Determining the Soft Anomalous Dimension from General Constraints

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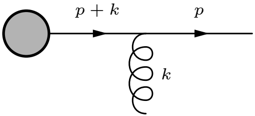
arXiv:1706.10162 [hep-ph]
with Ø. Almelid, C. Duhr, E. Gardi, and C. White

Soft Radiation

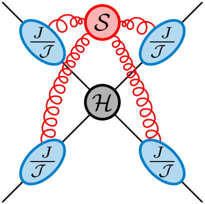
Eikonal Feynman rules are simpler than normal Feynman rules



$$ig\bar{u}(p)\gamma^{\mu}\frac{\not{p}+\not{k}}{2p\cdot k}\mathbf{T}^{a} \xrightarrow{k\ll p} ig\bar{u}(p)\frac{p^{\mu}}{p\cdot k}\mathbf{T}^{a}$$

- Soft gluon emission gives rise to a rescaling invariance on the external momenta p^{μ}, and thus only depends on the four-velocity $\beta^{\mu}=p^{\mu}/Q$
- However, we here are interested in the massless soft anomalous dimension, and this rescaling invariance is spoiled in the presence of cusps with light-like rays Dixon, Magnea, Sterman, arXiv:0805.3515 [hep-ph]

Soft-Collinear Factorization



At fixed-angle, where $p_i \cdot p_j \gg \Lambda_{\text{QCD}}$, the massless n-parton amplitude can be factorized in $d = 4 - 2\epsilon$ dimensions as

$$\mathcal{A}_n(\{p_i\}, \epsilon, \alpha_s) = \mathcal{S}(\{\beta_i\}, \{\mathbf{T}_i\}, \epsilon, \alpha_s)\, \mathcal{H}_n(\{p_i\}, \{n_i\}, \epsilon, \alpha_s) \prod_{i=1}^{n} \frac{J(p_i, n_i, \epsilon, \alpha_s)}{\mathcal{J}(\beta_i, n_i, \epsilon, \alpha_s)}$$

where $\mathcal{S}$ and J are the soft and jet functions, $\mathcal{H}_n$ is a process-dependent hard function, and $\mathcal{J}$ are eikonal jet functions.

- Collinear singularities cancel in the ratio

$$\frac{\mathcal{S}(\{\beta_i\}, \{\mathbf{T}_i\}, \epsilon, \alpha_s)}{\prod_{i=1}^{n} \mathcal{J}(\beta_i, n_i, \epsilon, \alpha_s)},$$

restoring the rescaling invariance in all four-velocities $\beta_i \to \kappa_i \beta_i$

Dixon, Magnea, Sterman, arXiv:0805.3515 [hep-ph]; Gardi, Magnea, arXiv:0901.1091 [hep-ph]

The Dipole Formula

This rescaling invariance gives rise to strong constraints on the kinematic dependence and functional form of the soft function

Gardi, Magnea, arXiv:0901.1091 [hep-ph]; Becher, Neubert, arXiv:0901.0722 [hep-ph]

- We can collect all singular contributions into a single factor Z_n

$$\mathcal{A}_n\left(\{p_i\},\epsilon,\alpha_s(\mu^2)\right) = Z_n\left(\{p_i\},\{\mathbf{T}_i\},\epsilon,\alpha_s(\mu_f^2)\right)\mathcal{H}_n\left(\{p_i\},\frac{\mu_f}{\mu},\epsilon,\alpha_s(\mu^2)\right)$$

  where the factor Z_n is renormalized multiplicatively by the soft anomalous dimension Γ_n

$$Z_n = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu_f^2}\frac{d\lambda^2}{\lambda^2}\Gamma_n(\{p_i\},\{\mathbf{T}_i\},\lambda,\alpha_s(\lambda^2))\right\}$$

- The minimal solution to the rescaling constraint $\beta_i \to \kappa_i\beta_i$ is provided by the all-loop dipole formula

$$\Gamma_n^{\text{dip.}} = -\frac{1}{2}\hat{\gamma}_K(\alpha_s)\sum_{i<j}\log\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}(\alpha_s)$$

  where $\hat{\gamma}_K(\alpha_s)$ is the part of the lightlike cusp anomalous dimension that admits Casimir scaling

Corrections to the Dipole Formula

- Casimir scaling fails starting at four loops
  Boels, Huber, Yang, arXiv:1705.03444 [hep-th]

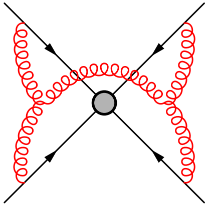
$$\gamma_K^{(i)}(\alpha_s) \equiv C_i \hat{\gamma}_K(\alpha_s) + \tilde{\gamma}_K^{(i)}(\alpha_s), \quad \tilde{\gamma}_K^{(i)}(\alpha_s) = \mathcal{O}(\alpha_s^4)$$

- Homogenous solutions to the rescaling-invariance constraint
  - The conformally invariant cross ratios

$$\rho_{ijkl} \equiv \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})} = \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$$

    are invariant under the rescaling of individual parton momenta, so any function of these variables is allowed

  - These variables can only appear in front of color factors that irreducibly connect four partons, which first happens at three loops



The Bootstrap Approach

$\Delta_n^{(3)}$ was recently computed and found to be nonzero by direct evaluation of the relevant Feynman integrals

Almelid, Duhr, Gardi, arXiv:1507.00047 [hep-ph]

However, could $\Delta_n^{(3)}$ have been ascertained without carrying out these integrals? Perhaps we can find a unique linear combination of (some relevant set of) functions that satisfies all known properties of $\Delta_n^{(3)}$

- Need to know the types of functions that can appear in $\Delta_n^{(3)}$
- Known color structure, symmetries, and transcendental properties
- Constrained behavior in special kinematic limits

The Space of Functions

To identify the class of functions that depend on n massless four-velocities in a rescaling-invariant way, we parametrize

$$\beta_i^\mu = \left(1 + \frac{z_i \bar{z}_i}{4}, \frac{z_i + \bar{z}_i}{2}, \frac{z_i - \bar{z}_i}{2i}, 1 - \frac{z_i \bar{z}_i}{4}\right)$$

in terms of variables $z_i, \bar{z}_i$ that can be thought of as living on the Riemann sphere.

- The space of iterated integrals on the Riemann sphere with n marked points is known to be expressible as multiple polylogarithms Brown, arXiv:math/0606419 [math.AG]
- Since our cross ratios only depend on the angles between Wilson lines, we have an $SL(2, \mathbb{C})$ symmetry that allows us to choose

  $$z_i \equiv z, \quad z_j = 0, \quad , z_k = \infty, \quad z_l = 1$$

  which implies that only polylogarithms of a single variable, namely harmonic polylogarithms (HPLs), are needed at three loops
- Plugging this parametrization into our cross ratios, they become simple rational combinations of z and $\bar{z}$

Single-Valued Harmonic Polylogarithms

Further simplifications occur in the Euclidean region, since branch cuts can only end where a Mandelstam invariant becomes zero or infinite

- In this region, the soft anomalous dimension must be single-valued, further restricting the relevant function space at three loops to single-valued harmonic polylogarithms (SVHPLs)

$$\mathcal{L}_{w_1 \cdots w_n}(z, \bar{z}), \quad w_i \in \{0, 1\}$$

- Singularities only occur when two marked points z_i and z_j coincide, which happens at $z = 0, 1, \infty$

The correction $\Delta_n^{(3)}$ is also insensitive to the matter content of the theory, and so is the same in QCD and $\mathcal{N} = 4$ SYM. This implies it must be a pure function of uniform transcendental weight five

- Each SVHPL has weight equal to its number of indices n. Riemann zeta values ζ_n can also appear and have weight n

Color Structure and Symmetries

The non-abelian exponentiation theorem, combined with rescaling invariance, Bose symmetry, and color conservation, leaves only two types of color factors that can appear in $\Delta_n^{(3)}$

$$\Delta_n^{(3)} = 16 f_{abe} f_{cde} \Bigg\{ - C \Bigg(\sum_{i=1}^{n} \sum_{1 \leq j < k \leq n; j,k \neq i} \{ \mathbf{T}_i^a, \mathbf{T}_i^d \} \mathbf{T}_j^b \mathbf{T}_k^c \Bigg)$$
$$+ \sum_{1 \leq i < j < k < l \leq n} \Big[\mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d \big(F(1-1/z) - F(1/z) \big)$$
$$+ \mathbf{T}_i^a \mathbf{T}_k^b \mathbf{T}_j^c \mathbf{T}_l^d \big(F(1-z) - F(z) \big)$$
$$+ \mathbf{T}_i^a \mathbf{T}_l^b \mathbf{T}_j^c \mathbf{T}_k^d \big(F(1/(1-z)) - F(z/(z-1)) \big) \Big] \Bigg\}$$

and implies that the objects multiplying the first of these color factors must be a constant, while the object multiplying the second must be antisymmetric under the parton indices associated with the same structure constant

Gardi, Almelid, Duhr, arXiv:1606.05697 [hep-ph]

An Ansatz at Three Loops

This reduces the problem to determining a set of rational coefficient in $F(z)$ and C

$$\begin{aligned} F(z) = F(z,\bar{z}) = F(\bar{z},z) \\ &= a_1\mathcal{L}_{00000} + a_2\mathcal{L}_{00100} + a_3\mathcal{L}_{10001} \\ &\quad + a_4\mathcal{L}_{10101} + a_5\left(\mathcal{L}_{01001} + \mathcal{L}_{10010}\right) \\ &\quad + a_6\left[\mathcal{L}_{00101} + \mathcal{L}_{10100} + 2(\mathcal{L}_{00011} + \mathcal{L}_{11000})\right] \\ &\quad + a_7\left[\mathcal{L}_{11010} + \mathcal{L}_{01011} + 3(\mathcal{L}_{00011} + \mathcal{L}_{11000})\right] \\ &\quad + a_8\,\zeta_2\mathcal{L}_{000} + a_9\,\zeta_2\left(\mathcal{L}_{001} + \mathcal{L}_{100}\right) + a_{10}\,\zeta_3\,\mathcal{L}_{00} + a_{11}\,\zeta_2^2\,\mathcal{L}_0 \end{aligned}$$

$$C = a_{12}\,\zeta_5 + a_{13}\,\zeta_2\,\zeta_3$$

- An understanding of the symmetries of $\Delta_n^{(3)}$ and the space of functions it can depend on determines its value up to only 13 rational coefficients

Constraints from Collinear Limits

In the limit that two partons become collinear, the n-parton amplitude is expected to factorize into an $(n-1)$-parton amplitude times a splitting function that only depends on the partons becoming collinear

- This translates into the condition that

$$\begin{aligned}\Delta^{(3)}_{\mathbf{Sp}}(\mathbf{T}_1,\mathbf{T}_2) &\equiv \Big[\Delta^{(3)}_n(\{\rho_{ijkl}\},\mathbf{T}_1,\mathbf{T}_2,\{\mathbf{T}_j\}) \\ &\qquad - \Delta^{(3)}_{n-1}(\{\rho_{ijkl}\},\mathbf{T}_1+\mathbf{T}_2,\{\mathbf{T}_j\})\Big]_{p_1\|p_2} \\ &= \Big[\Delta^{(3)}_3(-\mathbf{T}_1-\mathbf{T}_2,\mathbf{T}_1,\mathbf{T}_2)\Big]_{p_1\|p_2}\end{aligned}$$

  is a constant that depends only on the color of the partons becoming collinear, here taken to be partons 1 and 2
- This gives rise to the nontrivial constraint that

$$\begin{aligned}8C &= \Big[F\big(1/(1-z)\big)+F\big(1-z\big)-F\big(z\big)-F\big(z/(z-1)\big)\Big]_{p_1\|p_2} \\ &= -\frac{a_1}{60}\log^5(z\bar z)-\frac{a_8}{3}\zeta_2\log^3(z\bar z) \\ &\quad -(4a_7+a_{10})\zeta_3\log^2(z\bar z)-2a_{11}\zeta_2^2\log(z\bar z) \\ &\quad +8a_9\zeta_2\zeta_3+(24a_2-12a_3+8a_4+36a_5-12a_6+4a_7)\zeta_5\end{aligned}$$

Constraints from the Regge Limit

In the high-energy limit $s \gg -t$ the L-loop amplitude develops logarithmically-enhanced terms proportional to

$$\alpha_s^L \log^p\left(\frac{s}{-t}\right), \quad p \leq L$$

at leading order in $(-t)/s$

- These contributions can be independently calculated using rapidity evolution equations, and this has been done at three loops for

$$\alpha_s^3 \log^p\left(\frac{s}{-t}\right), \quad p \geq 1$$
$$i\alpha_s^3 \log^p\left(\frac{s}{-t}\right), \quad p \geq 2$$

arXiv:Caron-Huot, 1309.6521 [hep-th]; Caron-Huot, Gardi, Vernazza, arXiv:1701.05241 [hep-ph]

- The Regge limit of the dipole formula matches the predictions for all of the above terms, so these powers of logarithms must be absent from the Regge limit of $\Delta_n^{(3)}$

Constraints from the Regge Limit

- To compute one of the Regge limits of our ansatz in the case of four-parton scattering, we must choose two partons i and j to be incoming and analytically continue to the region where

$$s_{ij} = s_{kl} = s > 0$$

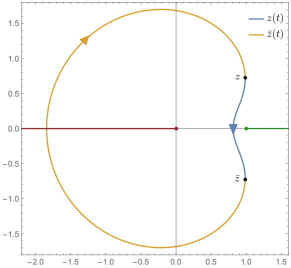
- For instance, taking partons 1 and 2 to be incoming, we take

$$z\bar{z} = \rho_{1234} = \frac{(-s_{12})(-s_{34})}{(-s_{13})(-s_{24})} = \left|\frac{s_{12}s_{34}}{s_{13}s_{24}}\right| e^{-2i\pi t},$$

$$(1-z)(1-\bar{z}) = \rho_{1432} = \frac{(-s_{14})(-s_{32})}{(-s_{13})(-s_{42})} = \left|\frac{s_{14}s_{23}}{s_{13}s_{24}}\right|,$$

  and advance t from 0 to 1

- This takes us out of the space of SVHPLs and into the space of HPLs with arguments z and $\bar{z}$, and gives rise to imaginary terms



Constraints from the Regge Limit

- To specialize to massless four-parton scattering, we impose momentum conservation and $s_{12} + s_{13} + s_{23} = 0$, which sets

$$z\bar{z} = \left(\frac{s_{12}}{s_{12} + s_{23}} \right)^2 ,$$
$$(1 - z)(1 - \bar{z}) = \left(\frac{s_{23}}{s_{12} + s_{23}} \right)^2 ,$$

  implying that $z = \bar{z}$ (up to opposite small imaginary parts)
- These relations tell us that the high-energy limit $s_{12} \gg -s_{23}$ requires sending $z = \bar{z} \to 1$, in which limits HPLs either vanish or logarithmically diverge, which leaves us with a polynomial in the expected large logarithm $\log(s/(-t))$

$$\log(1 - z) \xrightarrow{s_{12} \gg -s_{23}} -\log\left(\frac{s_{12}}{-s_{23}} \right) + i\pi$$
$$\log(1 - \bar{z}) \xrightarrow{s_{12} \gg -s_{23}} -\log\left(\frac{s_{12}}{-s_{23}} \right) - i\pi$$

- We can go into six different Regge limits, where we require that the predicted powers of large logarithms are always absent

Determining $\Delta_n^{(3)}$

The constraints from the Regge limit provide 8 relations between our coefficients, while those from collinear limits provide 6 relations

Together, they determine the function $\Delta_n^{(3)}$ up to an overall scale:

$$F(z) = a_4 \left(\mathcal{L}_{10101} + 2\zeta_2(\mathcal{L}_{100} + \mathcal{L}_{001})\right)$$
$$C = a_4(\zeta_5 + 2\zeta_2\zeta_3)$$

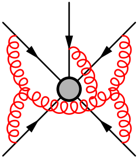
which exactly matches its calculated value when $a_4 = 1$

Almelid, Duhr, Gardi, arXiv:1507.00047 [hep-ph]

The Soft Anomalous Dimension at Higher Loops

Further complications will arise beyond three loops

- Color structures involving more partons can appear
- Casimir Scaling breaks down Boels, Huber, Yang, arXiv:1705.03444 [hep-th]
- Corrections to the dipole formula become sensitive to the matter content of the theory



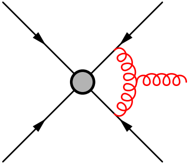
However, a bootstrap approach can still be employed

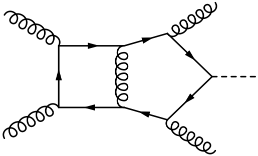
- Contributions depending only on conformally invariant cross ratios expressible in terms of single-valued multiple polylogarithms
- Two-particle collinear limits and Regge limits can again be applied
- Multi-particle collinear limits and multi-Regge limits may provide further constraints

Bootstrapping Other Quantities

We can also attempt to bootstrap other QCD quantities, such as the soft gluon current

- Known to be expressible in terms of SVHPLs through two loops
  Catani, Grazzini, arXiv:hep-ph/0007142; Duhr, Gehrmann, arXiv:1309.4393 [hep-ph]; Li, Zhu, arXiv:1309.4391 [hep-ph]; Dixon, Zhu, forthcoming





Eventually, we can hope to bootstrap QCD amplitudes themselves

- Essentially none of the amplitudes relevant for $2 \to 3$ hard scattering processes at the LHC known at two loops
- Need a better understanding of the relevant space of functions before this can be attempted

Thanks!