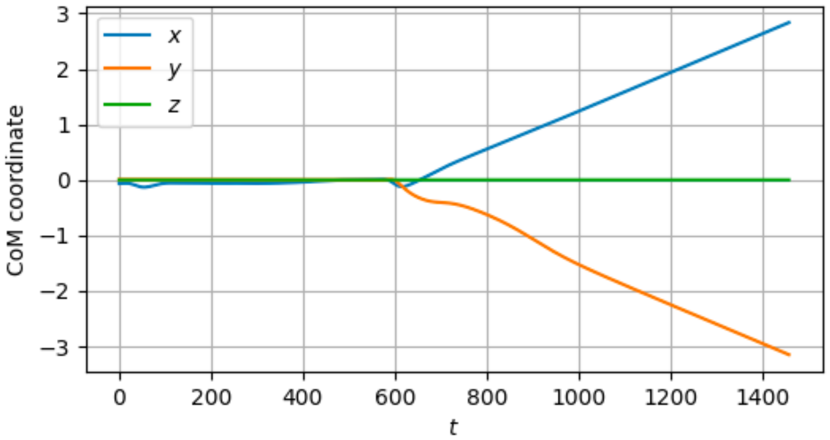
The outputs of Numerical Relativity

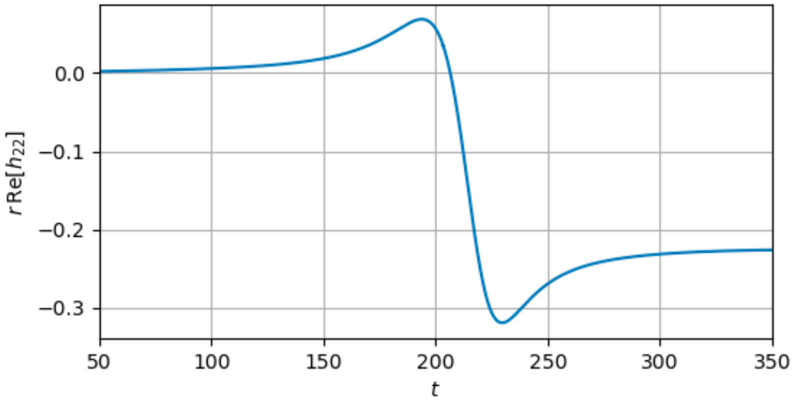
Spacetime:
- Curvature scalars.
- Black hole horizons.
- Induced spin.

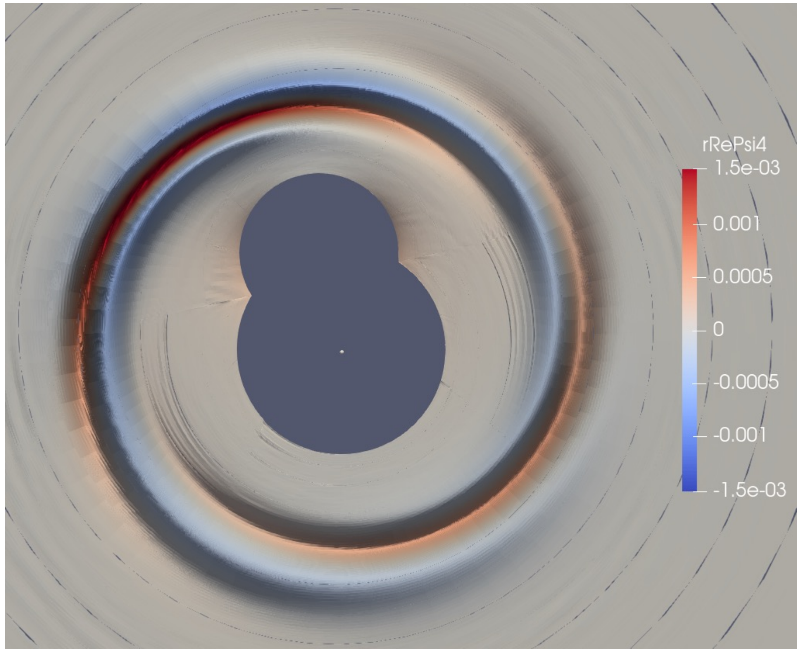
Dynamics:
- Scattering angle.
- Centre of mass (CoM) recoil velocity.

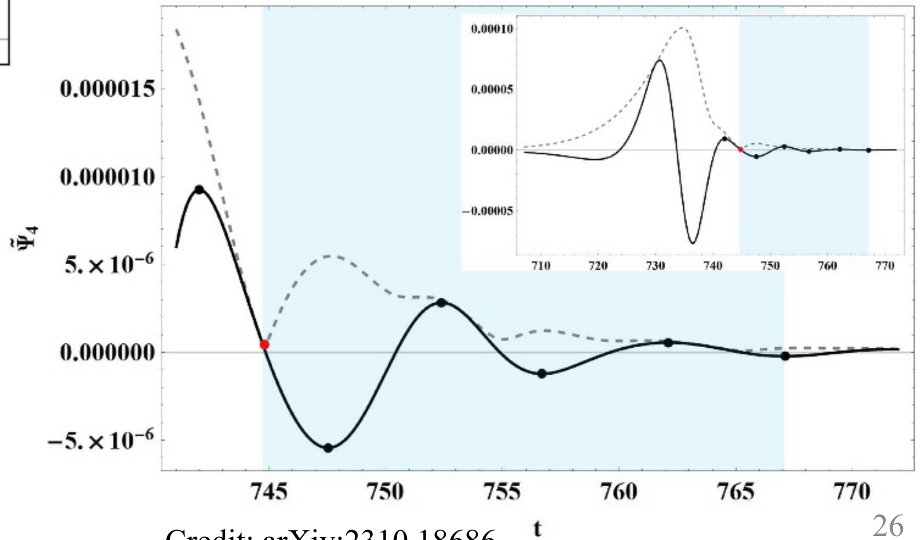
Asymptotics:
- Waveforms.
- Radiated energy and angular momentum.









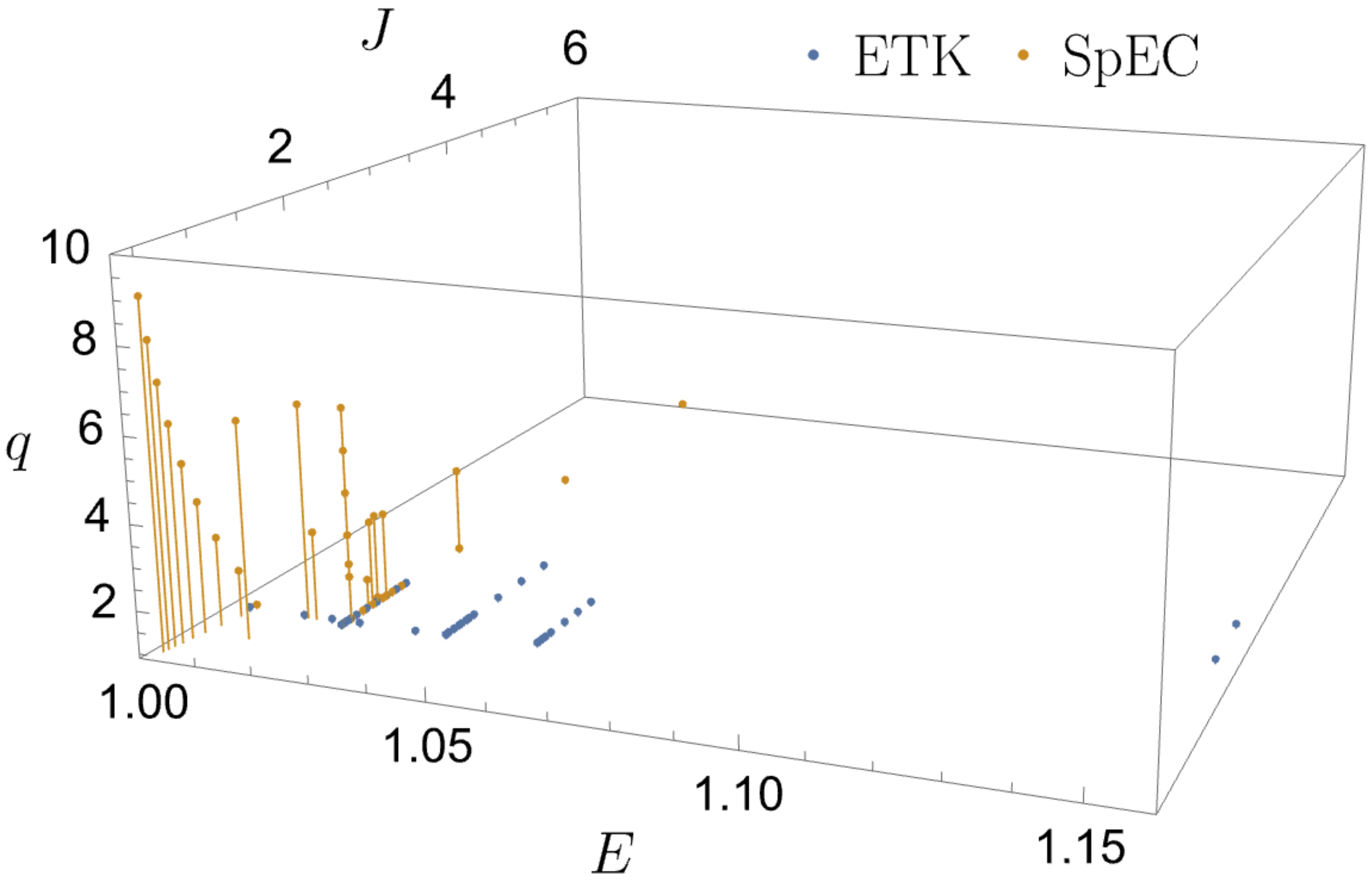


Credit: arXiv:2310.18686

Current status of hyperbolic NR

Two known codebases that can produce NR scattering simulations:

- Einstein toolkit (ETK) [Damour et al '14, Hopper et al '22, Rettegno et al '23, Yeong-Boket et al '23]:
  - Equal mass, non-spinning – 48
  - Equal mass, aligned spin – 34
- Spectral Einstein Code (SpEC):
  - Equal mass, non-spinning – 10
  - Equal mass, aligned spin – 2
  - Unequal mass, non-spinning – 27



E : Energy

J : Angular momentum

$$q := \frac{m_1}{m_2} > 1$$

Effective One Body models for scattering

1. EOB Hamiltonian and radiation-reaction (RR) forces built by re-summing PN results: (e.g. SEOBNRv5, TEOBResumS)

Dissipative terms

$$\dot{r} = \frac{\partial H_{\rm EOB}}{\partial p_r} \qquad \dot{\phi} = \frac{\partial H_{\rm EOB}}{\partial p_\phi} \qquad \dot{p}_r = -\frac{\partial H_{\rm EOB}}{\partial r} + \mathcal{F}_r \qquad \dot{p}_\phi = \frac{\partial H_{\rm EOB}}{\partial \phi} + \mathcal{F}_\phi$$

H : Hamiltonian

p_i : Momentum

2. Expanded or resummed PM potential from mass-shell constraint:

$$p_r^2 = p_\infty^2(E) - \frac{J^2}{r^2} + w(E, J, r) \qquad \theta = -\pi - 2\int_{r_{\min}}^{\infty} dr \frac{\partial}{\partial J} p_r$$

Fix coefficients of PM expanded w by matching to PM expanded scattering angle.

The potential w contains both conservative and dissipative information from the scattering angle

wPM	SEOB-PM
Uses PM-expanded w.	Uses PM-resummed w from SEOB-PM Hamiltonian, that reduces to the test-body limit.
[Damour & Rettegno '23, Rettegno et al '24]	[Buonanno, Jakobson & Mogull '24]

Extracting the scattering angle from NR

Only have trajectory for finite values for radius.

Can expand azimuthal angle as a function of radius:

$$\varphi = \varphi^{(0)} + \frac{\varphi^{(1)}}{r} + \frac{\varphi^{(2)}}{r^2} + \ldots$$

Asymptotic value

Fit polynomial to NR data on ingoing and outgoing legs of orbit to extrapolate to infinity:

$$\theta = \varphi^{(0)}_{\text{out}} - \varphi^{(0)}_{\text{in}} - \pi$$

Vary the range and order of the fit gives the dominant error on the scattering angle.

Current status of hyperbolic NR

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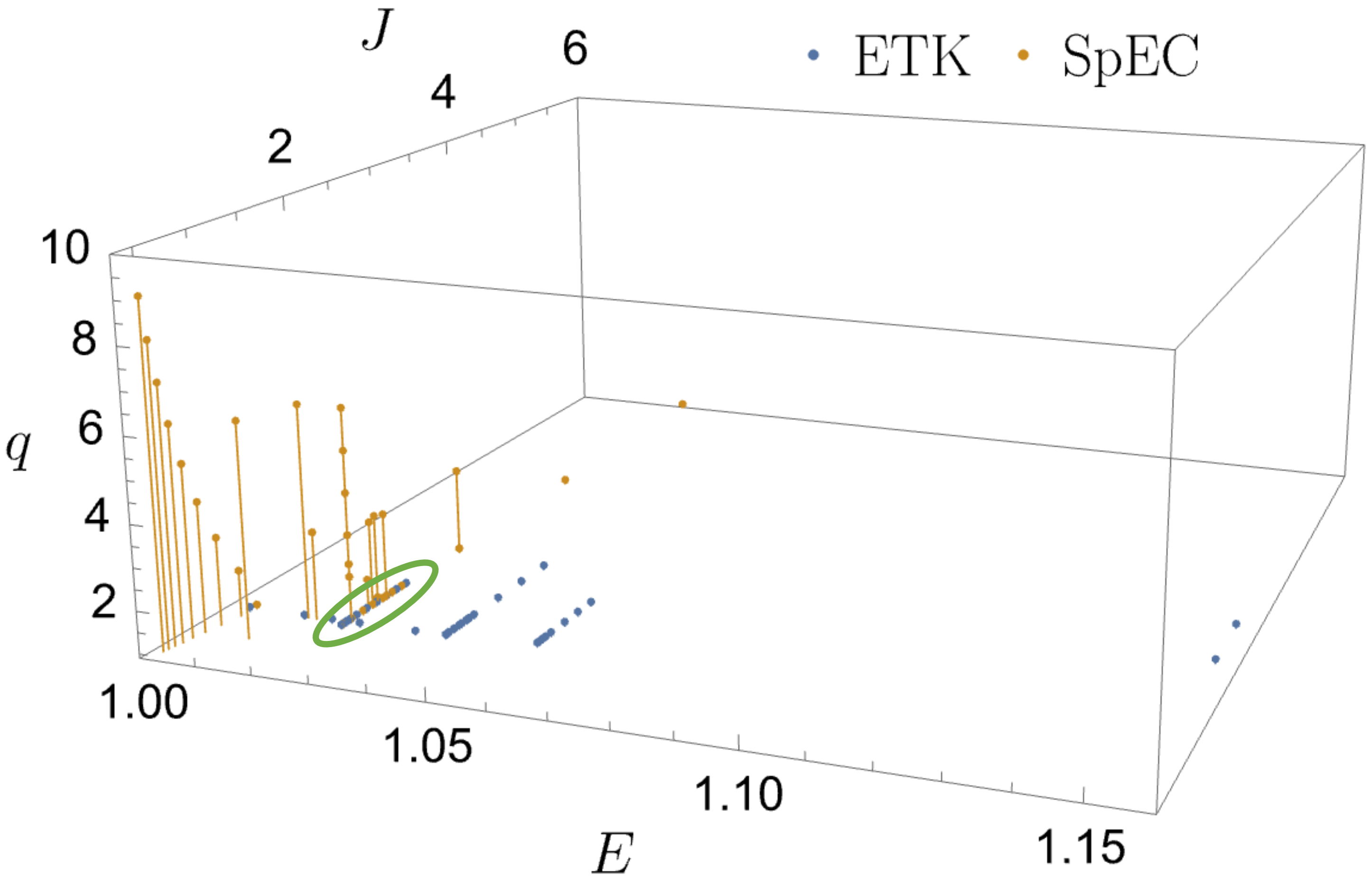
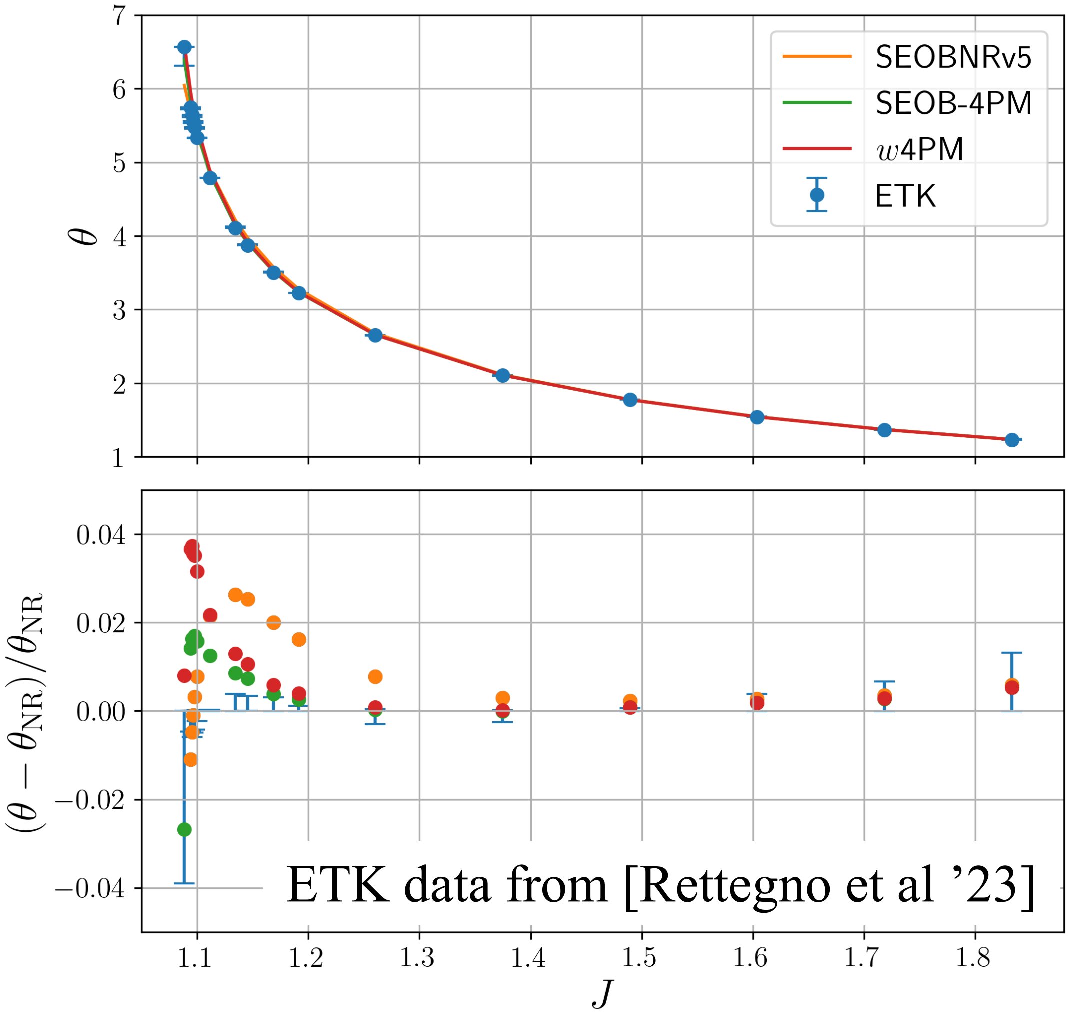
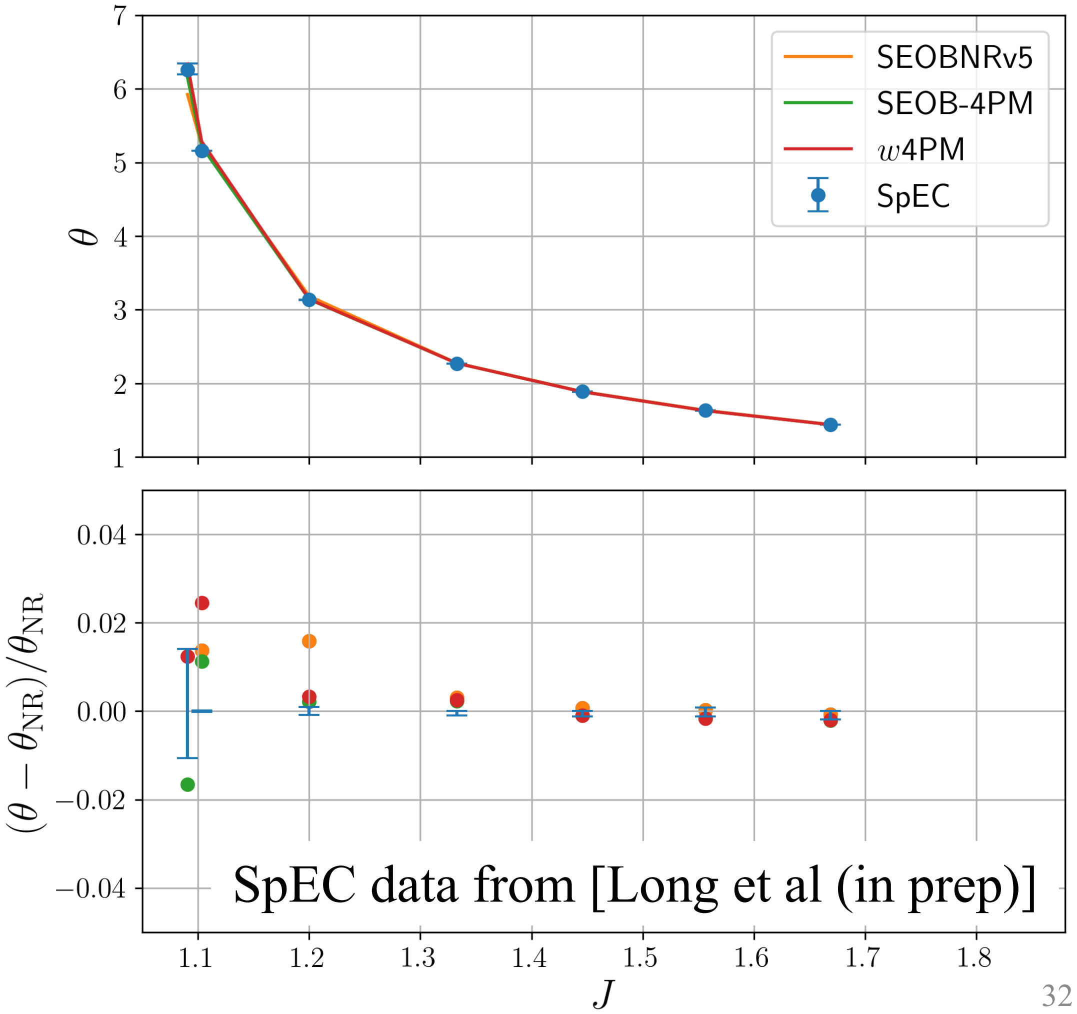


Equal mass comparison: $(E, q) = (1.02264, 1)$

Comparison with SEOB-4PM [Buonanno et al '23], w4PM [Damour & Rettegno '23, and SEOBNRv5 (with quasi-circular RR forces) [Ramos-Buades et al '23]:





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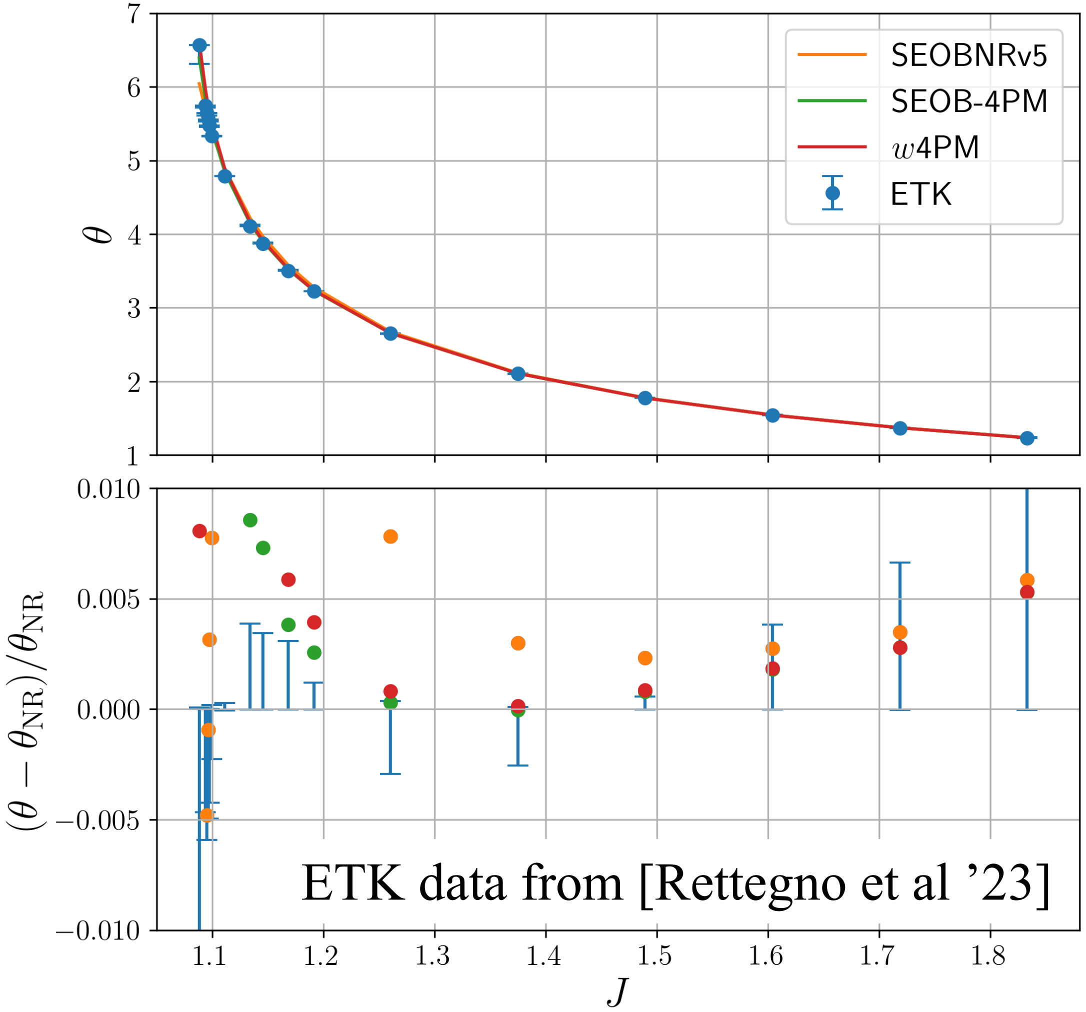
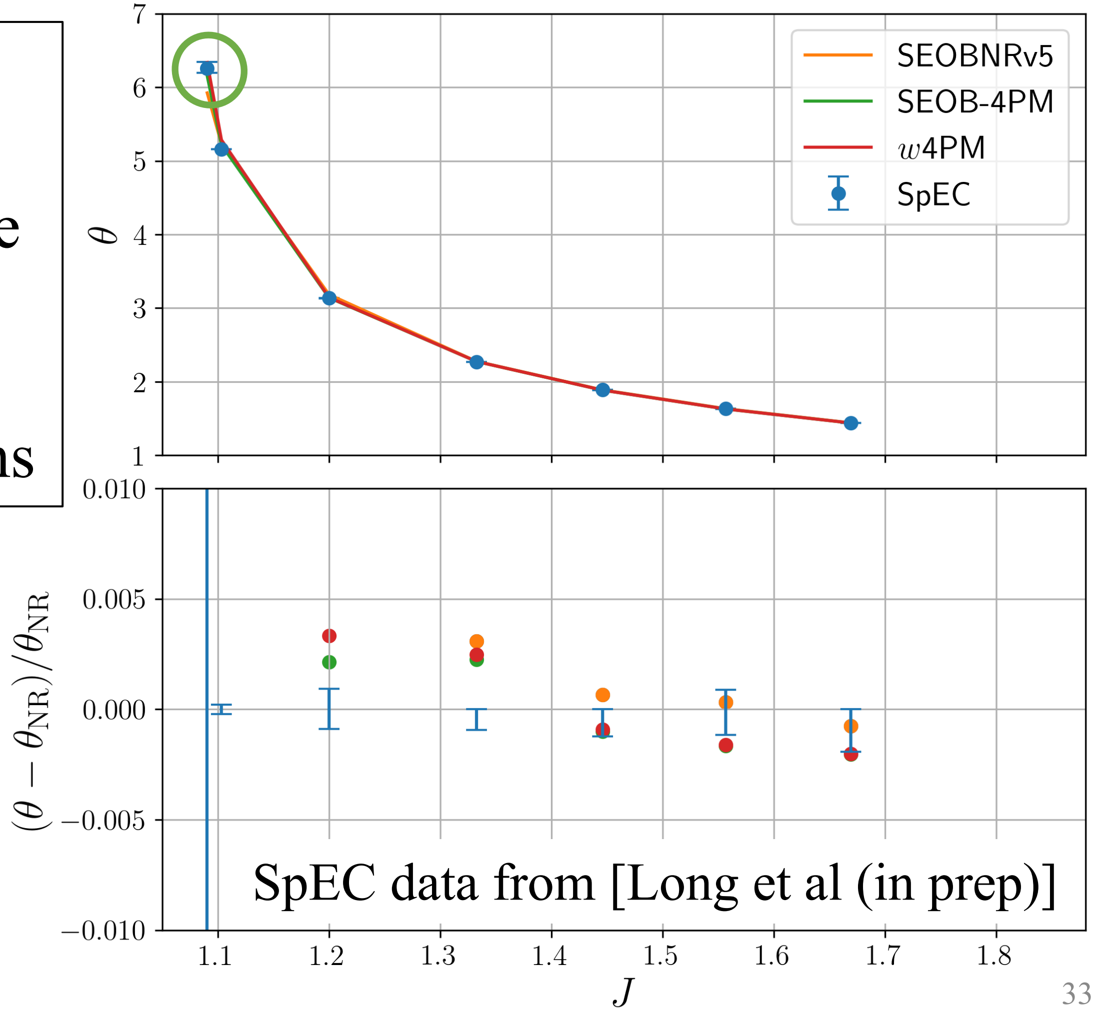
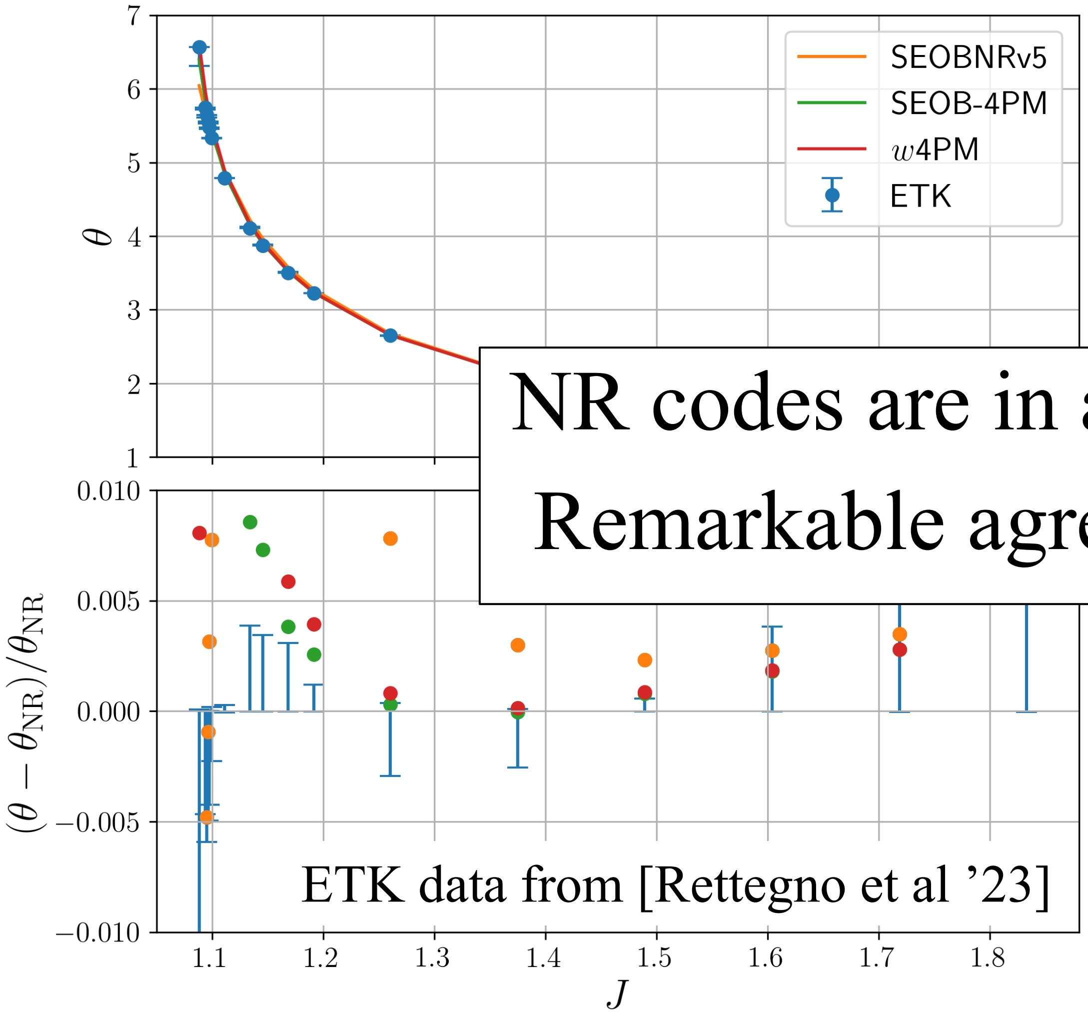


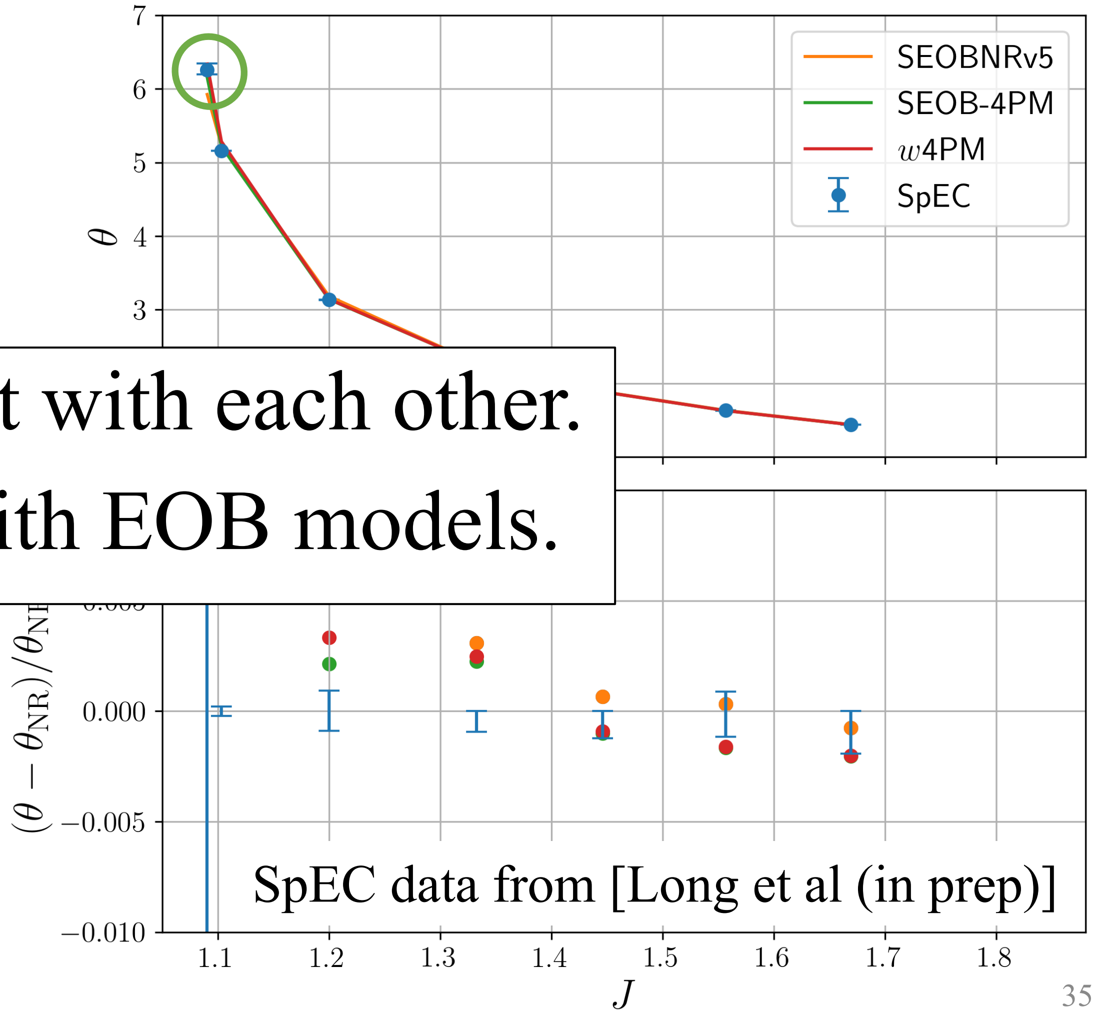
Smaller errors in SpEC due to larger initial separations



Equal mass comparison: $(E, q) = (1.02264, 1)$

Comparison with SEOB-4PM [Buonanno et al '23], w4PM [Damour & Rettegno '23, and SEOBNRv5 (with quasi-circular RR forces) [Ramos-Buades et al '23]:





NR codes are in agreement with each other.

Remarkable agreement with EOB models.

Challenges for unequal masses

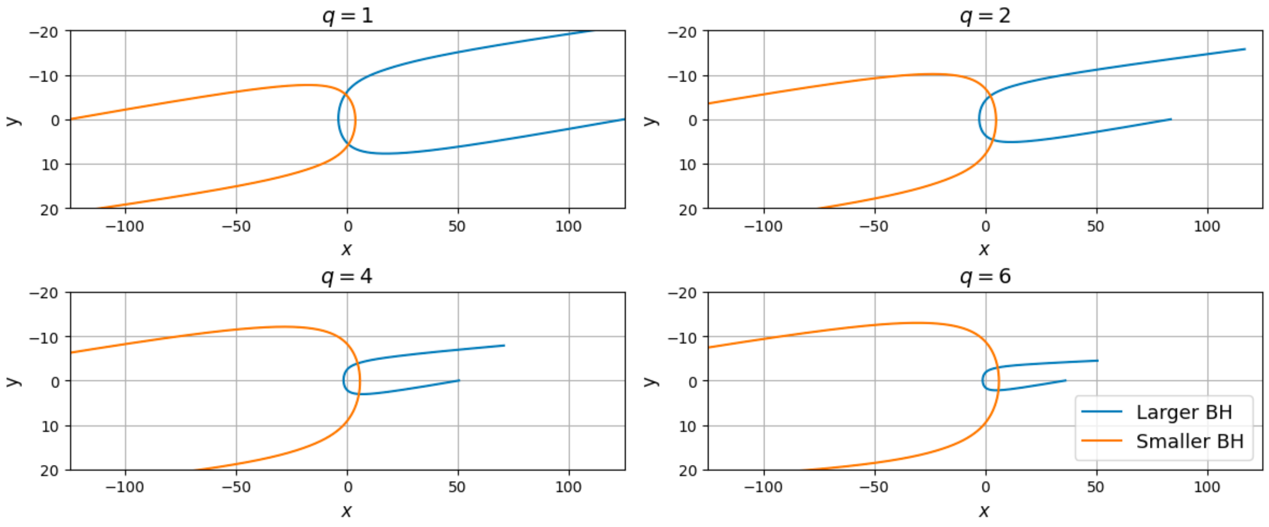
Simulations are in initial CoM frame.

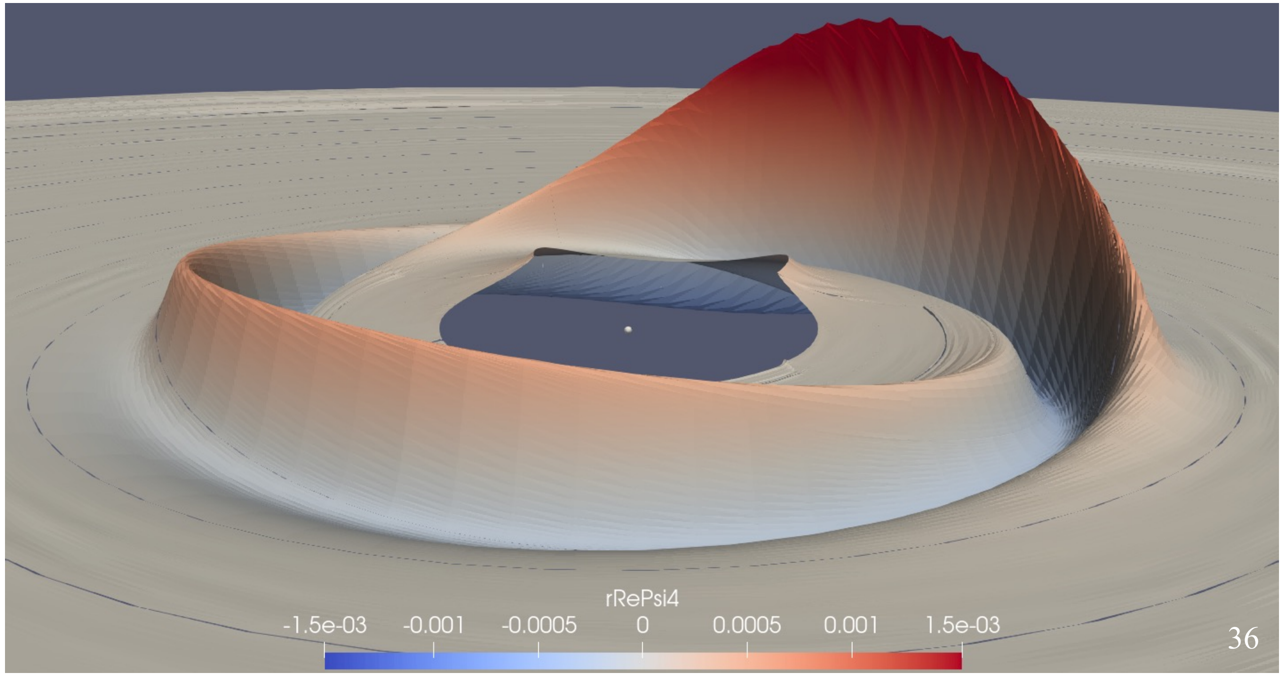
Length of orbits:

- Get a shorter trajectory for increasing mass ratio.
- Extracting angle of larger BH unfeasible.
- Have more data to extract angle of smaller BH .

Centre of mass recoil:

- Asymmetric emission causes a CoM ‘kick’.
- CoM recoil not in current PM/EOB models.





Current status of hyperbolic NR

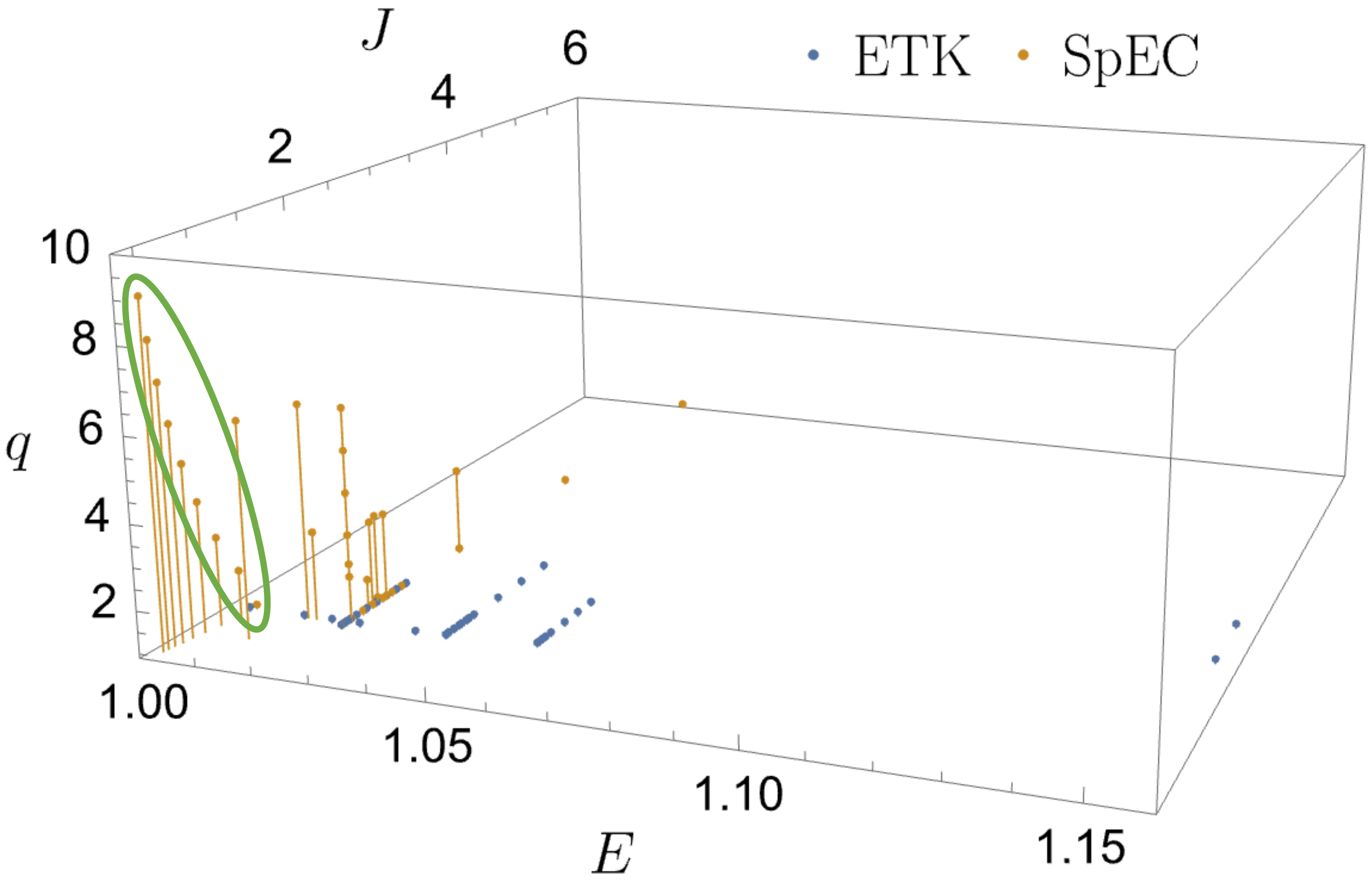
Two known codebases that can produce NR scattering simulations:

- Einstein toolkit (ETK) [Damour et al '14, Hopper et al '22, Rettegno et al '23, Yeong-Boket et al '23]:
  - Equal mass, non-spinning – 48
  - Equal mass, aligned spin – 34
- Spectral Einstein Code (SpEC):
  - Equal mass, non-spinning – 10
  - Equal mass, aligned spin – 2
  - Unequal mass, non-spinning – 27

E : Energy

J : Angular momentum

$$q := \frac{m_1}{m_2} > 1$$



Unequal mass comparison: $(\gamma, j) = (1.02, 4.8)$

Rescaled energy and angular momentum:

$$j := \frac{J}{\mu M}$$

$$\Gamma := \frac{E}{M} = \sqrt{1 + 2\nu(\gamma - 1)}$$

$$\gamma := \frac{1}{\sqrt{1 - v^2}}$$

$$M := m_A + m_B$$

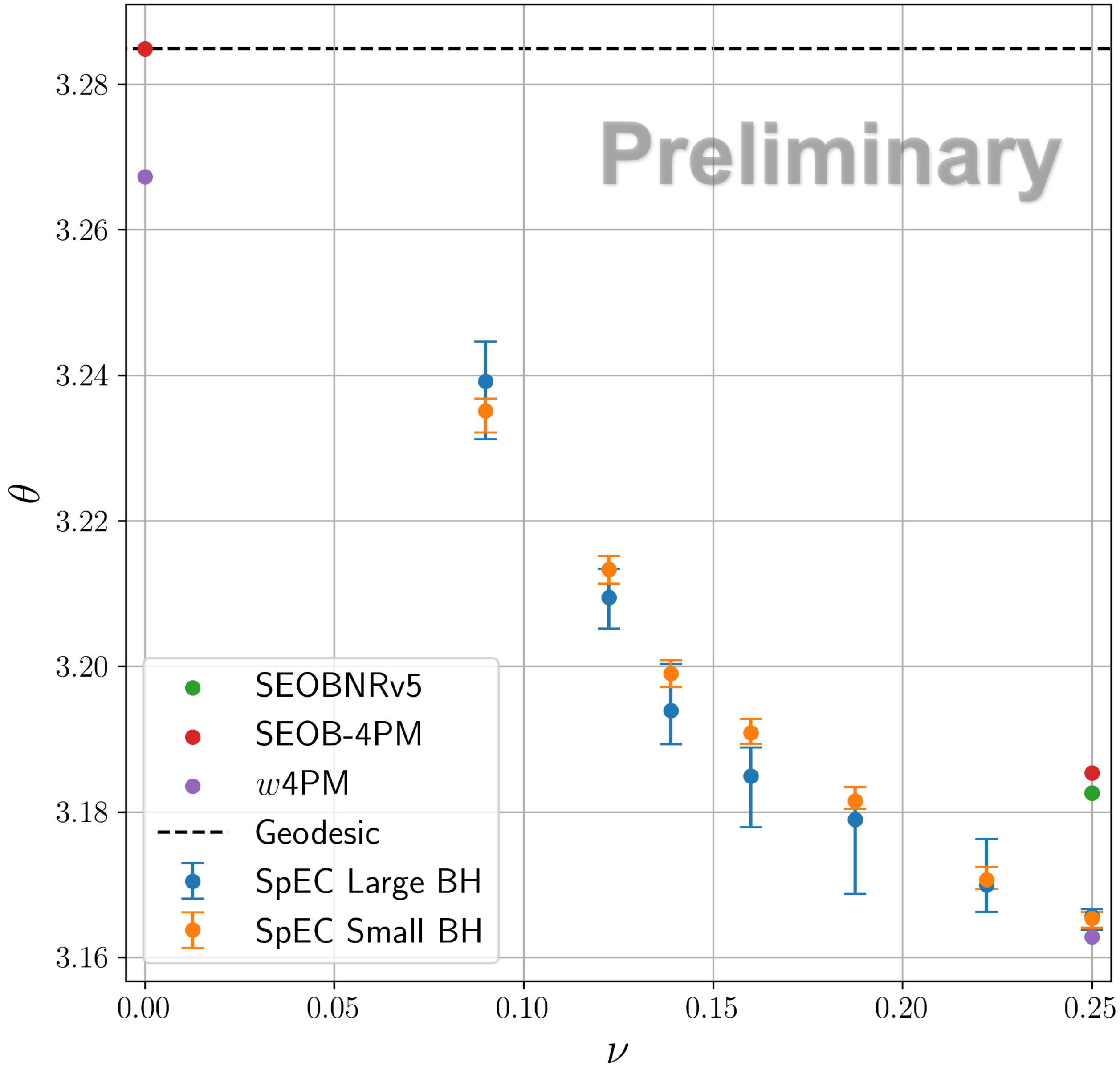
$$\mu := \frac{m_A m_B}{M}$$

$$\nu := \frac{\mu}{M}$$

Slices of constant (γ, j) correspond to the geodesic limit as $\nu \to 0$.

To compare to PM/EOB:

- Ensure the same angle definition.
- Account for CoM recoil.



Future work in NR

Comparison between NR and PM/EOB scattering angles for unequal masses.

Expand the NR parameter space:

- More unequal masses.
- More extreme energies/angular momenta.
- More extreme spins.
- Generic/precessing spins.

Waveform comparisons.

CoM recoil calculations.

Comparison to/extraction of the GSF correction to the scattering angle.

What is useful to the scattering amplitude community?

Summary

Scalar field toy model:

- Energy absorption appears at lower PM order in scalar toy model.
- Numerical self-force results agree to high order with PM results in the weak field.
- Higher numerical precision is needed to reliably extract higher-order PM.

Scattering in Numerical Relativity:

- Have ~100 NR scattering simulations.
- Equal mass, non-spinning results from different codes agree with each other.
- Equal mass, non-spinning Effective One Body agrees with NR to within a few percent.
- Starting to expand parameter space by adding spin and using unequal masses.