Optimal Transport Reconstruction of Large Scale Structures

Farnik Nikakhtar
YCAA Fellow, Yale University

In collaboration with:

Ravi K. Sheth (Penn)
Nikhil Padmanabhan (Yale)
Bruno Levy (Inria)
Roya Mohayaee (IAP)
Sasha Gaines (Yale)

The Higgs Centre for Theoretical Physics, Edinburgh
June 3rd, 2024

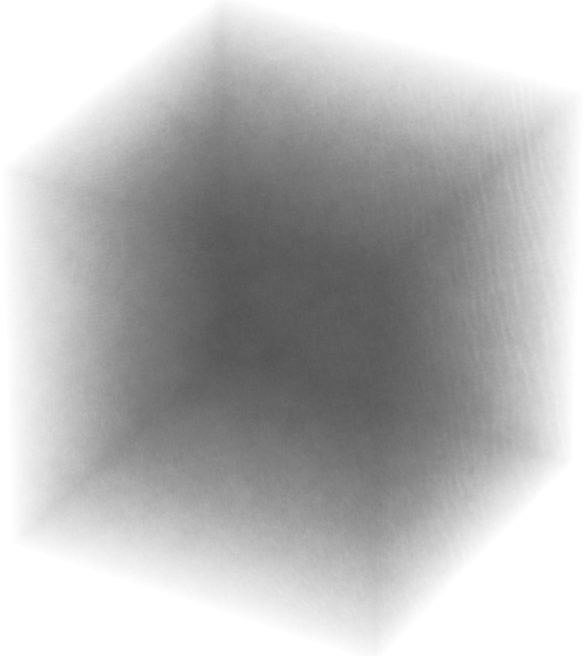
Jasper Johns (1930–), Mind/Mirror Exhibition
Philadelphia Museum of Art, Sep. 29, 2021–Feb. 13, 2022

Cosmological Reconstruction

Optimal Transport Theory

- How can we move back the evolution of cosmic structures?

Initial Condition

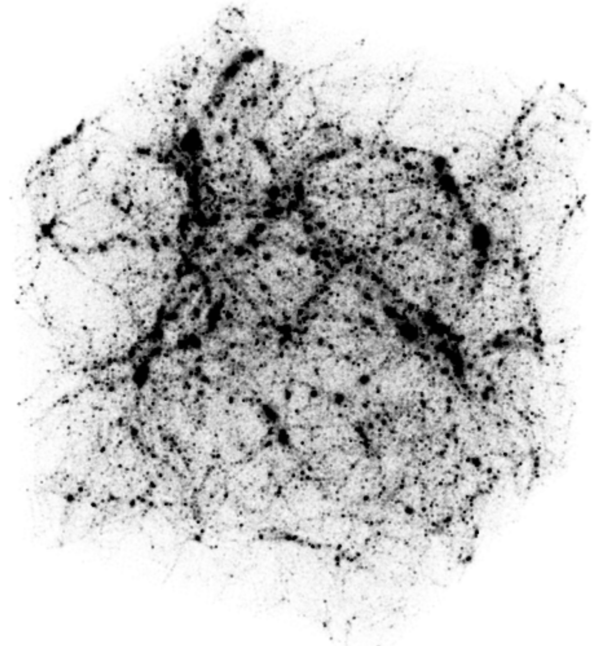


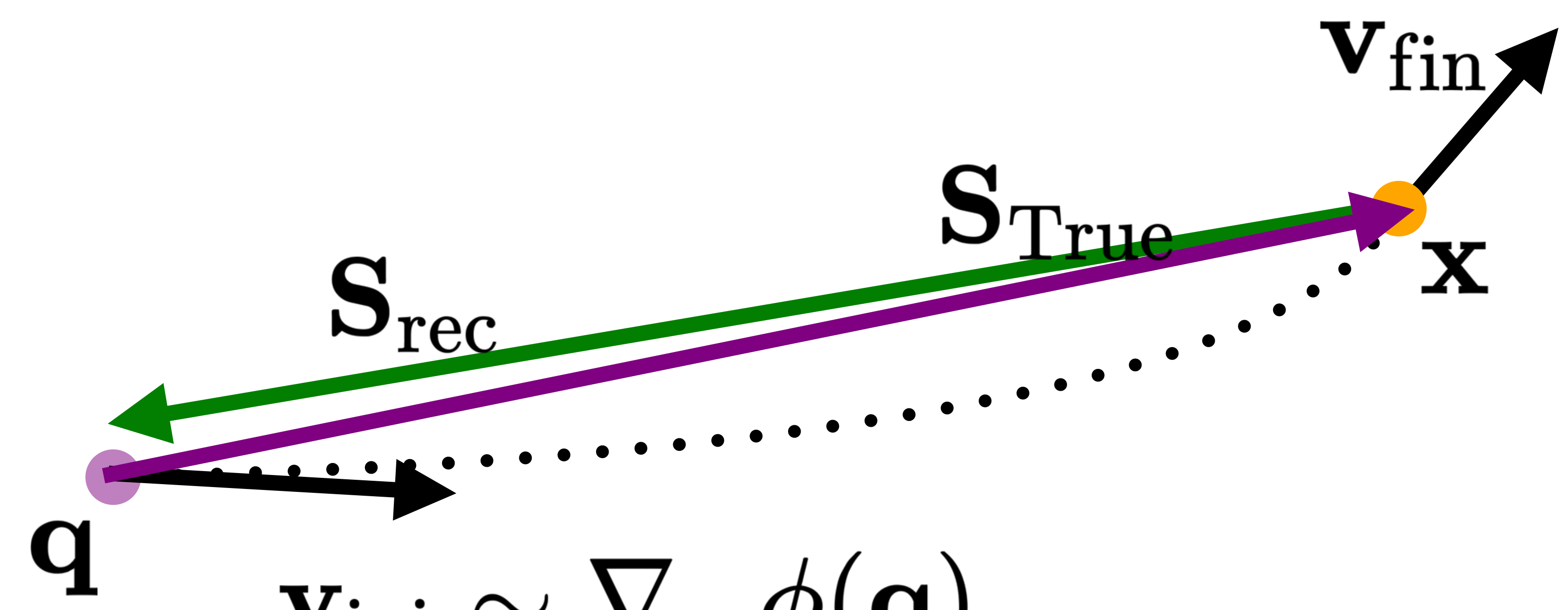
Cosmic Evolution

?

Reconstruction

Final Redshift



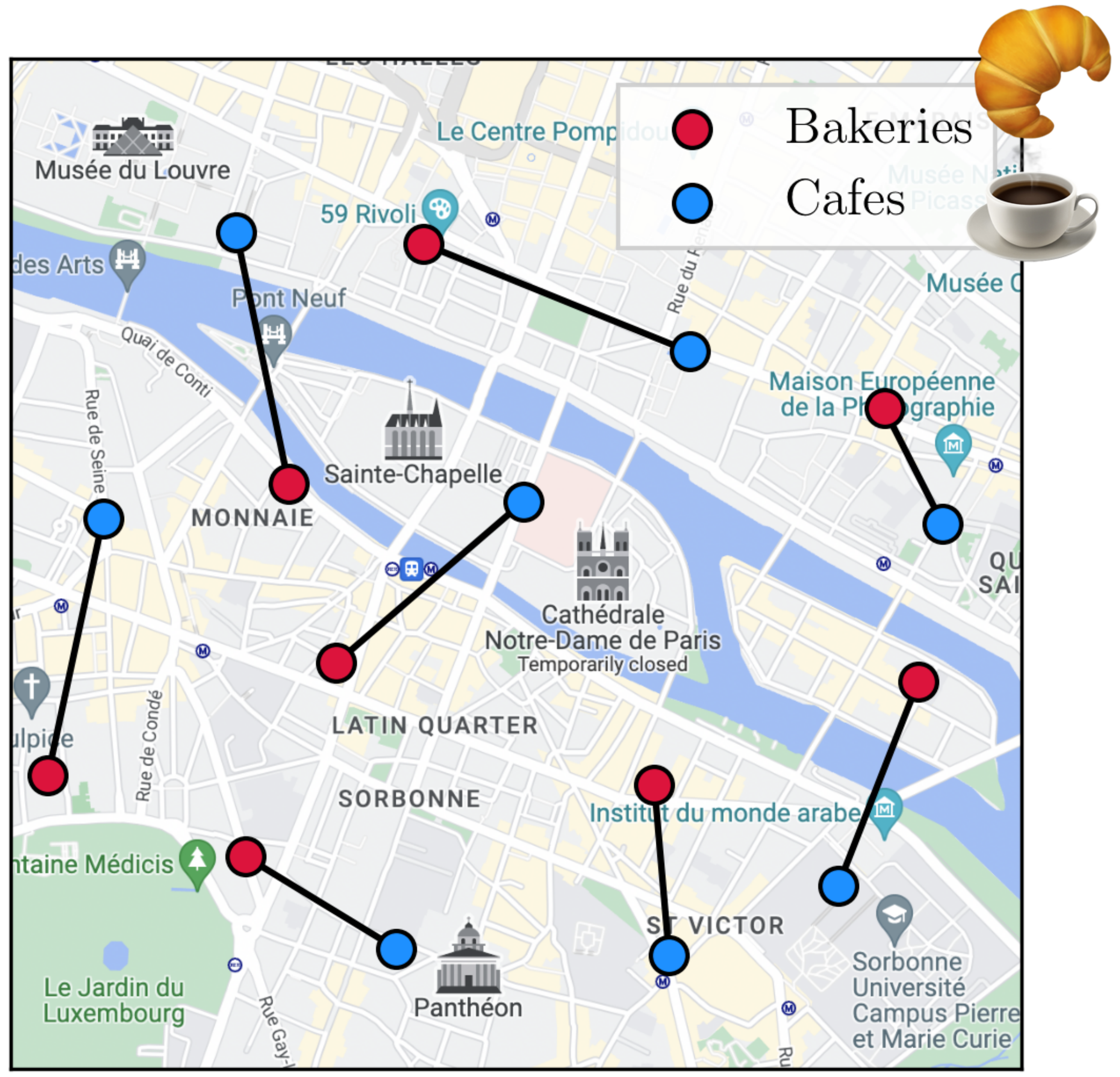


Lagrangian

Eulerian

$$\mathbf{v}_{\rm ini} \sim \nabla_{\mathbf{q}}\, \phi(\mathbf{q})$$

$$\nabla_{\mathbf{q}} \cdot \mathbf{v}_{\rm ini} \sim \nabla^2_{\mathbf{q}}\, \phi(\mathbf{q}) \sim \delta_{\rm ini}$$



Gaspard Monge
(1746 — 1818)

Leonid Kantorovich
(1912 – 1986)

Yann Brenier
Polar Factorization Thm.
1991

Cédric Villani
Fields Medal 2010

Cosmological Reconstruction

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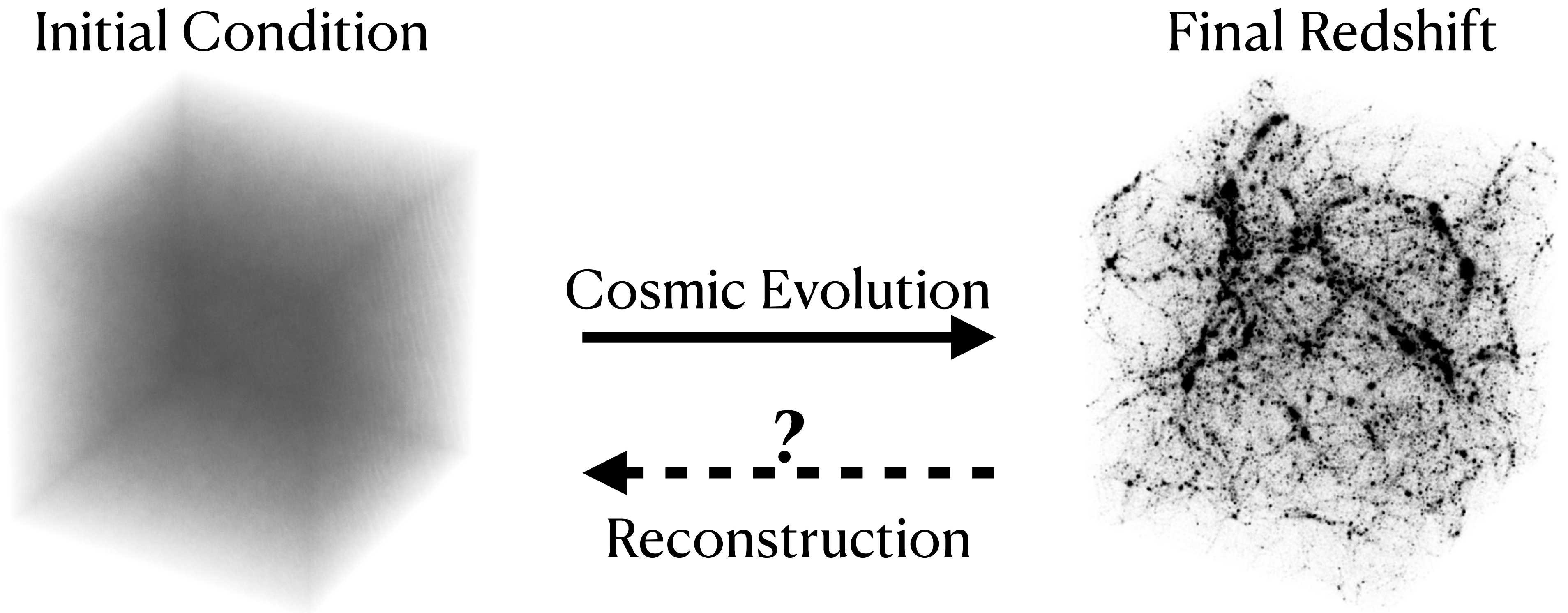

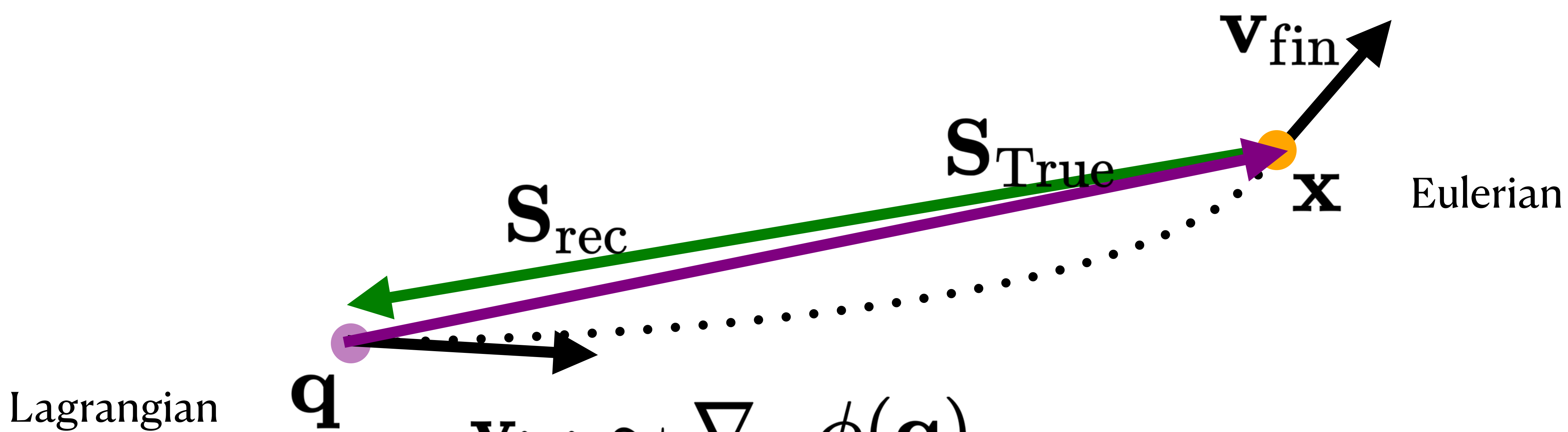

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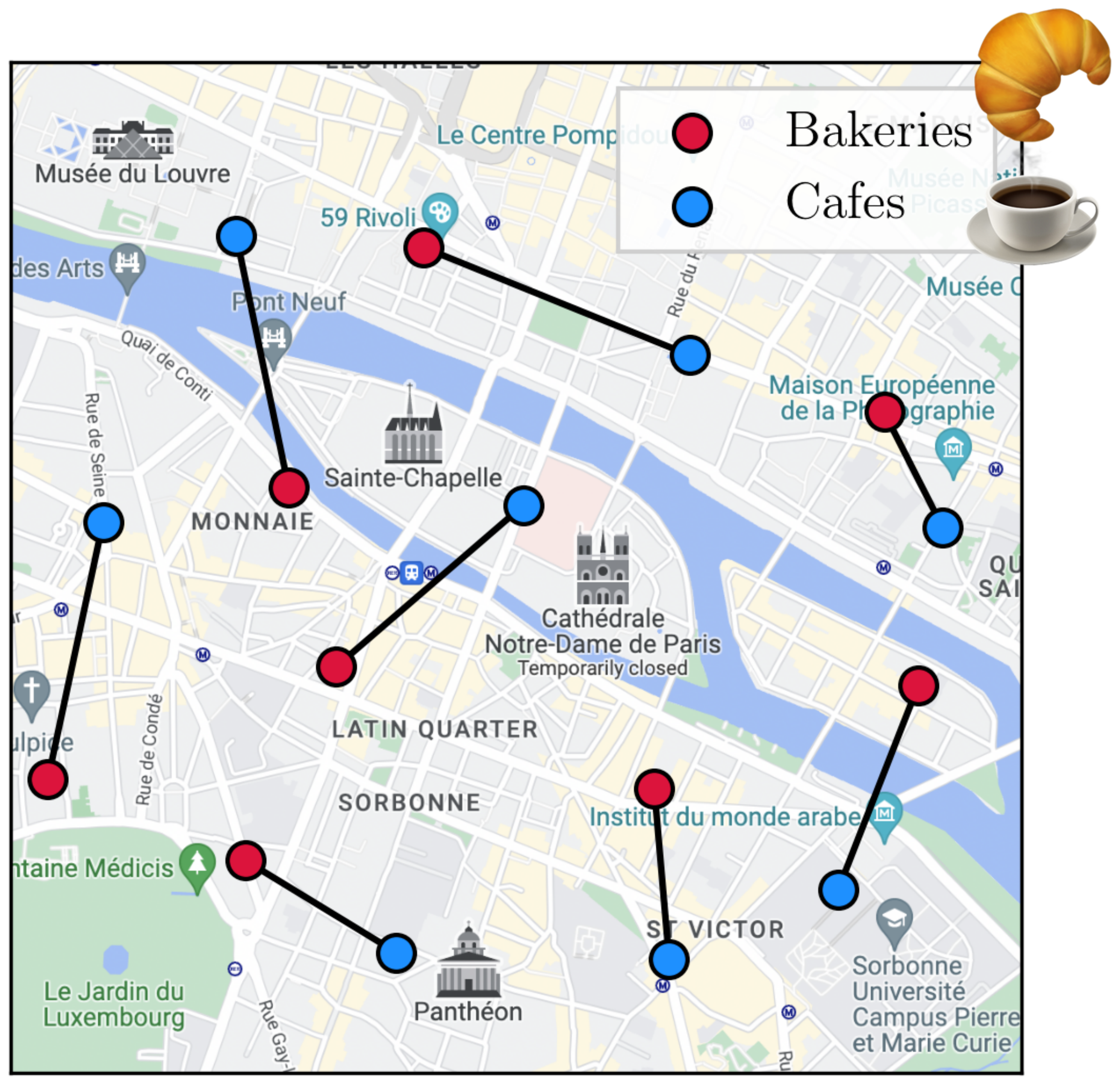
Optimal Transport Theory



OT: How to map a distribution into another one while preserving weights and minimizing the effort ?

preserving weights → Conservation law

minimizing the effort → Minimum action principle

Minimum action principle subject to conservation law!

Cosmological Reconstruction

Mass Conservation

$$\rho_{\rm fin}(\mathbf{x})\mathrm{d}^3\mathbf{x} = \rho_{\rm ini}(\mathbf{q})\mathrm{d}^3\mathbf{q}$$

$$\mathbf{q} = \mathbf{x} + \nabla\Theta(\mathbf{x})$$

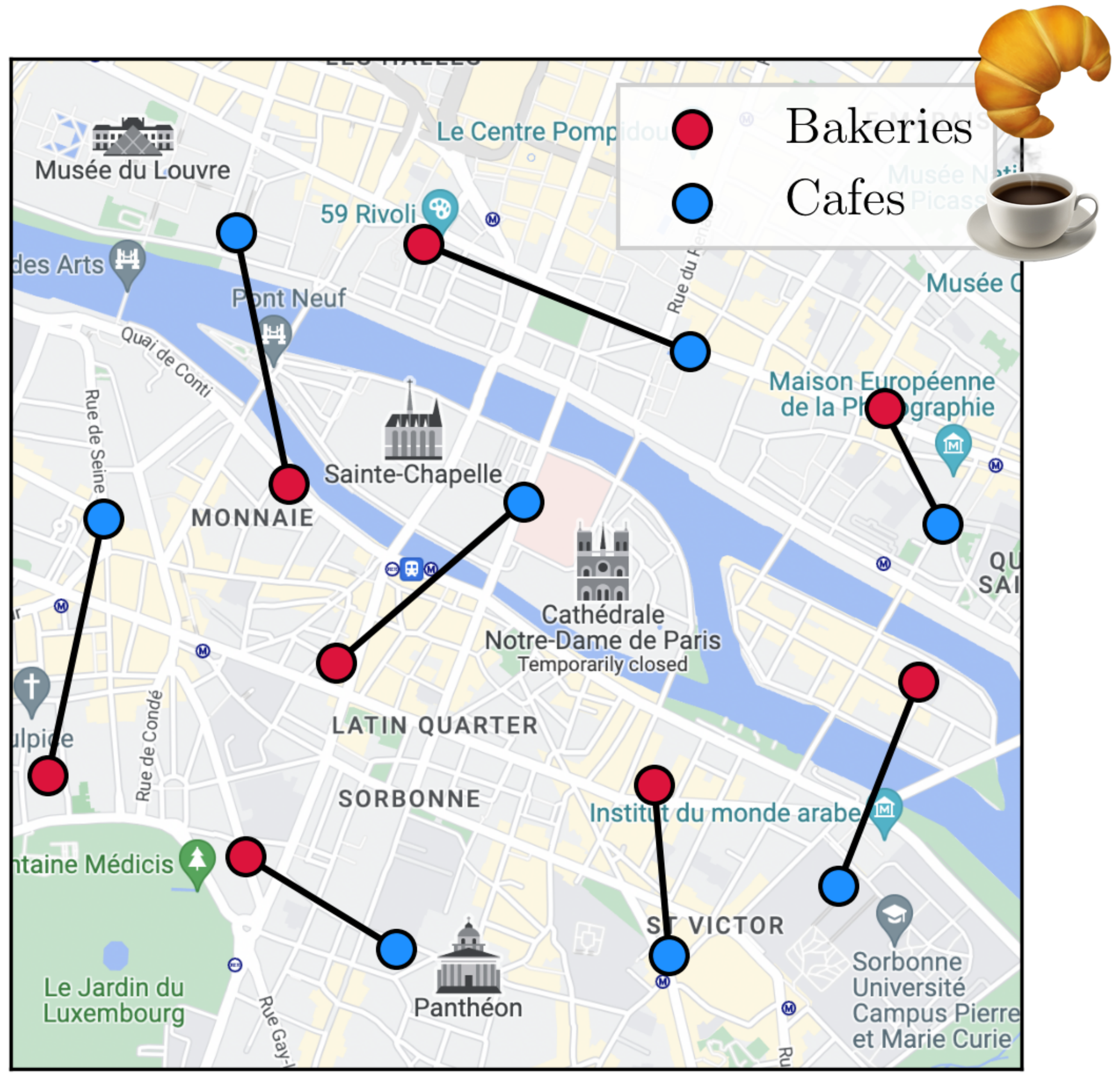
$$\frac{\rho_{\rm fin}(\mathbf{x})}{\bar{\rho}} = \left|\frac{\mathrm{d}^3\mathbf{q}}{\mathrm{d}^3\mathbf{x}}\right| = \det\left(\frac{\partial q^i}{\partial x^j}\right) = \det[1 + \partial_i\partial_j\Theta(\mathbf{x})]$$

Nonlinear continuity eq. (Monge–Ampère eq.)

Yann Brenier

Brenier's Theorem (1991):
For OT with a quadratic cost, there exists a unique gradient map of a convex potential that satisfies MA equation.

Optimal Transport Theory



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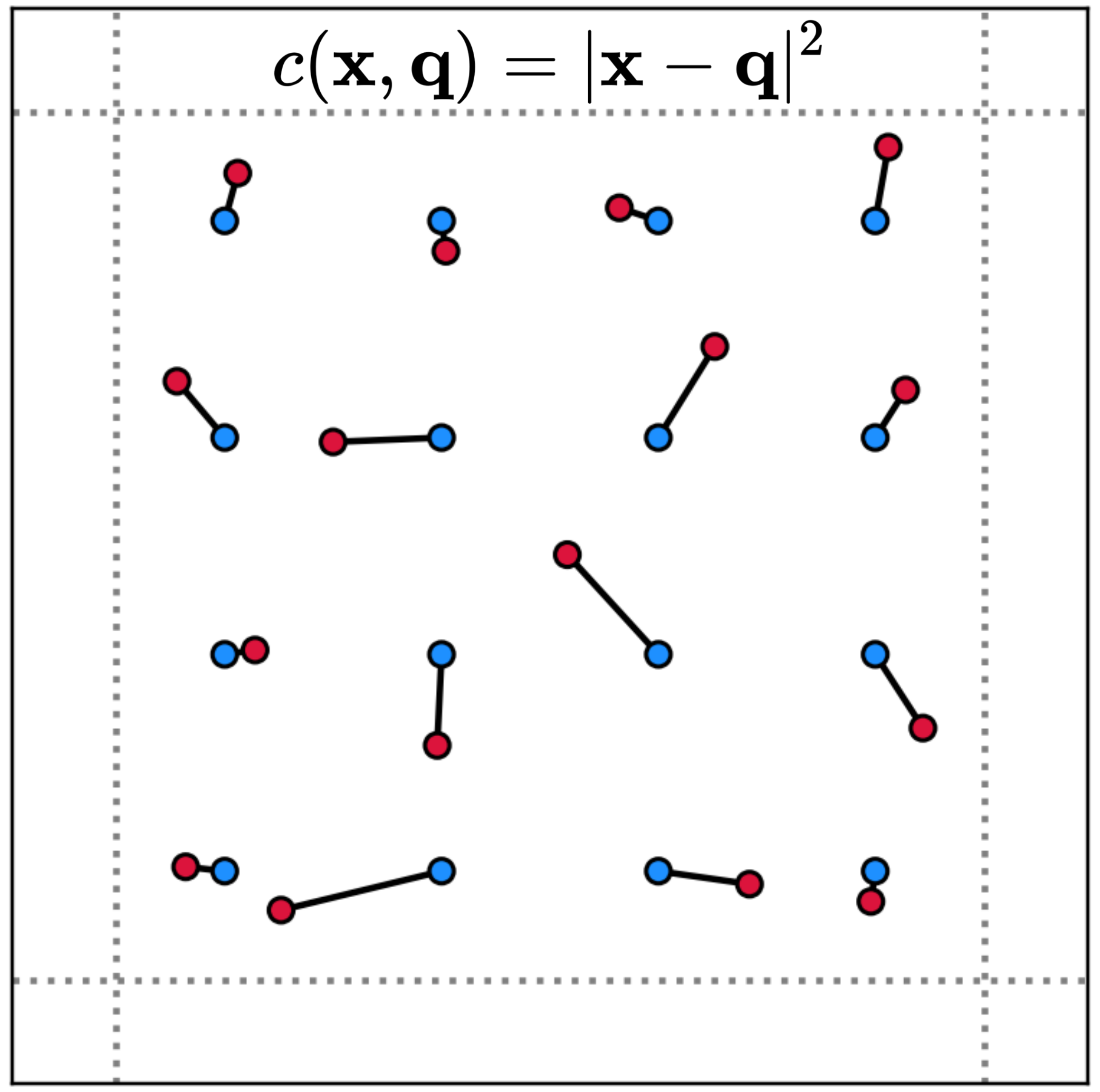

$\mathcal{O}(n^2 \log n)$

Frisch et al. (2002)
Brenier et al. (2003)

Yann Brenier

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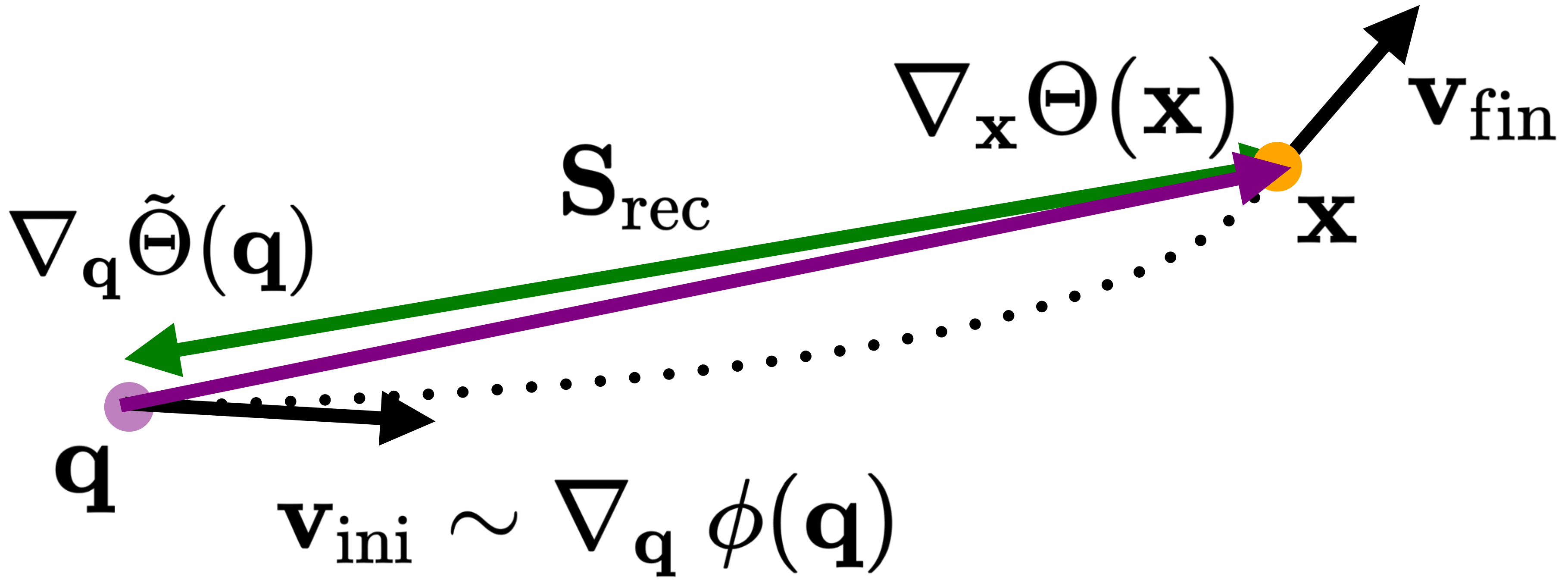

Legendre Transform $\Theta(\mathbf{x}),\ \tilde{\Theta}(\mathbf{q})$

Cosmological Reconstruction

Mass Conservation

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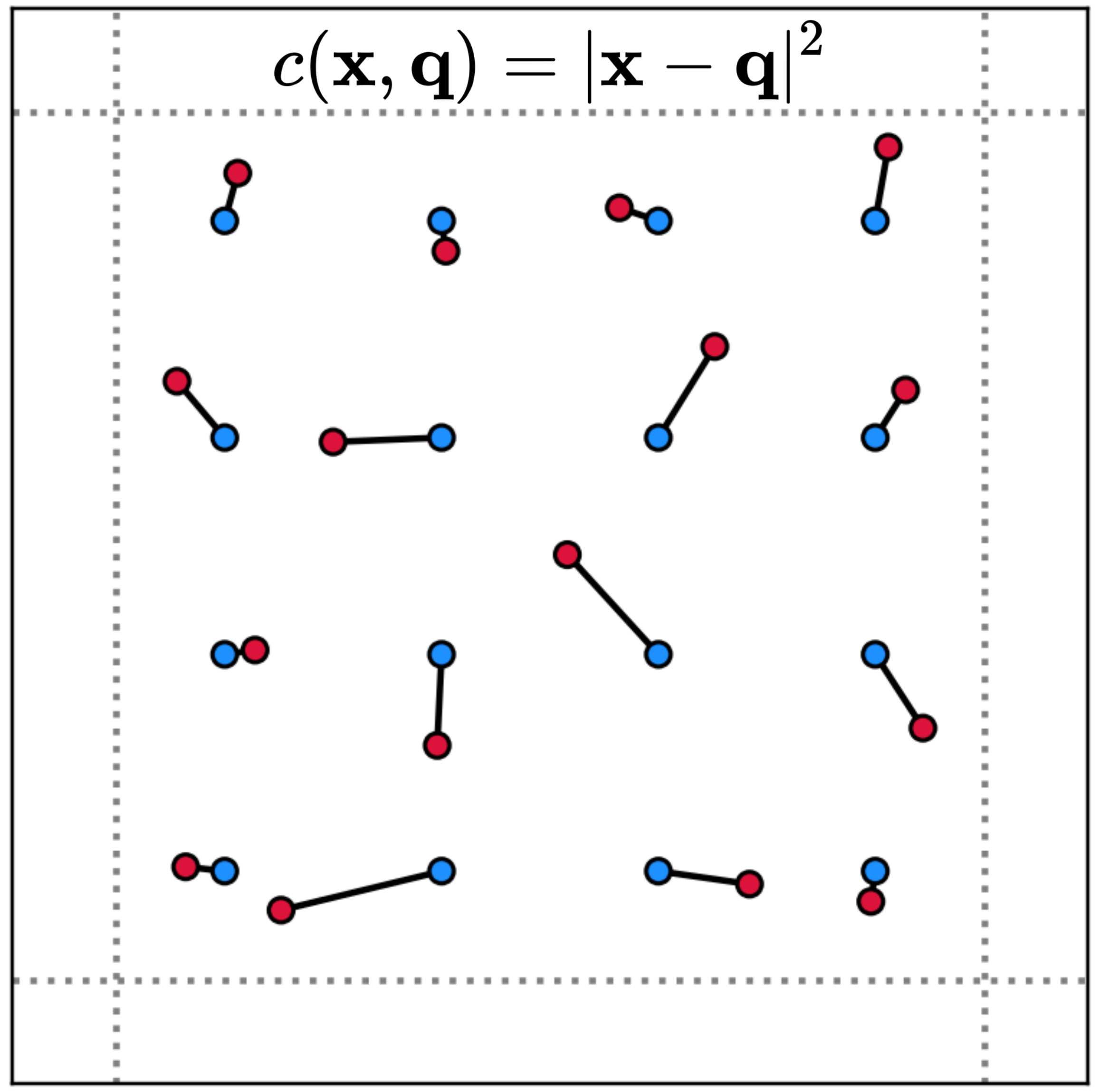

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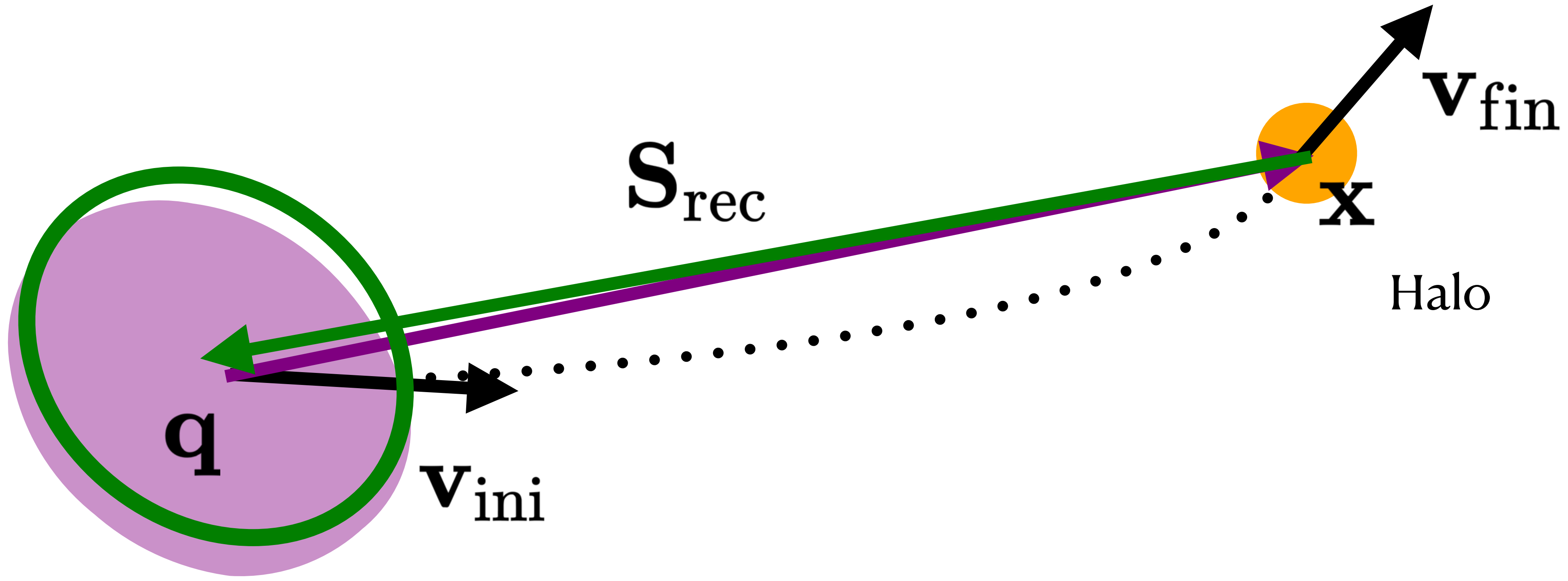
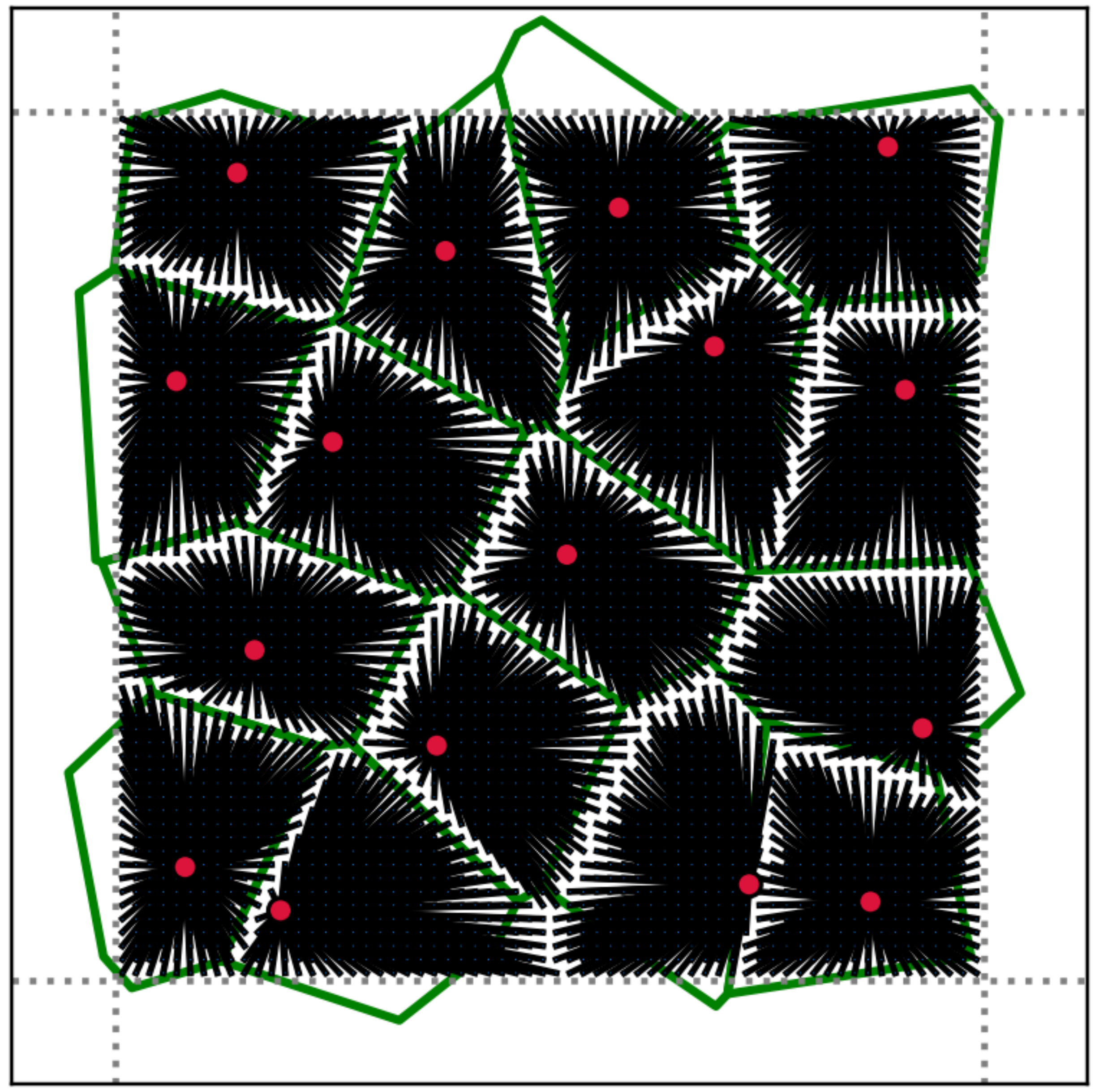


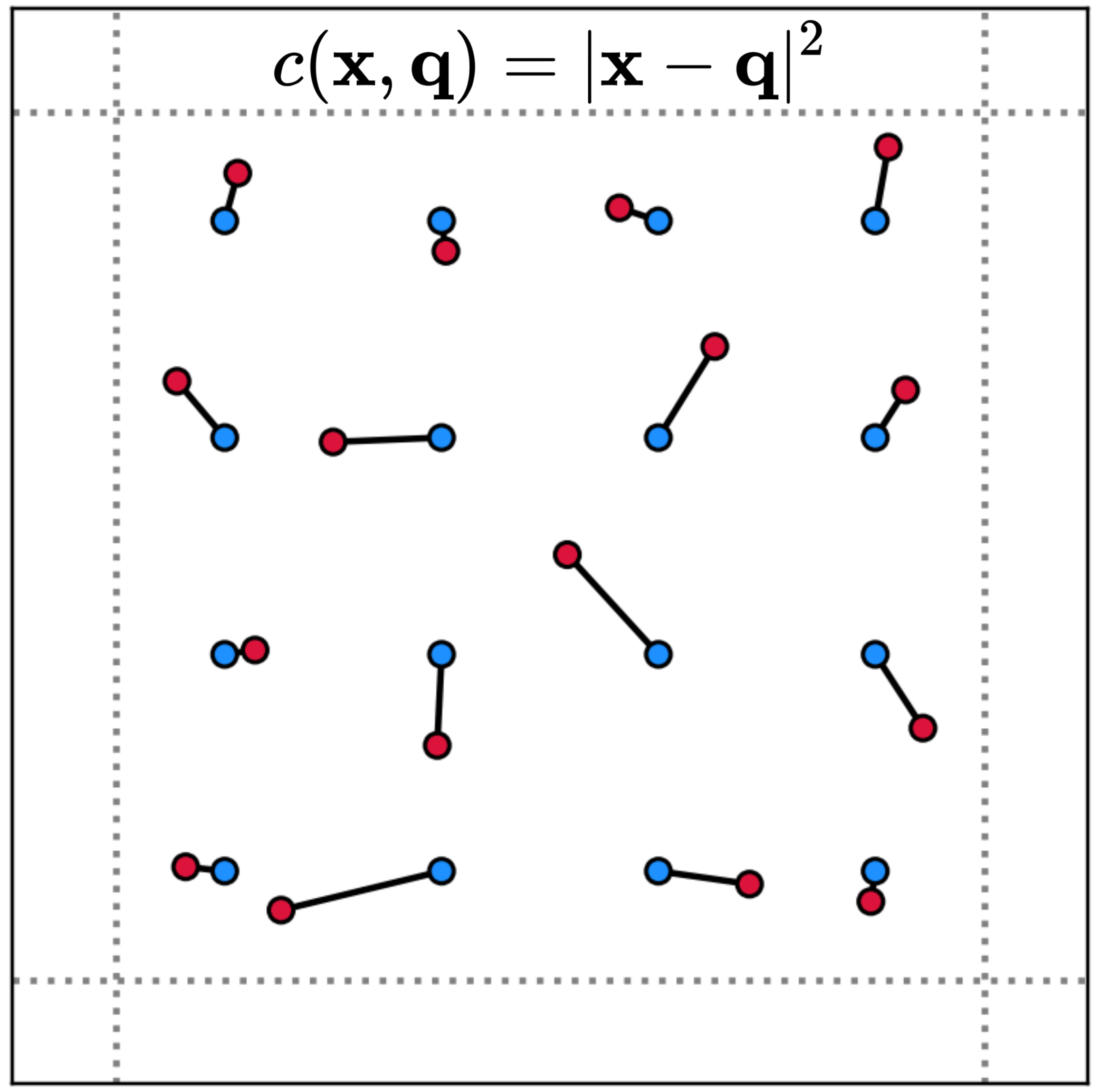
Semi-Discrete Optimal Transport



Partition space into Laguerre cells

(power diagram / modified Voronoi diagram)

$\mathcal{O}(n \log n)$



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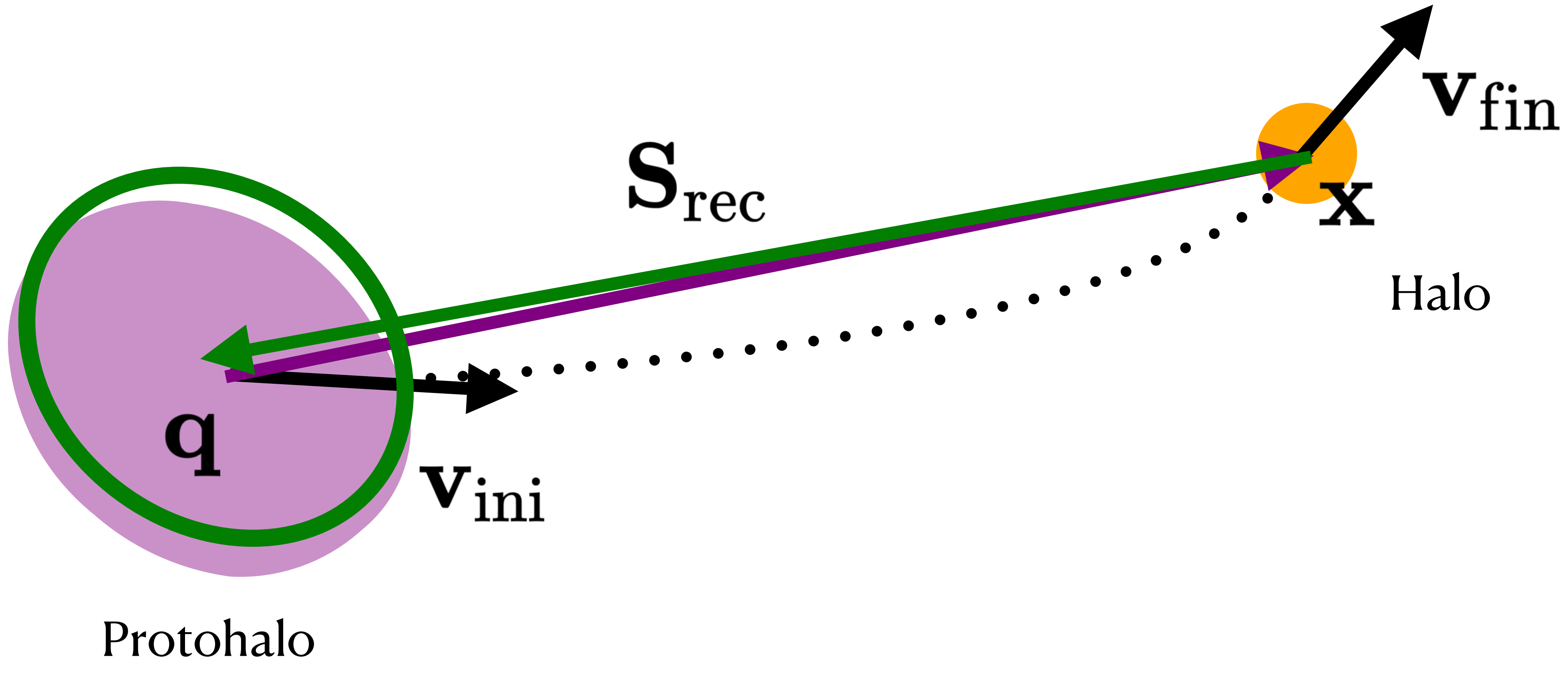
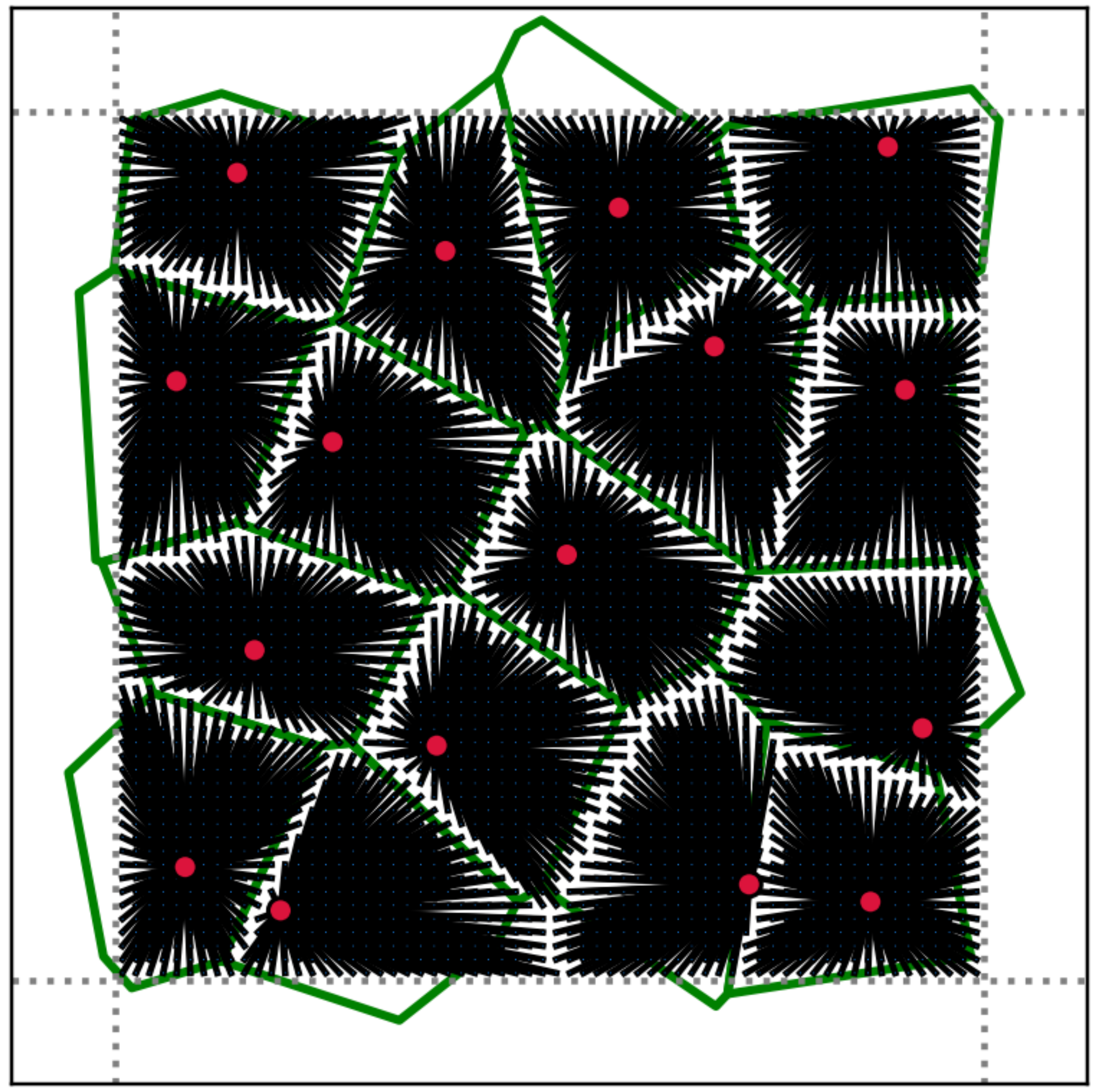


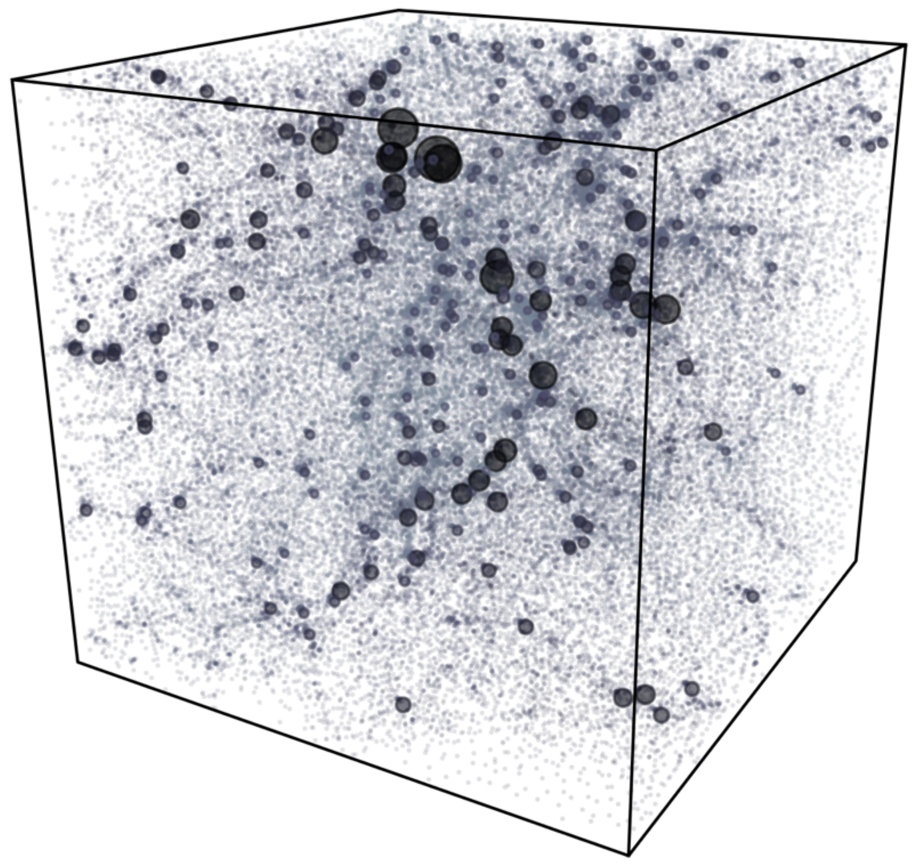
Semi-Discrete Optimal Transport



Partition space into Laguerre cells

(power diagram / modified Voronoi diagram)

Observed Field



Nikakhtar et al. PRL 129, 251101 (2022)

Optimal Transport Reconstruction

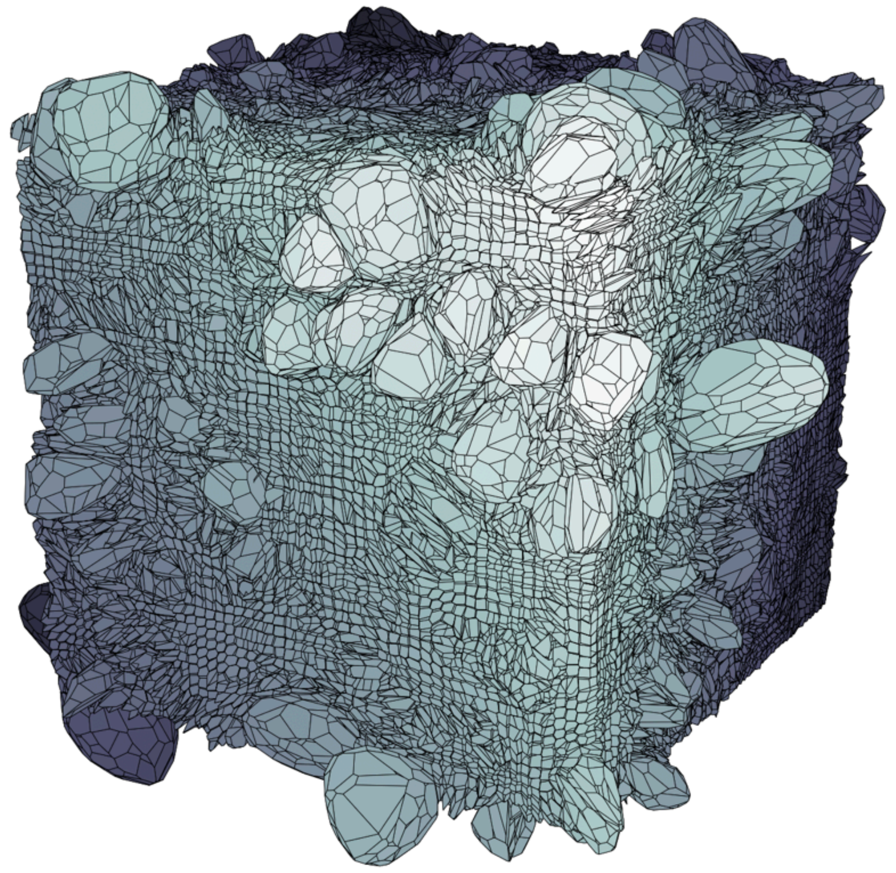
$$\underbrace{V_i^{\psi}}_{\text{Reconstructed Patches}} = \left\{ \mathbf{q} \;\middle|\; \underbrace{\frac{1}{2}|\mathbf{x}_i - \mathbf{q}|^2}_{\text{Kinetic}} - \underbrace{\psi_i}_{\text{Potential}} < \frac{1}{2}|\mathbf{x}_j - \mathbf{q}|^2 - \psi_j,\ \forall j \neq i \right\}$$

Geometrical Optimization in Mathematics

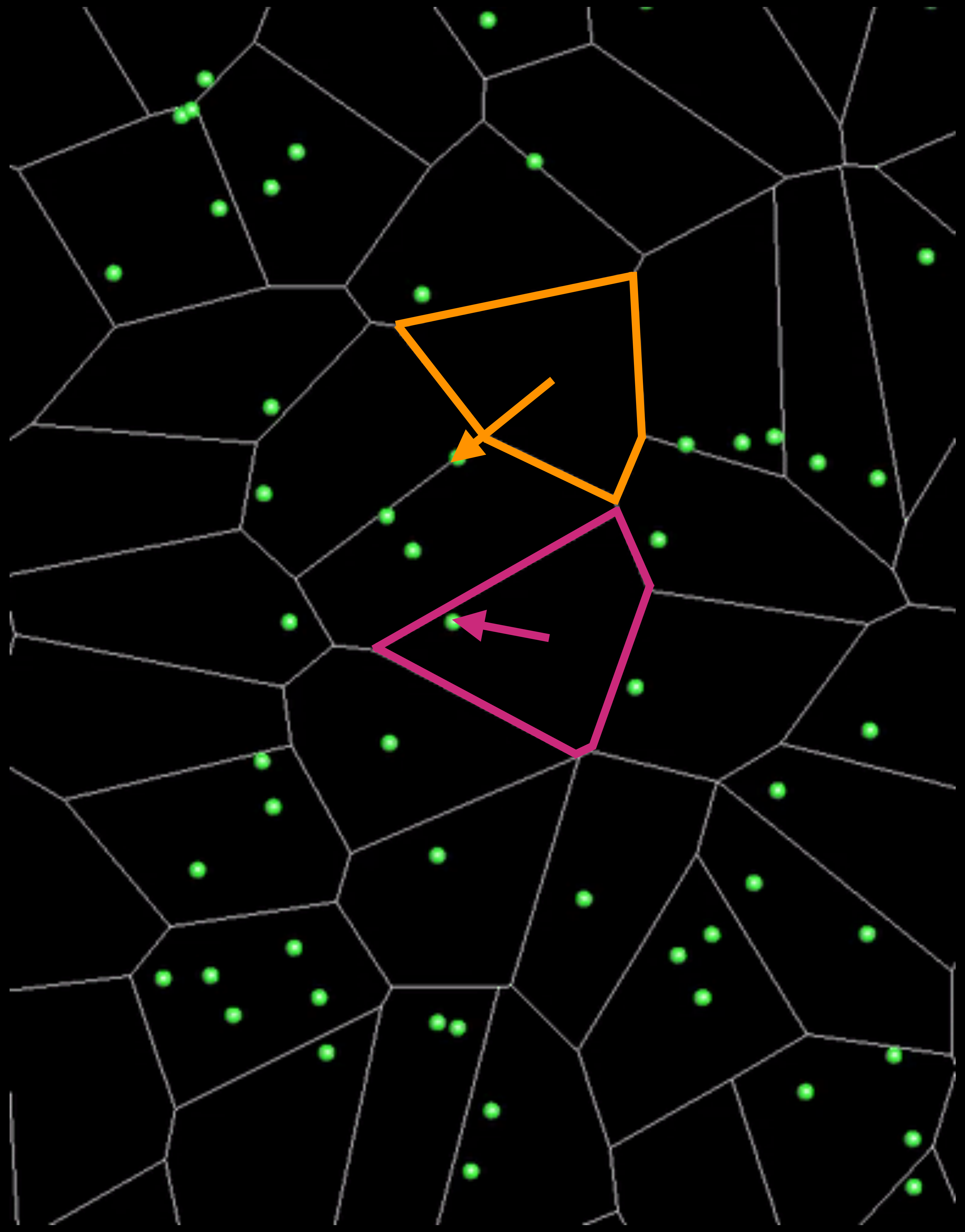
$\equiv$

Action Minimization in Physics

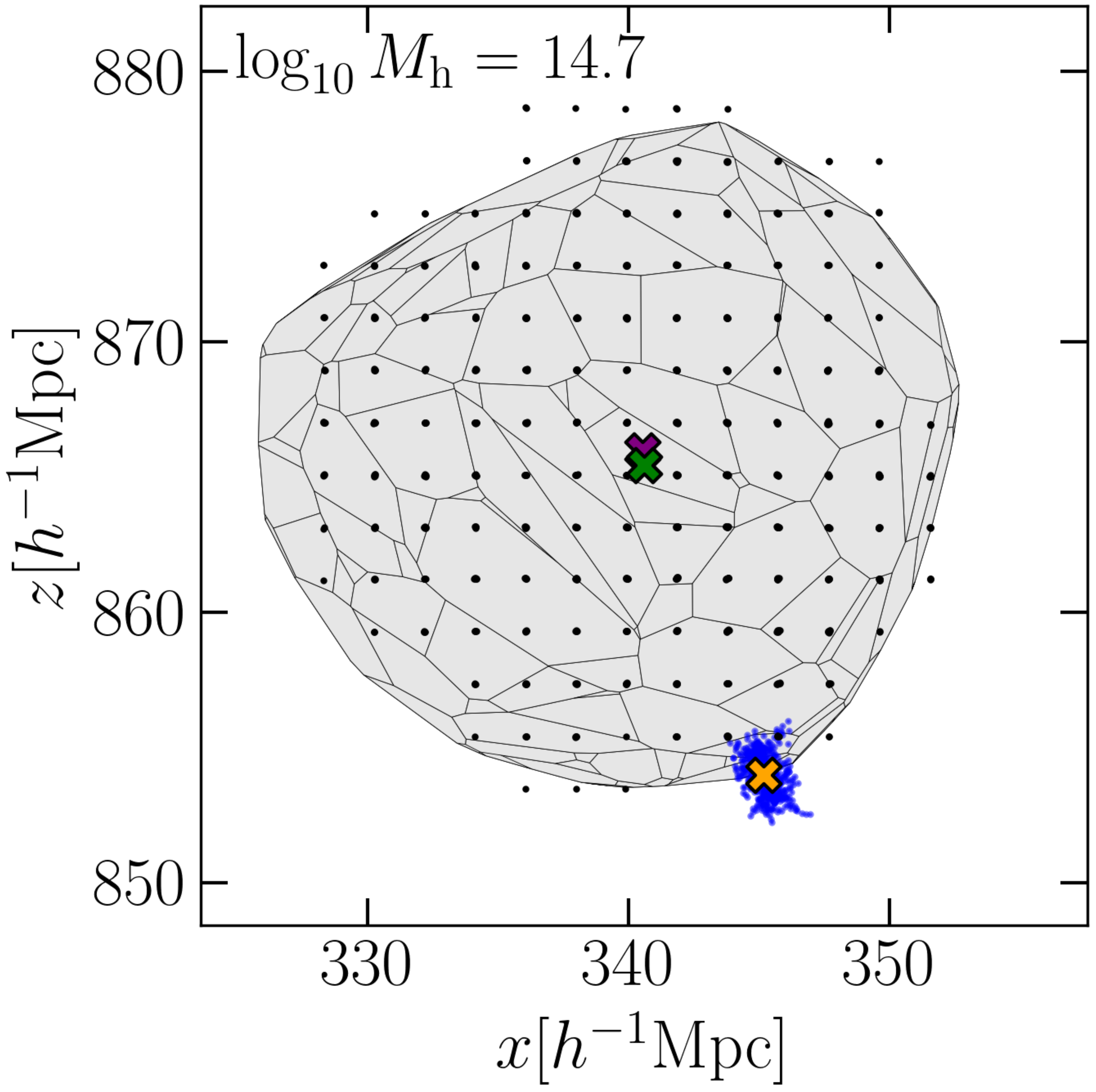
Reconstructed Field



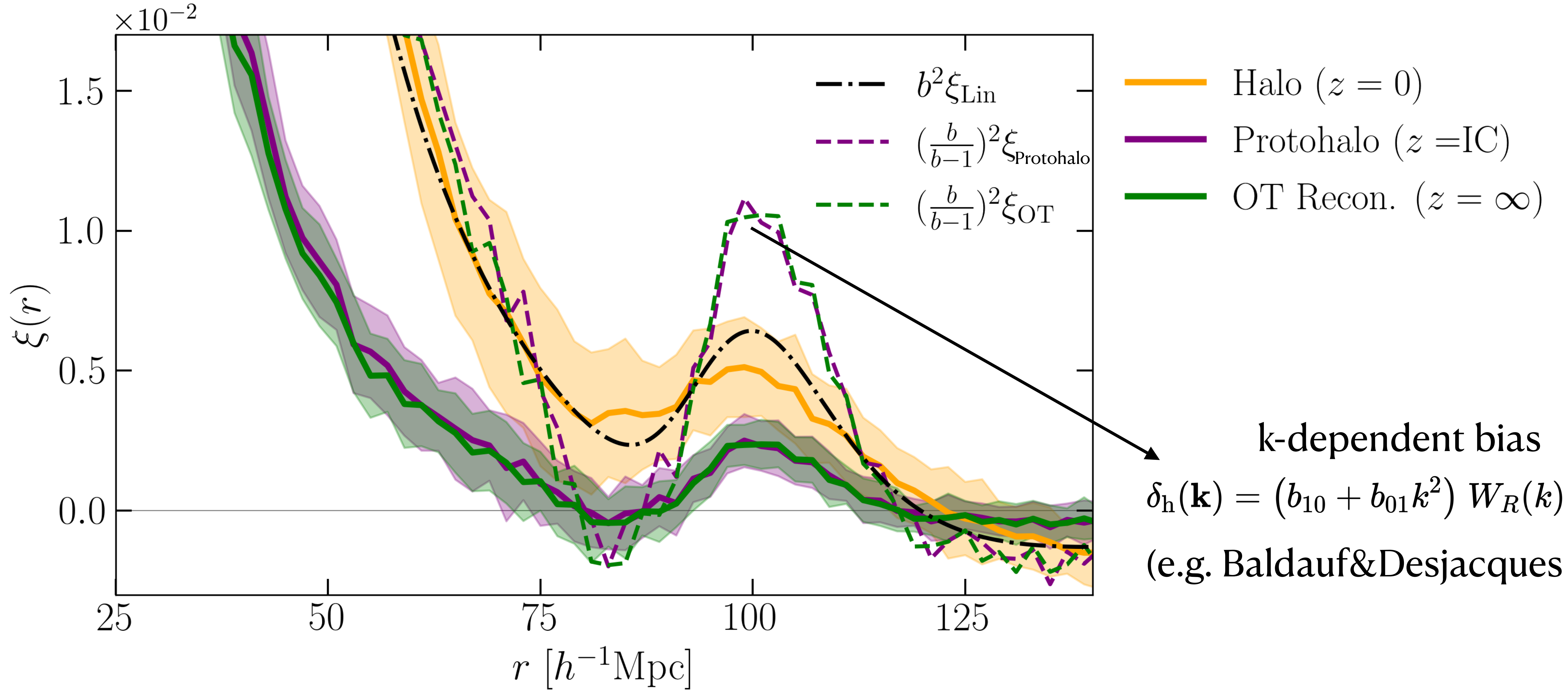
- Mass estimate of the biased tracers
- Distribution of the remaining mass (field particles)



Reconstructing Protohalo Positions + Shapes



Nikakhtar et al. (2022)

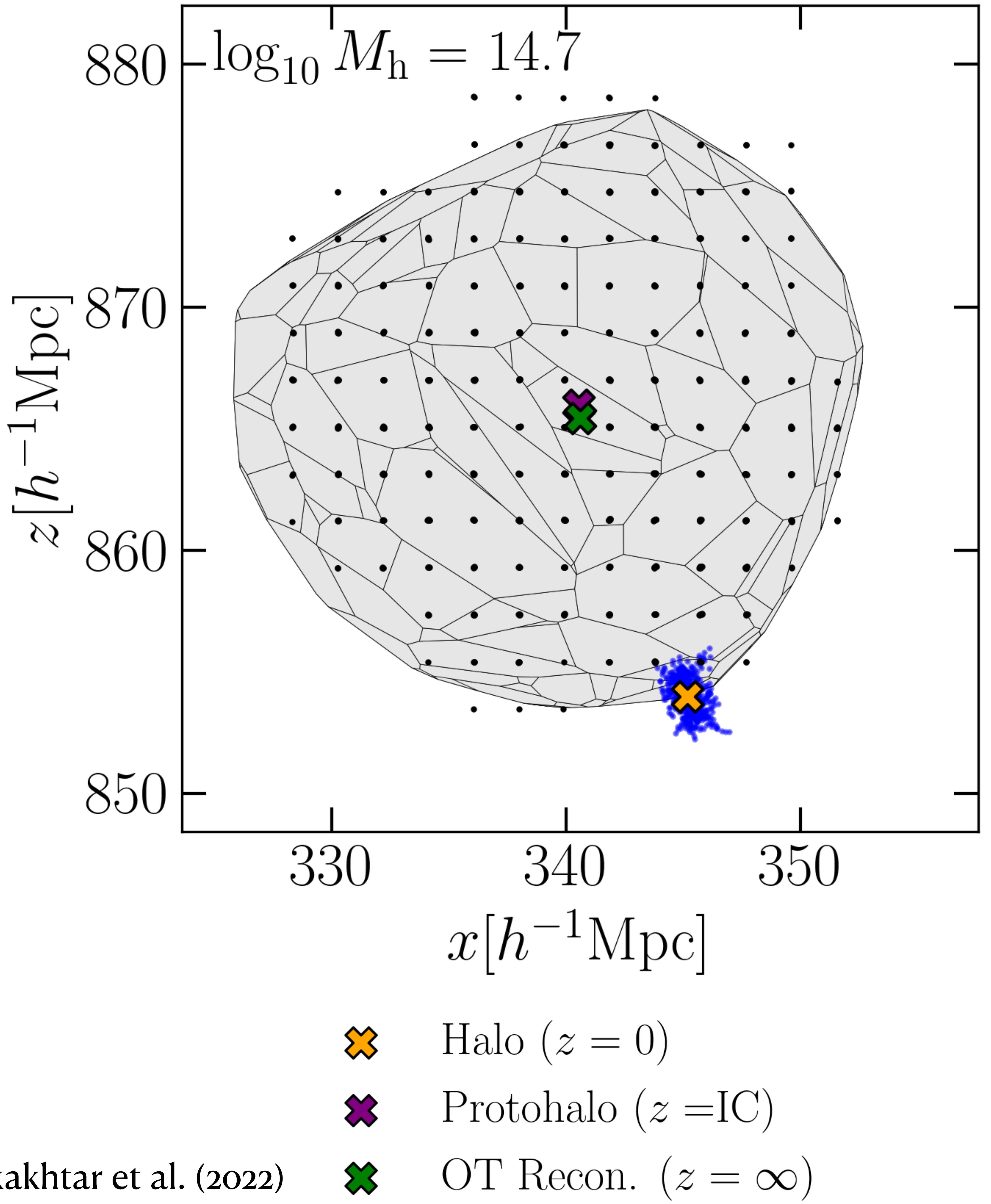


k-dependent bias

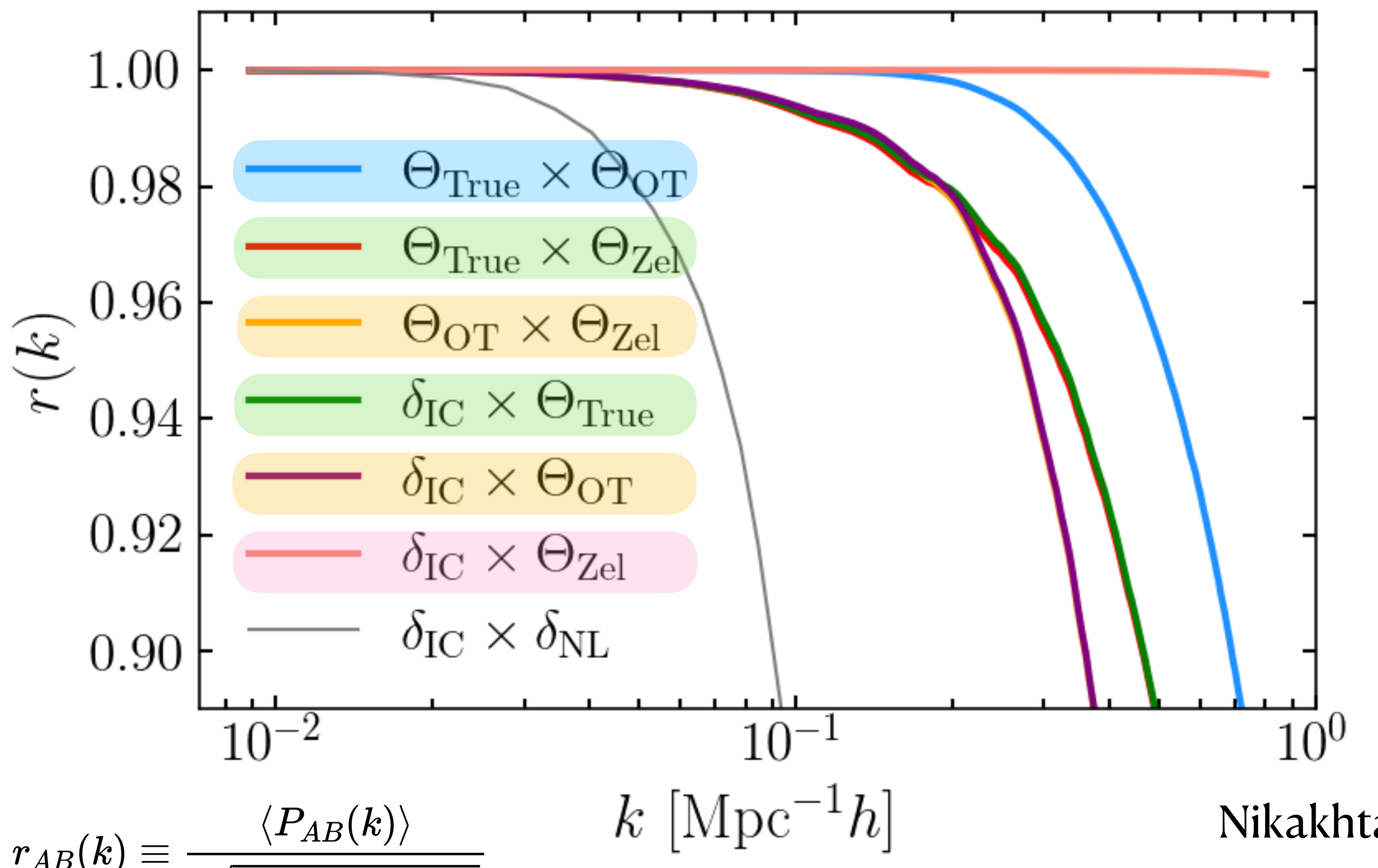
$$\delta_\mathrm{h}(\mathbf{k}) = \left(b_{10} + b_{01}k^2\right)\, W_R(k)\, \delta_\mathrm{L}(\mathbf{k})$$

(e.g. Baldauf&Desjacques 2016)

Reconstructing the Displacement Field

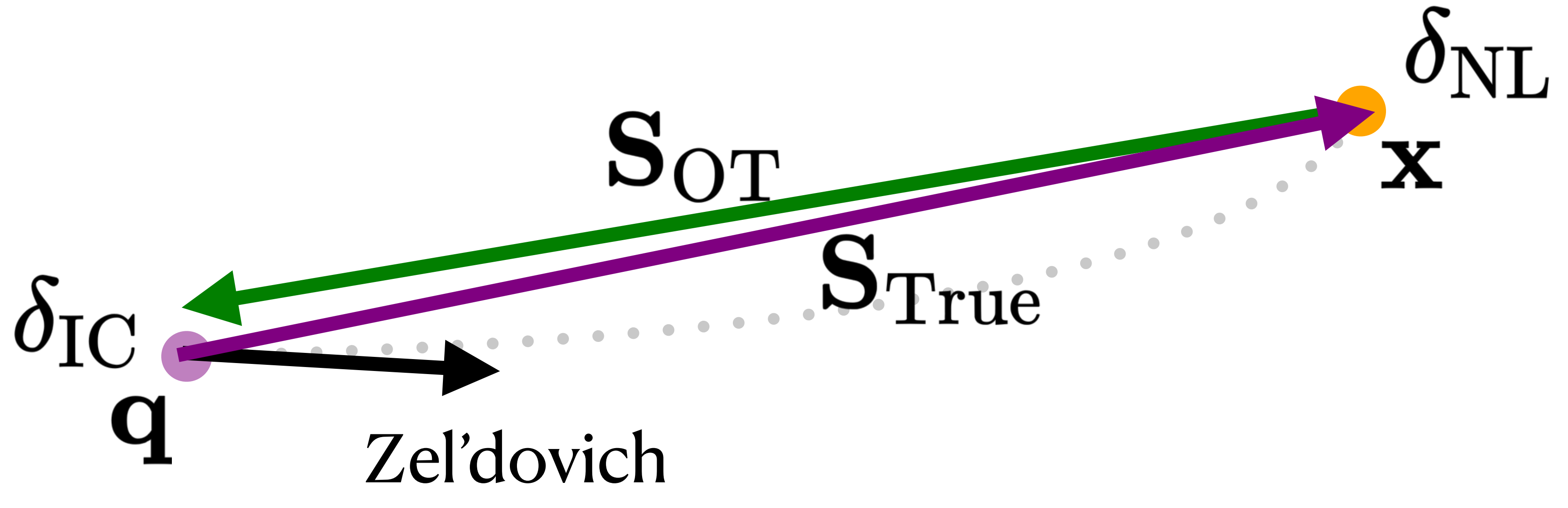


Nikakhtar et al. (2022)



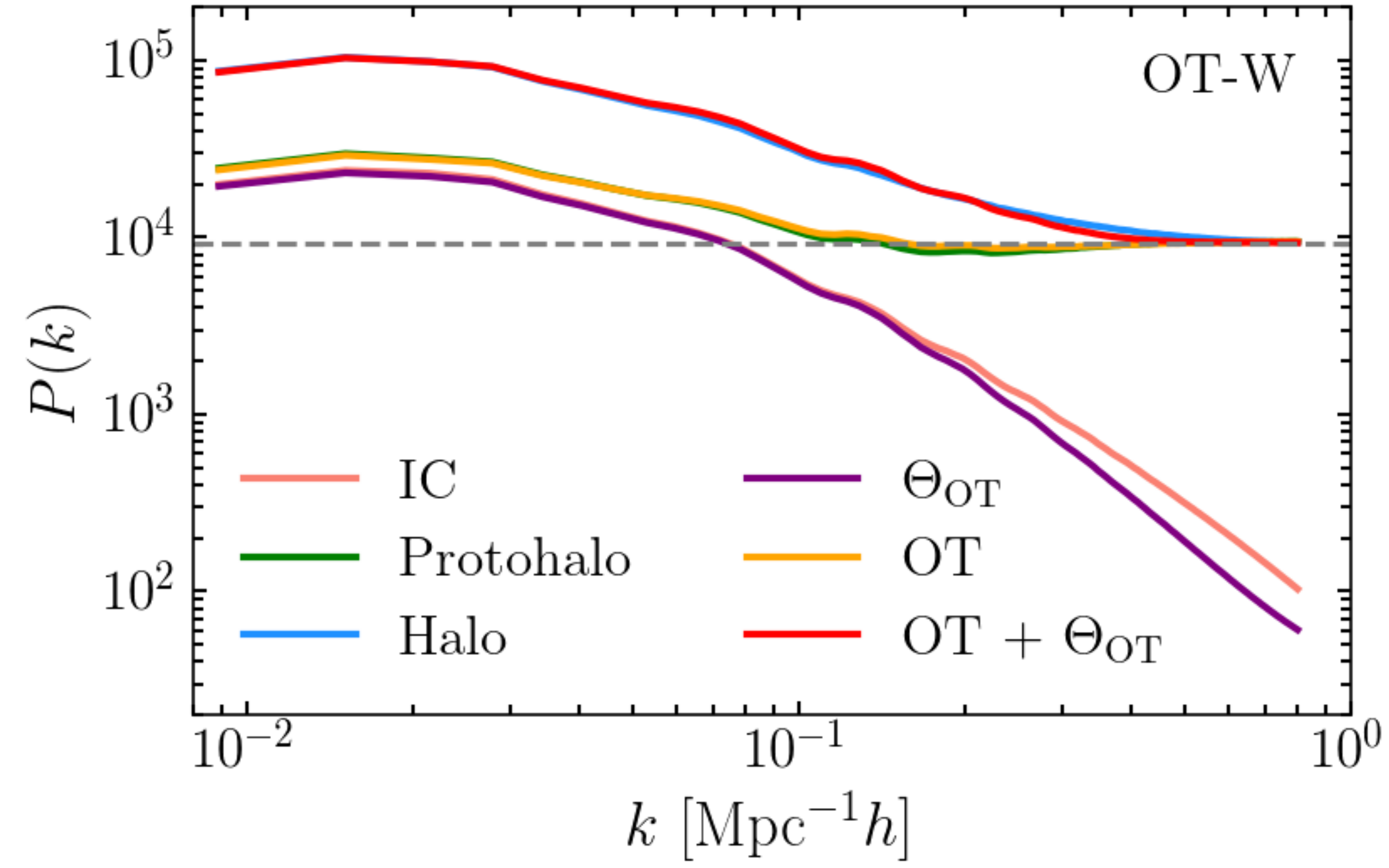
$$r_{AB}(k) \equiv \frac{\langle P_{AB}(k)\rangle}{\sqrt{P_{AA}(k)P_{BB}(k)}}$$

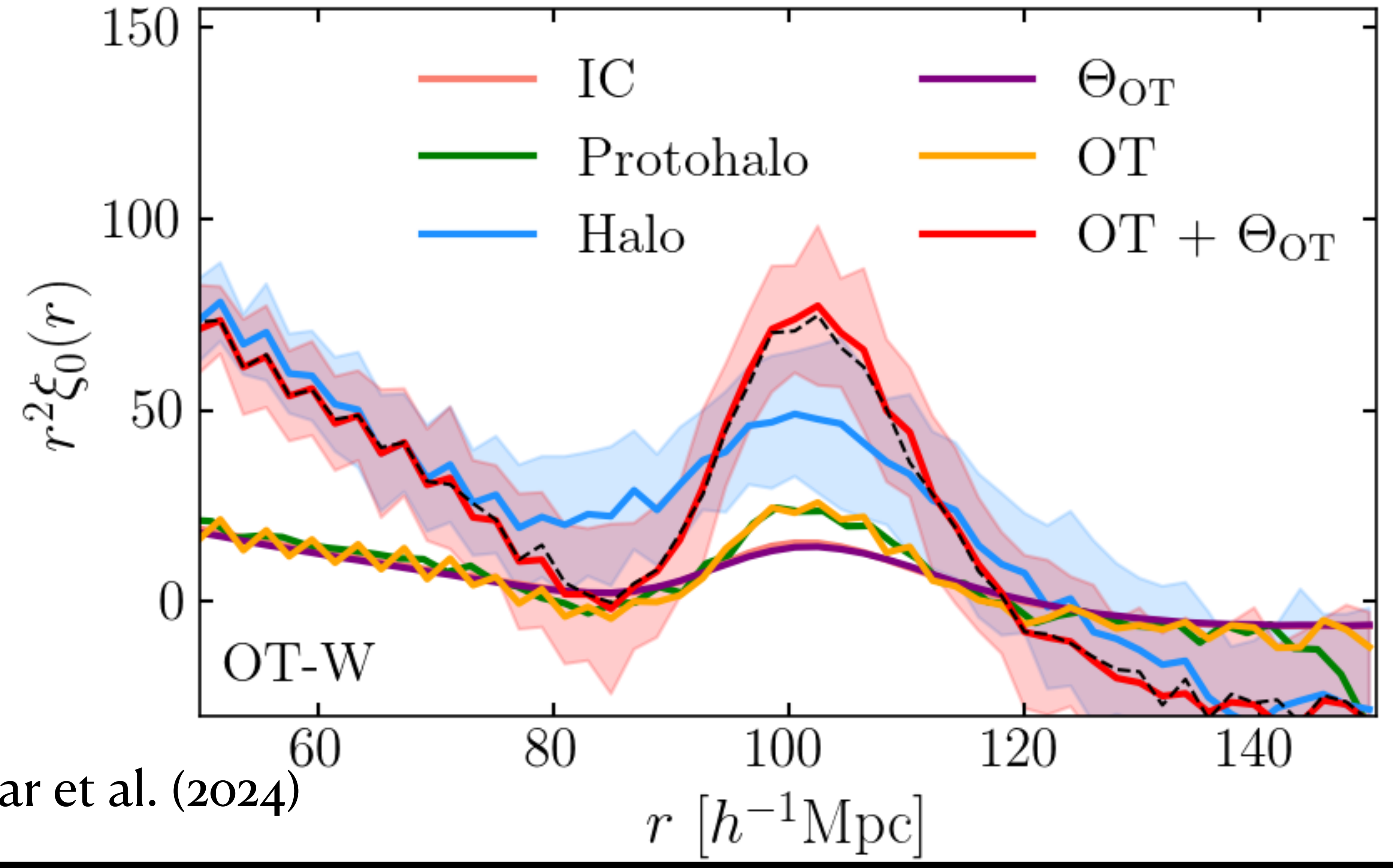
Nikakhtar et al. (2024)



$$\Theta_x \equiv -\nabla \cdot x$$

Reconstructing the Displacement Field

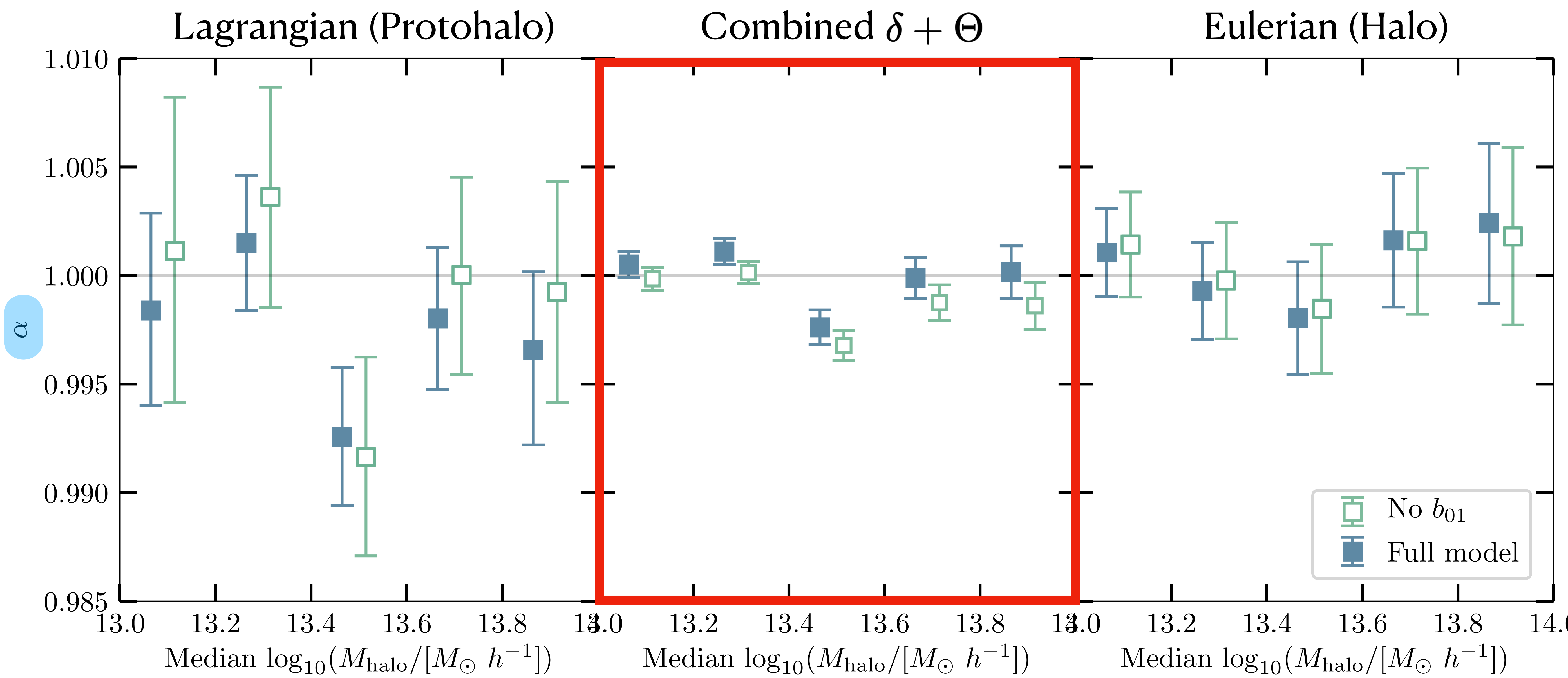




Nikakhtar et al. (2024)

Reconstruction of biased tracers:

$$\delta_{\rm OT(protohalo)} + \Theta_{\rm OT}$$



$$\xi_{\rm template}(r) = \xi_{\rm ph}(\alpha r, b_{01}, b_{10}) + \sum_{i=1}^{n} \frac{a_n}{r^{n-1}}$$

Sasha Gaines (Yale)

Gaines, Nikakhtar, Padmanabhan, Sheth (2024 in prep.)

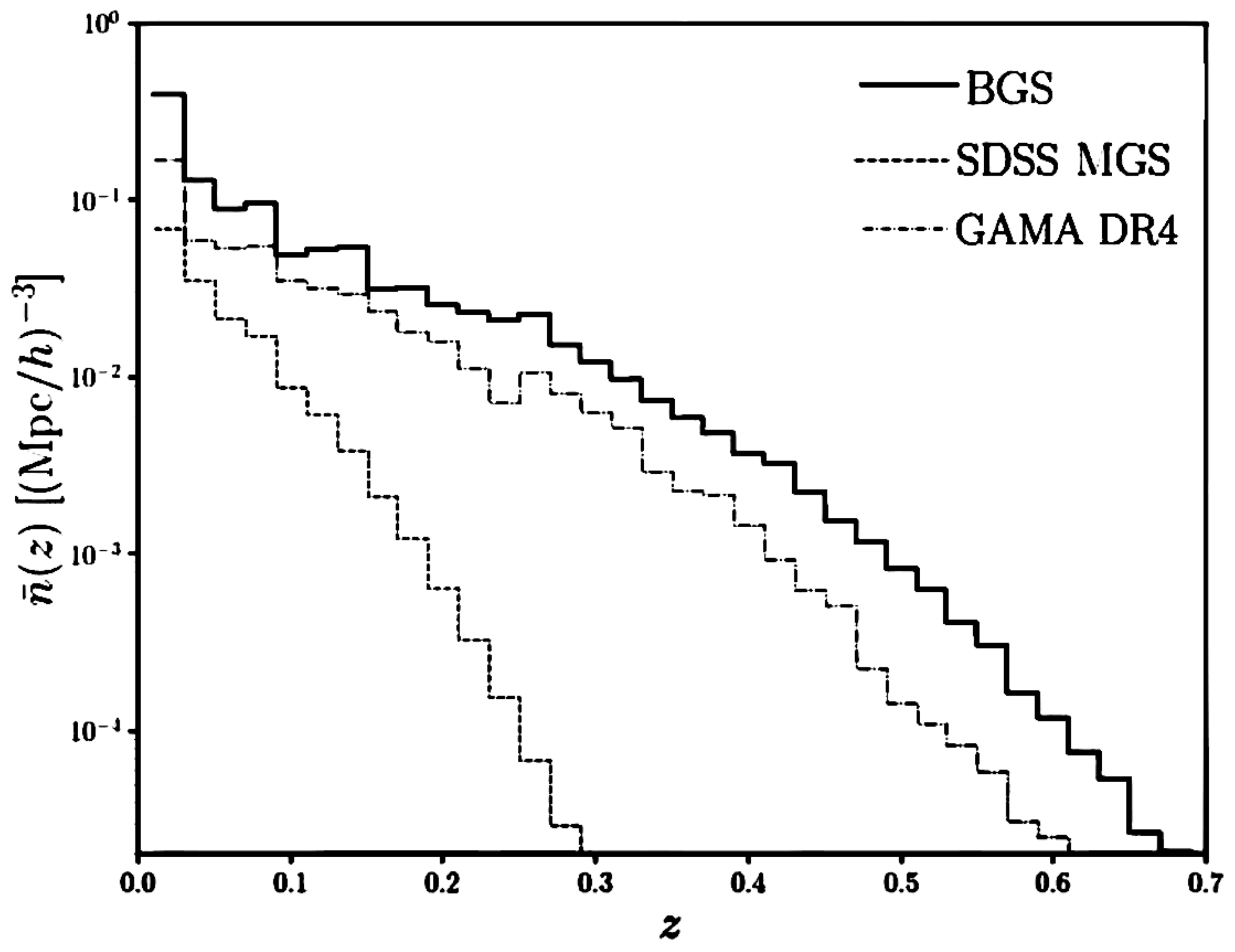
OT Reconstruction on DESI Bright Galaxy Survey

DARK ENERGY SPECTROSCOPIC INSTRUMENT
U.S. Department of Energy Office of Science

15 million galaxies at $z < 0.6$ in the **dark energy dominated epoch**

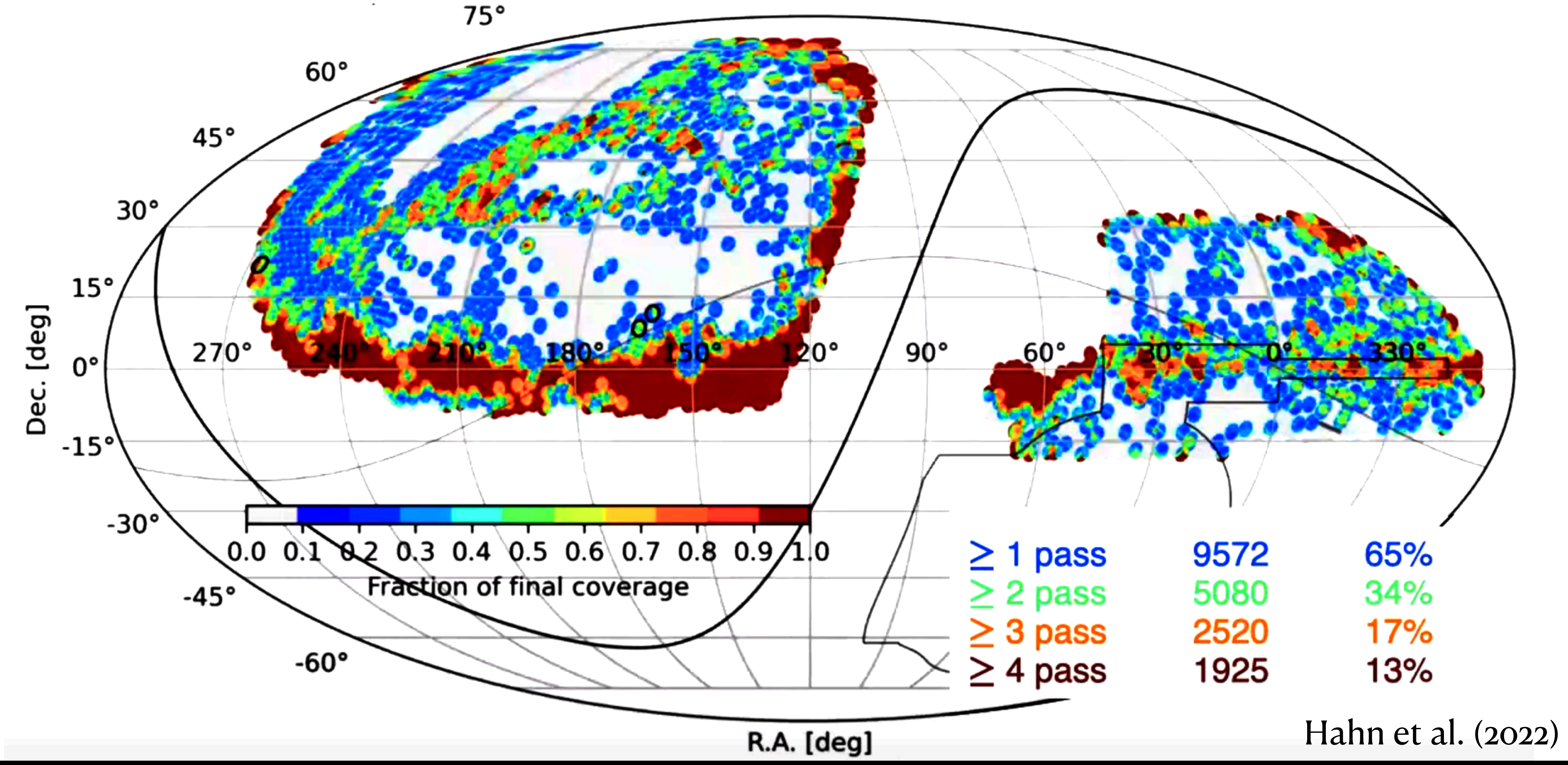
magnitude limited sample to $r < 19.5$, fainter sample to $r < 20.175$ **over 14,000 deg^2 footprint**

2 magnitude deeper than SDSS main survey



high density sample
more than **an order of magnitude** larger than previous surveys

in its 1st year, BGS is >33% complete



Hahn et al. (2022)

What else can we learn from reconstructed density & displacement?

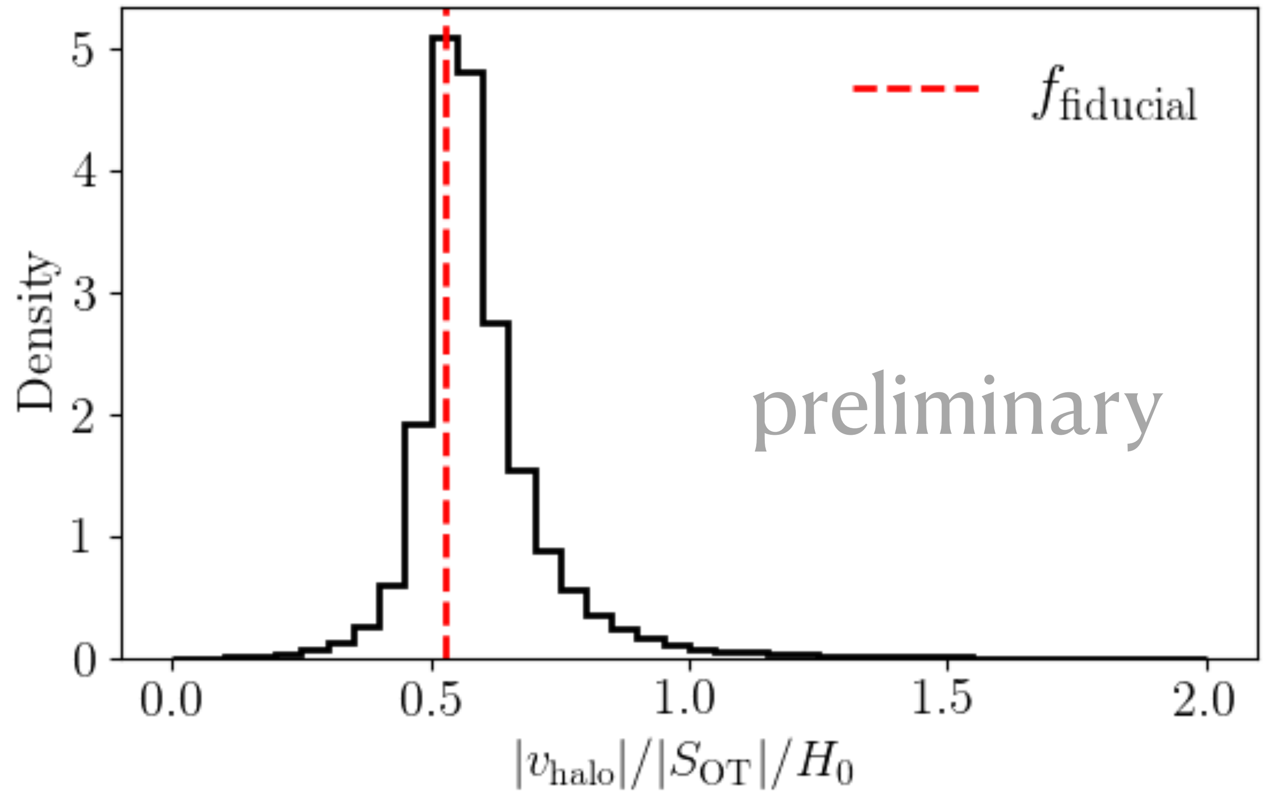
Future Research Avenues

Peculiar Velocity Surveys

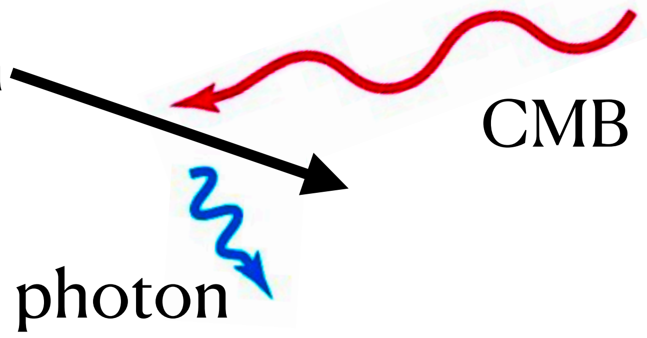
Tully-Fisher / Fundamental Plane relations

$$\boldsymbol{v}^{\rm rec} = a f H\, \mathbf{S}^{\rm rec}$$

Growth rate of structures



Baryonic feedback with the kSZ effect

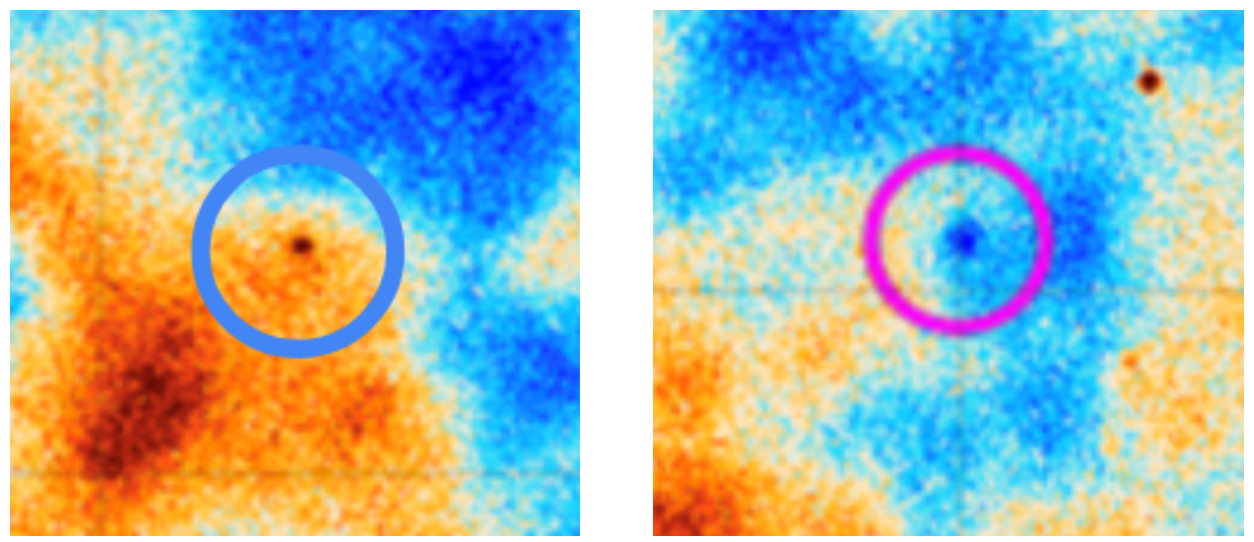


$$\frac{\delta T_{\rm kSZ}(\hat{\boldsymbol{n}})}{T_{\rm CMB}} = -\tau_{\rm gal} \left(\frac{v_{e,r}}{c}\right)$$

Optical depth
integral of gas density along LOS

Electron/halo velocity

cluster moving away | cluster moving inward



Madhavacheril et al. (2020)

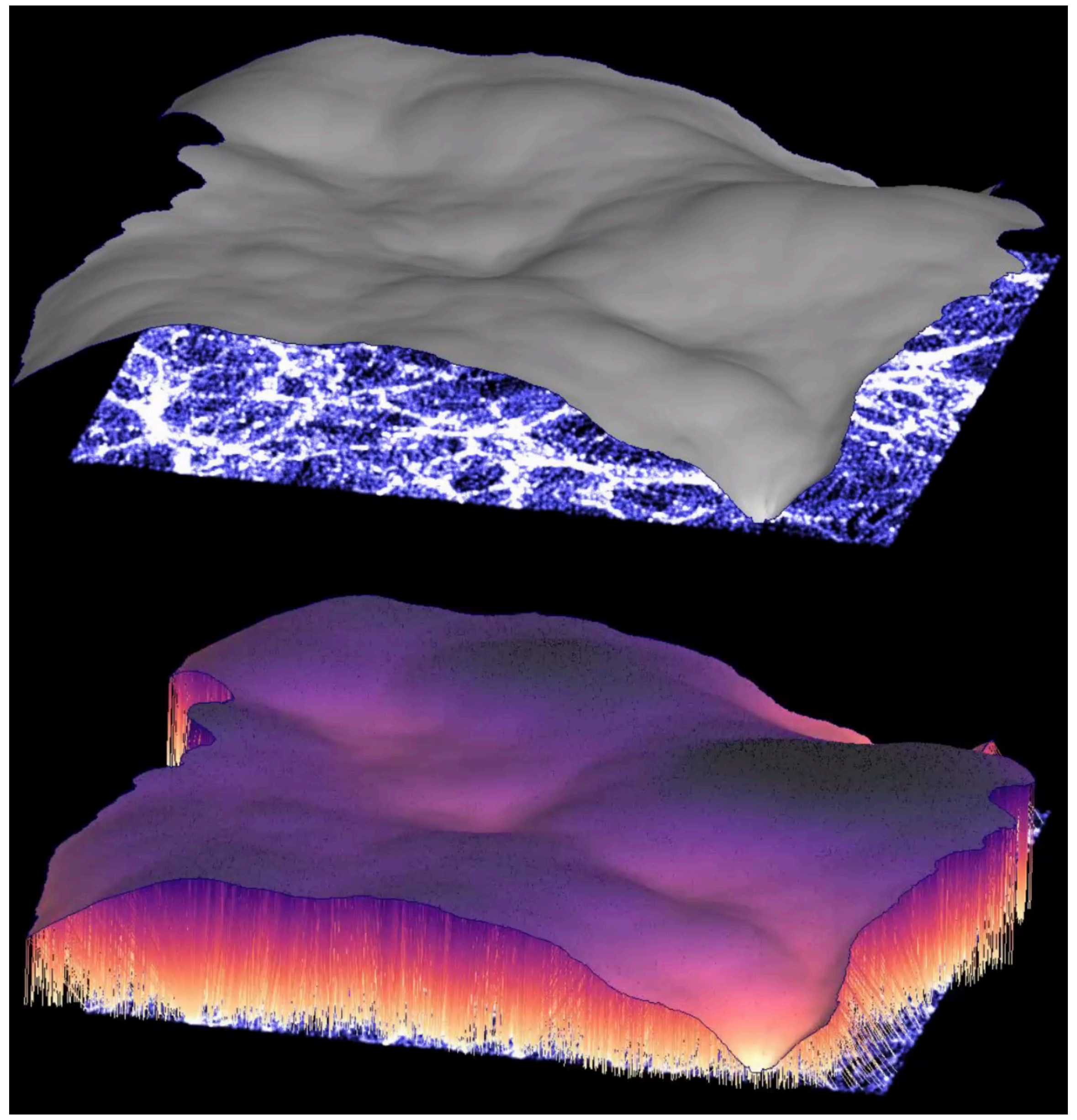
DESI x ACT

Forward modeling inspired by OT

$$\mathbf{q} = \mathbf{x} + \nabla\Theta_{\rm OT}(\mathbf{x}) \qquad \mathbf{x} = \mathbf{q} + \nabla\tilde{\Theta}_{\rm OT}(\mathbf{q})$$

$$\Theta_{\rm OT}(\mathbf{x}) = \max_{\mathbf{q}}\ \mathbf{x}\cdot\mathbf{q} - \tilde{\Theta}_{\rm OT}(\mathbf{q})$$

$$\tilde{\Theta}_{\rm OT}(\mathbf{q}) = \max_{\mathbf{x}}\ \mathbf{x}\cdot\mathbf{q} - \Theta_{\rm OT}(\mathbf{x})$$



Summary

Reconstruction of the *initial density & displacement* fields:
BAO scale, growth rate, baryon distribution, …

Optimal Transport theory is not merely a tool:
Minimum action principle subject to conservation law!

DESI: Exciting Data Ahead for both *Cosmology & Astrophysics*

DARK ENERGY SPECTROSCOPIC INSTRUMENT

U.S. Department of Energy Office of Science

Thank you!

For more details:

- Uberoi, **FN**, Padmanabhan et al. (2024 in prep.)
- Gaines, **FN**, Padmanabhan, Sheth (2024 in prep.)
- **FN**, N Padmanabhan, R Sheth, B Lévy, R Mohayaee; PRD accepted, 2024
- **FN**, N Padmanabhan, R Sheth, B Lévy, R Mohayaee; PRD, 108 (8), 083534
- **FN**, RK Sheth, B Lévy, R Mohayaee; PRL, 129 (25), 251101
- **FN**, RK Sheth, I Zehavi; PRD 105 (4), 043536
- **FN**, RK Sheth, I Zehavi; PRD 104 (6), 063504
- **FN**, RK Sheth, I Zehavi; PRD 104, 043530

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