Wonders of Kerr-Schild geometry

Abraham Harte

Centre for Astrophysics & Relativity (CfAR)
Dublin City University

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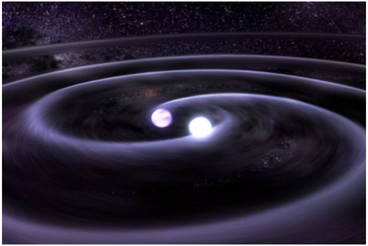
Spacetime tells matter how to move; matter tells spacetime how to curve.



- Elegant but computationally challenging: This drives nonlinearity.
- However, some interactions/nonlinearities are not as strong as one might expect. . .

Strong gravity $\not\Longrightarrow$ Strong nonlinearity

If ℓ_a is null and if $V\ell_a\ell_b$ solves the vacuum Einstein equation linearized about a vacuum background $\bar{g}_{ab}$,

$$g_{ab} = \bar{g}_{ab} + V\ell_a\ell_b$$

solves the exact vacuum Einstein equation. All nonlinearities vanish!

[Gürses and Gürsey (1975), Xanthopoulos (1978), ...]

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Kerr, exact plane waves, de Sitter, and are all linear in this sense (on flat backgrounds). There is no self-interaction.

Kerr-Schild double copy

For *some* Maxwell potentials $A_a = U\ell_a$ in flat spacetime, with ℓ_a null, $g_{ab} = \eta_{ab} + V\ell_a\ell_b$ is an exact solution to Einstein's equation.

[Monteiro, O'Connell, and White (2014), . . .]

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[Monteiro, O'Connell, and White (2014), ...]

Again, a relatively-simple linear equation can (sometimes) be used to solve a nonlinear equation:

1. Coulomb field $\mapsto$ Schwarzschild
2. EM plane waves $\mapsto$ Gravitational plane waves
3. EM field with const charge density $\mapsto$ de Sitter

Lesson #1

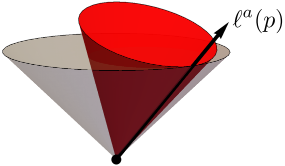
Simplifications can arise not only when (dimensionless magnitudes) $\ll 1$, but also when perturbations have a special tensorial structure.

Kerr-Schild transformations

Given some null ℓ_a, call

$$g_{ab} \mapsto g_{ab} + V \ell_a \ell_b$$

a Kerr-Schild transformation of g_{ab}.



Some of the most important metrics are Kerr-Schild transformations of flat spacetime: Schwarzschild, Kerr, plane waves, de Sitter, ...

Why is Kerr-Schild so simple?

1. Kerr-Schild perturbations $h_{ab} = V\ell_a\ell_b$ are nilpotent with degree 2, or "square roots of zero":
$$h_{ab}h^b{}_c = 0.$$
2. Metric inverses are linear:
$$g^{ab} = \bar{g}^{ab} - h^{ab}.$$
3. Volume elements are unperturbed:
$$\sqrt{-g} = \sqrt{-\bar{g}}.$$

Can Kerr-Schild be interesting *generically*?

- Few metrics are KS transformations of, say, η_{ab}.
- However, those which are can act as "attractors": Perturbed black holes settle down to Kerr, ultrarelativistic (Penrose) limits result in plane waves, etc. Spherically-symmetric metrics are also conformal KS.
- Can the reduced nonlinearity inherent in Kerr-Schild structures be exploited beyond these special cases? Look at perturbation theory.

Making approximate solutions exact

Let $h_{ab}^{(1)}$ be a solution to the vacuum linearized Einstein equation in any convenient gauge. If there exists (V, ℓ_a, ξ^a) such that

$$h_{ab}^{(1)} + \mathcal{L}_\xi \bar{g}_{ab} = V \ell_a \ell_b$$

with ℓ_a null,

$$g_{ab}^{\mathrm{exact}} = \bar{g}_{ab} + \mathcal{L}_\xi \bar{g}_{ab} + h_{ab}^{(1)}$$

is exact.

[AH & Vines (2016)]

Linearized approximations can contain all information necessary to construct exact solutions.

A guaranteed "resummation"

If one can transform to a "KS gauge," $h_{ab}^{(1)}$ and errors in the 1st-order gauge xform $\mathcal{L}_\xi \bar{g}_{ab}$ are *guaranteed* to cancel:

$$g_{ab}^{\mathrm{exact}} = \bar{g}_{ab} + (h_{ab}^{(1)} + \cancel{E_{ab}}) + (\mathcal{L}_\xi \bar{g}_{ab} - \cancel{E_{ab}}).$$

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Example: Exact Kerr can be obtained in this way from a 1st-order metric perturbation in Lorenz gauge (which is not exact).

Introducing matter

If a Kerr-Schild gauge exists, the *exact* stress-energy is

$$T^a{}_b = (\text{linear operator})[h_{\mathrm{KS}}].$$

Linearized stress-energy in original gauge becomes exact.

[AH & Vines (2016)]

1. Linearity $\implies$ Distributional stress-energies make sense "nonlinearly."
2. $\bar{\nabla}_a T^a{}_b = 0$, so "self-fields exert no force."

Lesson #2

Gauge choices in perturbation theory are not just a matter of computational or interpretational convenience. Gauge can also alter accuracy.

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- It may be useful to use (or to transform to) something as close as possible to a "Kerr-Schild gauge"
- Some work on this already via the "highly-regular gauge" [Pound (2017)]

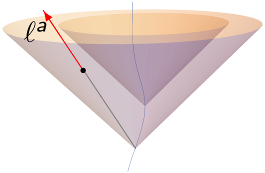
- These results are still not generic.
- Instead of asking for how different parts of the geometry might decouple from other parts of *itself*, ask for how parts of the geometry might decouple from test fields in that geometry.
- Kerr-Schild again plays a central role!

How much does geometry affect a *high-frequency* test field?

At high frequencies, fields look like

$$F_{ab}(x) = [\mathcal{A}_{ab}(x) + \mathcal{O}(\omega^{-1})]e^{i\omega\varphi(x)}$$

as $\omega \to \infty$, where $\ell_a = \nabla_a \varphi$ is tangent to a twist-free null geodesic congruence.



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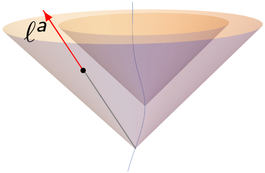

Focus first on these geodesics.

Given an eikonal φ which is compatible with some metric g_{ab}, is there another metric g'_{ab} in which φ is also a valid eikonal?

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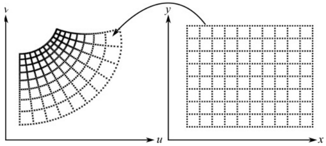

Conformal transformations $g_{ab} \mapsto \Omega^2 g_{ab}$ preserve eikonals. Is that all?

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Irrelevant parts of the metric:

- 1 component from 1 conformal transformation,
- 3 components from 3 Kerr-Schild transformations,
- 3 components from 1 "generalized Kerr-Schild" transformation.

Multiple Kerr-Schild transformations

One can preserve an eikonal φ with any Kerr-Schild transformation $g_{ab} \mapsto g_{ab} + V k_a k_b$ in which $k \cdot k = 0$ and $k \cdot \nabla\varphi = 0$.

1. One option is $k_a = \nabla_a \varphi$. [1 free function]

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1. One option is $k_a = \nabla_a \varphi$. [1 free function]

2. Another option is to choose k_a to be complex and orthogonal to $\nabla_a \varphi$. But $Vk_a k_b$ is then complex.

   Produce real metrics by performing *two* KS transformations, one with k_a and one with $\bar{k}_a$. [2 free functions]

Composing Kerr-Schild transformations

Multiple KS transformations *don't just add*; they must be *composed*.

1. If k_a and k'_a are both null wrt g_{ab}, it won't necessarily be true that k'_a is also null wrt $g_{ab} + V k_a k_b$: The "obvious" metric $g_{ab} + V k_a k_b + V' k'_a k'_b$ is not a KS transformation of $g_{ab} + V k_a k_b$.

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2. Find a linear combination of k'_a and k_a which *is* null wrt $g_{ab} + V k_a k_b$.

$$g_{ab} \mapsto g_{ab} + \frac{V k_a k_b + (k \cdot k') V V' k_{(a} k'_{b)} + V' k'_a k'_b}{1 - \frac{1}{4}(k \cdot k')^2 V V'}$$

Preserving high-frequency fields

If $m \cdot \nabla \varphi = 0$ and $m \cdot \bar{m} = 1$, the transformations

$$g_{ab} \mapsto \Omega^2 \left[g_{ab} + w_{(a} \nabla_{b)} \varphi + \frac{V m_a m_b + \bar{V} \bar{m}_a \bar{m}_b + |V|^2 m_{(a} \bar{m}_{b)}}{1 - \frac{1}{4}|V|^2} \right]$$

preserve the eikonal φ, where Ω, V, and w_a are arbitrary.

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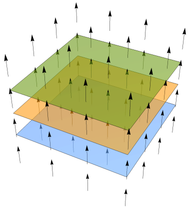
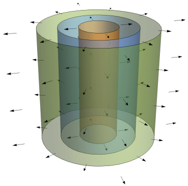
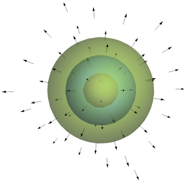
- These transformations also preserve all scalar amplitudes at high-frequencies
- At least the $V = 0$ transformations also preserve EM and gravitational amplitudes (but not in Lorenz gauge!)

Generating new solutions from old I

Given a known high-frequency field in one spacetime, generate another field in the *same* spacetime.





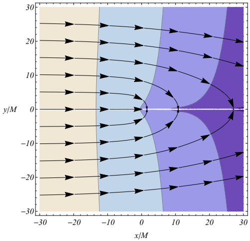


Example: Plane waves to spherical and cylindrical waves.

Generating new solutions from old II

Given a known high-frequency field in one spacetime, generate a field in a *different* spacetime.

Example: Plane wave in flat spacetime to initially-planar wave in Schwarzschild.



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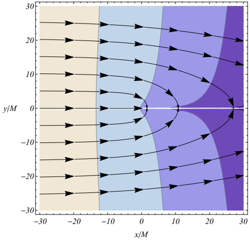

(7 free metric functions) + (4 gauge functions) > 10: A field in one spacetime can be turned into a field in any other spacetime!

Beyond geometric optics

Given an electromagnetic field F_{ab} which is known to be compatible with a metric g_{ab}, are there other metrics g'_{ab} with which it is also compatible?

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Given an electromagnetic field F_{ab} which is known to be compatible with a metric g_{ab}, are there other metrics g'_{ab} with which it is also compatible?

- The metric can be deformed with five free functions [AH (2017)]
- Yet again, Kerr-Schild transformations play a central role!

Null electromagnetic fields ($F^{ab}F_{ab} = F^{ab}F^{\star}_{ab} = 0$)

If F_{ab} is a null Maxwell field with metric g_{ab} and null eigenvector ℓ_a, it remains a solution under all transformations

$$g_{ab} \mapsto \Omega^2 (g_{ab} + \ell_{(a} w_{b)}),$$

where Ω and w_a are arbitrary.

[AH (2017)]

These are the $V = 0$ transformations from geometric optics without any complex Kerr-Schild transformations.

Non-null electromagnetic fields

General metric transformations for a *non-null* (generic) F_{ab} with null eigenvectors ℓ_a, k_a, m_a and $\bar{m}_a$ are

$$g_{ab} \mapsto \Omega^2(\mathrm{KS}[\ell] \circ \mathrm{KS}[k] \circ \mathrm{KS}[m] \circ \mathrm{KS}[\bar{m}])g_{ab}.$$

Except for the conformal factor, everything is a composition of Kerr-Schild transformations.

Example: Interaction between gravitational and EM waves

Plane-fronted electromagnetic wave

$$F_{ab} = \ell_{[a} \nabla_{b]} \alpha_{\mathrm{EM}}$$

is a solution in flat spacetime. It is also a solution in the plane-fronted gravitational-wave spacetimes

$$g_{ab} = \eta_{ab} + \alpha_{\mathrm{grav}} \ell_a \ell_b.$$

EM waves are not affected by gravitational waves which propagate in the same direction.

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- Physically, this is because TT-gauge coordinates are tied to *timelike* geodesics, and those aren't preserved in the same ways as EM fields, null geodesics, etc.

Conclusions

Lots of things in physics transform nicely, or decouple, with respect to Kerr-Schild transformations $g_{ab} \mapsto g_{ab} + V\ell_a\ell_b$.

1. EM fields, null geodesics, high-frequency gravitational waves, etc. decouple from KS geometry.
2. In full GR, nonlinearity in GR can be much weaker when perturbations have a KS (or related) form.
3. These results and others form a growing toolbox. Expand and apply!