Two-Loop Fundamental Colour-Kinematics Duality

Based on work done with Henrik Johansson & Gregor Kälin
[1706.09381]

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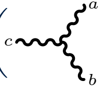
Cubic diagram expansion

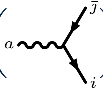
- Super-Yang-Mills (SYM) amplitudes with massless fundamental matter can be written as

$$\mathcal{A}_m^{L\text{-loop}} = \sum_{\text{cubic diagrams i}} \int \frac{\mathrm{d}^{DL}\ell}{(2\pi)^{DL}} \frac{1}{S_i} \frac{c_i n_i}{D_i}$$

- Colour factors c_i made from cubic structure constants $\tilde{f}^{abc}$ and fundamental generators $T^a_{i\bar{\jmath}}$:

$\tilde{f}^{abc} = c\Big($ 

$\Big)$

$T^a_{i\bar{\jmath}} = c\Big($ 

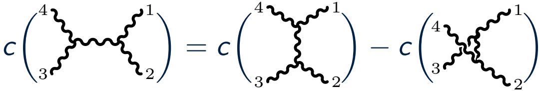
$\Big)$

- Kinematic numerators n_i are gauge-dependent objects.

Jacobi and commutation relations

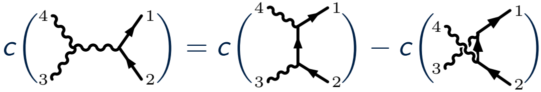
- Colour factors satisfy Jacobi identities:

$$\tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4} = \tilde{f}^{a_4 a_1 b} \tilde{f}^{b a_2 a_3} - \tilde{f}^{a_2 a_4 b} \tilde{f}^{b a_3 a_1}$$



- And commutation relations:

$$T^{b}_{i_1 \bar{\imath}_2} \tilde{f}^{b a_3 a_4} = T^{a_3}_{i_1 \bar{\jmath}} T^{a_4}_{j \bar{\imath}_2} - T^{a_4}_{i_1 \bar{\jmath}} T^{a_3}_{j \bar{\imath}_2} = [T^{a_3}, T^{a_4}]_{i_1 \bar{\imath}_2}$$



- Colour-dual numerators can be chosen such that [BCJ '08]

$$c_i = c_j - c_k \iff n_i = n_j - n_k$$

- One-loop fundamental matter [Johansson & Ochirov '15]

Adjoint double copy

- Supergravity (SG) amplitudes obtainable via double copy:

$$\mathcal{A}_m^{L\text{-loop}} = \sum_{\text{cubic diagrams i}} \int \frac{\mathrm{d}^{DL}\ell}{(2\pi)^{DL}} \frac{1}{S_i} \frac{c_i n_i}{D_i}$$

$$\mathcal{M}_m^{L\text{-loop}} = \sum_{\text{cubic diagrams i}} \int \frac{\mathrm{d}^{DL}\ell}{(2\pi)^{DL}} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{D_i}$$

- Different pairs of numerators give different SGs.
- E.g. pure $\mathcal{N} \geq 4$ SGs are factorizable

$$(\mathcal{N} = 8 \text{ SG}) = (\mathcal{N} = 4 \text{ SYM}) \otimes (\mathcal{N} = 4 \text{ SYM})$$

$$(\mathcal{N} = 4 + p \text{ SG}) = (\mathcal{N} = 4 \text{ SYM}) \otimes (\mathcal{N} = p \text{ (S)YM})$$

- $\mathcal{N} < 4$ SGs are non-factorizable. Why?

State counting

- $\mathcal{N}=4$ SYM has $2^{\mathcal{N}}$=16 states:

$$\mathcal{V}_{\mathcal{N}=4}(\eta^I) = A^+ + \eta^I\psi_I^+ + \frac{1}{2}\eta^I\eta^J\varphi_{IJ} + \cdots + \eta^1\eta^2\eta^3\eta^4 A_-$$

- $\mathcal{N}\leq 2$ SYM has chiral and anti-chiral multiplets, so $2\times 2^{\mathcal{N}}$ states. E.g.

$$V_{\mathcal{N}=2}(\eta^I) = A^+ + \eta^I\psi_I^+ + \eta^1\eta^2\varphi_{12}$$
$$\overline{V}_{\mathcal{N}=2}(\eta^I) = \varphi_{34} + \epsilon_{IJ}\eta^I\psi_-^J + \eta^1\eta^2 A_-$$

- The same counting also applies to SGs. With $p, q < 4$,

$$\begin{aligned} \mathcal{N}=8\text{ SG:} &\qquad 2^8 = 2^4\otimes 2^4 \\ \mathcal{N}=4+p\text{ SG:} &\qquad 2^{4+p+1} = 2^4\otimes 2^{p+1} \\ \mathcal{N}=p+q\text{ SG:} &\qquad 2^{p+q+1} \neq 2^{p+1}\otimes 2^{q+1} \end{aligned}$$

- When $\mathcal{N}=4$ SYM is not involved we need to include fundamental matter hypermultiplets.

$\mathcal{N} = 2$ super-QCD

- The full $\mathcal{N} = 4$ on-shell multiplet is decomposed into

$$\begin{aligned} \mathcal{V}_{\mathcal{N}=4}(\eta^I) &= A^+ + \eta^I \psi_I^+ + \frac{1}{2}\eta^I \eta^J \varphi_{IJ} + \cdots + \eta^1\eta^2\eta^3\eta^4 A_- \\ &= \mathcal{V}_{\mathcal{N}=2} + \Phi_{\mathcal{N}=2} + \bar{\Phi}_{\mathcal{N}=2} \end{aligned}$$

- The chiral and anti-chiral $\mathcal{N} = 2$ multiplets are combined into a single non-chiral multiplet:

$$\mathcal{V}_{\mathcal{N}=2} = V_{\mathcal{N}=2} + \eta^3\eta^4 \bar{V}_{\mathcal{N}=2}$$

- We also have CPT-conjugate hypermultiplets:

$$\begin{aligned} \Phi_{\mathcal{N}=2}(\eta^I) &= (\psi_3^+ + \eta^I \varphi_{I3} + \eta^1\eta^2 \psi_-^4)\eta^3 \\ \overline{\Phi}_{\mathcal{N}=2}(\eta^I) &= (\psi_4^+ + \eta^I \varphi_{I4} - \eta^1\eta^2 \psi_-^3)\eta^4 \end{aligned}$$

- We take hypermultiplets in the fundamental representation.

Fundamental double copy

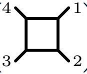
- The double copy with hypermultiplets gives pure $\mathcal{N}=4$ SG:

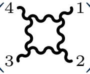
$$\mathcal{H}_{\mathcal{N}=4} \oplus 2\mathcal{V}_{\mathcal{N}=4} = \mathcal{V}_{\mathcal{N}=2} \otimes \mathcal{V}_{\mathcal{N}=2}$$
$$2\mathcal{V}_{\mathcal{N}=4} = (\Phi_{\mathcal{N}=2} \otimes \bar{\Phi}_{\mathcal{N}=2}) \oplus (\bar{\Phi}_{\mathcal{N}=2} \otimes \Phi_{\mathcal{N}=2})$$

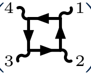
- The double copy becomes:

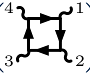
$$\mathcal{M}_m^{L\text{-loop}} = \sum_{\text{cubic diagrams i}} \int \frac{\mathrm{d}^{LD}\ell}{(2\pi)^{LD}} \frac{(-1)^{|i|}}{S_i} \frac{n_i(\ell)\bar{n}_i(\ell)}{\mathcal{D}_i(\ell)}$$

- For instance,

$N^{[\mathcal{N}=4\ \text{SG}]}\Big($ 

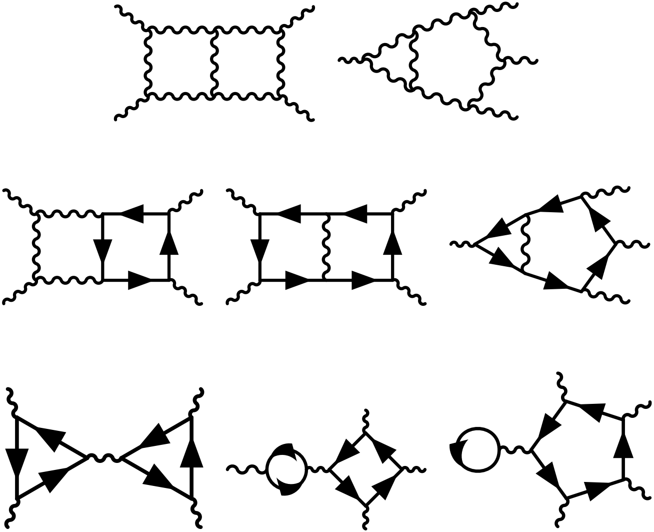
$\Big) = n\Big($ 

$\Big)^2 - 2n\Big($ 

$\Big)\, n\Big($ 

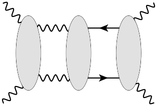
$\Big)$

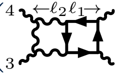
Two-loop four-point masters



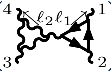
Finding colour-dual numerators

- Construct generalized unitarity cuts from tree amplitudes.
- Write cuts in terms of contributing numerators, e.g.



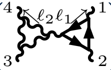
$$= \frac{1}{(\ell_1 - p_1)^2(\ell_2 - p_4)^2(\ell_1 + \ell_2)^2}\, n\Bigg($$


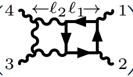
$$\Bigg)$$

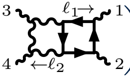
$$+ \frac{1}{s(\ell_1 - p_1)^2(\ell_2 - p_4)^2}\, n\Bigg($$


$$\Bigg) + \cdots$$

- Then express numerators in terms of the masters, e.g.

$$n\Bigg($$


$$\Bigg) = n\Bigg($$


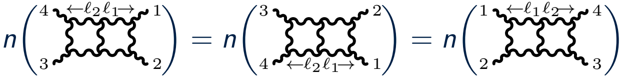
$$\Bigg) - n\Bigg($$


$$\Bigg)$$

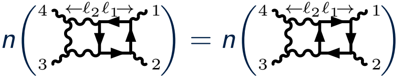
- Fit expressions to the master numerators using ansätze.

Off-shell constraints

- Cuts leave unfixed freedom for numerators. How to fix?
- Solve symmetry equations, e.g.



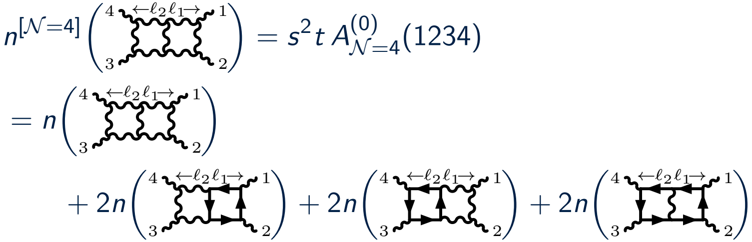
- Impose fermion reversal symmetry, e.g.



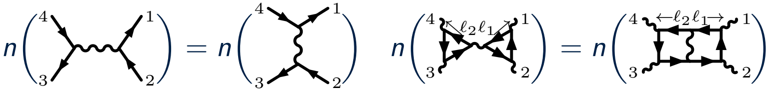
- There's still unfixed freedom. Is there anything else we can impose?

Matching to $\mathcal{N} = 4$ SYM

- Combine $\mathcal{N} = 2$ + matter content into $\mathcal{N} = 4$ off-shell, e.g.



- Goes together naturally with the 2-term identity:



- Reduces the problem to 3 masters.
- Setting a specific master to zero yields a unique solution.

Extending to $D = 6$

- Six dimensions provides a natural embedding for $D = 4 - 2\epsilon$:

$$D = 4, \mathcal{N} = 4 \text{ SYM} \iff D = 6, \mathcal{N} = (1,1) \text{ SYM}$$

- But, for $\mathcal{N} = 2$ SYM there is no unique map:

$$D = 4, \mathcal{N} = 2 \text{ SYM} \iff \begin{cases} D = 6, \mathcal{N} = (1,0) \text{ SYM} \\ D = 6, \mathcal{N} = (0,1) \text{ SYM} \end{cases}$$

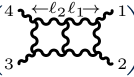
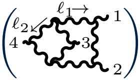
- So the numerators are unavoidably chiral!
- But the double copy is natural:

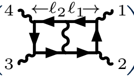
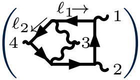
$$\mathcal{H}_{\mathcal{N}=(1,1)} = \mathcal{V}_{\mathcal{N}=(1,0)} \otimes \mathcal{V}_{\mathcal{N}=(0,1)}$$

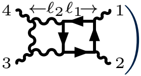
$$\mathcal{V}_{\mathcal{N}=(1,1)} = \Phi_{\mathcal{N}=(1,0)} \otimes \overline{\Phi}_{\mathcal{N}=(0,1)} = \overline{\Phi}_{\mathcal{N}=(1,0)} \otimes \Phi_{\mathcal{N}=(0,1)}$$

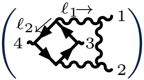
The all-chiral solution

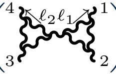
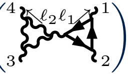
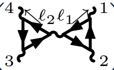
- With all externals in $V_{\mathcal{N}=2}$ there are 9 non-zero numerators:

$n\Big($  $\Big) = n\Big($  $\Big) = 2s(m_1^2 + m_2^2 + m_1 m_2)$

$n\Big($  $\Big) = n\Big($  $\Big) = s m_1 m_2$

$n\Big($  $\Big) = -s m_1 (m_1 + m_2)$

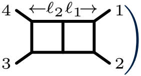
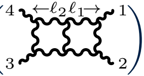
$n\Big($  $\Big) = -s m_2 (m_1 + m_2)$

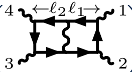
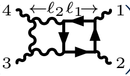
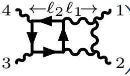
$n\Big($  $\Big) = -2n\Big($  $\Big) = 4n\Big($  $\Big) = 4 s m_1 m_2$

where $m_i = (\ell_i)_4 - i\,(\ell_i)_5$.

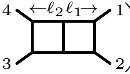
- All of these numerators integrate to zero.

The all-chiral double copy

$N^{[\mathcal{N}=4\ \mathrm{SG}]}\Big($  $\Big) = \Big| n\Big($  $\Big)\Big|^2$

$+ (D_s - 6)\Big(\Big| n\Big($  $\Big)\Big|^2 + \Big| n\Big($  $\Big)\Big|^2 + \Big| n\Big($  $\Big)\Big|^2 \Big)$

- We obtain $D = (4 - 2\epsilon)$-dimensional $\mathcal{N} = 4$ SG numerators,

$N^{[\mathcal{N}=4\ \mathrm{SG}]}\Big($  $\Big)$

$$= s^2 \left(16(\mu_{12}^2 - \mu_{11}\mu_{22}) + (D_s - 2)(\mu_{11}\mu_{22} + \mu_{11}\mu_{33} + \mu_{22}\mu_{33})\right)$$

- $\mu_{ij} = -\ell_i^{[D-4]} \cdot \ell_j^{[D-4]}$ and D_s is a state-counting parameter.

Conclusions and future directions

- We have found D-dimensional colour-dual numerators in the MHV sector!
- This is just a first step: we intend to consider other examples.
- We can think about lower-degree supersymmetries, eventually 2-loop pure gravity?
- Higher loop orders: forbids use of 6D spinors.

Thanks for listening!