Gravitational Two-body Dynamics at NNNLO in PM Approximation

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Gravitational Self-Force and Scattering Amplitudes

Higgs Centre for Theoretical Physics, Edinburg

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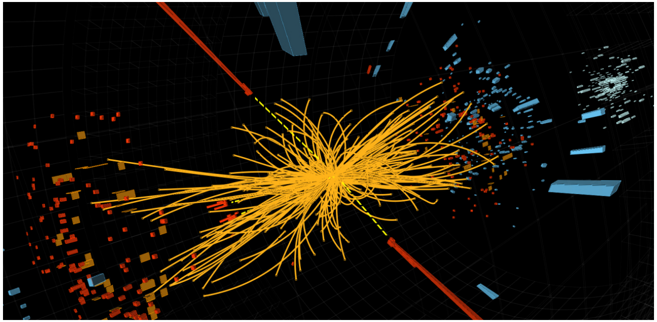
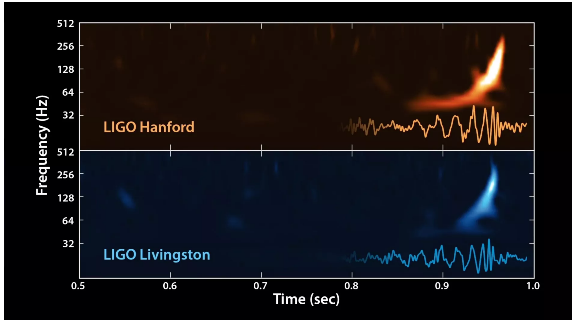
Precision era of fundamental physics

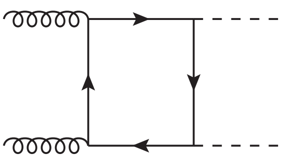


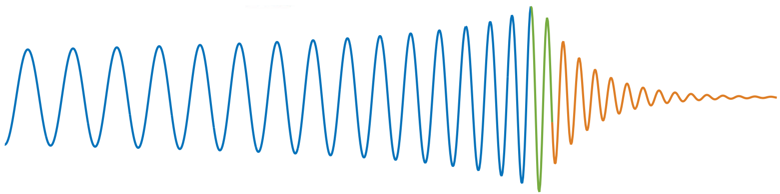


Two historic breakthroughs in science:

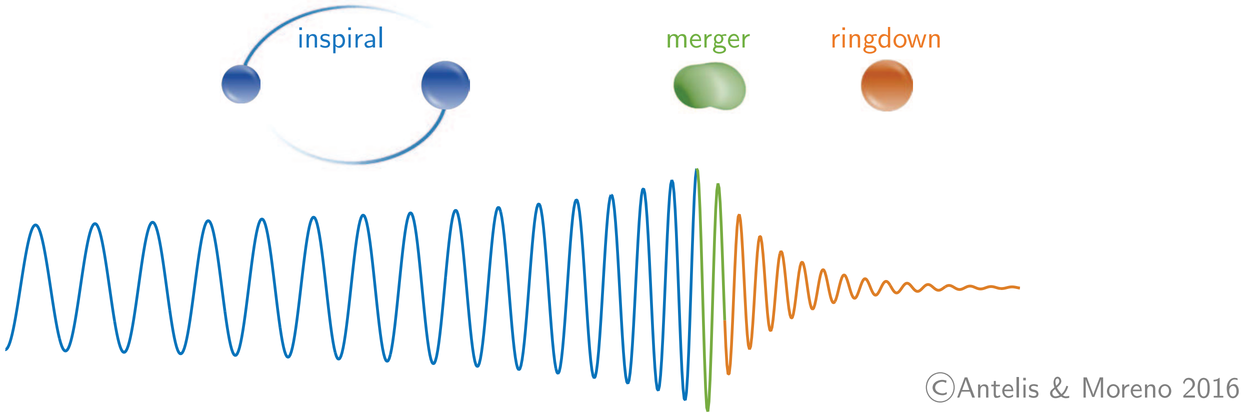
- Higgs bosons from the LHC (2012)
- Gravitational waves from the LIGO (2016)
- High-energy and gravitational physics entered a precision era!

Modern techniques from Higgs physics are playing a crucial role in precision GW physics!





Gravitational waves from binary coalescences



Merger: Numerical Relativity

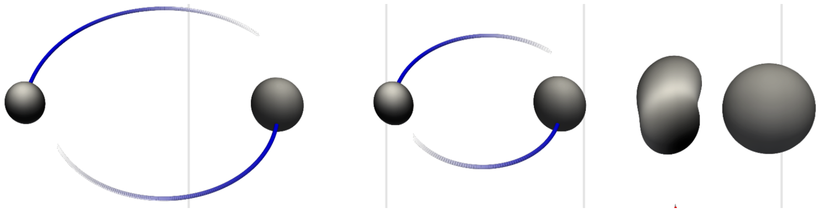
Ringdown: black hole perturbation theory

Inspiral: the interaction between two bodies is weak

$$v^2 \sim \frac{GM}{r} \ll 1$$

- Analytic perturbation methods work: post-Newtonian/Minkowskian, EOB...
- QFT methodology, combined with modern loop techniques, shown great power.

Effective Field Theory



- The gravitational two-body problem

$$\mathcal{S}_{\rm WL} = \sum_{i=1,2} \left[-\frac{m_i}{2} \int dt\, g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu + \cdots \right]$$

$$\mathcal{S}_{\rm GR} = \frac{-1}{16\pi G} \int d^4x \sqrt{-g}\, R + \cdots$$

- In the inspiral phase

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}$$

- Effective action for binary systems

Goldberger-Rothstein 2004

$$e^{i\mathcal{S}_{\rm eff}[x_a(\tau)]} = \int \mathcal{D}h_{\mu\nu}\, e^{i\mathcal{S}_{\rm WL} + i\mathcal{S}_{\rm GR}}$$

Effective Field Theory

- EFT description

$$e^{i\mathcal{S}_{\text{eff}}[x_a(\tau)]} = \int \mathcal{D}h_{\mu\nu}\, e^{i\mathcal{S}_{\text{WL}}+i\mathcal{S}_{\text{GR}}}$$

- Post-Minkowskian expand in powers of G

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + G\mathcal{L}_1 + G^2\mathcal{L}_2 + \cdots \qquad \mathcal{L}_0 = -\sum_i \frac{m_i}{2}\eta_{\mu\nu}\dot{x}_i^\mu \dot{x}_i^\nu$$

The equations of motion for trajectories:

$$m_i\ddot{x}_i^\mu = -\eta^{\mu\nu}\sum_{n=1}^{\infty} G^n \left(\frac{\partial \mathcal{L}_n}{\partial x_i^\nu} - \frac{d}{d\tau_i}\frac{\partial \mathcal{L}_n}{\partial \dot{x}_i^\nu}\right) \qquad x_i^\mu = b_i^\mu + u_i^\mu \tau_i + \delta x_i^\mu(\tau_i) + \cdots$$

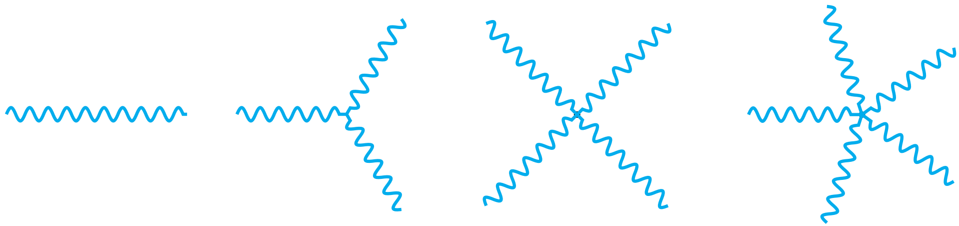
- Physical observables:

$$\Delta p_i^\mu = p_i^\mu(+\infty) - p_i^\mu(-\infty) = -\eta^{\mu\nu}\sum_{n=1}^{\infty} G^n \int_{-\infty}^{\infty} d\tau_i \left(\frac{\partial \mathcal{L}_n}{\partial x_i^\nu}\right)$$

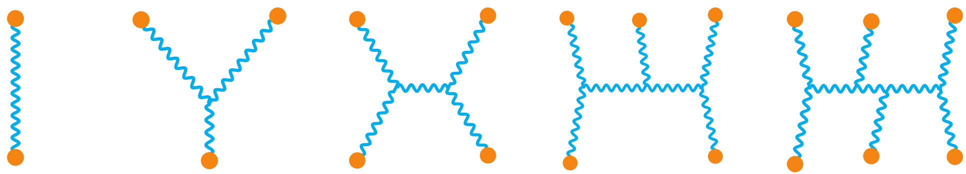
Effective Field Theory

- Worldlines as sources in path integral:

- Hilbert-Einstein: $\mathcal{L}_{\text{HE}} = \mathcal{L}_{hh} + \mathcal{L}_{hhh} + \mathcal{L}_{hhhh} + \cdots$



- Classical physics: we use the saddle-point approximation in path integrals.



- Enjoy the advantages of pure classical physics and quantum field theoretic methods.

Effective Field Theory

The in-in effective action is obtained by performing a closed-time-path integral

$$e^{i\mathcal{S}_{\text{eff}}[x_{a,1},x_{a,2}]} = \int \mathcal{D}h_1\mathcal{D}h_2\, e^{i(S_{\text{GR}}[h_1]-S_{\text{GR}}[h_2]+S_{\text{WL}}[h_1,x_{a,1}]-S_{\text{WL}}[h_2,x_{a,2}])}$$

It is convenient to use the Keldysh basis

$$\begin{aligned} h^-_{\mu\nu} &= \frac{1}{2}(h_{1\mu\nu} + h_{2\mu\nu}) \qquad & x^\alpha_{a,+} &= \frac{1}{2}(x^\alpha_{a,1} + x^\alpha_{a,2}) \\ h^+_{\mu\nu} &= h_{1\mu\nu} - h_{2\mu\nu} & x^\alpha_{a,-} &= x^\alpha_{a,1} - x^\alpha_{a,2} \end{aligned}$$

for which the matrix of (classical) propagators for the metric field becomes

$$i\begin{pmatrix} 0 & -\Delta_{\text{adv}}(x-y) \\ -\Delta_{\text{ret}}(x-y) & 0 \end{pmatrix}$$

The worldline equations of motion:

$$m_i \frac{d}{d\tau}\dot{x}^\mu_i(\tau) = -\eta^{\mu\nu}\frac{\delta\mathcal{S}_{\text{eff, int}}[x_{a,\pm}]}{\delta x^\nu_{i,-}(\tau)}\bigg|_{\text{PL}}, \qquad \Delta p^\mu_i = -\eta^{\mu\nu}\int_{-\infty}^{\infty} d\tau \frac{\delta\mathcal{S}_{\text{eff, int}}[x_{a,\pm}]}{\delta x^\nu_{i,-}(\tau)}\bigg|_{\text{PL}}$$

Physical Limit (PL): $x_{a,-} \to 0$, $x_{a,+} \to x_a$.

Effective Field Theory

- In practise, Feynman rules are still simple in the physical limit!
- Worldline source: $\downarrow k$  $= -\dfrac{im}{2M_{\rm Pl}} \int \mathrm{d}\tau\, e^{i\,k\cdot x} \dot{x}^\mu \dot{x}^\nu$
- Variation of worldline: $\downarrow k$  $= -\dfrac{im}{2M_{\rm Pl}} e^{i\,k\cdot x}\Big(i\,k^\alpha \dot{x}^\mu \dot{x}^\nu - i\,k\cdot\dot{x}\,\eta^{\mu\alpha}\dot{x}^\nu - \eta^{\mu\alpha}\ddot{x}^\nu - i\,k\cdot\dot{x}\,\eta^{\nu\alpha}\dot{x}^\mu - \eta^{\nu\alpha}\ddot{x}^\mu\Big)$
- Variation of effective action:

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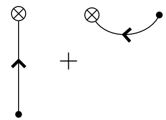
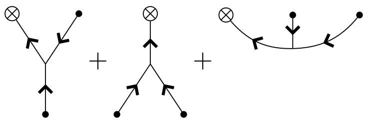
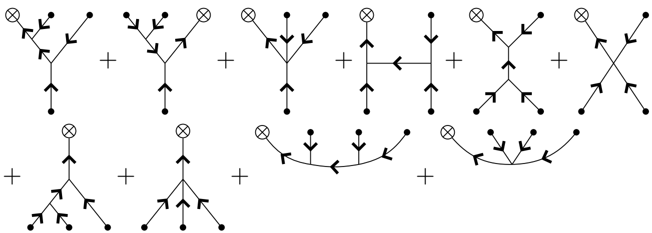
$$\frac{\delta \mathcal{S}_{\rm eff}[x_\pm]}{\delta x^\alpha_{1-}}\bigg|^{G}_{\rm PL} =$$



$$\frac{\delta \mathcal{S}_{\rm eff}[x_\pm]}{\delta x^\alpha_{1-}}\bigg|^{G^2}_{\rm PL} =$$



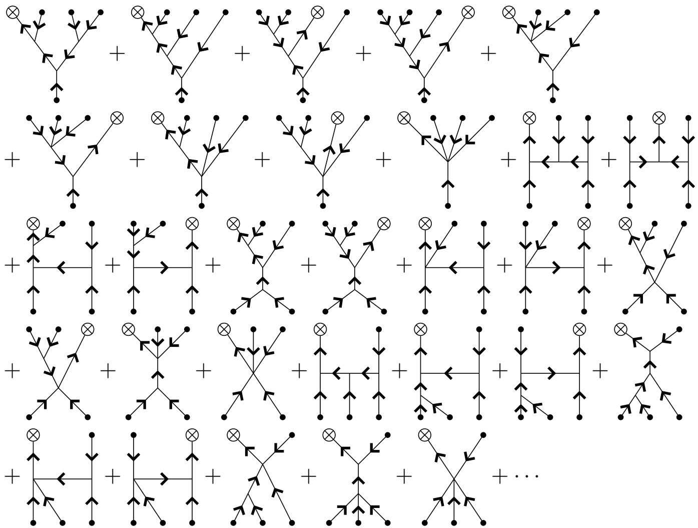
$$\frac{\delta \mathcal{S}_{\rm eff}[x_\pm]}{\delta x^\alpha_{1-}}\bigg|^{G^3}_{\rm PL} =$$



Effective Field Theory

- In practise, Feynman rules are still simple in the physical limit!
- Worldline source: $\downarrow k$  $= -\frac{im}{2M_{\rm Pl}} \int \mathrm{d}\tau\, e^{i\,k\cdot x} \dot{x}^{\mu} \dot{x}^{\nu}$
- Variation of worldline: $\downarrow k$  $= -\frac{im}{2M_{\rm Pl}} e^{i\,k\cdot x}\Big(i\,k^{\alpha}\dot{x}^{\mu}\dot{x}^{\nu} - i\,k\cdot\dot{x}\,\eta^{\mu\alpha}\dot{x}^{\nu} - \eta^{\mu\alpha}\ddot{x}^{\nu} - i\,k\cdot\dot{x}\,\eta^{\nu\alpha}\dot{x}^{\mu} - \eta^{\nu\alpha}\ddot{x}^{\mu}\Big)$
- Variation of effective action: 2207.00580 2304.01275

$$\frac{\delta \mathcal{S}_{\rm eff}[x_{\pm}]}{\delta x^{\alpha}_{1-}}\bigg|^{G^4}_{\rm PL} =$$



Effective Field Theory

- Impulse at $\mathcal{O}(G^N)$ 2304.01275

$$\Delta p_i^\mu \sim \int d^D q \, \frac{e^{iq\cdot b}\,\delta(q\cdot u_1)\delta(q\cdot u_2)}{|q^2|^\sharp} \int \left(\prod_{i=1}^{N-1} d^D\ell_i \, \frac{\delta(\ell_j\cdot u_a)}{(\ell_i\cdot u_b - i0)^{\nu_i}} \right) \frac{\mathcal{N}^\mu(q,u_a)}{D_1 D_2 D_3 \cdots}$$

  - Graviton propagators:

$$\frac{1}{D_i} \longrightarrow \frac{1}{(\ell^0 \pm i0)^2 - \vec{\ell}^{\,2}} \quad \text{or} \quad \frac{1}{\ell^2 + i0}$$

  - Cut: always one delta function $\delta(\ell_i\cdot u_a)$ for each loop
  - Kinematics: $q\cdot u_a = 0,\ u_a^2 = 1,\ u_1\cdot u_2 = \gamma \implies$ single scale γ to all orders

- Multi-loop technology from collider physics can be used to solve gravitational problems!

Collider physics toolbox

Post-Minkowskian Loop Integrals at $\mathcal{O}(G^N)$

$$\int \left(\prod_{i=1}^{N-1} d^D \ell_i \, \frac{\delta(\ell_j \cdot u_{a_i})}{(\ell_i \cdot u_{b_i} - i0)^{\alpha_i}} \right) \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots}$$

- Reverse Unitarity: replace the delta-function by the cut-propagator Anastasiou-Melnikov 2002

$$\delta(k_i \cdot u_a) \to \frac{1}{2\pi i} \left(\frac{1}{k_i \cdot u_a - i0} - \frac{1}{k_i \cdot u_a + i0} \right)$$

Then standard loop-integral techniques can be applied straightforwardly!

- IBP reduction: any integral = a linear combination of a small number of basis integrals

$$\vec{f} = \{I_1, I_2, \ldots\}$$

  - Publicly-available programs: `Reduze`, `FIRE`, `LiteRed`, `Kira`, `FiniteFlow`
  - New developments: `NeatIBP`, `FIRE6.5`, `Kira3`

Collider physics toolbox

- Differential equations:

$$\frac{d\vec{f}(x,\epsilon)}{dx} = A(x,\epsilon)\,\vec{f}(x,\epsilon) \qquad D = 4-2\epsilon \qquad \gamma = \frac{x^2+1}{2x}$$

- Canonical form

Henn 2013 Lee 2014

$$\vec{g} = T\cdot\vec{f} \qquad \Longrightarrow \qquad \frac{d\vec{g}(x,\epsilon)}{dx} = \epsilon\, M(x)\,\vec{g}(x,\epsilon)$$

- We can solve iteratively.

$$\vec{g}(x,\epsilon) = \sum_k \epsilon^k \vec{g}^{(k)}(x) \qquad \vec{g}^{(k)}(x) = \int_{x_0}^{x} M(t)\vec{g}^{(k-1)}(t)\,\mathrm{d}t + \vec{g}_0^{(k)}$$

- M is rational in x: multiple polylogarithms

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1}\,G(a_2,\ldots,a_n;t), \quad G(\,;z)=1$$

- Boundary constants $\vec{g}_0$ can be computed in PN limit using the method of regions.

potential: $\ell^\mu \sim (v,1)$ radiation: $\ell^\mu \sim (v,v)$ Beneke-Smirnov 1997

Elliptic differential equations

- The majority of 4PM integrals can be solved in terms of multiple polylogarithms.
- Elliptic integrals appear in post-Minkwskian gravity for the first time.

$$\frac{d}{dx}\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} \frac{1-x^2}{2x(1+x^2)} & \frac{1+x^2}{4x(1-x^2)} & \frac{3x}{(1-x^2)(1+x^2)} \\ -\frac{1-x^2}{x(1+x^2)} & \frac{3(1+x^2)}{2x(1-x^2)} & -\frac{6x}{(1-x^2)(1+x^2)} \\ \frac{1-x^2}{x(1+x^2)} & -\frac{1+x^2}{2x(1-x^2)} & -\frac{1-4x^2+x^4}{x(1-x^2)(1+x^2)} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} + \mathcal{O}(\epsilon)$$

It can then be written as a third-order differential equation:

$$\left[\frac{d^3}{dx^3} - \frac{6x}{1-x^2}\frac{d^2}{dx^2} - \frac{1-4x^2+7x^4}{x^2(1-x^2)^2}\frac{d}{dx} - \frac{1+x^2}{x^3(1-x^2)}\right] f_1(x) = 0$$

It is easy to find the three solutions:

$$x\,\mathrm{K}^2\,(1-x^2), \qquad x\,\mathrm{K}(1-x^2)\mathrm{K}(x^2), \qquad x\,\mathrm{K}^2(x^2)$$

Complete elliptic integrals: $\mathrm{K}(x) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-xt^2)}}$

Elliptic differential equations

With the knowledge of leading-ϵ solutions, one may transform the elliptic diagonal block into

$$\frac{d}{dx}\vec{g}(x,\epsilon) = \epsilon\, \tilde{D}_{\text{ell}}(x)\, \vec{g}(x,\epsilon) + \ldots$$

with

$$\tilde{D}_{\text{ell}} = \begin{pmatrix} -\frac{4\left(1+x^2\right)}{3x(1-x^2)} & \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)} & 0 \\ \frac{2\left(1+110x^2+x^4\right)\mathrm{K}^2\left(1-x^2\right)}{3\pi^2 x(1-x^2)} & -\frac{4\left(1+x^2\right)}{3x(1-x^2)} & \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)} \\ \frac{16\left(1+x^2\right)\left(1-18x+x^2\right)\left(1+18x+x^2\right)\mathrm{K}^4\left(1-x^2\right)}{27\pi^2 x(1-x^2)} & \frac{2\left(1+110x^2+x^4\right)\mathrm{K}^2\left(1-x^2\right)}{3\pi^2 x(1-x^2)} & -\frac{4\left(1+x^2\right)}{3x(1-x^2)} \end{pmatrix}$$

- Elliptic integrals appear in the transformation matrix: found by INITIAL
- Higer $\mathcal{O}(\epsilon)$: Iterated integrals involving elliptic kernels.

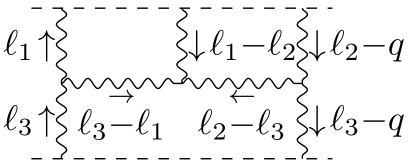
Method of Regions

- Regions: classical soft regions contains *potential* and *radiation* regions Beneke-Smirnov 1997

$$\text{potential:}\quad \ell^\mu \sim (w, \boldsymbol{\ell}) \sim (v, 1)|\boldsymbol{q}|$$

$$\text{radiation:}\quad \ell^\mu \sim (w, \boldsymbol{\ell}) \sim (v, v)|\boldsymbol{q}|$$

- A 4PM example:



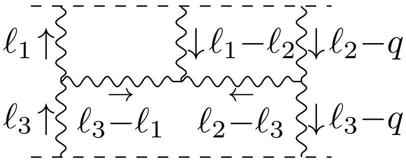
$$= \int_{\ell_1\ell_2\ell_3} \frac{\delta(\ell_1\cdot u_1)\,\delta(\ell_2\cdot u_1)\,\delta(\ell_3\cdot u_2)}{\ell_1^2\,\ell_3^2\,(\ell_2-q)^2\,(\ell_3-q)^2\,(\ell_1-\ell_2)^2\,(\ell_2-\ell_3)^2\,(\ell_3-\ell_1)^2}$$

Relabeling $k_1 = \ell_3 - \ell_1$, $k_2 = \ell_2 - \ell_3$, $\ell = \ell_3$, we found 2304.01275

$$\begin{aligned}
\text{pot (ppp)}:&\quad k_1 \sim (v,1)|\boldsymbol{q}|, \quad k_2 \sim (v,1)|\boldsymbol{q}|, \quad \ell \sim (v,1)|\boldsymbol{q}| \\
\text{1rad}^{(1)}\text{(rpp)}:&\quad k_1 \sim (v,v)|\boldsymbol{q}|, \quad k_2 \sim (v,1)|\boldsymbol{q}|, \quad \ell \sim (v,1)|\boldsymbol{q}| \\
\text{1rad}^{(2)}\text{(prp)}:&\quad k_1 \sim (v,1)|\boldsymbol{q}|, \quad k_2 \sim (v,v)|\boldsymbol{q}|, \quad \ell \sim (v,1)|\boldsymbol{q}| \\
\text{rad2 (rrp)}:&\quad k_1 \sim (v,v)|\boldsymbol{q}|, \quad k_2 \sim (v,v)|\boldsymbol{q}|, \quad \ell \sim (v,1)|\boldsymbol{q}|
\end{aligned}$$

Confirmed using `asy2.m` in Feynman parameterization.

Method of Regions



$$= \int_{\ell_1\ell_2\ell_3} \frac{\delta(\ell_1\cdot u_1)\,\delta(\ell_2\cdot u_1)\,\delta(\ell_3\cdot u_2)}{\ell_1^2\,\ell_3^2\,(\ell_2-q)^2\,(\ell_3-q)^2\,(\ell_1-\ell_2)^2\,(\ell_2-\ell_3)^2\,(\ell_3-\ell_1)^2}$$

Expanding around each region we have

$$I_{\rm pot} = \int_{\boldsymbol{\ell}\,\boldsymbol{k}_1\boldsymbol{k}_2} \frac{1}{[(\boldsymbol{\ell}-\boldsymbol{k}_1)^2]\,[\boldsymbol{\ell}^2]\,[(\boldsymbol{k}_2+\boldsymbol{\ell}-\boldsymbol{q})^2]\,[(\boldsymbol{\ell}-\boldsymbol{q})^2]\,[(\boldsymbol{k}_1+\boldsymbol{k}_2)^2]\,[\boldsymbol{k}_2^2]\,[\boldsymbol{k}_1^2]} + \mathcal{O}(v_\infty^2)$$

$$I_{\rm 1rad}^{(1)} = \int_{\boldsymbol{\ell}\,\boldsymbol{k}_2} \frac{1}{[\boldsymbol{\ell}^2]\,[\boldsymbol{\ell}^2]\,[(\boldsymbol{k}_2+\boldsymbol{\ell}-\boldsymbol{q})^2]\,[(\boldsymbol{\ell}-\boldsymbol{q})^2]\,[\boldsymbol{k}_2^2]^2} \int_{\boldsymbol{k}_1} \frac{v_\infty^{d-2}}{\boldsymbol{k}_1^2-(\ell^z)^2} + \mathcal{O}(v_\infty^d)$$

$$I_{\rm 1rad}^{(2)} = \int_{\boldsymbol{\ell}\,\boldsymbol{k}_1} \frac{1}{[(\boldsymbol{\ell}-\boldsymbol{k}_1)^2]\,[\boldsymbol{\ell}^2]\,[(\boldsymbol{\ell}-\boldsymbol{q})^2]\,[(\boldsymbol{\ell}-\boldsymbol{q})^2]\,[\boldsymbol{k}_1^2]^2} \int_{\boldsymbol{k}_2} \frac{v_\infty^{d-2}}{\boldsymbol{k}_2^2-(\ell^z)^2} + \mathcal{O}(v_\infty^d)$$

$$I_{\rm 2rad} = \int_{\boldsymbol{\ell}} \frac{1}{[\boldsymbol{\ell}^2]\,[\boldsymbol{\ell}^2]\,[(\boldsymbol{\ell}-\boldsymbol{q})^2]^2} \int_{\boldsymbol{k}_1\boldsymbol{k}_2} \frac{v_\infty^{2d-6}}{[(\boldsymbol{k}_1+\boldsymbol{k}_2)^2]\,[\boldsymbol{k}_2^2-(\ell^z)^2]\,[\boldsymbol{k}_1^2-(\ell^z)^2]} + \mathcal{O}(v_\infty^{2d-4})$$

- These integrals can be straightforwardly evaluated through direct integration.
- All regions added up leads to a finite result, in particular IR divergences cancel.

Inspiral dynamics at NNNLO

The full impulse at $\mathcal{O}(G^4)$: 2106.08276 2112.11296 2210.05541 2304.01275

$$\Delta p_1^\mu\Big|_{\text{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b\,\frac{b^\mu}{|b|} + c_1\,\frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2\,\frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}\right)$$

$$\begin{aligned}
\frac{c_b}{\pi} = &-\frac{3h_{34}m_2m_1(m_1^3+m_2^3)}{64v_\infty^5} + \frac{m_1^2m_{12}m_2^2}{4}\Bigg[\frac{3h_6\mathrm{K}^2(w_2)}{4v_\infty^3} - \frac{3h_8\mathrm{K}(w_2)\mathrm{E}(w_2)}{4v_\infty^3} + \frac{21h_5w_3\mathrm{E}^2(w_2)}{8v_\infty^3} - \frac{\pi^2h_{16}v_\infty}{4(\gamma+1)} + \frac{3\gamma h_{10}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{w_3v_\infty^2}\\
&+\log(v_\infty)\left(\frac{h_{32}}{2v_\infty^3} - \frac{3h_{14}\log(\frac{w_3}{2})}{v_\infty} - \frac{3\gamma h_{22}\log(w_1)}{2v_\infty^4}\right)\Bigg] + m_2^2m_1^3\Bigg[\frac{h_{52}}{48v_\infty^6} - \frac{h_{63}}{768\gamma^9w_3v_\infty^5} - \frac{3v_\infty(h_{40}\mathrm{Li}_2(w_2)+2w_3h_{33}\mathrm{Li}_2(-w_2))}{64w_3}\\
&+\frac{3h_{14}\log(\frac{w_3}{2})\log(w_3)}{4v_\infty} + \frac{\gamma h_{39}\log(w_1)}{8w_3^3v_\infty^2} + \frac{3\gamma h_{22}\log(w_3)\log(w_1) - h_{35}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{h_{56}\log(2) - h_{57}\log(w_3) + 2\gamma h_{55}\log(\gamma)}{32v_\infty^5} - \frac{\gamma h_{51}\log(w_1)}{16v_\infty^7}\Bigg]\\
&+m_1^2m_2^3\Bigg[\frac{h_{58}}{192\gamma^7v_\infty^5} + \frac{h_{53}}{48v_\infty^6} + \frac{\gamma h_{49}\log(w_1)}{16v_\infty^6} - \frac{2\gamma h_{50}\log(w_1) + 3\gamma^2h_{13}\log^2(w_1)}{32v_\infty^7} - \frac{h_{41}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{3\gamma\log(w_1)(5h_{26}\log(2)+8h_{12}\log(w_3))}{8v_\infty^4}\\
&-\frac{h_{36}\log(w_3)}{4v_\infty^3} + \frac{\gamma h_{30}\log(\gamma)}{2v_\infty^3} + \frac{h_{37}\log(2)}{8v_\infty^3} + \frac{3(h_{17}w_3\mathrm{Li}_2(w_2) - 2h_{23}\mathrm{Li}_2(-w_2) + h_{15}\log^2(w_3) - h_9\log^2(2))}{8v_\infty} - \frac{3h_7\log(2)\log(w_3)}{v_\infty}\Bigg]
\end{aligned}$$

$$c_1 = m_1m_2^2\left(\frac{2h_{46}m_{12s}}{v_\infty^6} + \frac{9\pi^2h_1m_{12}^2}{32v_\infty^2}\right) + m_1^2m_2^3\left(\frac{4\gamma h_{47}}{3v_\infty^6} - \frac{8\gamma h_2\log(w_1)}{v_\infty^6} + \frac{16h_{25}\log(w_1)}{v_\infty^3} - \frac{8h_3}{3v_\infty^5}\right)$$

$$\begin{aligned}
c_2 = &-m_1^4m_2\left(\frac{9\pi^2h_1}{32v_\infty^2} + \frac{2h_{46}}{v_\infty^6}\right) + m_2^2m_1^3\Bigg[+\frac{h_{60}}{705600\gamma^8v_\infty^5} - \frac{4\gamma h_{48}}{3v_\infty^6} + \frac{3h_{38}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2})) - \gamma h_{21}(\mathrm{Li}_2(-w_1^2)+2\log(\gamma)\log(w_1))}{16v_\infty^4}\\
&+\frac{3\gamma h_{31}(2\mathrm{Li}_2(-w_1)+\log(w_1)\log(w_3))}{8v_\infty^4} + \frac{h_{62}\log(w_1)}{6720\gamma^9v_\infty^6} + \frac{32\gamma^2h_{44}\log^2(w_1)}{v_\infty^7} + \frac{8\gamma(2h_4\log(2) - h_{27}\log(w_1))\log(w_1)}{v_\infty^4} - \frac{32h_{29}\log(w_1)}{3v_\infty^3} + \frac{\pi^2h_{42}}{192v_\infty^4}\Bigg]\\
&+m_2^3m_1^2\Bigg[\frac{h_{59}}{1440\gamma^7v_\infty^5} - \frac{h_{19}(\mathrm{Li}_2(-w_1^2)+2\log(\gamma)\log(w_1))}{8v_\infty^4} + \frac{h_{43}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{32v_\infty^4} - \frac{h_{20}(2\mathrm{Li}_2(-w_1)+\log(w_1)\log(w_3))}{4v_\infty^4}\\
&-\frac{h_{61}\log(w_1)}{480\gamma^8v_\infty^6} - \frac{16\gamma^2h_{11}\log^2(w_1)}{v_\infty^4} - \frac{32\gamma h_{45}\log^2(w_1)}{v_\infty^7} + \frac{16\gamma h_{28}\log(w_1)}{5v_\infty^3} - \frac{32h_{24}\log(2)\log(w_1)}{v_\infty^4} - \frac{\pi^2h_{18}}{48v_\infty^4} - \frac{2h_{54}}{45v_\infty^6}\Bigg]
\end{aligned}$$

with $\gamma \equiv u_1\cdot u_2$, $v_\infty = \sqrt{\gamma^2-1}$, $w_1 = \gamma - v_\infty$, $w_2 = \frac{\gamma-1}{\gamma+1}$, $w_3 = \gamma+1$, h_i = polynomial in γ.

$$\mathrm{Li}_2(z) \equiv \int_0^z dx\,\frac{\log(1-x)}{-x}$$

$$K(z) \equiv \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}}$$

$$E(k) \equiv \int_0^1 dx\,\frac{\sqrt{1-zx^2}}{\sqrt{1-x^2}}$$

Inspiral dynamics at NNNLO

The full impulse at $\mathcal{O}(G^4)$: 2106.08276 2112.11296 2210.05541

$$\Delta p_1^\mu\big|_{\text{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b \frac{b^\mu}{|b|} + c_1 \frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2 - 1} + c_2 \frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2 - 1}\right)$$

- We obtained the full dynamics of binary inspirals at $\mathcal{O}(G^4)$ for the first time.
- Conservative part agrees perfectly with Amplitudes' derivations.
  Bern-Parra-Martinez-Roiban-Ruf-Shen-Solon-Zeng 2021
- Perfect agreement with the state-of-the-art PN computations
  Cho-Dandapat-Gopakumar 2021 Cho 2022 Bini-Geralico 2021 2022 Bini-Damour 2022
- Very recently two new calculations confirmed our results.
  Damgaard-Hansen-Planté-Vanhove 2023 (exponentiation of amplitudes)
  Jakobsen-Mogull-Plefka-Sauer-Xu 2023 (worldline QFT)

Local-in-time part

- The full result cannot be used to describe generic elliptic-like motion due to nonlocal-in-time effects. Damour-Jaranowski-Schäfer 2014 Galley-Leibovich-Porto-Ross 2015 Cho-Kälin-Porto 2021

- The nonlocal-in-time radial action takes the form

$$\mathcal{S}_r^{(\mathrm{nloc})} = -\frac{GE}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{dE}{d\omega}\log\left(\frac{4\omega^2}{\mu^2}e^{2\gamma_E}\right)$$

E and $\frac{dE}{d\omega}$ are the total energy and emitted GW flux in the center-of-mass frame. Renormalization scale μ can be arbitrarily chosen, $4e^{2\gamma_E}$ follows the PN conventions.

- For scattering, the deflection angle is given by

$$\frac{\chi}{2\pi} = -\partial_j \mathcal{I}_r, \qquad \mathcal{I}_r \equiv \frac{\mathcal{S}_r}{GM^2\nu}, \quad j \equiv \frac{J}{GM^2\nu}$$

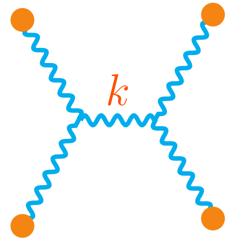
PM expansion

$$\frac{\chi}{2} = \sum_{n=1}\left(\chi_b^{(n)} + \chi_b^{(n)\log}\log\frac{\mu b}{\Gamma}\right)\left(\frac{GM}{b}\right)^n \qquad \Gamma \equiv \frac{E}{M} = \sqrt{1+2\nu(\gamma-1)}$$

Local-in-time dynamics

- The integrand can be built from 3PM diagrams.

$$\int d^D\ell_1 d^D\ell_2 \frac{\delta(\ell_1\cdot u_1)\delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2][\ell_2\cdot u_1]} \frac{\log(\omega^2)}{[\ell_1^2][\ell_2^2][(\ell_1+\ell_2-q)^2][(\ell_1-q)^2][(\ell_2-q)^2]}$$



with

$$\omega \equiv k\cdot u_{\rm com}, \quad k = \ell_1+\ell_2-q, \quad u_{\rm com} = \frac{m_1 u_1 + m_2 u_2}{M\Gamma}$$

- Integral family:

$$\int d^D\ell_1 d^D\ell_2 \frac{\delta(\ell_1\cdot u_1)\delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2][\ell_2\cdot u_1]} \frac{1}{\omega^{-2\tilde{\epsilon}}} \frac{1}{[\ell_1^2][\ell_2^2][(\ell_1+\ell_2-q)^2][(\ell_1-q)^2][(\ell_2-q)^2]}$$

  - IBP can be done using `LiteRed` and `FiniteFlow`: 17 MIs $\qquad Q \equiv m_2/m_1$
  - MPLs: $\{x, 1\pm x\}\cup\{y, 1\pm y, y-\frac{1+x}{1-x}, y-\frac{1-x}{1+x}, 1+2\frac{1-x}{1+x}y+y^2\}$ $\quad Q^{-1} = -\gamma - \frac{\sqrt{\gamma^2-1}}{2}(y+y^{-1})$
  - Complete elliptic integrals and iterated integrals of the elliptic integrals.

Iterated integrals of elliptic kernels

- Iterated integrals:

$$\mathrm{I}(h_1, h_2, \ldots, h_n; z) := \int_0^z dt\, \mathrm{I}(h_2, \ldots, h_n; t)$$

The following set appears in the result

$$\left\{\mathrm{I}(f_i; Q),\ \mathrm{I}(q^{-1}, f_i; Q),\ \mathrm{I}\left(\frac{2}{\sqrt{(Q+x)(Q+1/x)}}, f_5; Q\right),\ \mathrm{I}\left(\frac{2}{q\sqrt{(Q+x)(Q+1/x)}}, f_5; Q\right)\right\}$$

with

$$f_1(q) \equiv \frac{\mathrm{K}(-qx)\mathrm{K}\left(1+q/x\right) - \mathrm{K}\left(-q/x\right)\mathrm{K}(1+qx)}{\pi}$$

$$f_2(q) \equiv \frac{f_1(q)}{q}, \qquad f_3(q) \equiv \partial_x f_1(q), \qquad f_4(q) \equiv \frac{\partial_x f_1(q)}{q}$$

$$f_5(q) \equiv \left[\frac{1-x^2}{x}(1+q\,\partial_q) - \frac{1-q^2}{q} x\,\partial_x\right] \frac{f_1(q)}{\sqrt{(q+x)\,(q+1/x)}}$$

- The combination of complete elliptic integrals in f_1 has a simple PN expansion ($x \to 1$).
- All f_i's have (at most) simple poles $\Longrightarrow$ easy to evaluate in SF expansion ($Q \to 0$).

Iterated integrals of elliptic kernels

In[2]:= $f[1] = \dfrac{\text{EllipticK}[-Q\,x]\,\text{EllipticK}\left[\frac{Q+x}{x}\right] - \text{EllipticK}\left[-\frac{Q}{x}\right]\,\text{EllipticK}[1+Q\,x]}{\pi}$

In[3]:= `Series[f[1], {Q, 0, 5}] // Collect[#, {Q, _Log}, Simplify] &`

Out[3]=
$$\frac{\text{Log}[x]}{2} + Q\left(-\frac{-1+x^2}{8\,x} - \frac{(1+x^2)\,\text{Log}[x]}{8\,x}\right) + Q^2\left(\frac{21\,(-1+x^4)}{256\,x^2} + \frac{(9+4\,x^2+9\,x^4)\,\text{Log}[x]}{128\,x^2}\right) + Q^3\left(\frac{185+9\,x^2-9\,x^4-185\,x^6}{3072\,x^3} - \frac{(25+9\,x^2+9\,x^4+25\,x^6)\,\text{Log}[x]}{512\,x^3}\right) + Q^4\left(\frac{35\,(-533-32\,x^2+32\,x^6+533\,x^8)}{393\,216\,x^4} + \frac{(1225+400\,x^2+324\,x^4+400\,x^6+1225\,x^8)\,\text{Log}[x]}{32\,768\,x^4}\right) + Q^5\left(\frac{307\,503+19\,775\,x^2+3600\,x^4-3600\,x^6-19\,775\,x^8-307\,503\,x^{10}}{7\,864\,320\,x^5} - \frac{(3969+1225\,x^2+900\,x^4+900\,x^6+1225\,x^8+3969\,x^{10})\,\text{Log}[x]}{131\,072\,x^5}\right)$$

In[4]:= `Integrate[%, Q] // Collect[#, {Q, _Log}, Simplify] &`

Out[4]=
$$\frac{1}{2}\,Q\,\text{Log}[x] + Q^2\left(-\frac{-1+x^2}{16\,x} - \frac{(1+x^2)\,\text{Log}[x]}{16\,x}\right) + Q^3\left(\frac{7\,(-1+x^4)}{256\,x^2} + \frac{(9+4\,x^2+9\,x^4)\,\text{Log}[x]}{384\,x^2}\right) + Q^4\left(\frac{185+9\,x^2-9\,x^4-185\,x^6}{12\,288\,x^3} - \frac{(25+9\,x^2+9\,x^4+25\,x^6)\,\text{Log}[x]}{2048\,x^3}\right) + Q^5\left(\frac{7\,(-533-32\,x^2+32\,x^6+533\,x^8)}{393\,216\,x^4} + \frac{(1225+400\,x^2+324\,x^4+400\,x^6+1225\,x^8)\,\text{Log}[x]}{163\,840\,x^4}\right) + Q^6\left(\frac{307\,503+19\,775\,x^2+3600\,x^4-3600\,x^6-19\,775\,x^8-307\,503\,x^{10}}{47\,185\,920\,x^5} - \frac{(3969+1225\,x^2+900\,x^4+900\,x^6+1225\,x^8+3969\,x^{10})\,\text{Log}[x]}{786\,432\,x^5}\right)$$

Strategy II: SF expansion

- Integral family:

$$\int d^D\ell_1 d^D\ell_2 \frac{\delta(\ell_1\cdot u_1)\delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2][\ell_2\cdot u_1]} \frac{\log(\omega^2)}{[\ell_1^2][\ell_2^2][(\ell_1+\ell_2-q)^2][(\ell_1-q)^2][(\ell_2-q)^2]}$$

We first rewrite it as

$$\log\omega^2 = \log\left(\left(\frac{k\cdot u_1 + Qk\cdot u_2}{1+Q}\right)^2\right) = \log\left(\left(\ell_2\cdot u_1 + Q\ell_1\cdot u_2\right)^2\right) - 2\log(1+Q).$$

We can expand in $Q = m_2/m_1$

$$\log\left(\left(\ell_2\cdot u_1 + Q\ell_1\cdot u_2\right)^2\right) = \log\left(\left(\ell_2\cdot u_1\right)^2\right) - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{\ell_1\cdot u_2}{\ell_2\cdot u_1}Q\right)^n$$

- We factorised out the mass dependency Q (or ν)
- Ordinary 2-loop PM integrals, but with high powers for linear propagators
- `FIRE6.5` ($\oplus$`FLINT` $\oplus$ `LiteRed`) works weel to $\mathcal{O}(Q^{30})$ Smirnov-Zeng 2311.02370

Scattering angle

Nonlocalin-time contribution to the scattering angle:

$$\frac{1}{\pi\Gamma}\chi_{b(\text{nloc})}^{(4)\,\log} = -2\nu\chi_{2\epsilon}(\gamma) = \frac{-2\nu}{(\gamma^2-1)^2}\left[h_5 + h_9\log\frac{\gamma+1}{2} + h_{10}\frac{\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}\right]$$

$$\begin{aligned}\frac{1}{\pi\Gamma}\chi_{b(\text{nloc})}^{(4)(n\text{SF})} = \frac{\nu}{(\gamma^2-1)^2}\Bigg[&h_1 + \frac{\pi^2 h_2}{\sqrt{\gamma^2-1}} + h_3\log\frac{\gamma+1}{2} + \frac{h_4\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_5\log\frac{\gamma-1}{8}\\
&+ h_6\log^2\frac{\gamma+1}{2} + h_7\operatorname{arccosh}(\gamma)^2 + \frac{h_8\log(2)\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_9\log\frac{\gamma-1}{8}\log\frac{\gamma+1}{2}\\
&+\frac{h_{10}\log\frac{\gamma^2-1}{16}\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_{11}\text{Li}_2\frac{\gamma-1}{\gamma+1} + h_{12}\frac{\operatorname{arccosh}^2(\gamma)+4\text{Li}_2(\sqrt{\gamma^2-1}-\gamma)}{\sqrt{\gamma^2-1}}\Bigg]\end{aligned}$$

- We obtained exact-ν (iterated elliptic integrals) and SF-expanded (30SF) versions.

  h_i coefficients can found from the ancillary files in 2403.04853

- The result is in perfect agreement with the 6PN result in Bini-Damour-Geralico 2007.11239

Bound dynamics

- The total bound Hamiltonian up to 4PM:

$$\hat{H}^{\text{ell}}_{\text{4PM}} = \sum_{i=1}^{i=4} \frac{\hat{c}_{i(\text{loc})}}{\hat{r}^i} + \sum_{i=1}^{i=4} \frac{\hat{c}_{i(\text{nloc})}}{\hat{r}^i} + \frac{4\nu^2}{3\hat{r}^4} \frac{(\gamma^2-1)}{\Gamma^2 \xi} \chi_{2\epsilon} \log\left(\frac{\hat{r}}{e^{2\gamma_E}}\right)$$

$\hat{c}_{4(\text{loc})}$ is reported here for the first time.

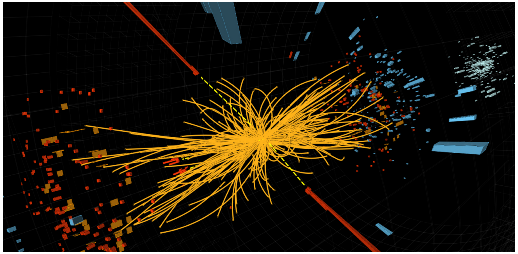
- Using the 6PN results (W_1-only) in [2007.11239], we obtained an improved bound Hamiltonian

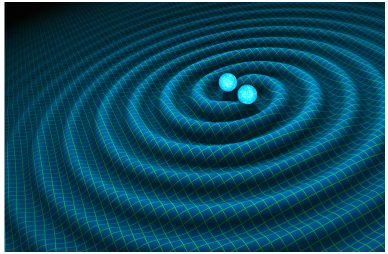
$$\hat{H}^{\text{ell}}(\hat{r}, \boldsymbol{p}^2, \nu) = \hat{E} + \sum_{i=1}^{i=4} \frac{\hat{c}_{i(\text{loc})}}{\hat{r}^i} + \frac{4\nu^2}{3\hat{r}^4} \frac{(\gamma^2-1)}{\Gamma^2 \xi} \chi_{2\epsilon}(\gamma) \log\left(\frac{\hat{r}}{e^{2\gamma_E}}\right)$$
$$+ \sum_{i=1}^{i=4} \frac{1}{\hat{r}^i} \left\{ \hat{c}^{\text{6PN}(e^8)}_{i(\text{nloc})} + \mathcal{O}\left(\hat{\boldsymbol{p}}^{2(8-i)}\right) \right\} + \frac{1}{\hat{r}^5} \left(\hat{c}^{\text{4PN}(e^8)}_{5(\text{loc+nloc})} - \frac{22\nu}{15} \log\left(\frac{\hat{r}}{e^{2\gamma_E}}\right) \right) + \mathcal{O}\left(\frac{\hat{\boldsymbol{p}}^2}{\hat{r}^5}\right)$$

- We find agreement with the $\hat{H}^{\text{ell}}_{\text{6PN(4PM)}}$ in Khalil-Buonanno-Steinhoff 2204.05047

Conclusion & Outlook

Modern techniques from collider physics have already proven useful to solve the gravitational two-body problem.





We have developed an efficient framework and obtained the full results at NNNLO, including conservative and dissipative parts, local/nonlocal separations.

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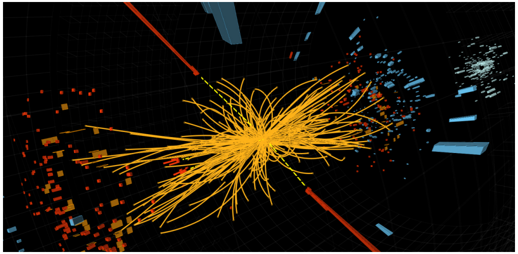
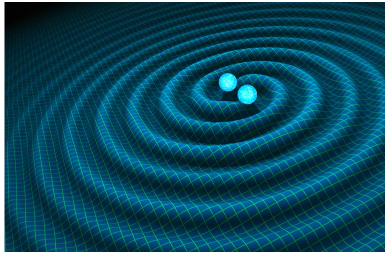
Modern techniques from collider physics have already proven useful to solve the gravitational two-body problem.





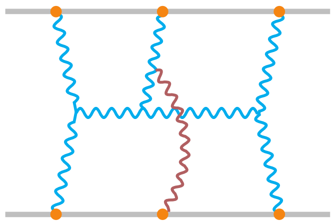
We have developed an efficient framework and obtained the full results at NNNLO, including conservative and dissipative parts, local/nonlocal separations.

Going to NNNNLO (5PM)

- Nonlocal conservative dynamics

  3-loop integrals (SF expand) `FIRE6.5`

- 2SF (1SF, Mogull's talk)



Thanks for your attention!

HORIZON 2020

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