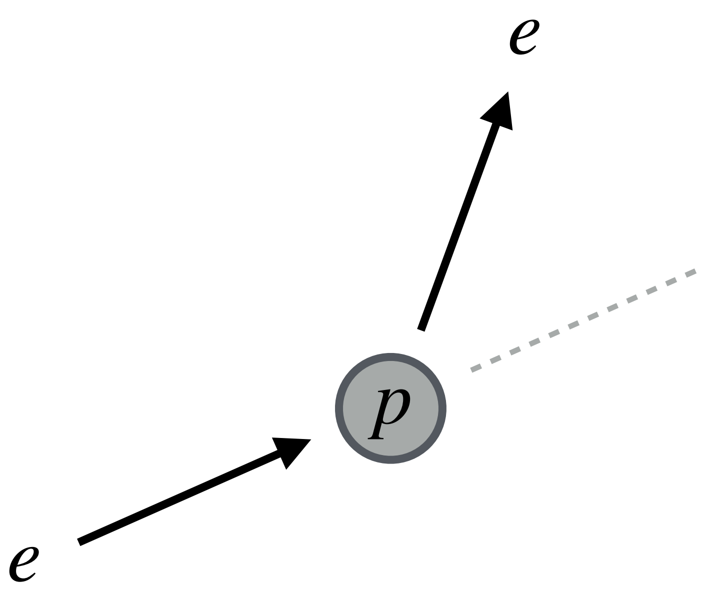
An Effective Field Theory For Extreme Mass Ratios

Clifford Cheung
Caltech

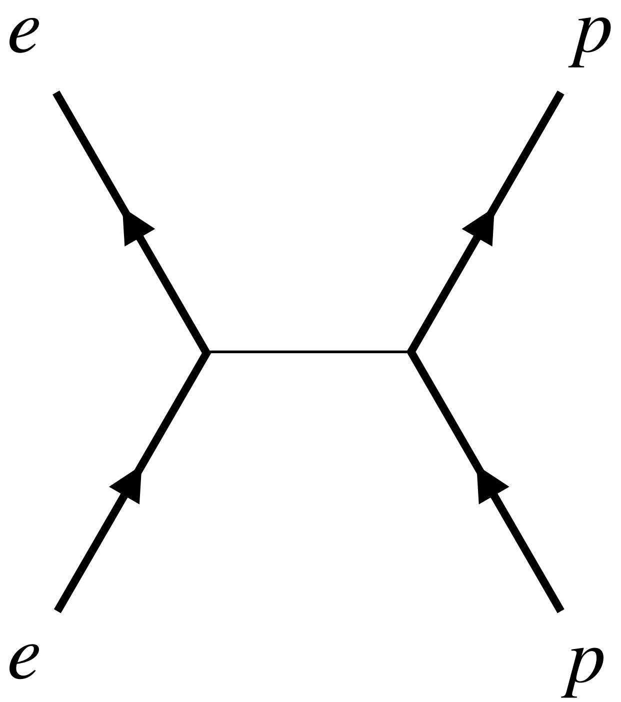
2308.14832 (CC, Parra-Martinez, Rothstein, Shah, Wilson-Gerow)
2404.xxxxx (CC, Parra-Martinez, Rothstein, Shah, Wilson-Gerow)

Two ways to approach electron-proton dynamics:

i) probe in $1/r$ background



ii) perturbation theory



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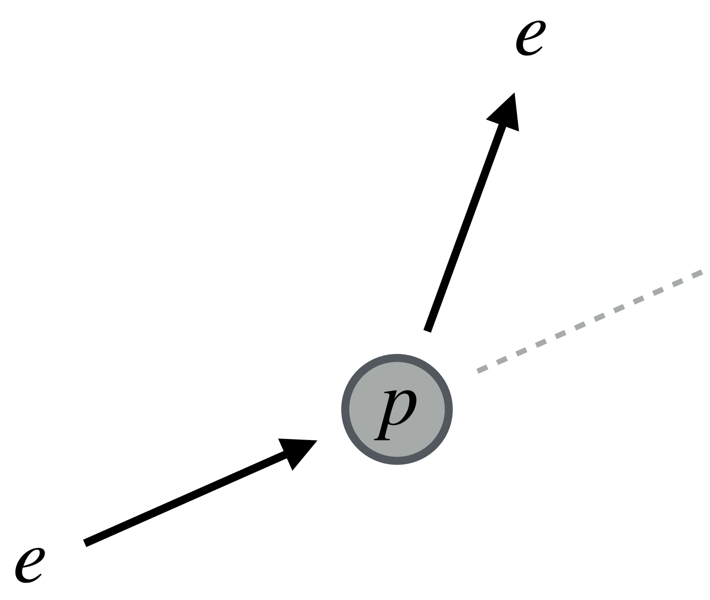

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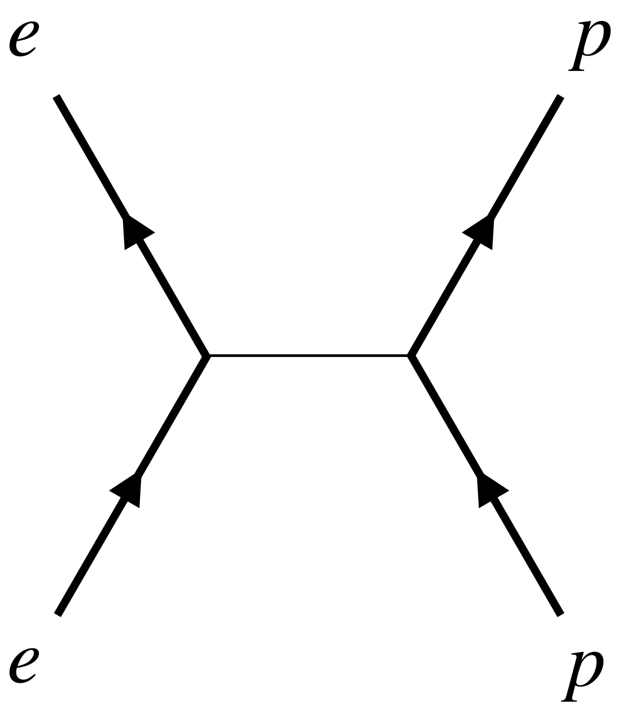


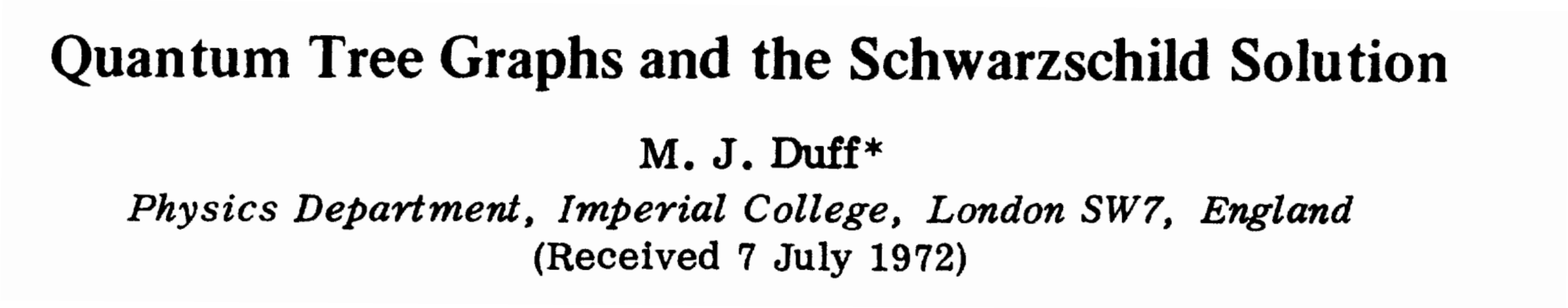
When does background field theory end and perturbation theory on a trivial vacuum begin?

The breakdown of the background field theory is signaled by the importance of corrections that:

- enter as powers of $\lambda = m_L/m_H$
- encode heavy particle “wobble”
- are perturbatively calculable

As in HQET or NRQCD, we integrate out the heavy state, but we also keep all orders in α data!

Schwarzschild metric describes the $\mathcal{O}(\lambda^0)$ limit.



Quantum Tree Graphs and the Schwarzschild Solution

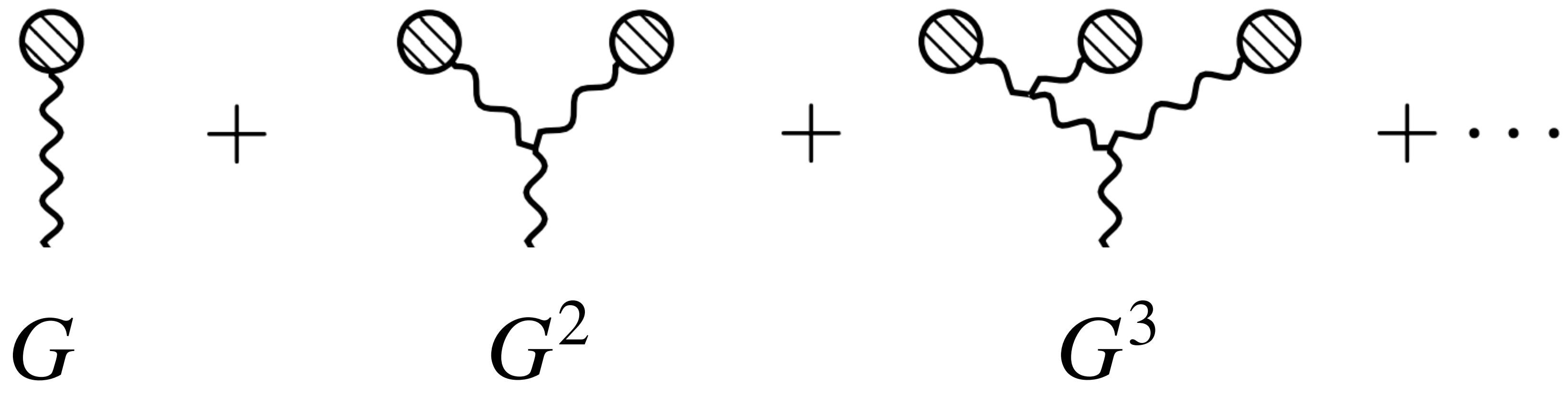
M. J. Duff*
Physics Department, Imperial College, London SW7, England

(Received 7 July 1972)

Schwarzschild metric describes the $\mathcal{O}(\lambda^0)$ limit.

static matter source

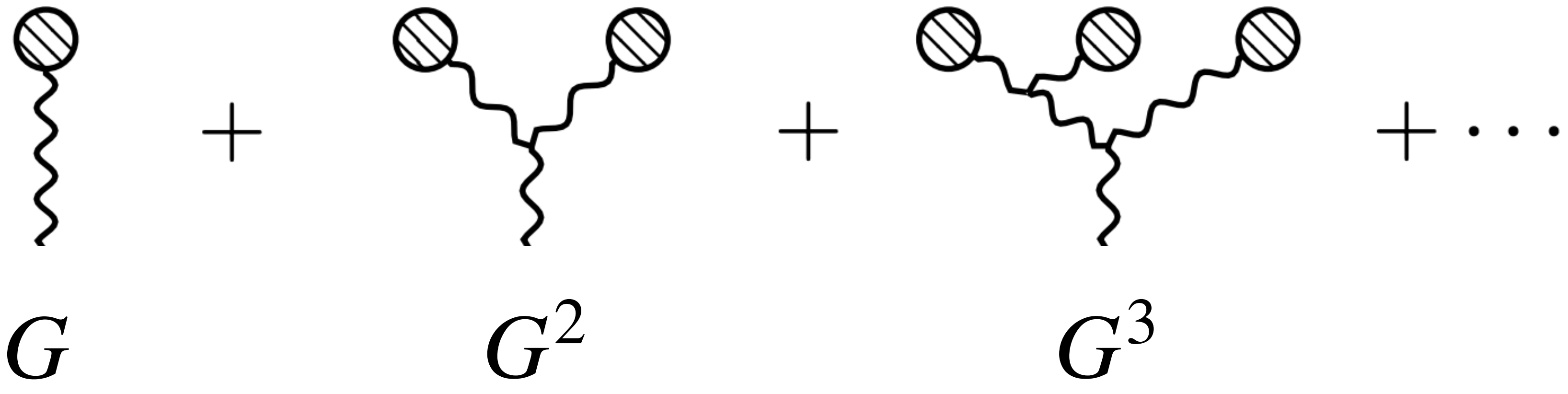
$g^{00} + 1 =$



graviton 1pt function

Schwarzschild metric describes the $\mathcal{O}(\lambda^0)$ limit.

$$g^{00} + 1 =$$



$$g^{00} = -1 - \frac{2GM}{r} - \frac{2G^2M^2}{r^2} + \cdots = -\frac{r+GM}{r-GM}$$

all orders
in coupling!

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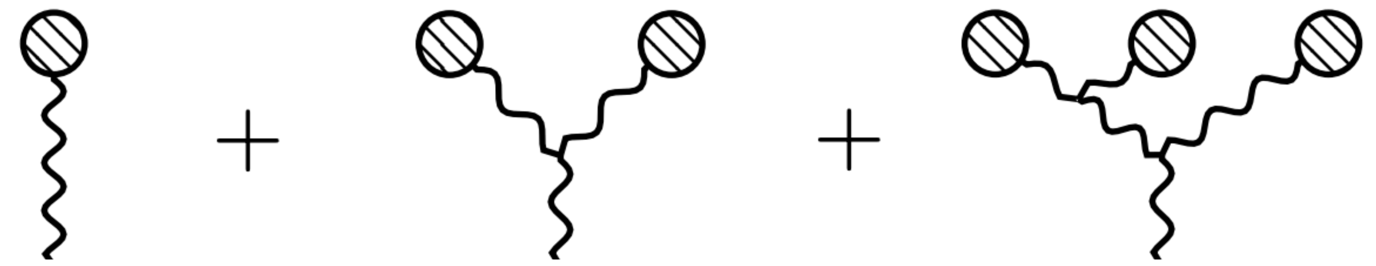

$+\cdots$

$$g^{00} = -1 - \frac{2GM}{r} - \frac{2G^2M^2}{r^2} + \cdots = -\frac{r+GM}{r-GM}$$

All orders in coupling is rare in perturbative QFT but commonplace in classical GR.

Probe geodesics also encode the $\mathcal{O}(\lambda^0)$ limit.

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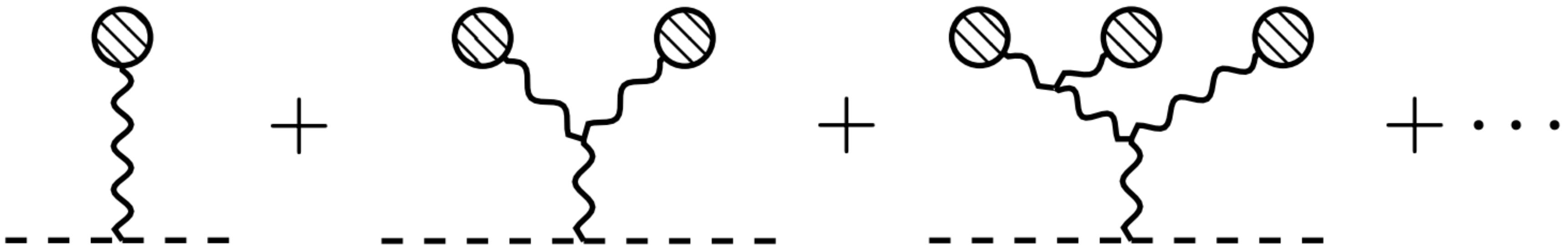
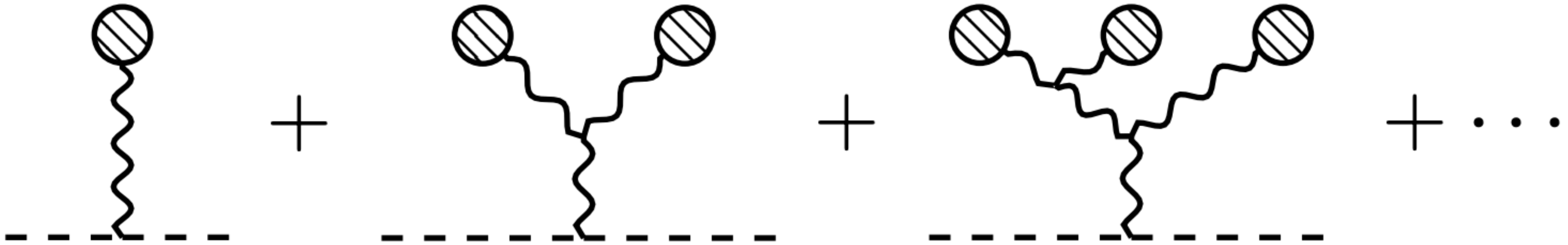
$x(\tau) \sim$



matter
worldline

Probe geodesics also encode the $\mathcal{O}(\lambda^0)$ limit.

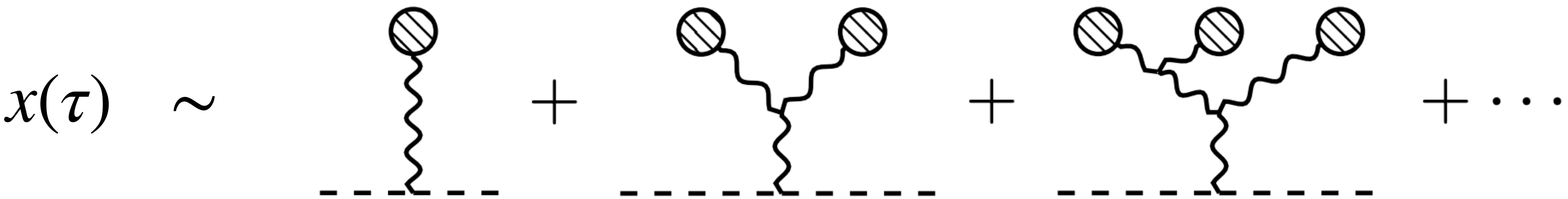
$x(\tau) \sim$



Flat space perturbation theory builds the metric and probe geodesics. Let's do the opposite.

flat space loops	$\Rrightarrow$	metric + geodesic
(hard)	(Duff)	(easy)

Probe geodesics also encode the $\mathcal{O}(\lambda^0)$ limit.



Flat space perturbation theory builds the metric and probe geodesics. Let's do the opposite.

flat space loops	⇚	metric + geodesic
(hard)	(this talk)	(easy)

Our goal: to maximally leverage all-orders-in-G classical results to flat space perturbation theory.

exact answer = background field method + everything else

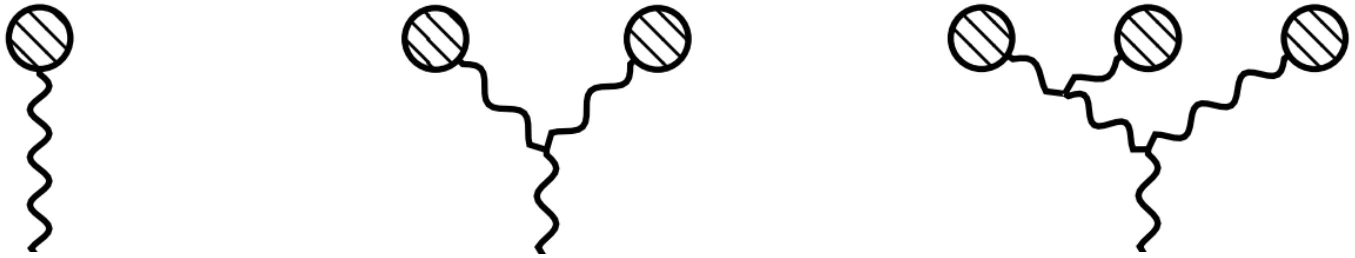
(Schwarzschild metric, geodesic trajectories)

("recoil operator")

Obviously this is just a reshuffling of perturbation theory. But metrics + geodesics nicely encode loop integrand contributions automatically!

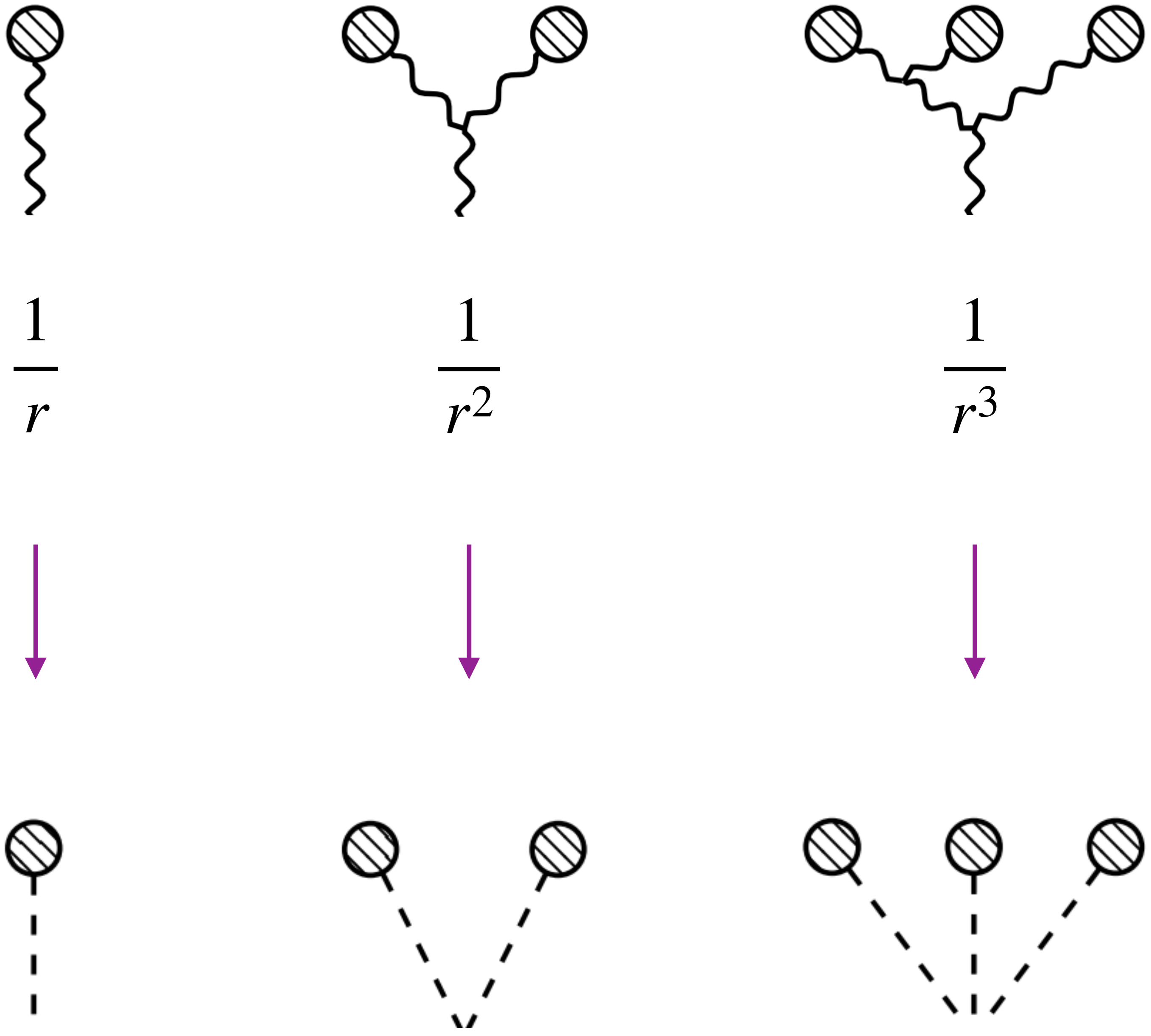
Classical solutions implement integral reduction.

tensor
numerators



$$\frac{1}{r} \qquad \frac{1}{r^2} \qquad \frac{1}{r^3}$$

Classical solutions implement integral reduction.



tensor
numerators

$$\frac{1}{r} \qquad \frac{1}{r^2} \qquad \frac{1}{r^3}$$

scalar
numerators

1. Electromagnetism

To warmup, take EM with a light + heavy particle,

$$S_{\text{EM}} = \int d^4x \left[-\frac{1}{4}F^2\right] + \sum_{i=L,H} m_i \int d\tau \left[-\frac{1}{2}\dot{x}_i^2 - z_i \dot{x}_i^\mu A_\mu\right]$$

where we fix charge-to-mass-ratios $z_i = q_i/m_i$ to maximally parallel GR. The SF parameter is:

$$\lambda = \frac{m_L}{m_H}$$

Expand about a $\mathcal{O}(\lambda^0)$ background and trajectory.

$$x_i^\mu = \underset{\nearrow \atop \text{closed-form trajectory}}{\overset{\mathcal{O}(\lambda^0)}{\bar{x}_i^\mu}} + \underset{\nwarrow \atop \text{deflection}}{\delta x_i^\mu}$$

$$A_\mu = \underset{\nearrow \atop \sim 1/r}{\overset{\mathcal{O}(\lambda^0)}{\bar{A}_\mu}} + \underset{\nwarrow \atop \text{photon}}{\delta A_\mu}$$

From full theory to background field theory:

$$S_{\mathrm{EM}}[\delta A, \delta x_L, \delta x_H] \xrightarrow[\text{integrate out heavy particle!}]{} S_{\mathrm{BF}}[\delta A, \delta x_L]$$

Now consider some explicit 0SF backgrounds.

- heavy trajectory $\bar{x}_H^\mu(\tau) = u_H^\mu \tau$

- EM field $\bar{A}_\mu(x) = \dfrac{z_H m_H u_{H\mu}}{4\pi r} \quad \text{s.t.} \quad r = \sqrt{(u_H x)^2 - x^2}$

- light trajectory $\bar{x}_L^\mu(\tau) \quad \text{s.t.} \quad \ddot{\bar{x}}_L^\mu(\tau) - z_L \bar{F}^{\mu\nu}(\bar{x}_L)\dot{\bar{x}}_{L\nu} = 0$

Perturbative contributions from EM field:

$$\bar{A}_\mu(x) = \frac{z_H m_H u_{H\mu}}{4\pi r} \quad \rightarrow \quad \tilde{A}_\mu(p) = \frac{z_H m_H u_{H\mu}}{p^2}\delta(u_H p)$$

What about the self-energy contributions?

$$\bar{A}_\mu(\bar{x}_H) \quad \sim \quad \lim_{r\to 0} \frac{1}{r^{1-\epsilon}} \quad \sim \quad \lim_{r\to 0} \int d^{3-\epsilon}q \, \frac{e^{iqr}}{q^2}$$

$$\rightarrow \quad \int d^{3-2\epsilon}q \, \frac{1}{q^2} \quad = \quad 0$$

dimensional
regulator

Perturbative contributions from EM field:

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$$\rightarrow \quad \bar{A}_\mu(\bar{x}_H) \quad = \quad \bar{F}_{\mu\nu}(\bar{x}_H) = 0$$

Terms in the action relevant to $\mathcal{O}(\lambda^1)$ are given by:

$$S_{\rm EM} = m_H \int d\tau \left[-\frac{1}{2}\delta\dot{x}_H^2 - z_H \delta x_H^\mu \dot{\bar{x}}_H^\nu \delta F_{\mu\nu}(\bar{x}_H) + \cdots \right]$$

$$+ \int d^4x \left[-\frac{1}{4}\delta F_{\mu\nu}\delta F^{\mu\nu} - \delta A_\mu \bar{J}_L^\mu \right]$$

with $\mathcal{O}(\lambda^1)$ current $\bar{J}_L^\mu(x) = z_L m_L \int d\tau\, \delta^4(x - \bar{x}_L)\dot{\bar{x}}_L^\mu$

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Integrating out δx_H yields the “recoil operator”.

$$m_H \int d\tau \left[-\frac{1}{2}\delta\dot{x}_H^2 - z_H \delta x_H^\mu \dot{\bar{x}}_H^\nu \delta F_{\mu\nu}(\bar{x}_H)\right]$$

$$\downarrow\downarrow\downarrow$$

$$S_{\text{recoil}} = -\frac{1}{2} z_H^2 m_H \int d\tau\, \dot{\bar{x}}_H^\alpha \delta F_{\alpha\mu}(\bar{x}_H)\frac{1}{\partial_\tau^2}\dot{\bar{x}}_H^\beta \delta F_{\beta}{}^{\mu}(\bar{x}_H)$$

$$\delta F_{\mu\nu} = F_{\mu\nu} - \bar{F}_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu$$

In summary, the EFT action for 1SF dynamics is

$$S_{\mathrm{EFT}} \quad = \quad S_{\mathrm{BF}} \quad + \quad S_{\mathrm{recoil}}$$

$$S_{\mathrm{BF}} = \int d^4x \left[-\tfrac{1}{4}\delta F_{\mu\nu}\delta F^{\mu\nu} - \delta A_\mu \bar{J}^\mu_L \right]$$

$$S_{\mathrm{recoil}} = -\frac{z_H^2 m_H}{2}\int d\tau\, \dot{\bar{x}}^\alpha_H \delta F_{\alpha\mu}(\bar{x}_H)\frac{1}{\partial_\tau^2}\dot{\bar{x}}^\beta_H \delta F_\beta{}^\mu(\bar{x}_H)$$

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$$S_{\text{BF}} = \int d^4x \left[-\frac{1}{4} \delta F_{\mu\nu} \delta F^{\mu\nu} - \delta A_\mu \bar{J}^\mu_L \right]$$

propagator

source

$$S_{\text{recoil}} = -\frac{z_H^2 m_H}{2} \int d\tau\, \dot{\bar{x}}_H^\alpha \delta F_{\alpha\mu}(\bar{x}_H) \frac{1}{\partial_\tau^2} \dot{\bar{x}}_H^\beta \delta F_\beta{}^\mu(\bar{x}_H)$$

recoil correction
to propagator

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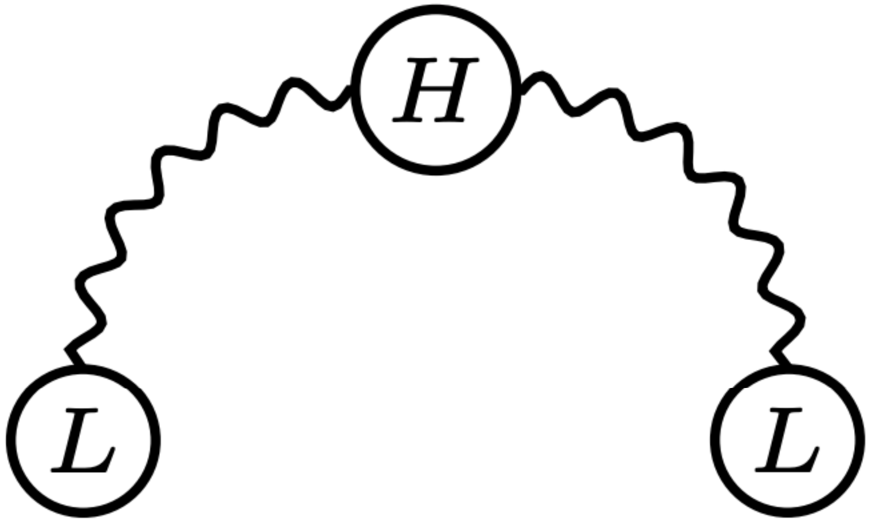

1SF contribution to
conservative radial action

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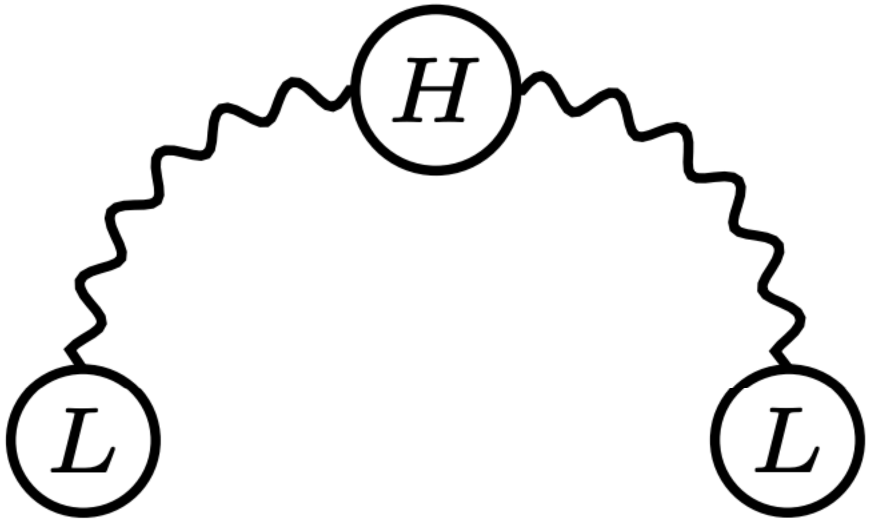

all orders in α
from trajectory

1SF contribution to
conservative radial action

The EFT Feynman rules for EM are given by:

$$= -\frac{i\eta_{\mu\nu}}{p^2}$$

propagator

$$= \lambda z_L m_H \int d\tau\, e^{-ip\bar{x}_L} \dot{\bar{x}}_L^{\mu}$$

source

$$= i z_H^2 m_H \frac{\delta(u_H p_1 + u_H p_2)}{(u_H p_1)(u_H p_2)} \mathcal{O}^{\alpha\mu_1}(u_H, p_1) \mathcal{O}_{\alpha}{}^{\mu_2}(u_H, p_2)$$

recoil

$$\mathcal{O}^{\alpha\mu}(u,p) = \eta^{\alpha\mu}(up) - u^{\mu}p^{\alpha}$$

The EFT Feynman rules for EM are given by:

$$= -\frac{i\eta_{\mu\nu}}{p^2}$$

propagator

probe trajectory encodes integrand

$$= \lambda z_L m_H \int d\tau\, e^{-ip\bar{x}_L}\dot{\bar{x}}_L^\mu$$

source

$$= i z_H^2 m_H \frac{\delta(u_H p_1 + u_H p_2)}{(u_H p_1)(u_H p_2)} \mathcal{O}^{\alpha\mu_1}(u_H, p_1)\mathcal{O}_\alpha{}^{\mu_2}(u_H, p_2)$$

recoil

$$\mathcal{O}^{\alpha\mu}(u,p) = \eta^{\alpha\mu}(up) - u^\mu p^\alpha$$

Starting from the EM light particle trajectory...

$$\bar{x}^\mu_L = \sum_{k=0}^{\infty} \bar{x}^\mu_k \longleftarrow$$

expansion in
fine structure
constant α

Starting from the EM light particle trajectory...

$$\bar{x}^\mu_L = \sum_{k=0}^{\infty} \bar{x}^\mu_k$$

$$\hat{x}^\mu_0 = \hat{b}^\mu + \frac{\hat{\tau}}{v}\check{u}^\mu_H + \frac{\hat{\tau}\sqrt{1-v^2}}{v}\check{u}^\mu_L$$

$$\hat{x}^\mu_1 = \hat{\alpha}\left(\operatorname{asinh}\hat{\tau}\,\check{u}^\mu_H - \frac{\hat{r}_0(\hat{\tau})}{v}\hat{b}^\mu\right)$$

$$\hat{x}^\mu_2 = \hat{\alpha}^2\left(\frac{\operatorname{asinh}\hat{\tau}}{v\hat{r}_0(\hat{\tau})}\check{u}^\mu_H + \frac{\hat{\tau} - v^2\operatorname{atan}\hat{\tau}}{2v\sqrt{1-v^2}}\check{u}^\mu_L - \frac{2\hat{\tau}\operatorname{asinh}\hat{\tau} + \hat{r}_0(\hat{\tau})\left(\hat{\tau}v^2\operatorname{atan}\hat{\tau} + v^2 - 1\right)}{2v^2\hat{r}_0(\hat{\tau})}\hat{b}^\mu\right)$$

Starting from the EM light particle trajectory...

$$\bar{x}_L^\mu = \sum_{k=0}^{\infty} \bar{x}_k^\mu$$

proper time domain

↓

$$\hat{x}_0^\mu = \hat{b}^\mu + \frac{\hat{\tau}}{v}\check{u}_H^\mu + \frac{\hat{\tau}\sqrt{1-v^2}}{v}\check{u}_L^\mu$$

$$\hat{x}_1^\mu = \hat{\alpha}\left(\operatorname{asinh}\hat{\tau}\,\check{u}_H^\mu - \frac{\hat{r}_0(\hat{\tau})}{v}\hat{b}^\mu\right)$$

$$\hat{x}_2^\mu = \hat{\alpha}^2\left(\frac{\operatorname{asinh}\hat{\tau}}{v\hat{r}_0(\hat{\tau})}\check{u}_H^\mu + \frac{\hat{\tau} - v^2\operatorname{atan}\hat{\tau}}{2v\sqrt{1-v^2}}\check{u}_L^\mu - \frac{2\hat{\tau}\operatorname{asinh}\hat{\tau} + \hat{r}_0(\hat{\tau})\left(\hat{\tau}v^2\operatorname{atan}\hat{\tau} + v^2 - 1\right)}{2v^2\hat{r}_0(\hat{\tau})}\hat{b}^\mu\right)$$

$$\hat{x}_k^\mu = \bar{x}_k^\mu/b \qquad \hat{\tau} = (\sigma^2-1)^{1/2}\tau/b \qquad \hat{r}_0(\hat{\tau}) = \sqrt{1+\hat{\tau}^2}$$

$$\check{u}_H^\mu = \frac{\sigma u_L - u_H}{\sigma^2 - 1} \qquad \check{u}_L^\mu = \frac{\sigma u_H - u_L}{\sigma^2 - 1} \qquad \hat{\alpha} = \alpha/b m_L(\sigma^2-1)^{1/2} \qquad \sigma = \frac{1}{\sqrt{1-v^2}}$$

…then by applying simple identities such as

$$\operatorname{asinh}(\hat{\tau}) = \frac{1}{\partial_{\hat{\tau}}}\left(\frac{1}{\hat{r}_0(\hat{\tau})}\right) \sim \frac{1}{\omega}\frac{1}{p^2}$$

yields EM integrands terms from trajectories:

$$\bar{x}_i^\mu = \frac{1}{\partial_{\hat{\tau}}^2}\left(\cdots\frac{1}{\partial_{\hat{\tau}}^2}\left(\cdots\frac{1}{\partial_{\hat{\tau}}^2}\left(\cdots\frac{1}{\partial_{\hat{\tau}}^2}(\cdots)\right)\right)\right)$$

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$$\bar{x}_i^{\mu} = \frac{1}{\partial_{\hat{\tau}}^2}\left(\cdots\frac{1}{\partial_{\hat{\tau}}^2}\left(\cdots\frac{1}{\partial_{\hat{\tau}}^2}\left(\cdots\frac{1}{\partial_{\hat{\tau}}^2}(\cdots)\right)\right)\right)$$

freedom here!

11. General Relativity

Now consider GR with a light + heavy particle,

$$S_{\rm GR} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \sum_{i=L,H}\frac{1}{2}m_i\int d\tau\, \dot{x}_i^\mu \dot{x}_i^\nu g_{\mu\nu}(x_i)$$

after gauge fixing the einbein to one. As before, mass ratio defines the SF expansion parameter:

$$\lambda = \frac{m_L}{m_H}$$

Expand about a $\mathcal{O}(\lambda^0)$ background and trajectory.

$$x_i^\mu = \underset{\substack{\nearrow \\ \text{probe geodesic}}}{\overset{\mathcal{O}(\lambda^0)}{\bar{x}_i^\mu}} + \underset{\substack{\nwarrow \\ \text{deviation}}}{\delta x_i^\mu}$$

$$g_{\mu\nu} = \underset{\substack{\nearrow \\ \text{Schwarzschild}}}{\overset{\mathcal{O}(\lambda^0)}{\bar{g}_{\mu\nu}}} + \underset{\substack{\nwarrow \\ \text{graviton}}}{\delta g_{\mu\nu}}$$

From full theory to background field theory:

$$S_{\mathrm{GR}}[\delta g, \delta x_L, \delta x_H] \xrightarrow[\substack{\text{integrate out} \\ \text{heavy particle!}}]{\qquad\qquad} S_{\mathrm{BF}}[\delta g, \delta x_L]$$

Now consider some explicit 0SF backgrounds.

- heavy trajectory $\bar{x}^\mu_H(\tau) = u^\mu_H \tau$

- metric $$\bar{g}_{\mu\nu}(x) = f_+(r)^4 \eta_{\mu\nu} + \left[\frac{f_-(r)^2}{f_+(r)^2} - f_+(r)^4 \right] u_{H\mu} u_{H\nu}$$

$$f_\pm(r) = 1 \pm \frac{r_S}{4r}$$

- light trajectory $\bar{x}^\mu_L(\tau) \quad \text{s.t.} \quad \ddot{\bar{x}}^\mu_L(\tau) + \bar{\Gamma}^\mu{}_{\alpha\beta}(\bar{x}_L)\dot{\bar{x}}^\alpha_L \dot{\bar{x}}^\beta_L = 0$

Perturbative terms from Schwarzschild metric:

$$\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \bar{\gamma}_{\mu\nu}(x)$$

$$\bar{\gamma}_{\mu\nu}(x) = \frac{r_S}{r}(\eta_{\mu\nu} - 2u_{H\mu}u_{H\nu}) + \frac{1}{8}\left(\frac{r_S}{r}\right)^2(3\eta_{\mu\nu} + u_{H\mu}u_{H\nu}) + \cdots$$

$$\begin{aligned}\tilde{\gamma}_{\mu\nu}(p) = &-\frac{4\pi r_S}{p^2}(\eta_{\mu\nu} - 2u_{H\mu}u_{H\nu})\delta(u_H p) \\ &-\frac{2\pi r_S^2}{\sqrt{-p^2}}(3\eta_{\mu\nu} + u_{H\mu}u_{H\nu})\delta(u_H p) + \cdots\end{aligned}$$

Terms in the action relevant to $\mathcal{O}(\lambda^1)$ are given by:

$$S_{\mathrm{GR}} = m_H \int d\tau \left[-\frac{1}{2}\delta\dot{x}_H^2 + \delta x_H^\rho \dot{\bar{x}}_H^\mu \dot{\bar{x}}_H^\nu \delta\Gamma_{\rho\mu\nu}(\bar{x}_H) + \cdots \right]$$

$$+ \int d^4x \sqrt{-\bar{g}} \left[\mathcal{O}(\delta g^2) - \frac{1}{2}\delta g_{\mu\nu}\bar{T}_L^{\mu\nu} \right]$$

with $\mathcal{O}(\lambda^1)$ current $\bar{T}_L^{\mu\nu}(x) = m_L \int d\tau \, \frac{\delta^4(x-\bar{x}_L)}{\sqrt{-\bar{g}}} \dot{\bar{x}}_L^\mu \dot{\bar{x}}_L^\nu$

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$$S_{\mathrm{GR}} = m_H \int d\tau \left[\overset{\text{Gaussian in } \delta x_H}{\boxed{-\frac{1}{2}\delta \dot{x}_H^2 + \delta x_H^\rho \dot{\bar{x}}_H^\mu \dot{\bar{x}}_H^\nu \delta\Gamma_{\rho\mu\nu}(\bar{x}_H)}} + \cdots \right]$$

$$+ \int d^4x \sqrt{-\bar{g}} \left[\mathcal{O}(\delta g^2) - \frac{1}{2}\delta g_{\mu\nu} \bar{T}_L^{\mu\nu} \right]$$

with $\mathcal{O}(\lambda^1)$ current $\bar{T}_L^{\mu\nu}(x) = m_L \int d\tau \, \dfrac{\delta^4(x - \bar{x}_L)}{\sqrt{-\bar{g}}} \dot{\bar{x}}_L^\mu \dot{\bar{x}}_L^\nu$

Integrating out δx_H yields the "recoil operator".

$$m_H \int d\tau \left[-\tfrac{1}{2} \delta \dot{x}_H^2 + \delta x_H^\rho \dot{\bar{x}}_H^\mu \dot{\bar{x}}_H^\nu \delta \Gamma_{\rho\mu\nu}(\bar{x}_H) \right]$$

$$\downarrow\downarrow\downarrow$$

$$S_{\text{recoil}} = -\tfrac{1}{2} m_H \int d\tau \, \dot{\bar{x}}_H^\alpha \dot{\bar{x}}_H^\beta \delta \Gamma^\mu{}_{\alpha\beta}(\bar{x}_H) \frac{1}{\partial_\tau^2} \dot{\bar{x}}_H^\gamma \dot{\bar{x}}_H^\delta \delta \Gamma_{\mu\gamma\delta}(\bar{x}_H)$$

$$\delta \Gamma_{\rho\mu\nu} = \Gamma_{\rho\mu\nu} - \bar{\Gamma}_{\rho\mu\nu} = \tfrac{1}{2}(\partial_\mu \delta g_{\nu\rho} + \partial_\nu \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\nu})$$

In summary, the EFT action for 1SF dynamics is

$$S_{\mathrm{EFT}} \quad = \quad S_{\mathrm{BF}} \quad + \quad S_{\mathrm{recoil}}$$

$$S_{\mathrm{BF}} = \int d^4x \sqrt{-\bar{g}} \left[\mathcal{O}(\delta g^2) - \frac{1}{2} \delta g_{\mu\nu} \bar{T}_L^{\mu\nu} \right]$$

$$S_{\mathrm{recoil}} = -\frac{1}{2} m_H \int d\tau \, \dot{\bar{x}}_H^{\alpha} \dot{\bar{x}}_H^{\beta} \delta\Gamma^{\mu}{}_{\alpha\beta}(\bar{x}_H) \frac{1}{\partial_\tau^2} \dot{\bar{x}}_H^{\gamma} \dot{\bar{x}}_H^{\delta} \delta\Gamma_{\mu\gamma\delta}(\bar{x}_H)$$

In summary, the EFT action for 1SF dynamics is

$$S_{\text{EFT}} \quad = \quad S_{\text{BF}} \quad + \quad S_{\text{recoil}}$$

$$S_{\text{BF}} = \int d^4x \sqrt{-\bar{g}} \left[\mathcal{O}(\delta g^2) - \tfrac{1}{2} \delta g_{\mu\nu} \bar{T}^{\mu\nu}_L \right]$$

recall that
$\bar{g}_{\mu\nu}(\bar{x}_H) = \eta_{\mu\nu}$
in dim reg

$$S_{\text{recoil}} = -\tfrac{1}{2} m_H \int d\tau \, \dot{\bar{x}}^{\alpha}_H \dot{\bar{x}}^{\beta}_H \delta\Gamma^{\mu}{}_{\alpha\beta}(\bar{x}_H) \frac{1}{\partial_\tau^2} \dot{\bar{x}}^{\gamma}_H \dot{\bar{x}}^{\delta}_H \delta\Gamma_{\mu\gamma\delta}(\bar{x}_H)$$

In summary, the EFT action for 1SF dynamics is

$$S_{\rm EFT} \quad = \quad S_{\rm BF} \quad + \quad S_{\rm recoil}$$

$$S_{\rm BF} = \int d^4x \sqrt{-\bar{g}} \left[\mathcal{O}(\delta g^2) - \frac{1}{2} \delta g_{\mu\nu} \bar{T}^{\mu\nu}_{L} \right]$$

$$S_{\rm recoil} = -\frac{1}{2} m_H \int d\tau \, \dot{\bar{x}}^{\alpha}_{H} \dot{\bar{x}}^{\beta}_{H} \delta\Gamma^{\mu}{}_{\alpha\beta}(\bar{x}_H) \frac{1}{\partial^2_\tau} \dot{\bar{x}}^{\gamma}_{H} \dot{\bar{x}}^{\delta}_{H} \delta\Gamma_{\mu\gamma\delta}(\bar{x}_H)$$

In summary, the EFT action for 1SF dynamics is

$$S_{\text{EFT}} = S_{\text{BF}} + S_{\text{recoil}}$$

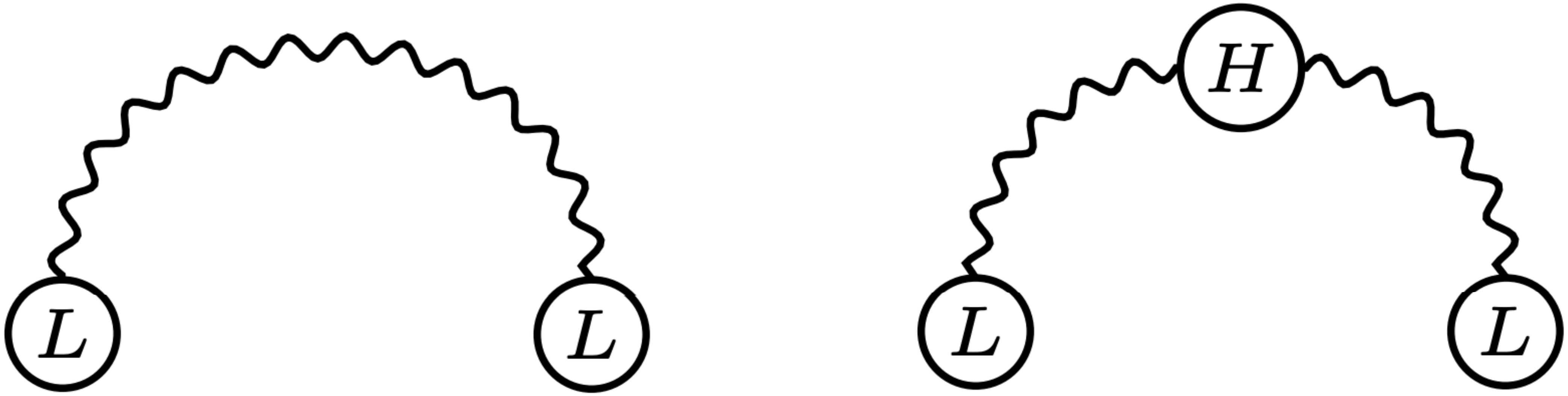
curved space propagator

source

recoil correction to propagator

In summary, the EFT action for 1SF dynamics is

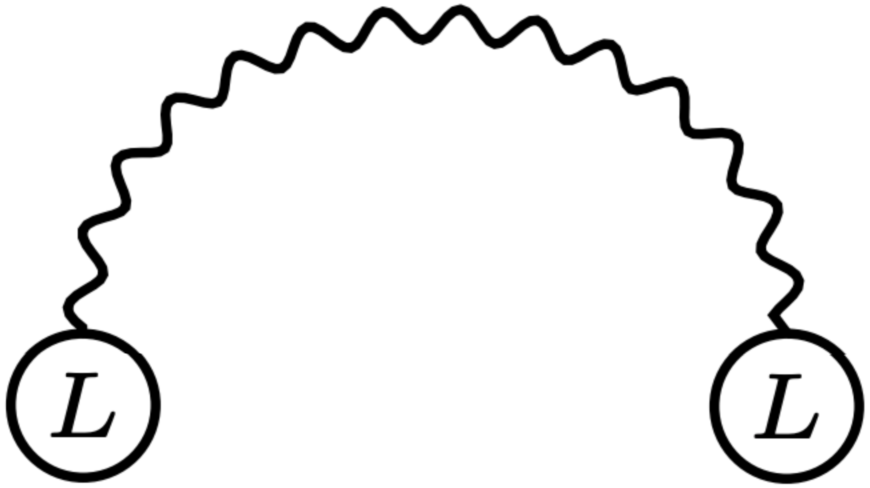
$$S_{\mathrm{EFT}} = S_{\mathrm{BF}} + S_{\mathrm{recoil}}$$

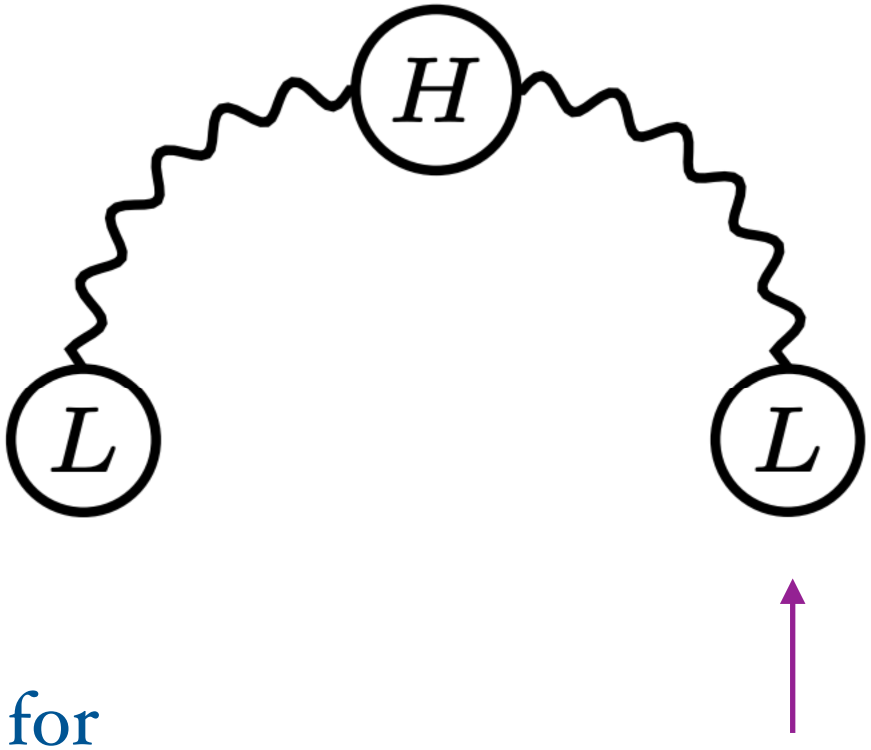


1SF diagrams for
conservative radial action

In summary, the EFT action for 1SF dynamics is

$$S_{\mathrm{EFT}} \quad = \quad S_{\mathrm{BF}} \quad + \quad S_{\mathrm{recoil}}$$





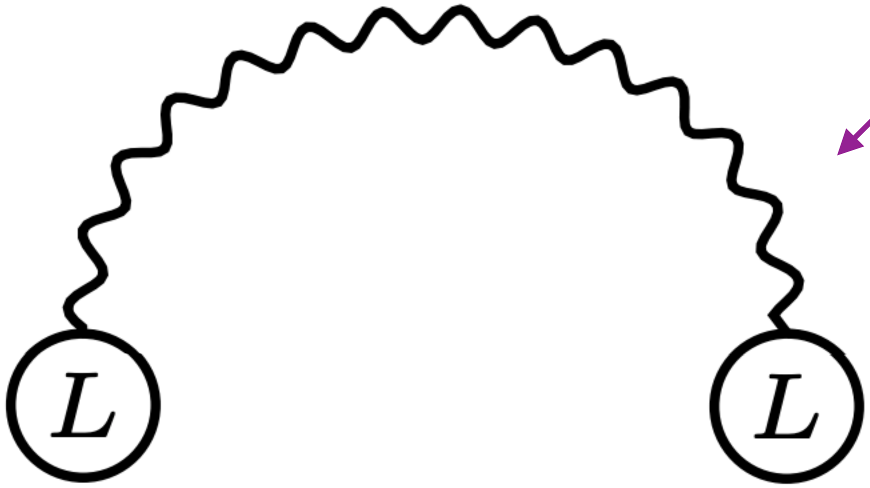
1SF diagrams for
conservative radial action

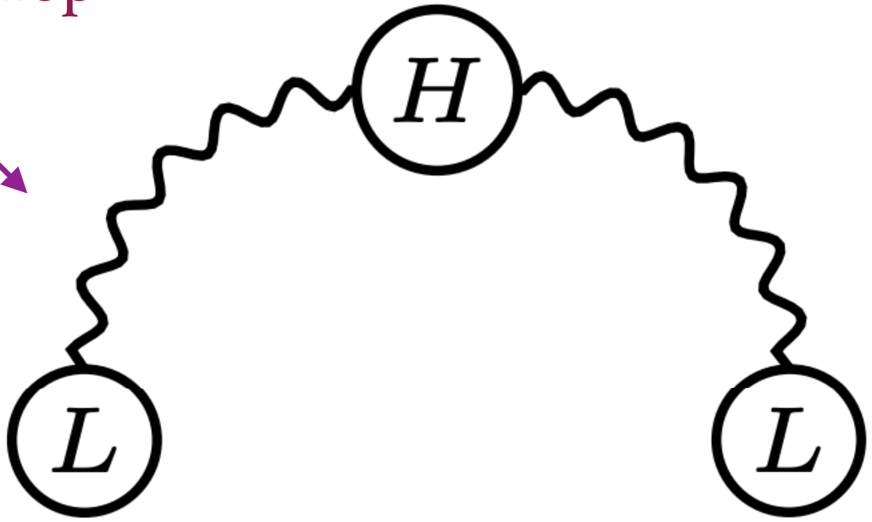
all orders in G
from trajectory

In summary, the EFT action for 1SF dynamics is

$$S_{\text{EFT}} \quad = \quad S_{\text{BF}} \quad + \quad S_{\text{recoil}}$$

all orders in G from
Schwarzschild prop





1SF diagrams for
conservative radial action

all orders in G
from trajectory

The EFT Feynman rules for GR are given by:

$$= \frac{32\pi i G}{p^2}\left(\frac{\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}}{2}-\frac{\eta_{\mu\nu}\eta_{\rho\sigma}}{2}\right)$$ + PM corrections

propagator

$$= \lambda m_H \int d\tau\, e^{-ip\bar{x}_L}\dot{x}_L^{\mu}\dot{x}_L^{\nu}$$

source

$$= \frac{i m_H}{2}\frac{\delta(u_H p_1 + u_H p_2)}{(u_H p_1)(u_H p_2)}\mathcal{O}^{\alpha\mu_1\nu_1}(u_H, p_1)\mathcal{O}_{\alpha}{}^{\mu_2\nu_2}(u_H, p_2)$$

recoil

$$\mathcal{O}^{\alpha\mu\nu}(u,p) = \tfrac{1}{2}\left((u^\mu\eta^{\nu\alpha}+u^\nu\eta^{\mu\alpha})(up) - u^\mu u^\nu p^\alpha\right)$$

Loop integrands via GR light particle trajectory:

$$\bar{x}^\mu_L = \sum_{k=0}^{\infty} \bar{x}^\mu_k \longleftarrow$$

expansion in
Schwarzschild
radius r_S

Loop integrands via GR light particle trajectory:

$$\bar{x}_L^\mu = \sum_{k=0}^{\infty} \bar{x}_k^\mu$$

$$\hat{x}_0^\mu = \hat{b}^\mu + \frac{\hat{\tau}}{v}\check{u}_H^\mu + \frac{\hat{\tau}\sqrt{1-v^2}}{v}\check{u}_L^\mu$$

$$\hat{x}_1^\mu = \frac{r_S}{bv^2}\left(- \frac{\left(v^2+1\right)\hat{r}_0(\hat{\tau})}{2}\hat{b}^\mu + v\mathrm{asinh}\,\hat{\tau}\check{u}_H^\mu + \frac{v\left(v^2+1\right)\mathrm{asinh}\,\hat{\tau}}{2\sqrt{1-v^2}}\check{u}_L^\mu \right)$$

$$\begin{aligned}\hat{x}_2^\mu = \left(\frac{r_S}{bv^2}\right)^2 \Bigg(&\left[\frac{3v^3\,\mathrm{atan}\,\hat{\tau}}{2} - \frac{v\left(v^2-1\right)\mathrm{asinh}\,\hat{\tau}}{2\hat{r}_0(\hat{\tau})}\right]\check{u}_H^\mu \\ &+ \left[\frac{\sqrt{1-v^2}v\left(v^2+1\right)\mathrm{asinh}\,\hat{\tau}}{4\hat{r}_0(\hat{\tau})} + \frac{2\hat{\tau}v\left(v^2+1\right)^2 + 3\left(v^2+4\right)v^3\mathrm{atan}\,\hat{\tau}}{16\sqrt{1-v^2}}\right]\check{u}_L^\mu \\ &+ \left[\frac{\hat{\tau}\left(v^4-1\right)\mathrm{asinh}\,\hat{\tau}}{4\hat{r}_0(\hat{\tau})} - \frac{3\hat{\tau}\left(v^2+4\right)v^2\mathrm{atan}\,\hat{\tau} + v^4 + 8v^2 - 2}{16}\right]\hat{b}^\mu \Bigg)\end{aligned}$$

Loop integrands via GR light particle trajectory:

$$\bar{x}_L^\mu = \sum_{k=0}^{\infty} \bar{x}_k^\mu$$

rewrite as worldline propagators

$$\operatorname{asinh}\hat{\tau} = \frac{1}{\partial_{\hat{\tau}}}\partial_{\hat{\tau}}\operatorname{asinh}\hat{\tau} = \frac{1}{\partial_{\hat{\tau}}}\left(\frac{1}{(1+\hat{\tau}^2)^{1/2}}\right)$$

$$\hat{x}_0^\mu = \hat{b}^\mu + \frac{\hat{\tau}}{v}\check{u}_H^\mu + \frac{\hat{\tau}\sqrt{1-v^2}}{v}\check{u}_L^\mu$$

$$\hat{x}_1^\mu = \frac{r_S}{bv^2}\left(-\frac{(v^2+1)\,\hat{r}_0(\hat{\tau})}{2}\hat{b}^\mu + v\operatorname{asinh}\hat{\tau}\,\check{u}_H^\mu + \frac{v\,(v^2+1)\operatorname{asinh}\hat{\tau}}{2\sqrt{1-v^2}}\check{u}_L^\mu\right)$$

$$\begin{aligned}\hat{x}_2^\mu = \left(\frac{r_S}{bv^2}\right)^2 \Bigg(&\left[\frac{3v^3\operatorname{atan}\hat{\tau}}{2} - \frac{v\,(v^2-1)\operatorname{asinh}\hat{\tau}}{2\hat{r}_0(\hat{\tau})}\right]\check{u}_H^\mu \\ &+ \left[\frac{\sqrt{1-v^2}v\,(v^2+1)\operatorname{asinh}\hat{\tau}}{4\hat{r}_0(\hat{\tau})} + \frac{2\hat{\tau}v\,(v^2+1)^2 + 3\,(v^2+4)\,v^3\operatorname{atan}\hat{\tau}}{16\sqrt{1-v^2}}\right]\check{u}_L^\mu \\ &+ \left[\frac{\hat{\tau}\,(v^4-1)\operatorname{asinh}\hat{\tau}}{4\hat{r}_0(\hat{\tau})} - \frac{3\hat{\tau}\,(v^2+4)\,v^2\operatorname{atan}\hat{\tau} + v^4 + 8v^2 - 2}{16}\right]\hat{b}^\mu\Bigg)\end{aligned}$$

III. Perturbative Calculations

We use this EFT to compute old and new results for the conservative radial action in EM and GR.

1*a*) $\mathcal{O}(\alpha^3)$ in EM (electric charges)

1*b*) $\mathcal{O}(\alpha^3)$ in EM (dyons)

2*a*) $\mathcal{O}(G^3)$ in GR (spinless masses)

2*b*) $\mathcal{O}(G^3)$ in GR + ϕ + A_μ (spinless masses)

2*c*) $\mathcal{O}(G^3)$ in GR + EM (RN, DCSG, EGB)

We use this EFT to compute old and new results for the conservative radial action in EM and GR.

1a) $\mathcal{O}(\alpha^3)$ in EM (electric charges) ← check

1b) $\mathcal{O}(\alpha^3)$ in EM (dyons)

2a) $\mathcal{O}(G^3)$ in GR (spinless masses) ← check

2b) $\mathcal{O}(G^3)$ in GR + ϕ + A_μ (spinless masses)

2c) $\mathcal{O}(G^3)$ in GR + EM (RN, DCSG, EGB)

We use this EFT to compute old and new results for the conservative radial action in EM and GR.

1a) $\mathcal{O}(\alpha^3)$ in EM (electric charges)

1b) $\mathcal{O}(\alpha^3)$ in EM (dyons) ← new!

2a) $\mathcal{O}(G^3)$ in GR (spinless masses)

2b) $\mathcal{O}(G^3)$ in GR + ϕ + A_μ (spinless masses) ← new!

2c) $\mathcal{O}(G^3)$ in GR + EM (RN, DCSG, EGB) ← new!

Example: $\mathcal{O}(\alpha^3)$ in EM (electric charges)

$$I_{\text{EM}}^{(i,j)} = \lambda^i \, m_L r_c \left(\frac{r_c}{b}\right)^{j-1} \mathcal{I}_{\text{EM}}^{(i,j)}(\sigma)$$

$$\begin{aligned} I_{\text{EM}}^{(1,2)} &= \frac{\lambda^2}{2} z_L^2 z_H^2 m_H^3 \int_{p_1,p_2} \frac{e^{-i(p_1+p_2)b}\delta(u_H(p_1+p_2))\delta(u_L p_1)\delta(u_L p_2)}{p_1^2 p_2^2} \left(1+\frac{\sigma^2(p_1p_2)}{(u_H p_1)(u_H p_2)}\right) \\ &= \lambda m_L r_c \frac{r_c}{b} \frac{\pi}{2\sqrt{\sigma^2-1}} \, . \end{aligned}$$

$$\begin{aligned} I_{\text{EM}}^{(1,3)} &= -(m_H m_L)^2 (z_H z_L)^3 \int_{q,p_1,p_2,p_3} e^{iqb} \delta(u_H q)\delta(u_L q) \frac{\delta(q-p_1-p_2-p_3)\delta(u_H p_2)\delta(u_L p_1)}{p_1^2 p_2^2 p_3^2 (u_H p_1)^2 (u_L p_2)^2} \\ &\times \Big(-(p_1p_3)(p_2p_3)\sigma^3 - \tfrac{1}{2}q^2(u_H p_1)(u_L p_2)\sigma^2 + (p_2p_3)(u_H p_1)^2\sigma + (p_1p_3)(u_L p_2)^2\sigma \\ &\quad + (u_L p_2)(u_H p_1)^3 + (u_H p_1)(u_H p_2)^3\Big) \\ &= -\lambda m_L r_c \left(\frac{r_c}{b}\right)^2 \frac{2\left(\sigma^4 - 3\sigma^2 + 3\right)}{3\left(\sigma^2-1\right)^{5/2}} , \end{aligned}$$

Example: $\mathcal{O}(\alpha^3)$ in EM (electric charges)

$$I^{(i,j)}_{\rm EM} = \lambda^i\, m_L r_c \left(\frac{r_c}{b}\right)^{j-1} \mathcal{I}^{(i,j)}_{\rm EM}(\sigma)$$

$\longleftarrow$ (i)-SF and (j)-PM

$$\begin{aligned}
I^{(1,2)}_{\rm EM} &= \frac{\lambda^2}{2} z_L^2 z_H^2 m_H^3 \int_{p_1,p_2} \frac{e^{-i(p_1+p_2)b}\delta(u_H(p_1+p_2))\delta(u_L p_1)\delta(u_L p_2)}{p_1^2 p_2^2}\left(1+\frac{\sigma^2(p_1 p_2)}{(u_H p_1)(u_H p_2)}\right) \\
&= \lambda m_L r_c \frac{r_c}{b}\frac{\pi}{2\sqrt{\sigma^2-1}}\,.
\end{aligned}$$

$$\begin{aligned}
I^{(1,3)}_{\rm EM} &= -(m_H m_L)^2 (z_H z_L)^3 \int_{q,p_1,p_2,p_3} e^{iqb}\delta(u_H q)\delta(u_L q)\frac{\delta(q-p_1-p_2-p_3)\delta(u_H p_2)\delta(u_L p_1)}{p_1^2 p_2^2 p_3^2 (u_H p_1)^2 (u_L p_2)^2} \\
&\times \Big(-(p_1 p_3)(p_2 p_3)\sigma^3 - \tfrac{1}{2}q^2(u_H p_1)(u_L p_2)\sigma^2 + (p_2 p_3)(u_H p_1)^2\sigma + (p_1 p_3)(u_L p_2)^2\sigma \\
&\quad + (u_L p_2)(u_H p_1)^3 + (u_H p_1)(u_H p_2)^3\Big) \\
&= -\lambda m_L r_c \left(\frac{r_c}{b}\right)^2 \frac{2\left(\sigma^4-3\sigma^2+3\right)}{3\left(\sigma^2-1\right)^{5/2}},
\end{aligned}$$

Example: $\mathcal{O}(G^3)$ in GR (spinless masses)

$$I_{\rm GR}^{(i,j)} = \lambda^i \, m_L r_S \left(\frac{r_S}{b}\right)^{j-1} \mathcal{I}_{\rm GR}^{(i,j)}(\sigma)$$

$$I_{\rm GR}^{(1,2)} = \lambda m_L r_S \frac{r_S}{b} \frac{3\pi(5\sigma^2-1)}{4\sqrt{\sigma^2-1}}$$

$$I_{\rm GR}^{(1,3)} = \lambda m_L r_S \left(\frac{r_S}{b}\right)^2 \left(\frac{\sigma\,(36\sigma^6-114\sigma^4+132\sigma^2-55)}{12\,(\sigma^2-1)^{5/2}} - \frac{(4\sigma^4-12\sigma^2-3)\,\operatorname{arccosh}\sigma}{2\,(\sigma^2-1)}\right)$$

In this EFT it is relatively easy to add additional fields to GR, including spectator scalars / vector matter, or gravitational axion / dilaton.

Example: $\mathcal{O}(G^3)$ in GR + ϕ + A_μ (spinless masses)

$$S_{\text{matt}} = \int d^4x\sqrt{-g}\left[\frac{1}{2}\nabla\phi^2+\frac{1}{2}\xi R\phi^2-\frac{1}{4}F^2\right]-m_L\int d\tau\left[y_L\phi(x_L)+z_L A_\mu(x_L)\dot{x}^\mu_L\right]$$

$$I^{(1,2)}_{\text{scalar}} = -\lambda m_L r_S\left(\frac{r_\Phi}{b}\right)\left(\frac{\pi}{8}\frac{\sigma^2-1+4\xi}{\sqrt{\sigma^2-1}}\right)$$

$$I^{(1,3)}_{\text{scalar}} = -\lambda m_L r_S\left(\frac{r_S r_\Phi}{b^2}\right)\frac{\sigma\left(2\sigma^4-\sigma^2-1+\xi\left(6\sigma^2-3\right)\right)}{6\left(\sigma^2-1\right)^{3/2}}$$

$$I^{(1,2)}_{\text{vector}} = -\lambda m_L r_S\left(\frac{r_A}{b}\right)\left(\frac{\pi}{8}\frac{3\sigma^2-1}{\sqrt{\sigma^2-1}}\right)$$

$$I^{(1,3)}_{\text{vector}} = -\lambda m_L r_S\left(\frac{r_S r_A}{b^2}\right)\left(\frac{\sigma\left(8\sigma^4-28\sigma^2+23\right)}{12\left(\sigma^2-1\right)^{3/2}}+\frac{\left(2\sigma^2+1\right)\operatorname{arccosh}\sigma}{\left(\sigma^2-1\right)}\right)$$

Conclusions

We derive an EFT for SF corrections to probe dynamics by integrating out a heavy particle.

Using Schwarzschild and geodesic trajectories, we mine any order in G data for loop integrands.

We perform several old and new calculations of the conservative radial action in EM and GR.

Thank You!