Selected Developments in BAO++ Theory

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Outline

Review of the BAO in LPT

BAO Reconstruction and IR resummation

The Bispectrum in LPT

- Efficient IR Safe Resummation of Zeldovich Displacements
- Wiggle/No-Wiggle Split
- Primordial Features?

Lagrangian Perturbation Theory in a quick minute

Can think of Lagrangian displacement as an added phase in overdensity

$$\delta_g(\mathbf{k}) = \int d^3\mathbf{q}\; e^{-i\mathbf{k}\cdot\mathbf{q}} \;\left((1+\delta_{g,0}(\mathbf{q}))\; e^{-ik\cdot\Psi(\mathbf{q})} - 1 \right)$$

such that for example the power spectrum is $\Delta = \Psi(\mathbf{q}_1) - \Psi(\mathbf{q}_2)$:

$$P(\mathbf{k}) = \int d^3\mathbf{q}\; e^{i\mathbf{k}\cdot\mathbf{q} - \frac{1}{2}k_i k_j \left\langle \Delta_i^{(1)} \Delta_j^{(1)} \right\rangle} \left(1 + 2ib_1 k_i \left\langle \delta_1 \Delta_i^{(1)} \right\rangle + 2b_1 b_2 i k_i \left\langle \delta_1 \Delta_i^{(1)} \right\rangle \langle \delta_1 \delta_2 \rangle + \ldots \right)$$

Extension to redshift space is “easy”: $\Psi \rightarrow \Psi_s = \Psi + \hat{n}\dot{\Psi}_{\hat{n}}$.

Note Galilean symmetry of the expression!

Matsubara 2008, Carlson et al 2013, others.

Nonlinear BAO in another 30 seconds

Imagine these linear pieces have a sharp peak $\delta_D(q - r_s)$

Most significant effects

$$P(\mathbf{k}) = \int d^3\mathbf{q}\; e^{i\mathbf{k}\cdot\mathbf{q} - \frac{1}{2}k_i k_j \left\langle \Delta_i^{(1)} \Delta_j^{(1)} \right\rangle} \left(1 + 2ib_1 k_i \left\langle \delta_1 \Delta_i^{(1)} \right\rangle + 2b_1 b_2 i k_i \left\langle \delta_1 \Delta_i^{(1)} \right\rangle \langle \delta_1 \delta_2 \rangle + \ldots \right)$$

Damping

(Perturbative) Shift

These effects are controlled by simple parameters

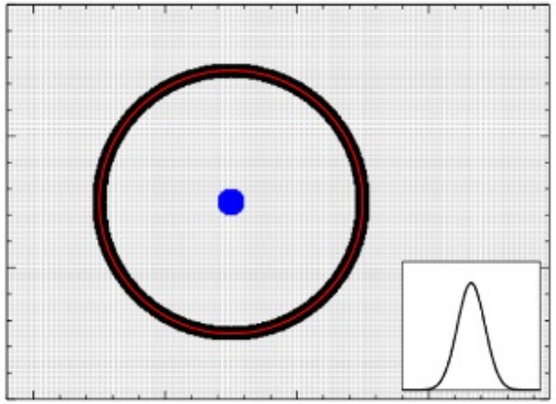
$$-\frac{1}{2}k_i k_j \left\langle \Delta_i^{(1)} \Delta_j^{(1)} \right\rangle_{q=r_s} \approx -\frac{1}{2}k^2 \int \frac{dk}{2\pi^2}\; P_{\rm lin}(k)\; \left(1 - j_0(kr_s) + 2j_2(kr_s)\right)$$

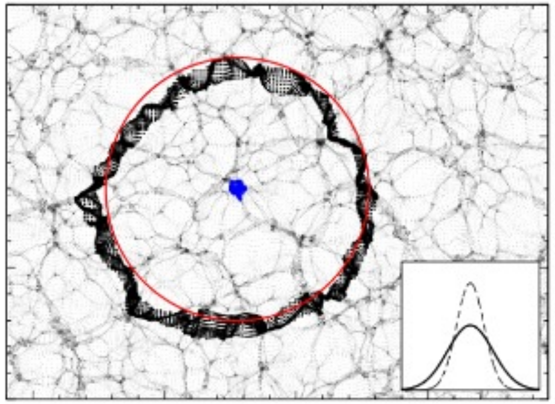
$$\left\langle \delta \Delta_i^{(1)} \right\rangle_{q=r_s} = -\frac{1}{3} q_i\; \sigma^2_{R=r_s}$$

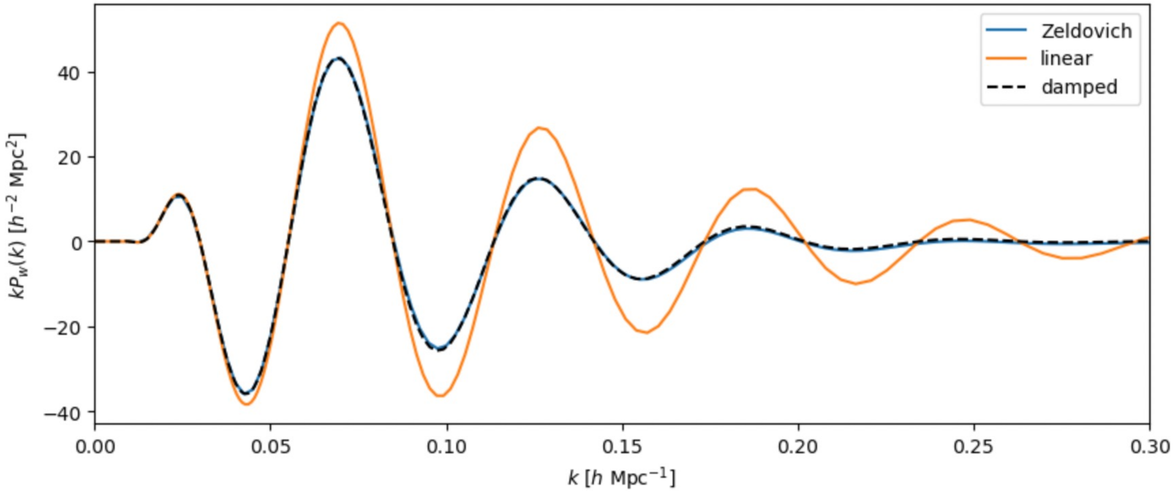
In LPT: Vlah et al 2016, Chen et al 2024

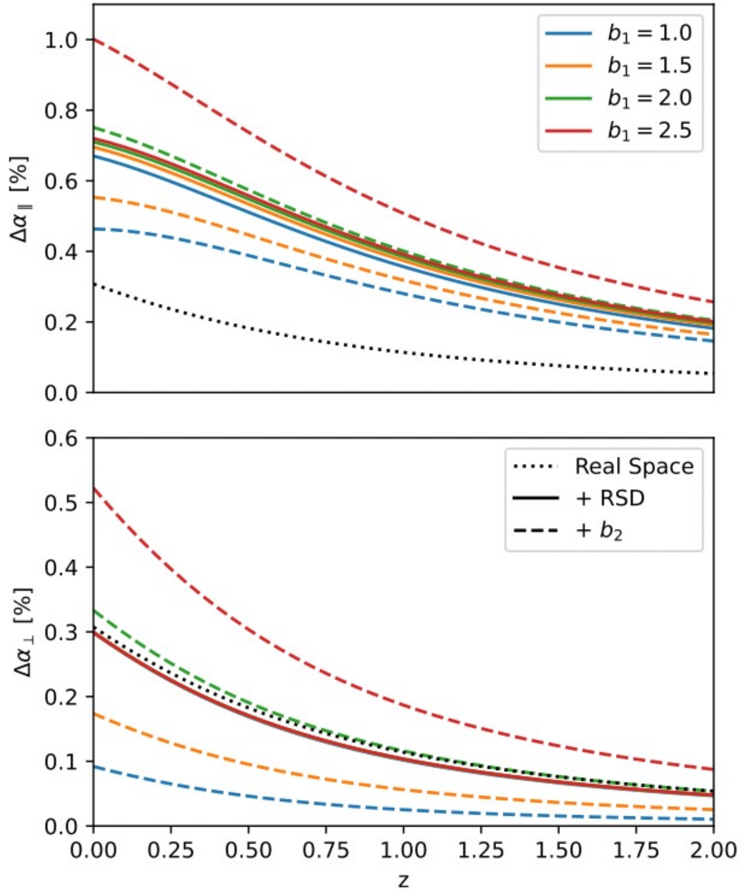
Lots of earlier/other work! e.g. Senatore et al 2015, Blas et al 2016

Nonlinear BAO in Pictures and Plots









Analytically, we understand the damping very well!*

*up to small-scale effects we have to fit

(Recent?) Application: Reconstruction and Algorithm Choice

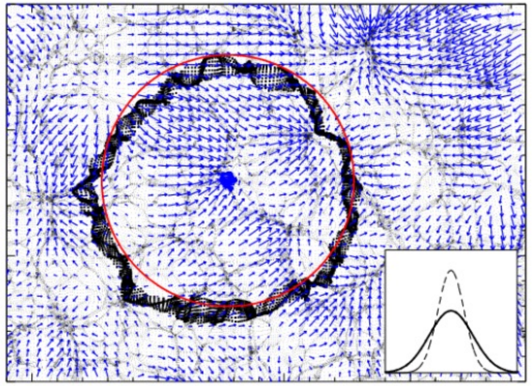
Known, quasi-magical way to undo the damping ("reconstruction"):

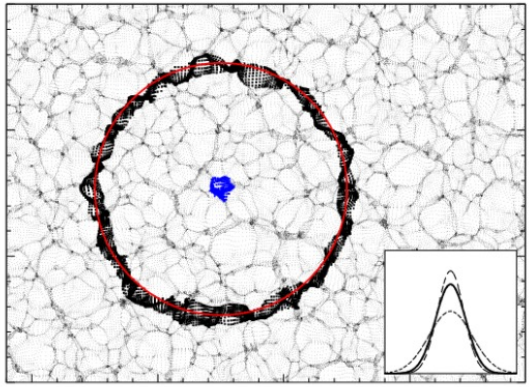
1. Smooth the density field with (Gaussian) filter S(k).
2. Reconstruct Zeldovich displacement with smoothed field:

$$\Psi_{\rm rec} = \frac{i\mathbf{k}}{k^2}\frac{\mathcal{S}(k)\ \delta_g}{b + f\mu^2}$$

3. Shift galaxies ('d') and randoms ('s') by the *negative* displacement.
4. Construct the reconstructed density field as $\delta_{\rm rec} = \delta_d - \delta_s$.

In redshift space, two options: **RecIso** (shift randoms in real space) or **RecSym** (shift randoms in redshift space too).





Eisenstein et al 2006, Padmanabhan et al 2012, White 2015.

Reconstruction (II)

Important point: BAO in randoms "s" subject to the same IR effects as galaxies *moved by the same amount*—because they are made from the galaxy density!

In fact a little calculation shows that e.g. the damping for both from IR displacements is given in **RecSym** by (cf. Chen et al 2020)

$$\Sigma^2_{s,\mathrm{rec}} = \int \frac{dk}{3\pi^2}(1 - j_0(kr_s) + 2j_2(kr_s))\ P_{\mathrm{lin}}(k)(1 - \mathcal{S}(k))^2$$

Not so in **RecIso**! Similarly the BAO shift is also fully not cancelled in RecIso, which therefore has more shift and less signal (in principle). (cf. also using Wiener filter instead of Gaussian)

Chen and Howlett et al (2024), see also Sugiyama (2024)

Beyond the Power Spectrum: Bispectrum in LPT

Extension of the Zeldovich exponent from the power spectrum ($q_{nm} = q_n - q_m$):

$$\exp\left[-\frac{1}{2}k_i k_j A_{ij}(\mathbf{q})\right] \to \exp\left[\frac{1}{2}\Big(k_{1,i}k_{3,j}A_{ij}(\mathbf{q}_{13}) + k_{2,i}k_{3,j}A_{ij}(\mathbf{q}_{23}) + k_{1,i}k_{2,j}A_{ij}(\mathbf{q}_{12})\Big)\right]$$

such that the bispectrum is

$$\int d^3\mathbf{q}_{13}d^3\mathbf{q}_{23}\ e^{-i\mathbf{k}_1\cdot\mathbf{q}_{13}-i\mathbf{k}_2\cdot\mathbf{q}_{23}}\exp\Big[\text{Zeldovich terms}\Big]\,(1+\text{bias terms})$$

Again, manifestly Galilean invariant, and you can probably guess the N-point function extension!

How to compute it (efficiently)

This calculation has historically been (numerically) difficult.

- (But see Tassev (2013) in configuration space).

However we can use techniques developed for the power spectrum and compute:

$$\mathcal{E}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}) = \int d^3\mathbf{q}\ e^{-i\mathbf{p}\cdot\mathbf{q} + \frac{1}{2}k_{1,i}k_{2,j}A_{ij}(\mathbf{q})}$$

(i.e. FFTLog)

such that the bispectrum is in terms of pairwise displacements

$$B(\mathbf{k}_1, \mathbf{k}_2) = \int_{\mathbf{p}} \mathcal{E}_{13}(\mathbf{k}_1 - \mathbf{p})\mathcal{E}_{23}(\mathbf{k}_2 + \mathbf{p})\mathcal{E}_{12}(\mathbf{p})$$

How to compute it (safely)

However we need to take a bit of care because

$$\mathcal{E}_{12}(\mathbf{p}) = \mathcal{E}_{12}^{\rm fin}(\mathbf{p}) + \left(1 - \int_{\mathbf{p}'} \mathcal{E}_{12}^{\rm fin}(\mathbf{p}')\right)(2\pi)^3\delta_D(\mathbf{p})$$

has a finite and infinite piece that obey an integral equation.

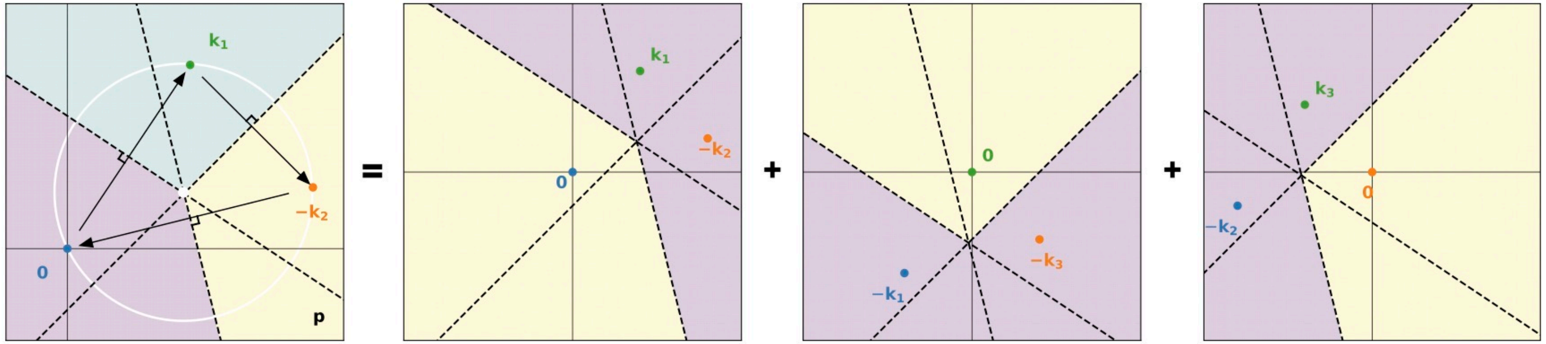
In linear theory

$$\mathcal{E}_{12}^{\rm lin}(\mathbf{p}) = -\frac{(\mathbf{k}_1\cdot\mathbf{p})(\mathbf{k}_2\cdot\mathbf{p})}{p^4}P_{\rm L}(p) + \left(1 + \frac{1}{2}(\mathbf{k}_1\cdot\mathbf{k}_2)\Sigma^2\right)(2\pi)^3\delta_D(\mathbf{p})$$

So we have to be quite careful when integrating in **p**.

How to compute it (safely) II

However, we can use the same procedure as the 1-loop bispectrum to make things IR safe, splitting into 3 IR regions and subtracting off the integral contribution



$$B(\mathbf{k}_1,\mathbf{k}_2) = \mathcal{E}^{\rm fin}_{13}(\mathbf{k}_1)\mathcal{E}^{\rm fin}_{23}(\mathbf{k}_2) + \int_{\mathbf{p}} \Big\{\mathcal{E}^{\rm fin}_{12}(\mathbf{p})\mathcal{E}^{\rm fin}_{13}(\mathbf{k}_1-\mathbf{p})\mathcal{E}^{\rm fin}_{23}(\mathbf{k}_2+\mathbf{p})\ \Theta(|\mathbf{k}_1-\mathbf{p}|-p)\Theta(|\mathbf{k}_2+\mathbf{p}|-p) - \mathcal{E}^{\rm fin}_{13}(\mathbf{p})\mathcal{E}^{\rm fin}_{12}(\mathbf{k}_1)\mathcal{E}^{\rm fin}_{23}(\mathbf{k}_3)\Big\} + (2\ \text{cycle})$$

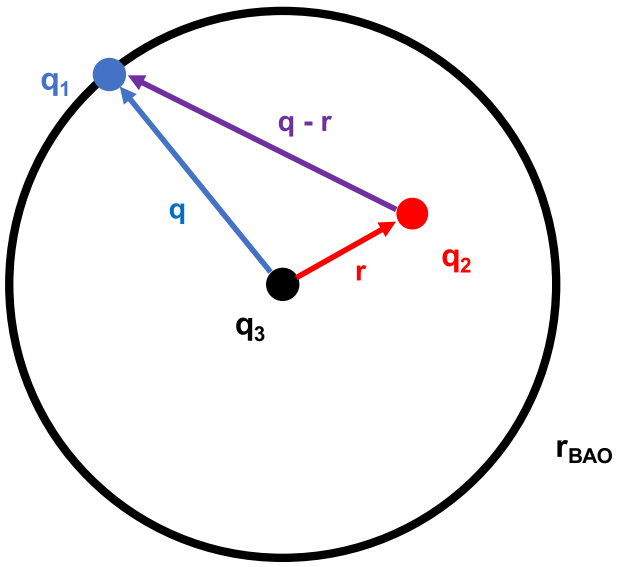
Regions also naturally include k/2 sphere around each pole.

(Baldauf et al (2015), see also Angulo et al (2016))

Wiggles in the Bispectrum

In the bispectrum the freedom of the third vertex/other two legs means the saddle point requires UV/IR separation, unlike in the power spectrum where it is "just" a saddle point.

$$\mathcal{E}_{13}^{\rm fin}(\mathbf{k}_1)\mathcal{E}_{23}^{\rm fin}(\mathbf{k}_2) + (\text{IR Region of } k_{1,2}) = \int_{\mathbf{q}_{1,2}} e^{-i\mathbf{k}_1\cdot\mathbf{q}_1 - i\mathbf{k}_2\cdot\mathbf{q}_2} \left(e^{\frac{1}{2}k_{1,i}k_{2,j}A_{ij}(\mathbf{q}_1-\mathbf{q}_2)}\right)_{\rm IR} \tilde{\mathcal{E}}_{13}^{\rm fin}(\mathbf{q}_1)\tilde{\mathcal{E}}_{23}^{\rm fin}(\mathbf{q}_2)$$



If BAO is in $\mathbf{q}_1$, dropping $\mathbf{q}_2$ dependence gives damping

$$e^{\frac{1}{2}(\mathbf{k}_1\cdot\mathbf{k}_2)\Sigma_s^2} e^{\frac{1}{2}(\mathbf{k}_1\cdot\mathbf{k}_3)\Sigma_s^2} P_{w,\rm lin}(k_1) = e^{-\frac{1}{2}k_1^2\Sigma_s^2} P_{w,\rm lin}(k_1)$$

as shown in TSPT.

This ignored dof incurs (perturbative) corrections due to IR modes on BAO scale due to derivatives of A_{ij}.

Blas et al (2016)

Wiggles in the N-point functions

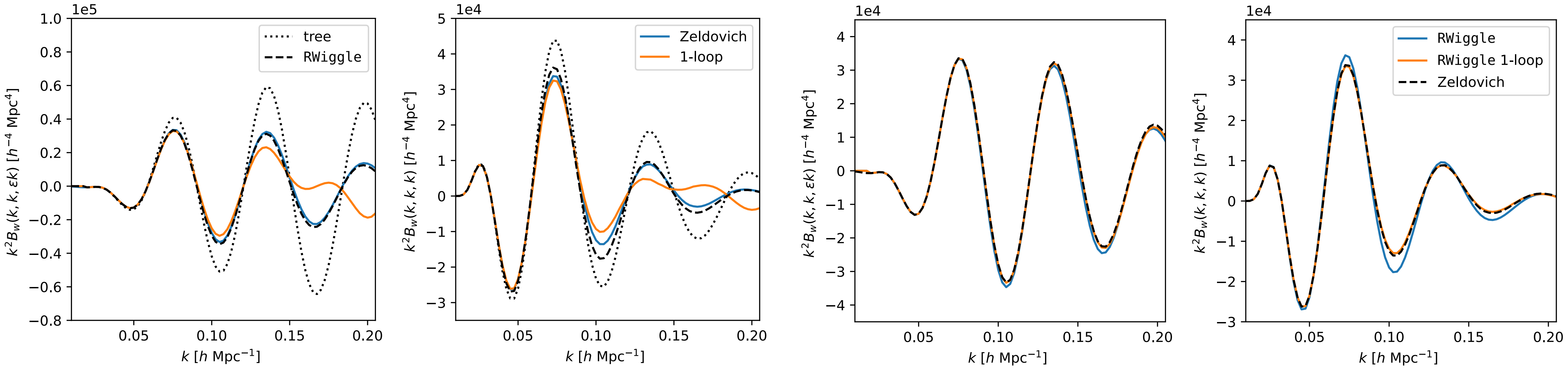
It's the same!

$$C_w \sim \left(\prod_{m=2}^{N-1} \int_{\mathbf{q}_m} e^{-i\mathbf{k}_m\cdot\mathbf{q}_m}\right) \exp\left\{\frac{1}{2}\sum_{n,m>1}^{n\neq m} k_{n,i}k_{m,j}A_{ij}(\mathbf{q}_{nm})\right\}$$

$$\int_{\mathbf{q}_1} e^{-i\mathbf{k}_1\cdot\mathbf{q}_1} \exp\left\{\frac{1}{2}\sum_{m\neq 1} k_{1,i}k_{m,j}A_{ij}(\mathbf{q}_{1m})\right\}\xi_w(\mathbf{q}_1)$$

$$\Rightarrow \quad \frac{1}{2}k_{1,i}\left(\sum_{m\neq n} k_{m,j}\right)A_{ij}(\mathbf{q}_s) = -\frac{1}{2}k_1^2\Sigma_s^2$$

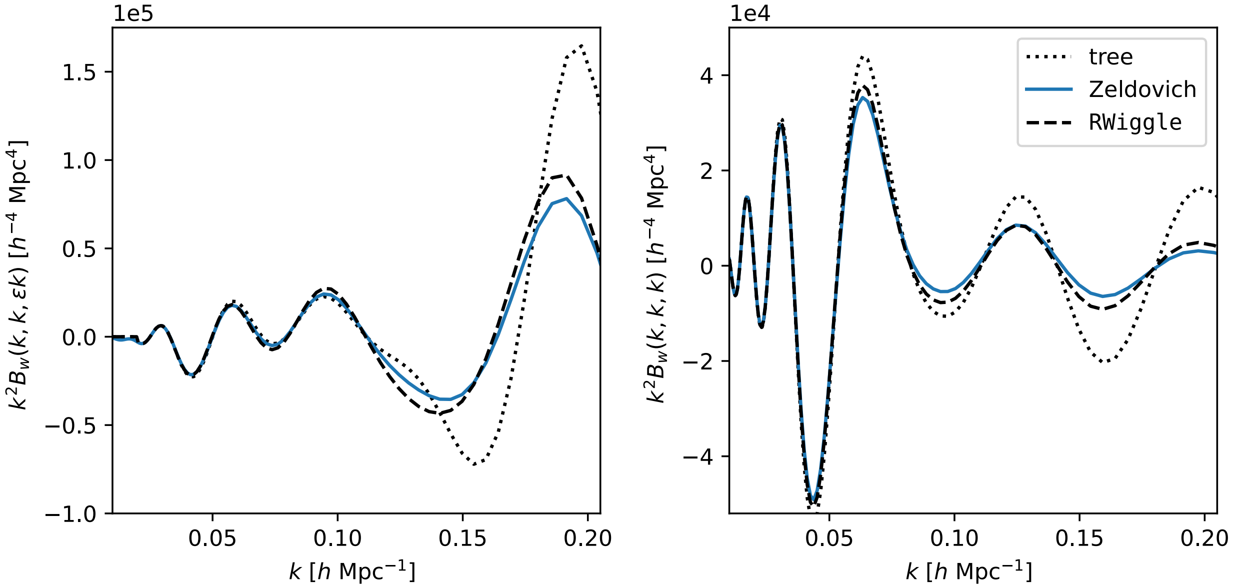
Comparisons with 1-loop



As could've been foreseen, adding loops naively is bad, but adding loops smartly is very good!
(i.e. corrections to Gaussian damping from Zeldovich are due to perturbative IR densities)

Primordial Features

Can also apply to the case of non-BAO features, e.g. for logarithmic wiggles



Can show that features in the power spectrum with nonlinear dispersion have damping ϕ(k) have k-dependent cutoff r = ϕ’(k) though this depends on a well-localized approximation and wiggle/no-wiggle split.

Beutler et al (2019), Vasudevan et al (2019), Chen et al (2020)

Conclusions

The state of the BAO is good.

Modern galaxy surveys can take advantage of what we know and e.g. do BAO reconstruction the “right” way.

The bispectrum at 1-loop is in excellent agreement between LPT and EPT, but need to be careful about primordial feature amplitude if only IR resuming at tree level.

triceratops

(Image courtesy of GPT-4o)

A Python code to compute bispectra in LPT. A cousin of velocileptors and spinosaurus.

Requires NumPy, SciPy, pyFFTW and velocileptors to run.