On-shell approaches to self-force using amplitudes on backgrounds

Sonja Klisch

University of Edinburgh

Gravitational Self Force and Scattering Amplitudes, Higgs Centre, 21/03/2024

Based on `2210.04696`, `2307.00431` with T. Adamo, A. Cristofoli, and A. Ilderton and upcoming work with T. Adamo

A quick summary of self-force (1)

- In a two-body problem where the mass ratio $\epsilon = m/V$, with V a mass scale associate to the background, is small, it's possible to expand the full metric

$$\mathrm{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \mathcal{O}(\epsilon^2)$$

  where $g_{\mu\nu}$ is the background metric
- Similarly it's possible to expand the stress energy tensor of the body moving on the *full* metric $\mathrm{g}_{\mu\nu}$

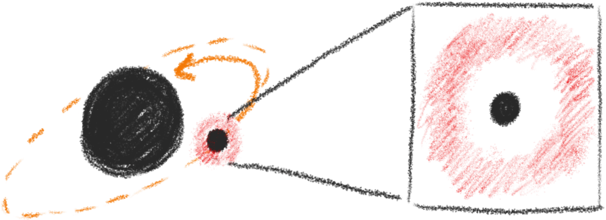
$$T_{\mu\nu} = \epsilon T^{(1)}_{\mu\nu} + \epsilon^2 T^{(2)}_{\mu\nu} + \mathcal{O}(\epsilon^3)$$

  where $T^{(1)}_{\mu\nu}$ is the stress-energy of a point mass moving in the background $g_{\mu\nu}$
- At leading order in the self force for radiation we simply solve the usual linearised Einstein equation on the background $\mathrm{g}_{\mu\nu}$

$$\delta G_{\mu\nu}[h^{(1)}] = 8\pi T^{(1)}_{\mu\nu}$$

A quick summary of self-force (2)

- At higher orders we see the failure of the point-particle treatment as the equations become too singular to be integrated
- It then becomes necessary to consider the finite size effects of the small body, and deal with the singularities, for example using matching. See review by [Barack, Pound: '18] and previous talks



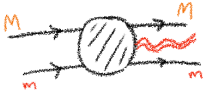
- In this talk we will take inspiration from the leading order contributions $\delta G_{\mu\nu}[h^{(1)}] = 8\pi T^{(1)}_{\mu\nu}$

An amplitudes-based approach to two-body mechanics

- The KMOC [Kosower,Maybee,O'Connell: '18] formalism allows us to express classical observables in terms of the classical limit of quantities built from amplitudes
- The initial states are given by wavepackets $\phi(p_1), \phi(p_2)$, with a well-defined notion of classical particle dynamics, to form $|\Psi_{in}\rangle$
- For example, the waveform [Cristofoli, Gonzo, Kosower, O'Connell: 2021] is then constructed out of the classical limit of the expectation value $\langle\Psi_{in}|S^\dagger\mathbb{R}_{\mu\nu\rho\sigma}(x)S|\Psi_{in}\rangle$ where

$$\mathbb{R}_{\mu\nu\rho\sigma}(x) = \frac{\kappa}{2}(\partial_\sigma\partial_{[\mu}\mathbf{h}_{\nu]\rho} - \partial_\rho\partial_{[\mu}\mathbf{h}_{\nu]\sigma}),$$

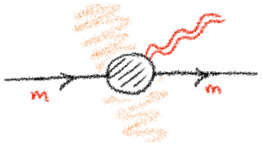
$$\mathbf{h}_{\mu\nu}(x) = \frac{1}{\sqrt{\hbar}}\sum_{\eta=\pm}\int \mathrm{d}\Phi(k)\Big[\mathbf{a}_\eta(k)\epsilon^{(\eta)*}_{\mu\nu}e^{-i\bar{k}\cdot x} + \text{h.c.}\Big]$$



Another approach

Amplitudes evaluated on a background spacetime?

- In the classical limit the particles follow geodesics on the background
- We can expect some properties of flat space amplitudes to still hold on backgrounds [Adamo,Casali,Mason,Nekovar: '17; Adamo,Ilderton: '20; Ilderton,Macleod: '20, Adamo,Bu,Zhu: '23]
- It's an on-shell approach to self-force observables



- This has also been considered recently from an EFT and worldline perspective [Kosmopoulos, Solon: '23; Cheung, Parra-Martinez, Rothstein, Shah, Wilson-Gerow: '23; Driesse, Jakokobsen, Mogull, Plefka, Sauer, Usovitch: '24]

Classical observables in curved spacetimes

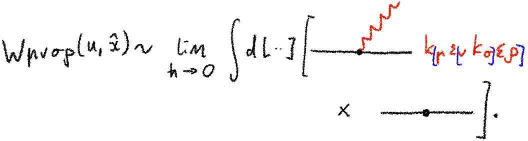
- On backgrounds, we can consider single particle scattering with initial states [Adamo, Cristofoli, Ilderton: '22]

$$|\Psi_{in}\rangle = \int \mathrm{d}\Phi(p)\phi(p)e^{ip\cdot b/\hbar}|p_{in}\rangle$$

- Just as before we construct the expectation value of our observable, for example:

$$\langle\Psi_{in}|S^{\dagger}\mathbb{R}_{\mu\nu\rho\sigma}(x)S|\Psi_{in}\rangle = \int \mathrm{d}\Phi(p')\langle\Psi_{in}|S^{\dagger}|p'\rangle\langle p'|\mathbb{R}_{\mu\nu\rho\sigma}(x)S|\Psi_{in}\rangle + h.o.t$$

- Has contributions coming from 3-point amplitude



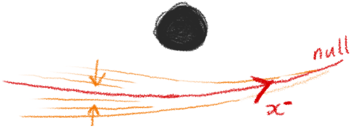
- What backgrounds?

A motivation for gravitational plane waves

A gravitational plane wave (GPW) [Brinkmann: '25; Einstein, Rosen: '37] metric has a metric of the form

$$\mathrm{d}s^2 = 2\mathrm{d}x^-\mathrm{d}x^+ - H_{ab}(x^-)x^a x^b(\mathrm{d}x^-)^2 + \mathrm{d}x_a\mathrm{d}x^a.$$

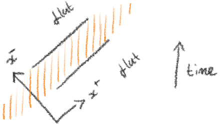
- Penrose: Any spacetime looks like a plane wave spacetime when viewed from along a null geodesic



- For example, a Schwarzschild black hole metric when viewed along a null geodesic with radial component $r(x^-)$ limits to

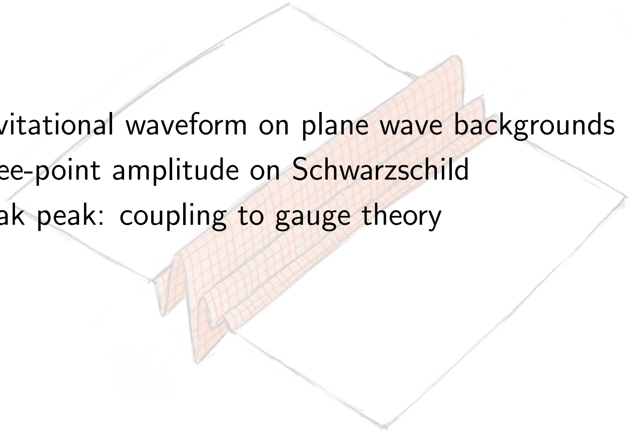
$$\mathrm{d}s^2 = 2\mathrm{d}x^-\mathrm{d}x^+ + \frac{3ML^2}{r(x^-)^2}(x_1^2 - x_2^2)(\mathrm{d}x^-)^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2$$

- We will be considering sandwich plane waves:

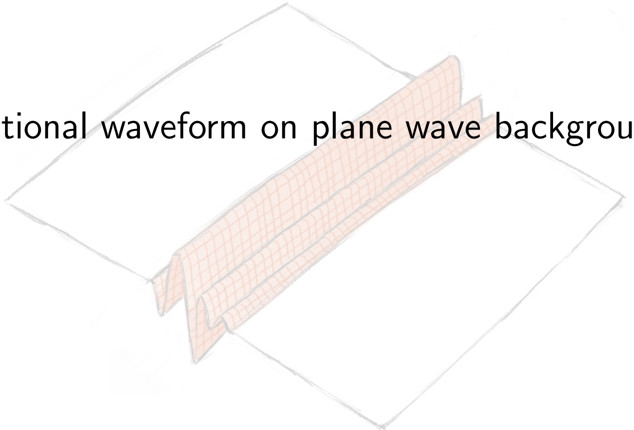


Rest of the talk

- Gravitational waveform on plane wave backgrounds
- Three-point amplitude on Schwarzschild
- Sneak peak: coupling to gauge theory



Gravitational waveform on plane wave backgrounds



Gravitational plane waves

Brinkmann (1925):

$$\mathrm{d}s^2 = 2\mathrm{d}x^-\mathrm{d}x^+ + H_{ab}(x^-)x^a x^b(\mathrm{d}x^-)^2 + \mathrm{d}x_a\mathrm{d}x^a, \quad H_a^a = 0$$

Einstein-Rosen (1937): $\mathrm{d}s^2 = 2\mathrm{d}X^-\mathrm{d}X^+ - \gamma_{ij}(X^-)\mathrm{d}y^i\mathrm{d}y^j$

- These metrics are related by the (non-unique) coordinate transformation

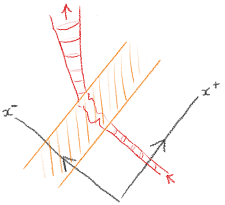
$$X^- = x^-, \quad X^+ = x^+ + \frac{1}{2}\sigma_{ab}x^a x^b, \quad y^i = E_a^i(x^-)x^a$$

where $\sigma_{ab} = \dot{E}_a^i E_{b\,i}$ and

$$\ddot{E}_{a\,i} = H_{ab}E_i^b, \qquad \gamma_{ij} = E_{(i}^a E_{|a|\,j)},$$

and E_b^i is the inverse of $E_{i\,a}$ in the sense that $E_a^i E_{i\,b} = \delta_{ab}$

Sandwich plane waves



- For sandwich plane waves there exist x_i^-, x_f^- so that $H_{ab}(x^- < x_i^-) = 0$ and $H_{ab}(x^- > x_f^-) = 0$
- There are therefore two natural boundary conditions one can impose on the 2nd order differential equation $\ddot{E}_{a\,i} = H_{ab}E_i^b$:

$$E^{in}_{a\,i}(x^- < x_i^-) = \delta_{a\,i}, \qquad E^{out}_{a\,i}(x^- > x_f^-) = \delta_{a\,i},$$

  with generic behaviour on the other side

$$E^{in}_{a\,i}(x^- > x_f^-) = c_{a\,i} + x^- b_{a\,i}, \qquad E^{out}_{a\,i}(x^- < x_i^-) = \tilde{c}_{a\,i} + x^- \tilde{b}_{a\,i}$$

- The solutions to this define the other geometric quantities $\sigma^{in/out}_{ab}, \gamma^{in/out}_{ij}$

Impulsive plane waves

Here the metric is

$$\mathrm{d}s^2 = 2\mathrm{d}x^-\mathrm{d}x^+ + \mathbf{H}_{ab}\delta(x^-)x^a x^b(\mathrm{d}x^-)^2 + \mathrm{d}x_a\mathrm{d}x^a, \quad \mathbf{H} = \mathrm{diag}(\lambda, -\lambda).$$

The geometric quantities related to this metric are then

$$E^{in}_{a\,i} = \delta_{ai} + \mathbf{H}_{ai}\Theta(x^-)x^-, \qquad E^{out}_{a\,i} = \delta_{ai} - \mathbf{H}_{ai}\Theta(x^-)x^-,$$

$$E^{i,in}_{a,>} = \begin{pmatrix} \frac{1}{1+\lambda x^-} & 0 \\ 0 & \frac{1}{1-\lambda x^-} \end{pmatrix}, \qquad \gamma^{in}_{ij,>} = \begin{pmatrix} (1+\lambda x^-)^2 & 0 \\ 0 & (1-\lambda x^-)^2 \end{pmatrix},$$

$$\sigma^{in}_{ab} = \begin{pmatrix} \frac{-\lambda}{1+\lambda x^-} & 0 \\ 0 & \frac{\lambda}{1-\lambda x^-} \end{pmatrix} \Theta(x^-)$$

States on a GPW background

The key building block of amplitudes in flat space are the wavefunctions $e^{ik\cdot x}$. In planewaves these get dressed by the background [Gibbons: '75; Ward: '87; Adamo, Casali, Mason, Nekovar: '17], so instead we have the dressed scalar wavefunction $\Phi(x) = \Omega(x^-)e^{i\phi_k}$ where

$$\phi_k := \frac{k_+}{2}\sigma_{ab}x^a x^b + k_i E^i_a x^a + k_+ x^+ + \frac{1}{2k_+}(m^2 + k_i k_j F^{ij}),$$

$\Omega(x^-) = |E|^{-1/2}$, $F^{ij} = \int^{x^-}\gamma^{ij}$. These have an associated 'dressed momentum' $P_\mu \mathrm{d}x^\mu = \mathrm{d}\phi_k$.

- Can be defined as either "in" or "out" depending on the boundary conditions we're considering
- There's no mixing of positive and negative frequencies between these two prescriptions, and so there's no pair production and the vacua of the two regions can be identified [cf. Aoki, Cristofoli: '24]

Metric perturbations on a GPW

- Solutions to the linearised Einstein equation

$$g^{\mu\nu}\partial_\mu\partial_\nu h_{\rho\sigma}+4n_{(\rho}n^{\mu}\partial_\mu h_{\sigma)a}H^a_b x^b-n_\rho n_\sigma H^{ab}h_{ab}=0,\quad n_\mu \mathrm{d}x^\mu=\mathrm{d}x^-$$

- These can be constructed using spin-raising operators [Mason: '89]

$$h_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu=\left((\varepsilon(x^-)\cdot\mathrm{d}x)^2-\frac{i}{k_+}\epsilon_a\epsilon_b\sigma^{ab}(\mathrm{d}x^-)^2\right)\Phi(x)$$
$$:=\mathcal{E}_{\mu\nu}(k;x)\Phi(x)\mathrm{d}x^\mu\mathrm{d}x^\nu.$$

where $\mathrm{d}x^\mu\varepsilon_\mu(x^-)=\epsilon^a(k_jE^j_a/k_++\sigma_{ab}x_b)\mathrm{d}x^-+\epsilon_a\mathrm{d}x^a=:\mathbb{P}_{\mu\nu}\epsilon^\mu\mathrm{d}x^\nu.$

- The tail can be associated with the tail effect in the gravitational Green's function of this spacetime, from waves scattering off the background metric [Friedlander: '75; Harte: '13]

Constructing the waveform on a GPW

- Recall that we want to find the classical limit of

$$\langle\Psi_{in}|S^\dagger\mathbb{R}_{\mu\nu\rho\sigma}(x)S|\Psi_{in}\rangle = \int \mathrm{d}\Phi(p')\langle\Psi_{in}|S^\dagger|p'\rangle\langle p'|\mathbb{R}_{\mu\nu\rho\sigma}(x)S|\Psi_{in}\rangle + h.o.t$$

where

$$\mathbb{R}_{\mu\nu\rho\sigma}(x) = \frac{\kappa}{2}(\partial_\sigma\partial_{[\mu}\mathbf{h}_{\nu]\rho} - \partial_\rho\partial_{[\mu}\mathbf{h}_{\nu]\sigma}),$$

$$\mathbf{h}_{\mu\nu}(x) = \frac{1}{\sqrt{\hbar}}\sum_{\eta=\pm}\int \mathrm{d}\Phi(k)\left[\mathbf{a}_\eta(k)\epsilon^{(\eta)*}_{\mu\nu}e^{-i\bar{k}\cdot x} + \text{h.c.}\right]$$

- The leading order terms corresponds to calculating

$$\lim_{r\to 0} r \times \frac{\kappa}{2\hbar}\sum_{\eta=\pm}\int \mathrm{d}\Phi(k,p,p')\phi(p)e^{-i\bar{k}\cdot x}k_{[\mu}\epsilon^\eta_{\nu]}k_{[\sigma}\epsilon^\eta_{\rho]} \times \langle p|S^\dagger|p'\rangle\langle p'|\mathbf{a}_\eta(k)S|\Psi_{in}\rangle$$

Neglecting velocity memory effect

- To simplify the calculation of the waveform we will assume that the velocity memory effect is parametrically small, and can be neglected
- This means that
$$E_a^{i,in}(x^- > x_f^-) = \delta_a^i$$
but everything is still non-trivial *in* the wave
- The two-point is then
$$\langle \Psi_{in}|\mathcal{S}|p'\rangle \to e^{i\theta(p')}\phi(p')$$
- Absorbing this shift into a redefinition of x^-, we now have
$$\frac{\kappa}{2\hbar}\sum_{\eta=\pm}\int \mathrm{d}\Phi(k,p,p')\phi(p,p')e^{-i\bar{k}\cdot x}k_{[\mu}\epsilon^{\eta}_{\nu]}k_{[\sigma}\epsilon^{\eta}_{\rho]}\langle p'|\mathbf{a}_\eta(k)S|\Psi_{in}\rangle$$
- Upcoming work with A. Cristofoli tracking full memory effects

Three-point amplitude on a GPW background (1)

Tree-level amplitudes can be computed using the 'perturbiner method'.
For example, the 3-point amplitude can be calculated from the cubic $\phi\phi h$ part of the action

$$S[g] \propto \int \mathrm{d}^d X \sqrt{-|g|} g^{\mu\nu} T_{\mu\nu}[\phi]$$

The final amplitude depends whether we're considering 'ingoing' or 'outgoing' states.
Here [Adamo,Ilderton:2020] for all in-going states

$$\mathcal{A}_3 = -\frac{2i\kappa}{\hbar^{3/2}} \delta^{+,\perp}(p'+k-p) \int_{-\infty}^{\infty} \mathrm{d}y^- \frac{\exp[i\mathcal{V}(y^-)]}{\sqrt{|E(y^-)|}} \mathcal{E}_{\mu\nu}(k; y^-) P^\mu(y^-) P'^\nu(y^-)$$

where

$$\mathcal{V}(y^-) := \frac{1}{\hbar} \int_{-\infty}^{y^-} \mathrm{d}x \frac{P_\mu(z) K_\nu(z) g^{\mu\nu}(z)}{p_+ - k_+}.$$

Three-point amplitude on a GPW background (2)

$$\mathcal{A}_3 = -\frac{2i\kappa}{\hbar^{3/2}}\delta^{+,\perp}(p'+k-p)\int_{-\infty}^{\infty}\mathrm{d}y^{-}\frac{\exp[i\mathcal{V}(y^{-})]}{\sqrt{|E(y^{-})|}}\mathcal{E}_{\mu\nu}(k;y^{-})P^{\mu}(y^{-})P'^{\nu}(y^{-})$$

- From the definition of the polarisation, we see that the integrand has two structurally different terms

$$\left[\mathbb{P}_{\mu\rho}(k;y^{-})\mathbb{P}_{\nu\sigma}(k;y) - \frac{i\hbar}{k_{+}}n_{\mu}n_{\nu}\delta^{a}_{\rho}\delta^{b}_{\sigma}\sigma_{ab}(y)\right]\epsilon_{\eta}^{\sigma\rho}P^{\mu}(y^{-})P'^{\nu}(y^{-})$$

where $\mathbb{P}_{\mu\nu} = g_{\mu\nu}(y) - 2K_{(\mu}(y)n_{\nu)}/k_{+}$

Waveformm calculation (1)

- We can now construct the full expression out of our ingredients

$$\langle \Psi_{in} | \mathbb{R}_{\mu\nu\rho\sigma}(x) S | \Psi_{in} \rangle = -\frac{2i\kappa}{\hbar^{3/2}} \mathrm{Re} \int \mathrm{d}\Phi(k) \mathrm{d}\Phi(p) \mathrm{d}\Phi(p') \phi(p) \phi(p')$$

$$\times e^{-ik\cdot x} k_{[\mu} \epsilon_{\nu]}^{-\eta} k_{[\sigma} \epsilon_{\rho]}^{-\eta} \delta^{+,\perp}(p' + k - p) \int_{-\infty}^{\infty} \mathrm{d}y^- \frac{e^{-\mathcal{V}(y^-)}}{\sqrt{|E(y^-)|}} \mathcal{E}^{\eta}_{\mu\nu} P^{\mu} P'^{\nu}$$

- Note that in general we can use stationary phase to evaluate

$$\lim_{r\to\infty} \int \mathrm{d}\Phi(k) e^{-ik\cdot x} \hat{a}(k) \to -\frac{i}{4\pi r} \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega u} \hat{a}(\omega \hat{x}_{\mu})$$

$$+ \text{ complex conjugate}$$

Waveform calculation (2)

- Evaluating the classical limit (essentially setting $p = p'$ whilst keeping k free) we arrive at

$$W_{\mu\nu\rho\sigma}(u,\hat{x}) = -\frac{\kappa}{\pi\hbar^{1/2}}\mathrm{Re}\int_0^\infty \frac{\mathrm{d}\omega\,\mathrm{d}y^-}{2\pi\sqrt{|E(y^-)|}} e^{-i\omega(u-\bar{\mathcal{V}}(y^-))} \times k_{[\mu}\epsilon_{\nu]}^{-\eta} k_{[\sigma}\epsilon_{\rho]}^{-\eta} \mathcal{E}^{\eta}_{\mu\nu}(k,y^-)P^\mu(y^-)P^\nu(y^-)\Big|_{k=\omega\hat{x}}$$

- The integrand now has (schematic) scaling behaviour coming from the graviton polarisation

$$\sim \omega^2 T^0 - i\omega T^1$$

Waveform calculation (3)

Doing the integrals the final waveform is then

$$W_{\mu\nu\rho\sigma}(u,\hat{x}) = -\frac{\kappa^2}{\pi}\hat{x}_{[\mu}\hat{x}_{[\sigma}\int \mathrm{d}y\,\delta(u-\bar{\mathcal{V}}(y))\left[\mathcal{D}^2\, T^0_{\rho]\nu]}(\hat{x},y) - \mathcal{D}\, T^1_{\rho]\nu]}(\hat{x},y)\right]$$

- Here

$$T^0_{\nu\rho}(\hat{x},y^-) := \frac{\mathbb{P}_{\nu\alpha}(\hat{x},y^-)\mathbb{P}_{\rho\beta}(\hat{x},y^-)P^\alpha P^\beta(y^-) - \frac{1}{2}\eta_{\nu\rho}m^2}{\sqrt{|E(y)|}},$$

$$T^1_{\nu\rho}(\hat{x},y^-) := \frac{\delta^a_\nu \delta^b_\rho \sigma_{ab}(y)}{\hat{x}_+\sqrt{|E(y)|}}p_+^2$$

- The classical orbit of the particle is encoded in $\bar{\mathcal{V}}(y^-) = \hat{x}\cdot X(y^-)$
- We introduce

$$\mathcal{D}f(y) := \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{f(y)}{\partial_-\bar{\mathcal{V}}(y)}\right).$$

On impulsive plane waves

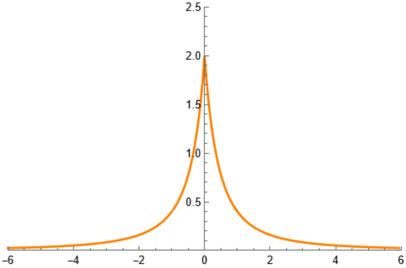
For example, in an impulsive wave where the metric becomes

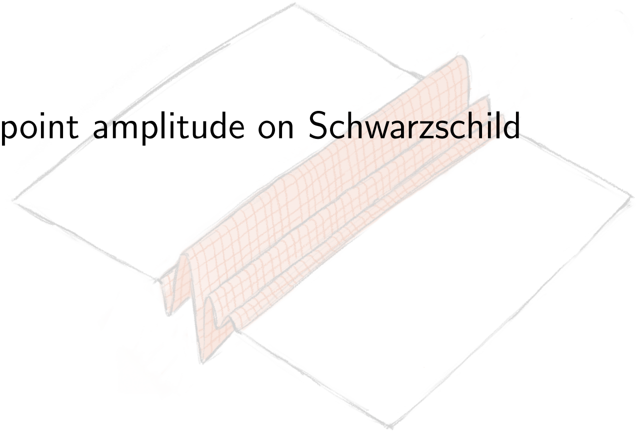
$$\mathrm{d}s^2 = 2\mathrm{d}x^+\mathrm{d}x^- - \mathrm{d}x_a\mathrm{d}x^a + \delta(x^-)H_{ab}x^a x^b(\mathrm{d}x^-)^2, \qquad H_{ab} = \mathrm{diag}(\lambda, -\lambda)$$

the waveform can be computed explicitly in certain kinematics ($p_\perp = 0$)

$$W_{\mu\nu\rho\sigma} \sim \kappa^2 p_+ \delta^+_{[\mu}\delta^+_{[\sigma}\delta^a_{\rho]}\delta^a_{\nu]}\frac{\partial^2}{\partial u^2}\left(\frac{\nu\log(\nu+\sqrt{\nu^2-1})}{\sqrt{\nu^2-1}}\right)$$

where $\nu := \lambda\sqrt{2}\frac{p_+^2}{m^2}|u|$.





Three-point amplitude on Schwarzschild

On Schwarzschild

- With the metric given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

  the massive scalar and metric perturbations are solved respectively using spherical harmonics by confluent Heun functions and solutions to the Regge-Wheeler-Zerilli equations
- Taking these as ingoing and outgoing states, we seek to calculate the 3-point amplitude

$$\langle \Phi_p h | T | \Phi_{p'} \rangle$$

- This is given by the integral over spacetime of the cubic part of the action

$$M_3^0(\phi_1, \phi_2, h) = \kappa \int \mathrm{d}^4 y \sqrt{-g}\, h_{\mu\nu} \partial^\mu \phi \partial^\nu \phi$$

Semi-classical scattering on linearised Schwarzschild (1)

- For a tractable calculation we make a WKB ansatz $\phi(p;x) = e^{iS_p(x)/\hbar}$ solving the Klein-Gordon equation

$$\big(g^{\mu\nu}\partial_\mu\partial_\nu - \frac{m^2}{\hbar^2}\big)\phi(x) = 0$$

  on the background.
- In the classical limit, to first order in the WKB expansion, we solve the Hamilton-Jacobi equations for the background $g^{\mu\nu}\partial_\mu S\partial_\nu S = m^2$
- For linearised Schwarzschild we have to first order in G

$$S_p(x) = p\cdot x + \frac{G\mathcal{P}^{\mu\nu}p_\mu p_\nu}{|\vec{p}|}\log(|\vec{p}|r + \vec{p}\cdot\vec{r}) + \dots$$

- We then match this onto the full solution as $|r| \to \infty$ to extract matching coefficients

$$\phi(p;x) = \int \mathrm{d}\Phi(l)\Lambda^p(l)e^{iS_p(x)}$$

Matching conditions on the scalar wavefunction

- We match onto the general solutions of the Klein-Gordon equation

$$\phi_p(x) = \frac{4\pi e^{iEt}}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\hat{x}) \bar{Y}_l^m(\hat{p}) R_{lm}(r)$$

- Only require the asymptotic behaviour of $R_{lm}(r)$ which after some analysis means that we match the WKB ansatz to the full solution using

$$\phi_p(x) \overset{r\to\infty}{=} -i|\vec{p}| \int \mathrm{d}^2\Omega_\ell f^p(|\vec{p}|, \hat{\ell}) e^{iS_\ell(x)} \Big|_{\ell^0=p^0}$$

where

$$f^p(|\vec{l}|, \hat{l}) = i|\vec{p}| \int \mathrm{d}^2 x^\perp e^{ix^\perp\cdot(\hat{\ell}-\hat{p})} e^{i(2I(|x^\perp|)-\pi|\vec{\ell}||x^\perp|)}$$

and $I(|x^\perp|) := I_{|\vec{p}||x^\perp|-\frac{1}{2}}(r=\infty)$, in terms of the radial action.

Semi-classical scattering on linearised Schwarzschild (2)

- Graviton can be constructed similarly by making the ansatz

$$\bar{h}_{\mu\nu}(x) = \mathcal{E}_{\mu\nu}(x) e^{iS_k(x)}$$

solving the linearised Einstein equations

$$\nabla^2 \bar{h}_{\mu\nu} + 2R_{\mu\rho\nu\sigma}\bar{h}^{\rho\sigma} = 0$$

- Using the same solution to the Hamilton-Jacobi equation

$$S_k(x) = k \cdot x + \frac{G\mathcal{P}^{\mu\nu}k_\mu k_\nu}{|\vec{k}|}\log(|\vec{k}|r + \vec{k}\cdot\vec{r}) + \mathcal{O}(G^2)$$

the dressed polarisation is given by
$\mathcal{E}_{\mu\nu}(x) = \epsilon_{\mu\nu} + G\mathcal{E}^{(1)}_{\mu\nu}(x) + \mathcal{O}(G^2)$ where

$$\begin{aligned}
\mathcal{E}^{(1)}_{\mu\nu}(x) = -2\,\varepsilon_{\beta(\mu}k^\alpha \int \hat{\mathrm{d}}^4\ell\, \frac{\mathrm{e}^{-\mathrm{i}\,\ell\cdot x}}{\ell^2 - 2\ell\cdot k + \mathrm{i}\,\epsilon}\left(\tilde{H}^{\beta}_{\nu)}\,\ell_\alpha + \ell_{\nu)}\,\tilde{H}^{\beta}_{\alpha} - \tilde{H}_{\nu)\alpha}\,\ell^\beta\right) \\
- \mathrm{i}\,\varepsilon_{\mu\nu}\int \hat{\mathrm{d}}^4\ell\,\frac{|\vec{\ell}|^2\,\mathrm{e}^{-\mathrm{i}\,\ell\cdot x}}{\ell^2 - 2\ell\cdot k + \mathrm{i}\,\epsilon}\,\tilde{S}(k;\ell) \\
- \varepsilon^{\rho\sigma}\int \hat{\mathrm{d}}^4\ell\,\frac{\mathrm{e}^{-\mathrm{i}\,\ell\cdot x}}{\ell^2 - 2\ell\cdot k + \mathrm{i}\,\epsilon}\left(-\ell_\mu\ell_\nu\,\tilde{H}_{\rho\sigma} + \ell_\nu\ell_\rho\,\tilde{H}_{\mu\sigma} + \ell_\sigma\ell_\mu\,\tilde{H}_{\rho\nu} - \ell_\sigma\ell_\rho\,\tilde{H}_{\mu\nu}\right)
\end{aligned}$$

Semi-classical scattering on linearised Schwarzschild (3)

- Can then match onto the correct asymptotics again,using Regge-Wheeler-Zerilli in Lorenz gauge [Berndtson: 2009]

$$\bar{h}_{\mu\nu}(x) = \int \mathrm{d}\Phi(k')\Lambda^{k'}(k)\mathcal{E}_{\rho\sigma}(k;x)e^{iS_k(x)}$$

- The 'semiclassical' graviton emission amplitude is then constructed just as before, with new versions of dressed polarisation and momentum

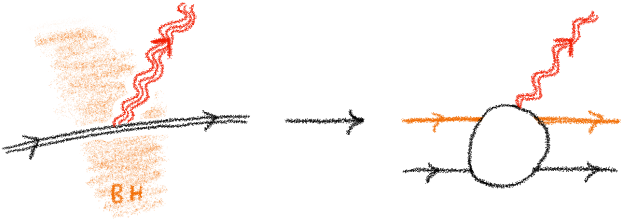
$$\langle p', k|\mathcal{S}|p\rangle = -2\kappa \int \mathrm{d}^4x\mathrm{d}\Phi(\ell,\ell',k')\sqrt{-|g|}\, \Lambda^{\bar{p}}(\ell)\Lambda^{p'}(\ell')\Lambda^{k}(k') \times \mathcal{E}_{\mu\nu}\partial^{\mu}S_{l'}\partial^{\nu}S_{\ell}e^{i(S_{k'}+S_{\ell'}-S_{\ell})}$$

Semi-classical scattering on linearised Schwarzschild (3)

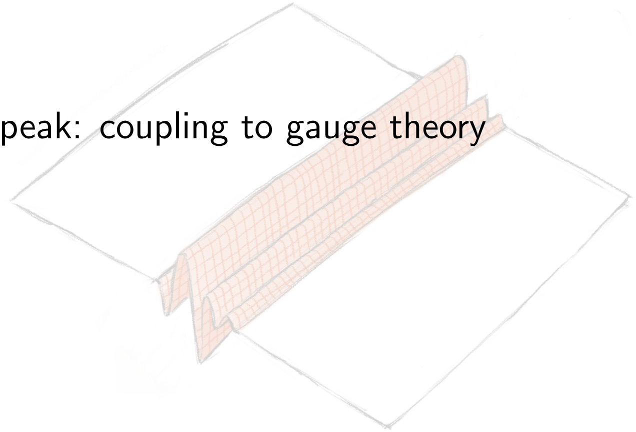
- In the weak field limit we have the explicit contribution

$$\langle p', k|\,\mathcal{S}\,|p\rangle\,|_{\kappa^3} = -2\kappa\Bigg[\varepsilon_{\mu\nu}\,p^\mu\,p^\nu\,\Big(\tilde{S}(k;k+q)+\tilde{S}(p+q;k+q)-\tilde{S}(p;k+q)\Big)$$
$$-\,\varepsilon_{\mu\nu}\,p^\mu\,q^\nu\,\Big(\tilde{S}(p;k+q)+\tilde{S}(p+q;k+q)\Big)+\tilde{\mathcal{E}}^{(1)}_{\mu\nu}(k+q)\,p^\mu\,(p+q)^\nu$$
$$+\,\frac{1}{2}\,\tilde{H}^\sigma_\sigma(k+q)\,\varepsilon_{\mu\nu}\,p^\mu\,p^\nu-\tilde{H}^{\mu\sigma}(k+q)\,\varepsilon_{\mu\nu}\,p^\nu\,(p+q)_\sigma-\tilde{H}^{\nu\sigma}(k+q)\,\varepsilon_{\mu\nu}\,p^\mu\,p_\sigma\Bigg]$$

- This matches the tree-level 5-point in the classical weak-field limit, neglecting the recoil of the background



Sneak peak: coupling to gauge theory



Plane waves in Einstein-Maxwell [In progress with T. Adamo]

- The metric is now coupled to electromagnetism via Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{\kappa} T_{\mu\nu}, \quad T_{\mu\nu} = \frac{1}{\mu_0} \left[F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$$

- In plane waves this means that for an electromagnetic field with potential $\mathcal{A}_\mu = n_\mu \dot{A}_a(x^-) x^a$ (and field strength $F_{\mu\nu} = 2 n_{[\mu} \dot{A}_{\nu]}$), the metric must now satisfy [cf. talk by Abraham]

$$\mathrm{d}s^2 = 2\mathrm{d}x^+ \mathrm{d}x^- + H_{ab}(x^-) x^a x^b (\mathrm{d}x^-)^2 + \mathrm{d}x_a \mathrm{d}x^a,$$
$$H_a^a(x^-) = -\frac{8\pi}{\mu_0 \kappa} \dot{A}_a \dot{A}^a(x^-)$$

- All other geometric quantities $E_{a\,i}, \sigma_{ab}, \gamma_{ij}$ continue satisfying the same relations ($\ddot{E} = HE, \sigma = \dot{E}^{-1} E, \gamma = EE$) as before
- We will again consider sandwich plane waves

Example: impulsive plane waves in Einstein-Maxwell

- The impulsive solution is

$$\dot{A}(x^-) = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \delta(x^-), \qquad H_{ab}(x^-) = \begin{pmatrix} \lambda & 0 \\ 0 & -\tilde{\lambda} \end{pmatrix} \delta(x^-),$$

$$\text{where} \quad \tilde{\lambda} = \lambda + \frac{8\pi}{\mu_0 \kappa}(\mu_1^2 + \mu_2^2)$$

- The related geometric quantities are

$$E^{in}_{a\,i} = \begin{pmatrix} 1 + \lambda x^- & 0 \\ 0 & 1 - \tilde{\lambda} x^- \end{pmatrix}, \quad \gamma^{in}_{ij} = \begin{pmatrix} (1 + \lambda x^-)^2 & 0 \\ 0 & (1 - \tilde{\lambda} x^-)^2 \end{pmatrix}$$

$$E^{in\,i}_{a} = \begin{pmatrix} \frac{1}{1+\lambda x^-} & 0 \\ 0 & \frac{1}{1-\tilde{\lambda} x^-} \end{pmatrix}, \quad \sigma^{in}_{ab} = \begin{pmatrix} \frac{\lambda}{1+\lambda x^-} & 0 \\ 0 & \frac{\tilde{\lambda}}{1-\tilde{\lambda} x^-} \end{pmatrix}$$

Free fields in EM plane waves

- The expressions for the gauge and metric perturbations follow easily from before:

$$h_{\mu\nu}(x^-) = \mathcal{E}_{\mu\nu}(x^-)\Phi(x^-), \qquad a_\mu(x^-) = \mathcal{E}_\mu(x^-)\Phi(x^-),$$

  where $\mathcal{E}_\mu \mathrm{d}x^\mu = (k_j E^j_a/k_+ + \sigma_{ab}x^b)\epsilon^a \mathrm{d}x^- + \epsilon_a \mathrm{d}x^a$.
- The only difference is for charged matter which also pick up contributions from the gauge background in addition to the gravitational background. E.g. for a charged massive scalar with charge q and mass m

$$\phi(x) = \frac{1}{\sqrt{|E|}} \exp i \left(k_+ x^+ + \frac{k_+}{2}\sigma_{ab}x^a x^b + E^i_a(k_i + qA_i)x^a + \frac{1}{2k_+}\int^{x^-} \mathrm{d}s \left[m^2 + (k_i + qA_i)(k_j + qA_j)\gamma^{ij}(s) \right] \right)$$

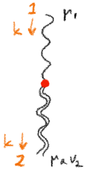
Feynman rules in EM on a background

- We can derive the Feynman rules governing the amplitudes either by using the perturbiner or looking at the action

$$S = \int \mathrm{d}^4 x \sqrt{-g[h]}\Big[R[h] - g_{\mu\nu}[h] T^{\mu\nu}[\phi, a, h]\Big];$$

this now contains an aAh contribution

- We obtain a new Feynman rule describing the 'back-reaction' of gluons and gravitons in the background

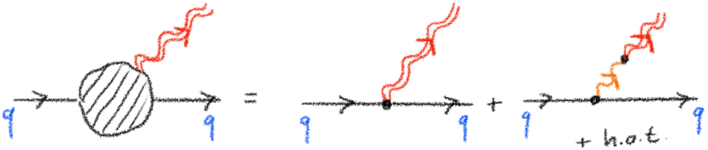


$$\sim \frac{\kappa}{g}\dot{A}_a\Big[(K_{\mu_2} n_{\mu_1} - k_+ g_{\mu_2\mu_1})\delta^a_{\nu_2} - (K_{\mu_2}\delta^a_{\mu_1} - K_a g_{\mu_2\mu_1})n_{\nu_2} + \frac{1}{2} g_{\mu_2\nu_2}(K_a n_{\mu_1} - k_+\delta^a_{\mu_1})\Big]$$

- Other massless interaction vertices are also dressed by the background

Amplitudes in EM plane waves (1)

- The existence of this extra Feynman rule is that we have an extra contribution to the 3-point describing scalar scattering with graviton emission



- This explicitly captures the presence of a "backreaction" term in these amplitudes, from the radiation backreacting on the charged background
- Requires the calculation of the photon propagator in this theory

Amplitudes in EM plane waves (2)

- The photon propagator dressed on the background is given by the following object:

$$G_{\mu\nu}(x,y) = \int \frac{\mathrm{d}^4 l}{l^2 + i\epsilon} \frac{e^{i\phi_l(x) - i\phi_l(y)}}{\sqrt{|E(x)||E(y)|}} \times \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & -\delta_{ab} & \frac{L_a(x)-L_a(y)}{l_+} \\ 1 & \frac{-L_b(x)+L_b(y)}{l_+} & \frac{(L(x)-L(y))^2}{2l_+^2} - \frac{\sum H}{2} + \frac{i\Delta S}{8l_+} \end{pmatrix}}_{D_{\mu\nu}(x,y)}$$

  where $\sum H = H_{ab}(x)x^a x^b + H_{ab}(y)y^a y^b$ and $S(x) = \mathrm{Tr}(\gamma^{-1}\dot{\gamma}\gamma^{-1}\dot{\gamma}(x))$

- Has some nice contraction properties such as

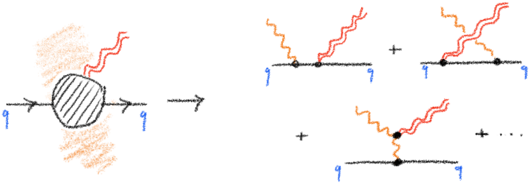
$$D_{\mu\nu}(x,y)\dot{A}^{\rho}(x)\mathcal{E}_{\rho\mu}(x) = -(\dot{A}(x)\cdot\epsilon)\mathcal{E}_{\nu}(y)$$

Amplitudes in EM plane waves (3)

- The momentum conserving δ-function in the 2-point means that the propagator should be evaluated with on-shell momenta
- Evaluating these expressions, the 3-point amplitude is

$$\mathcal{A}_3(p, p', k) \propto \int \mathrm{d}x^- \Bigg[\mathcal{E}_{\mu\nu} P^\mu P'^\nu(x^-) - q\,(P + P'))_\mu \mathcal{E}^\mu(x^-) \underbrace{\int \mathrm{d}y^- (\dot{A}(y^-) \cdot \epsilon)}_{A(\infty)\cdot\epsilon} \Bigg] e^{iV_{k,p,p'}(x^-)}$$

- In the weak-field limit for the electromagnetic background this would correspond to

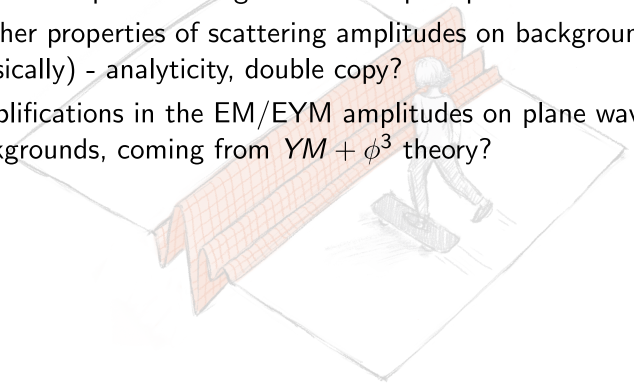


Summary

- We propose natural objects to study in the context of self-force: amplitudes on backgrounds and classical observables built from them
- We constructed the 3-point amplitude in Schwarzschild, schemetically using the exact solutions, and explicitly using WKB and weak-field approximations
- We calculated the waveform on a gravitational planewave, with full non-linear contributions from the background
- We looked at introducing charged matter, in an Einstein-Maxwell plane background

Outlook

- More work needed to connect these amplitudes explicitly to the self-force expansion - e.g. how do loop amplitudes contribute?
- Further properties of scattering amplitudes on backgrounds (not just classically) - analyticity, double copy?
- Simplifications in the EM/EYM amplitudes on plane wave backgrounds, coming from $YM + \phi^3$ theory?



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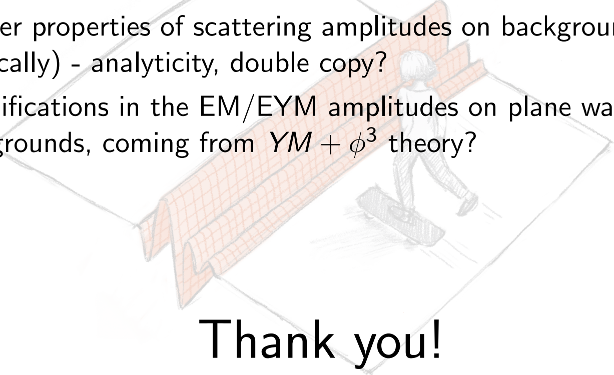

Thank you!