Statistics of tensor fields in observational cosmology and nonlinear perturbation theory

Selected topics from:
T. Matsubara, arXiv:2210.10435 (Paper I)
T. Matsubara, arXiv:2210.11085 (Paper II)
T. Matsubara, arXiv:2304.13304 (Paper III)
T. Matsubara, arXiv:2405.09038 (Paper IV)

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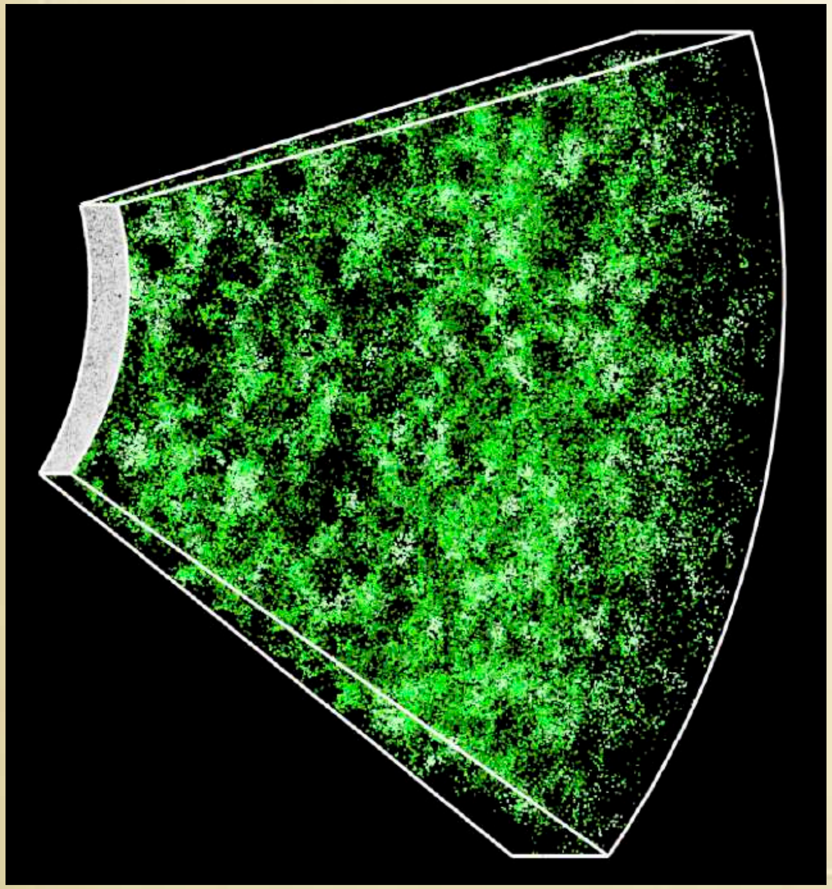
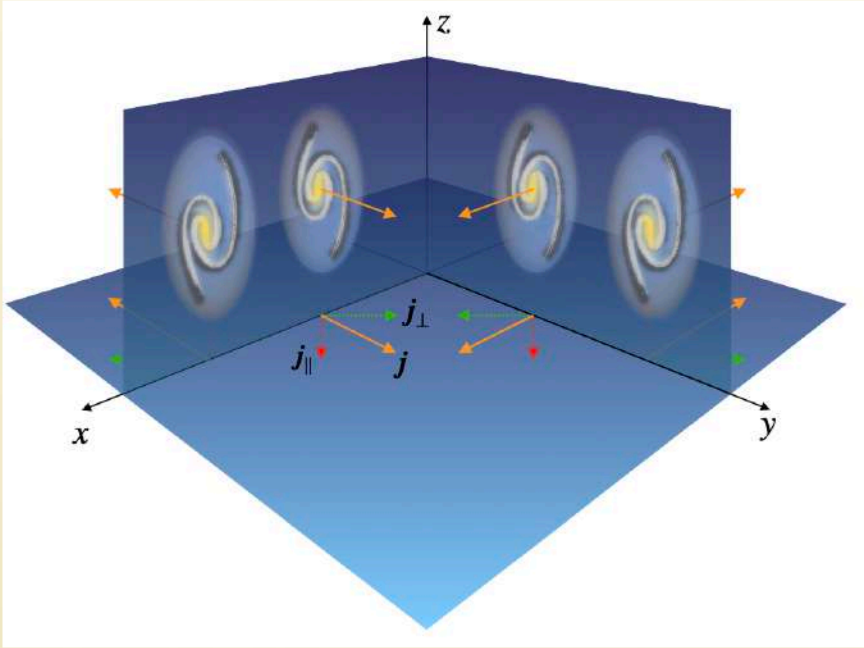
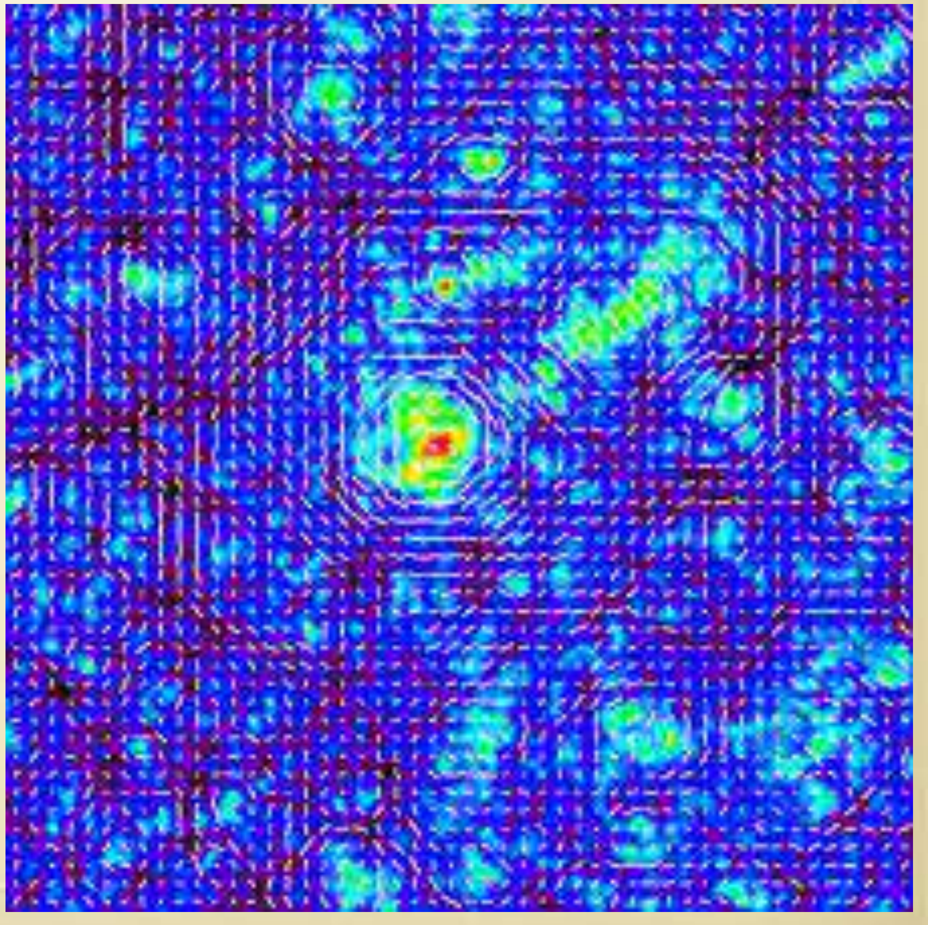
Tensor fields in cosmology

- Large-scale structure (LSS), weak lensing
  - galaxy density field: 3D rank-0 scalar field
  - galaxy angular momentum: 3D rank-1 vector field
  - galaxy shape field: 3D rank-2 (or higher) tensor fields







Example: galaxy shape alignment

- The spatial patterns of galaxy shapes:
    - The alignments are statistically correlated to the initial condition of the Universe, and thus to the large-scale structure of the universe
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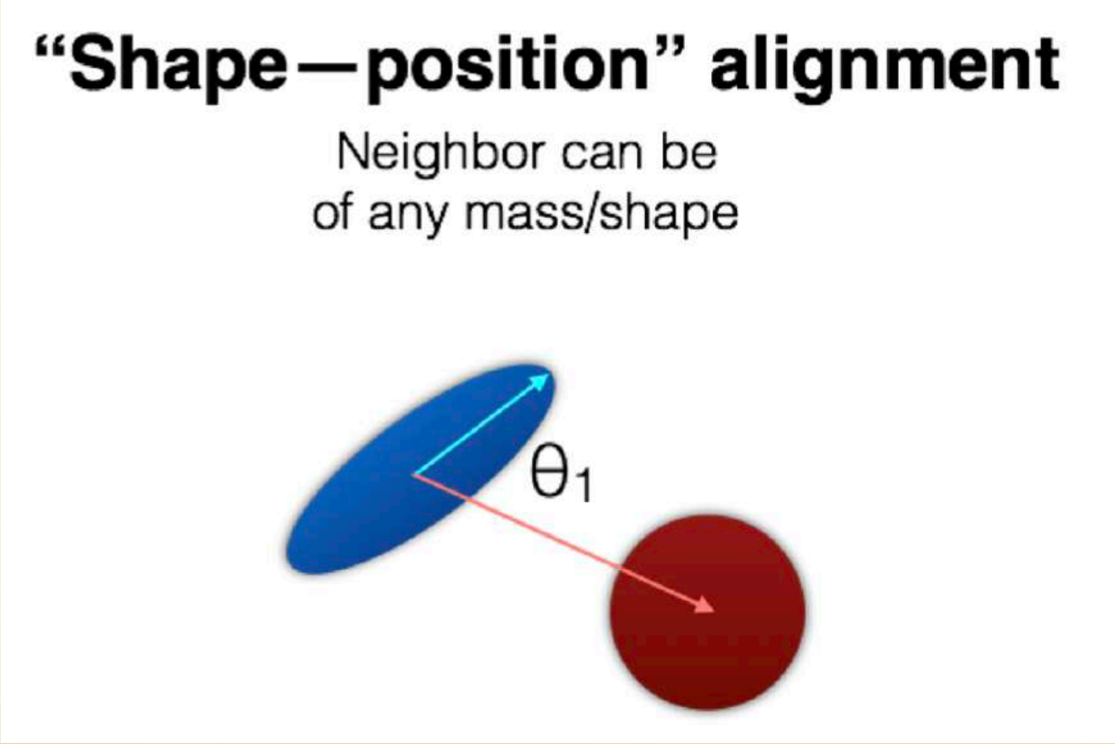
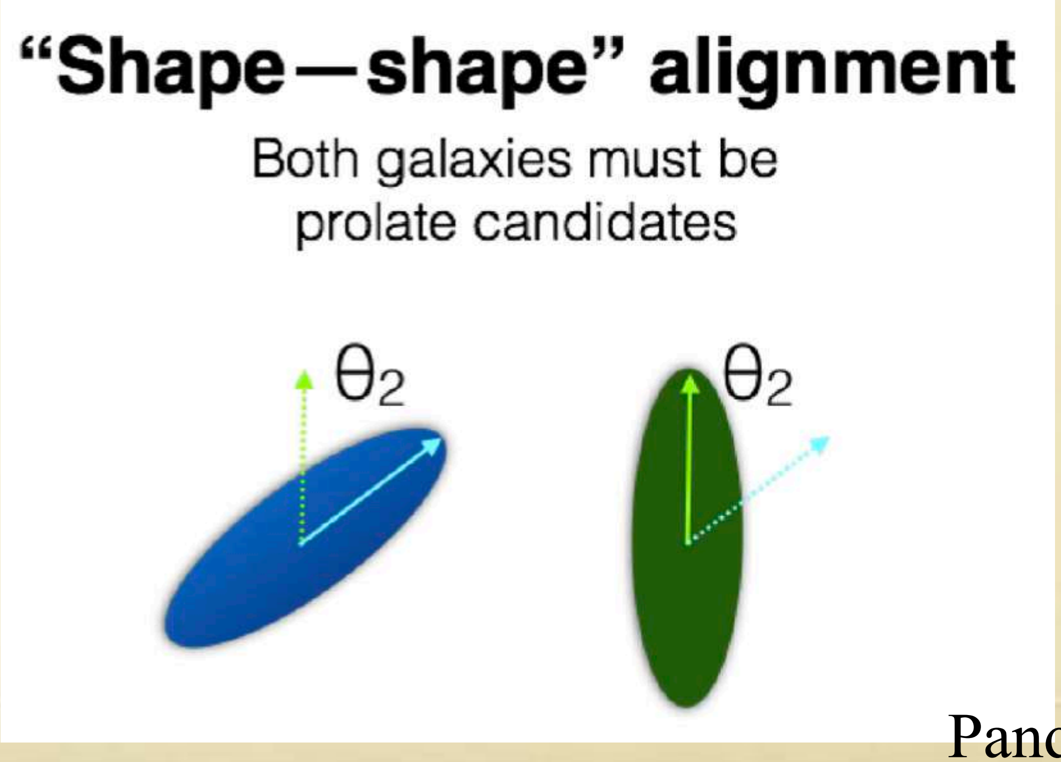
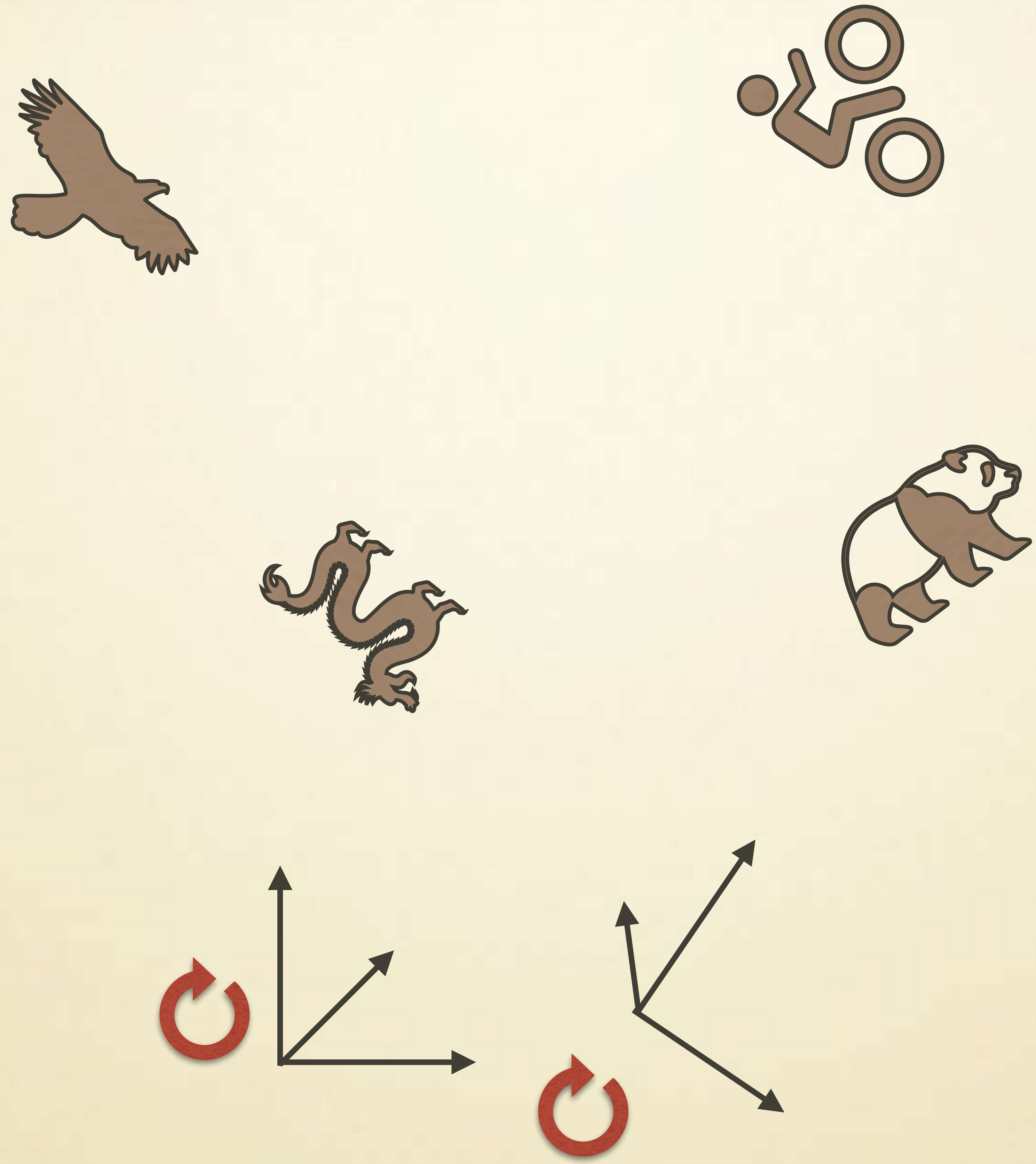




Pandya+ 2019



Moments of galaxy shapes

- One can define higher-order shape fields from higher-order moments (c.f. Kogai+ 2021)

$$I_{i_1\cdots i_l}(\boldsymbol{x}) = \frac{\int d^3x'\,(x'_{i_1} - x_{i_1})\cdots(x'_{i_l} - x_{i_l})\rho(\boldsymbol{x}')}{\int d^3x'\,\rho(\boldsymbol{x}')}$$

- The higher-order shape field

$$I_{i_i\cdots i_l}(\boldsymbol{x}) = \sum_{a\in\text{galaxies}} I_{i_1\cdots i_l}(\boldsymbol{x}_a)\delta^3_{\text{D}}(\boldsymbol{x} - \boldsymbol{x}_a)$$

  - (Normalization is arbitrary)

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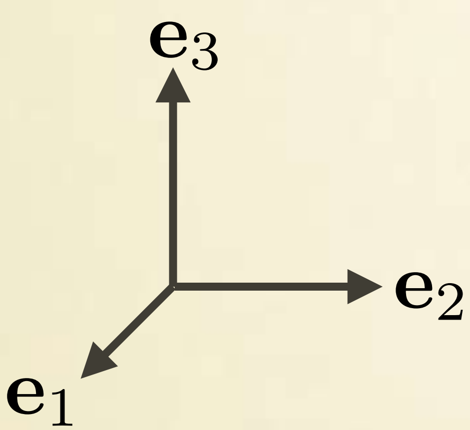
Spherical basis

- Spherical basis in Cartesian coordinates



$$\mathbf{e}^0 \equiv \mathbf{e}_3, \quad \mathbf{e}^{\pm} \equiv \mp \frac{\mathbf{e}_1 \mp i\mathbf{e}_2}{\sqrt{2}}$$

- Traceless spherical tensor basis

$$\text{rank-0} : Y^{(0)} \equiv 1$$

$$\text{rank-1} : Y_i^{(0)} = \mathrm{e}^0{}_i,\ Y_i^{(\pm 1)} = \mathrm{e}^{\pm}{}_i$$

$$\text{rank-2} : Y_{ij}^{(0)} = \sqrt{\frac{3}{2}}\left(\mathrm{e}^0{}_i \mathrm{e}^0{}_j - \frac{\delta_{ij}}{3}\right),\ Y_{ij}^{(\pm 1)} = \sqrt{2}\mathrm{e}^0{}_{(i}\mathrm{e}^{\pm}{}_{j)},\ Y_{ij}^{(\pm 2)} = \mathrm{e}^{\pm}{}_{(i},\mathrm{e}^{\pm}{}_{j)}$$

$$\cdots$$

$$\text{rank-}l : Y^{(m)}_{i_1 \cdots i_l} = \cdots \quad (m = 0, \pm 1, \ldots, \pm l)$$

Decomposition of tensor into spherical basis

- Any symmetric tensor can be decomposed into traceless tensors

$$T_{i_1 i_2 \cdots i_l} = T^{(l)}_{i_1 i_2 \cdots i_l} + \frac{l(l-1)}{2(2L-1)} \delta_{(i_1 i_2} T^{(l-2)}_{i_3 \cdots i_l)} + \cdots$$

- Decomposition of traceless tensor field into spherical basis:

$$F^{(l)}_{X i_1 \cdots i_l}(\boldsymbol{x}) = F_{Xlm}(\boldsymbol{x}) Y^{(m)}_{i_1 \cdots i_l}$$

Power spectrum of tensor field

- Definition of the power spectrum in Fourier space

$$\langle F_{X_1 l_1 m_1}(\boldsymbol{k}) F_{X_2 l_2 m_2}(\boldsymbol{k}') \rangle = (2\pi)^3 \delta_{\rm D}^3(\boldsymbol{k}+\boldsymbol{k}') P^{(l_1 l_2)}_{X_1 X_2 m_1 m_2}(\boldsymbol{k})$$

- Statistical isotropy
  - When the Universe is statistically isotropic, the power spectrum should be invariant under the coordinate rotation
  - In this case, the power spectrum should take the following form,

$$P^{(l_1 l_2)}_{X_1 X_2 m_1 m_2}(\boldsymbol{k}) = \sum_l \underbrace{(l_1\, l_2\, l)_{m_1 m_2}{}^{m}}_{\text{3j-symbol}} \underbrace{C_{lm}(\hat{\boldsymbol{k}})}_{\text{Spherical harmonics (Racah's normalization)}} \underbrace{P^{l_1 l_2; l}_{X_1 X_2}(k)}_{\text{Invariant spectrum}}$$

3j-symbol

Spherical harmonics
(Racah's normalization)

Invariant spectrum

Symmetries of invariant spectrum

- Complex conjugate

$$P^{l_1 l_2;l\,*}_{X_1 X_2}(k) = P^{l_1 l_2;l}_{X_1 X_2}(k) \quad \text{i.e., real function}$$

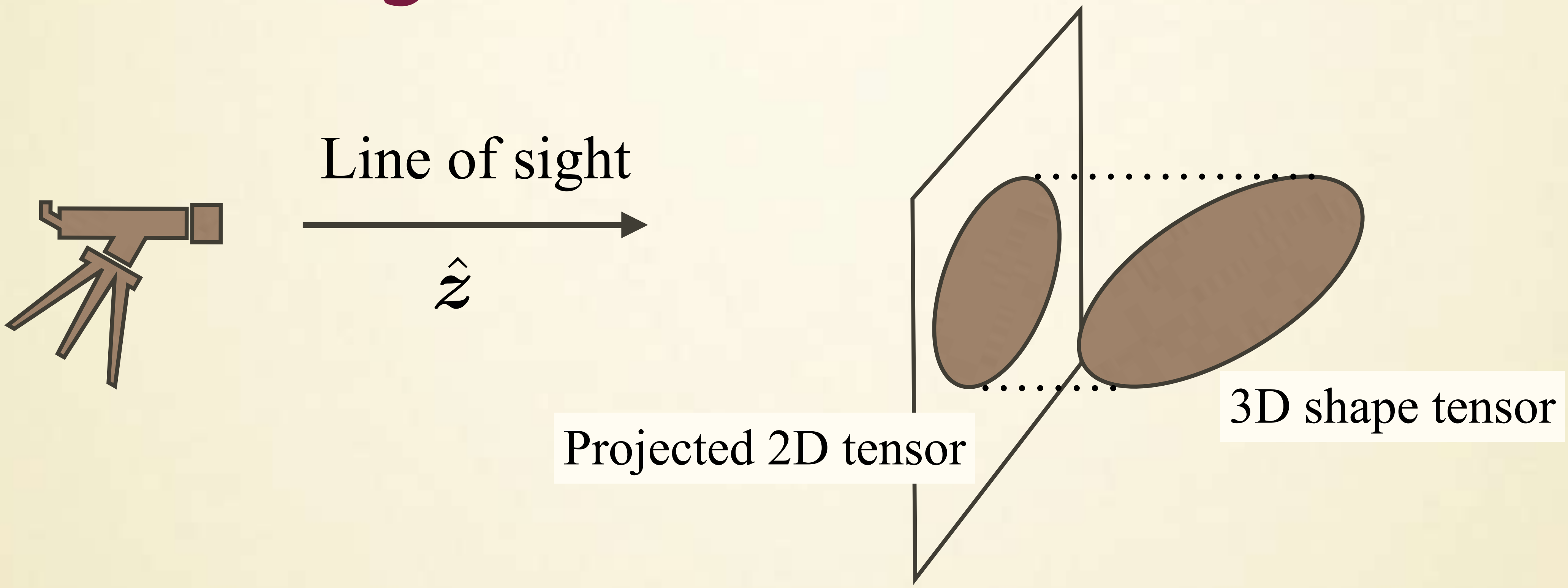
- Parity

$$P^{l_1 l_2;l}_{X_1 X_2}(k) \xrightarrow{\mathbb{P}} (-1)^{p_{X_1}+p_{X_2}+l_1+l_2+l} P^{l_1 l_2;l}_{X_1 X_2}(k)$$

- interchange

$$P^{l_2 l_1;l}_{X_2 X_1}(k) = (-1)^{l_1+l_2} P^{l_1 l_2;l}_{X_1 X_2}(k)$$

Projection effects



- Measurable tensors in realistic observations
    - 2D projected tensor on the sky

$$f_{X i_1 \cdots i_s}(\boldsymbol{x}) = \mathcal{P}_{i_1 j_1} \cdots \mathcal{P}_{i_s j_s} F_{X j_1 \cdots j_s}(\boldsymbol{x})$$

$$\mathcal{P}_{ij} = \delta_{ij} - \hat{z}_i \hat{z}_j \quad \text{(projection tensor)}$$

(distant-observer approximation applied)

2D irreducible decomposition

- 2D spherical basis

$$\mathbf{m}^{\pm} \equiv \mp \frac{\mathbf{e}_1 \mp i\mathbf{e}_2}{\sqrt{2}}$$

- Decomposition of 2D traceless tensor into 2D spherical basis

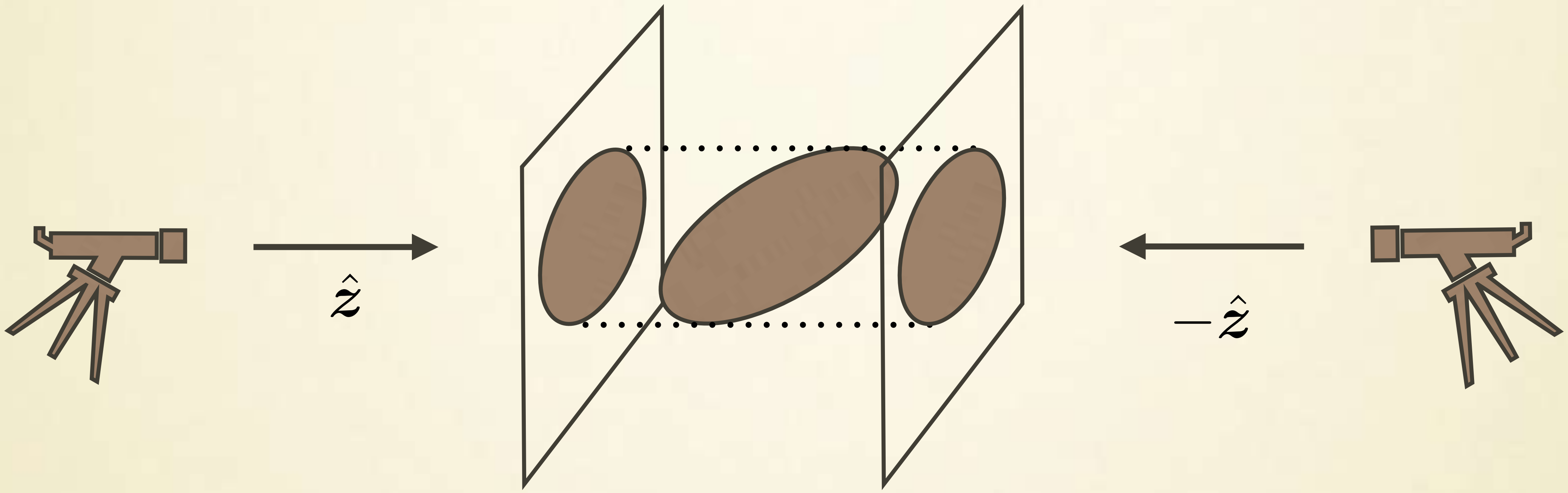
$$f^{(s)}_{X\,i_1\cdots i_s}(\boldsymbol{x}) = f^{(+s)}_{X}(\boldsymbol{x})\mathrm{m}^{+}_{i_1}\cdots\mathrm{m}^{+}_{i_s} + f^{(-s)}_{X}(\boldsymbol{x})\mathrm{m}^{-}_{i_1}\cdots\mathrm{m}^{-}_{i_s}$$

- Relation between 3D & 2D irreducible tensors

$$f^{(\pm s)}_{X}(\boldsymbol{x}) = i^s\sqrt{\frac{s!}{(2s-1)!}}\,\underbrace{F_{X\,s,\pm s}(\boldsymbol{x})}_{= \left.F_{X\,lm}(\boldsymbol{x})\right|_{l=s,m=\pm s}}$$

  - (The last eq. shows that the projected tensor from a 3D traceless tensor remains traceless also in the projected 2D space)

Flip symmetry of projected field



- Flip symmetry: invariance under $\hat{\boldsymbol{z}} \xrightarrow{\mathbb{F}} -\hat{\boldsymbol{z}}$

$$f_X^{(\pm s)}(\boldsymbol{x}) \xrightarrow{\mathbb{F}} f_X'^{(\pm s)}(\boldsymbol{x}) = e^{\pm 2is\phi} f_X^{(\mp s)}(-\boldsymbol{x}), \quad [\boldsymbol{x} : (x, \theta, \phi)]$$

- Parity+flip

$$f_X^{(\pm s)}(\boldsymbol{x}) \xrightarrow{\mathbb{PF}} f_X'^{(\pm s)}(\boldsymbol{x}) = (-1)^{p_X + s} e^{\pm 2is\phi} f_X^{(\mp s)}(\boldsymbol{x})$$

  - Parity+flip in distant-observer limit is more similar to the “parity” in full-sky spherical coordinates

E/B decomposition of projected field

- In Fourier space,

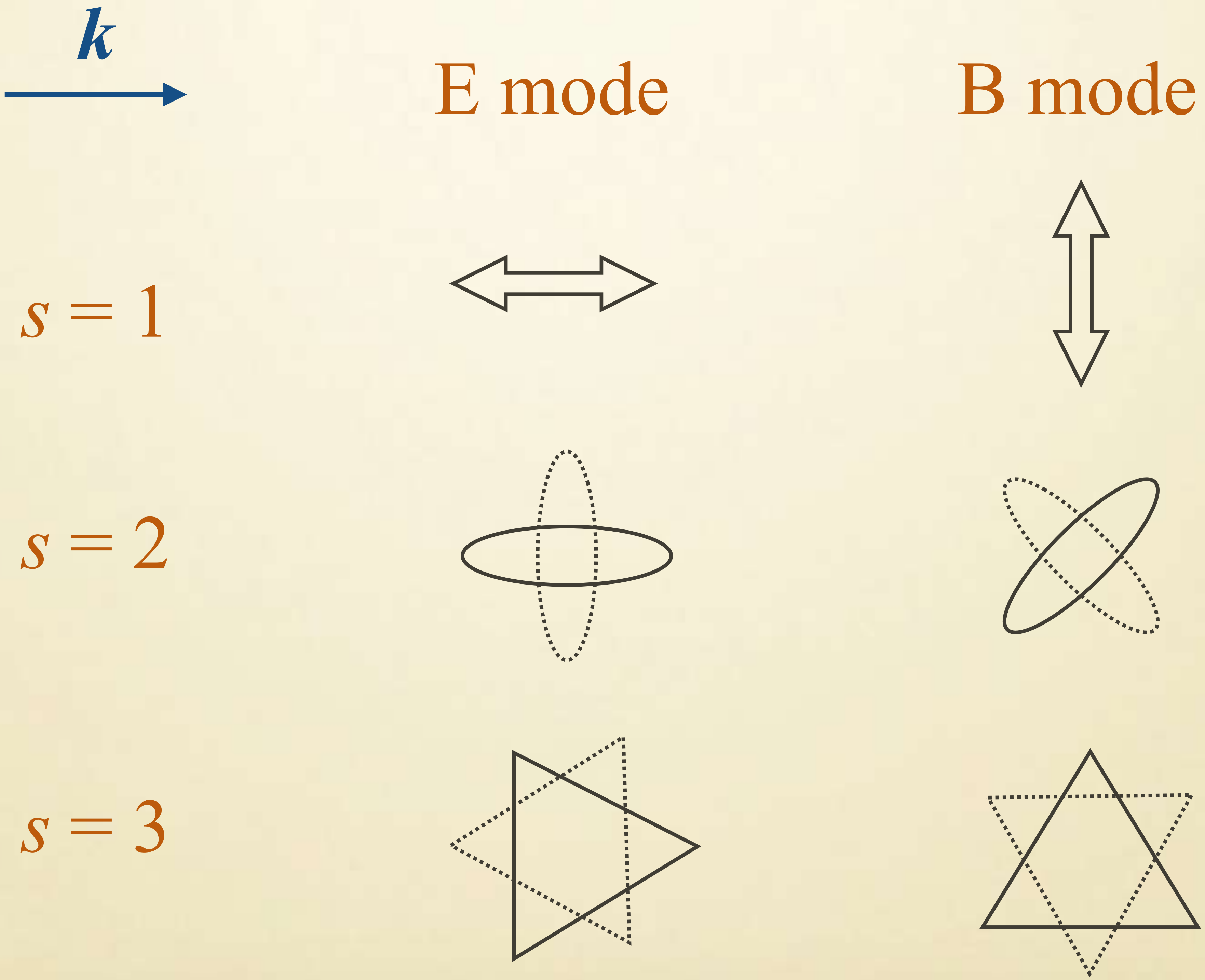
$$f_X^{(\pm s)}(\boldsymbol{k}) = (\mp i)^s e^{\pm is\phi} \left[f_X^{\mathrm{E}(s)}(\boldsymbol{k}) \pm i f_X^{\mathrm{B}(s)}(\boldsymbol{k}) \right]$$

- PF symmetry is simply given in the E/B modes

$$f_X^{\mathrm{E}(s)}(\boldsymbol{k}) \xrightarrow{\mathbb{PF}} (-1)^{p_X+s} f_X^{\mathrm{E}(s)}(\boldsymbol{k})$$
$$f_X^{\mathrm{B}(s)}(\boldsymbol{k}) \xrightarrow{\mathbb{PF}} (-1)^{p_X+s+1} f_X^{\mathrm{B}(s)}(\boldsymbol{k})$$

  - When p_X+s=even, E mode is parity even, B mode is parity odd
  - When p_X+s=odd, E mode is parity odd, B mode is parity even

E/B decomposition



Relations to the invariant spectrum

- Useful combinations:

$$P^{\pm} \equiv P^{\mathrm{EE}} \pm P^{\mathrm{BB}}$$
$$Q^{\pm} \equiv -i(P^{\mathrm{EB}} \pm P^{\mathrm{BE}})$$

$$P^{\mathrm{EE}} \sim \left\langle f_{X_1}^{\mathrm{E}(s_1)}(\boldsymbol{k}) f_{X_2}^{\mathrm{E}(s_2)}(-\boldsymbol{k}) \right\rangle$$
$$P^{\mathrm{EB}} \sim \left\langle f_{X_1}^{\mathrm{E}(s_1)}(\boldsymbol{k}) f_{X_2}^{\mathrm{B}(s_2)}(-\boldsymbol{k}) \right\rangle$$
$$P^{\mathrm{BE}} \sim \left\langle f_{X_1}^{\mathrm{B}(s_1)}(\boldsymbol{k}) f_{X_2}^{\mathrm{E}(s_2)}(-\boldsymbol{k}) \right\rangle$$
$$P^{\mathrm{EB}} \sim \left\langle f_{X_1}^{\mathrm{B}(s_1)}(\boldsymbol{k}) f_{X_2}^{\mathrm{B}(s_2)}(-\boldsymbol{k}) \right\rangle$$

- We derive

$$\begin{Bmatrix} P^+ \\ Q^- \end{Bmatrix} = \frac{1}{\sqrt{2\pi}} \sum_l \frac{1 \pm (-1)^{s_{12}^+ + l}}{2} \Theta_l^{s_{12}^-}(\mu) \begin{pmatrix} s_1 & s_2 & l \\ s_1 & -s_2 & -s_{12}^- \end{pmatrix} \hat{P}_{X_1 X_2}^{l_1 l_2; l}(k)$$

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$$P^- = \frac{1}{\sqrt{2\pi}} \frac{(-1)^{s_2}}{\sqrt{2s_{12}^+ + 1}} \Theta_{s_{12}^+}^{s_{12}^+}(\mu) \hat{P}_{X_1 X_2}^{s_1 s_2; s_{12}^+}(k), \qquad Q^+ = 0$$

$$s_{12}^{\pm} \equiv s_1 \pm s_2 \qquad \hat{P}_{X_1 X_2}^{l_1 l_2; l}(k) \equiv \sqrt{\frac{4\pi}{2l+1} \frac{l_1!\, l_2!}{(2l_1 - 1)!!\,(2l_2 - 1)!!}} P_{X_1 X_2}^{l_1 l_2; l}(k) \qquad \Theta_l^m(\mu) \equiv \sqrt{\frac{2l+1}{2} \frac{(l-m)!}{(l+m)!}} P_l^m(\mu)$$

Application of the perturbation theory

- Theoretical predictions from perturbation theory
  - In Paper I, systematic methods to derive the invariant spectrum from the “integrated Perturbation Theory” (iPT, Matsubara 2011) are formulated
  - In Paper II, further methods to calculate nonlinear corrections in perturbation theory are developed
  - In Paper III, the iPT is applied to the formulas of projection effects
  - In Paper IV, the formulas are generalized to those in the full-sky, without assuming the flat-sky, distant-observer limit.

Bias renormalization Schemes

Bias Renormalization Schemes

- Bias renormalization in conventional PT (orig., McDonald 2006)
  - For illustration, let's consider the simplest LIMD model (Lagrangian space, Gaussian initial condition):

$$\delta_X^{\mathrm{L}} = a_1\delta_R + \frac{a_2}{2!}\left({\delta_R}^2 - \sigma^2\right) + \frac{a_3}{3!}{\delta_R}^3 + \frac{a_2}{4!}\left({\delta_R}^4 - 3\sigma^2\right) + \cdots ,$$

  - Straightforward correlation function:

$$\xi_X^{\mathrm{L}}(q) = \left({a_1}^2 + a_1a_3\sigma^2 + \cdots\right)\xi_0(q) + \frac{1}{2}\left({a_2}^2 + a_2a_4\sigma^2 + \cdots\right)\left[\xi_0(q)\right]^2 + \cdots ,$$

  - Defining renormalized bias parameters:

$$b_1^{\mathrm{L}} = a_1 + \frac{a_3\sigma^2}{2} + \cdots , \quad b_2^{\mathrm{L}} = a_2 + \frac{a_4\sigma^2}{2} + \cdots ,$$

  - We have

$$\xi_X^{\mathrm{L}}(q) = (b_1^{\mathrm{L}})^2\xi_0(q) + \frac{1}{2}(b_2^{\mathrm{L}})^2\left[\xi_0(q)\right]^2 + \cdots .$$

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Bias Renormalization Schemes

- Bias renormalization in iPT
  - Renormalized bias functions

(when $\delta_X^{\mathrm{L}}(\delta_R)$ is a local function at the same position)

$$c_X^{(n)}(\boldsymbol{k}_1,\ldots\boldsymbol{k}_n) = (2\pi)^{3n}\int\frac{d^3k}{(2\pi)^3}\left\langle\frac{\delta^n\delta_X^{\mathrm{L}}(\boldsymbol{k})}{\delta\delta_{\mathrm{L}}(\boldsymbol{k}_1)\cdots\delta\delta_{\mathrm{L}}(\boldsymbol{k}_n)}\right\rangle = W(k_1R)\cdots W(k_nR)\left\langle\frac{\partial^n\delta_X^{\mathrm{L}}}{\partial{\delta_{\mathrm{R}}}^n}\right\rangle$$

  - iPT directly calculates the correlation function

$$\xi_X^{\mathrm{L}}(q) = \sum_{n=1}^{\infty}\frac{1}{n!}\int\frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}e^{i(\boldsymbol{k}_1+\cdots+\boldsymbol{k}_n)\cdot\boldsymbol{q}}\left[c_X^{(n)}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)\right]^2 P_{\mathrm{L}}(k_1)\cdots P_{\mathrm{L}}(k_n)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\left\langle\frac{\partial^n\delta_X^{\mathrm{L}}}{\partial{\delta_R}^n}\right\rangle^2[\xi_0(q)]^n$$

  - If we apply Taylor expansion as in conventional PT

$$\longleftarrow\quad \langle(\delta_R)^p\rangle = \sigma^2(p-1)!!$$

$$\delta_X^{\mathrm{L}} = \sum_{n=1}^{\infty}\frac{a_n}{n!}{\delta_R}^n \;\Rightarrow\; \left\langle\frac{\partial^n\delta_X^{\mathrm{L}}}{\partial{\delta_R}^n}\right\rangle = \sum_{m=0}^{\infty}\frac{a_{2m+n}\sigma^{2m}}{(2m-1)!!} \equiv b_n^{\mathrm{L}}$$

- Exactly the same as renormalized bias parameters in conventional PT
- Contains full-order nonlinear effects
- non-perturbative, no need for order-by-order renormalizations