Amplitude School 2017

Unifying Relations for Scattering Amplitudes

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BCJ Zoo of Theories

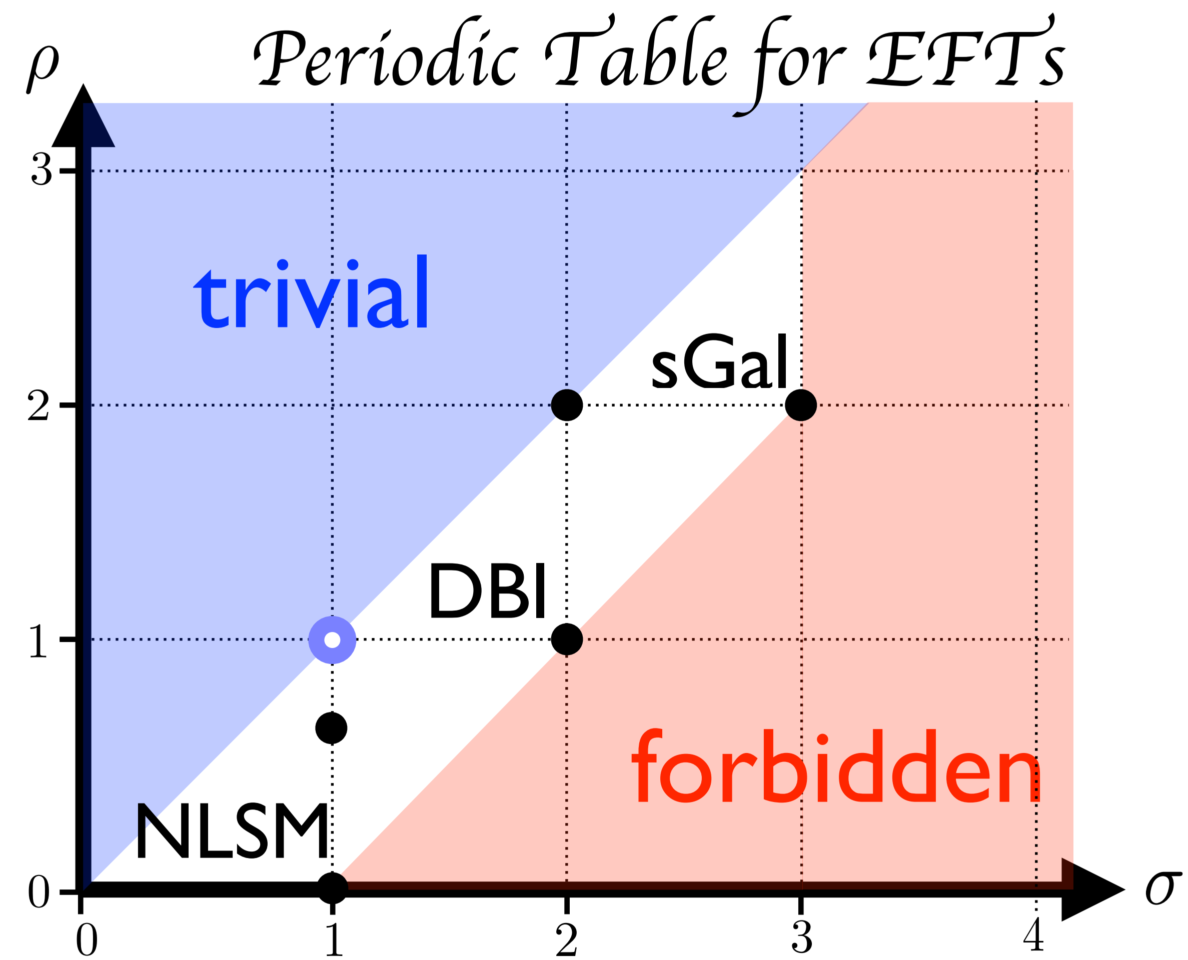
[Slide from Carrasco, talk at String 2017]

Key Point: **MANY Theories are Double Copies**

Bi-Adjoint Scalar:	color $\otimes$ color
Bern, de Freitas, Wong ('99); Bern, Dennen, Huang; Du, Feng, Fu; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell	
(S)YM (…(S)QCD…):	color $\otimes$ spin-1
BCJ ('08) Bjerrum-Bohr, Damgaard, Vanhove; Steiberger; Feng et al; Mafra, Schlotterer, ('08-'11); Johansson, Ochirov	
(S)Gr (…(S)Einstein-YM…):	spin-1 $\otimes$ spin-1
KLT('86); BCJ ('08); Chiodaroli, Gunaydin, Johansson, Roiban; Johansson, Ochirov; Johansson, Kälin, Mogull	
NLSM / Chiral Lagrangian:	"color" $\otimes$ even-spin-0
Chen, Du '13 Cachazo, He, Yuan '14 Cheung, Shen '16	
(S)Born-Infeld:	spin-1 $\otimes$ even-spin-0
Cachazo, He, Yuan '14	
Special Galileon:	even-spin-0 $\otimes$ even-spin-0
Cachazo, He, Yuan '14 Cheung, Shen '16	
Open String:	α' $\otimes$ spin-1
Broedel, Schlotterer, Stieberger	
Closed String:	spin-1 $\otimes$ α' corrected spin-1
Broedel, Schlotterer, Stieberger;	
Z-theory:	α' $\otimes$ "color"
Broedel, Schlotterer, Stieberger; JJMC, Mafra, Schlotterer	

Soft Theorems

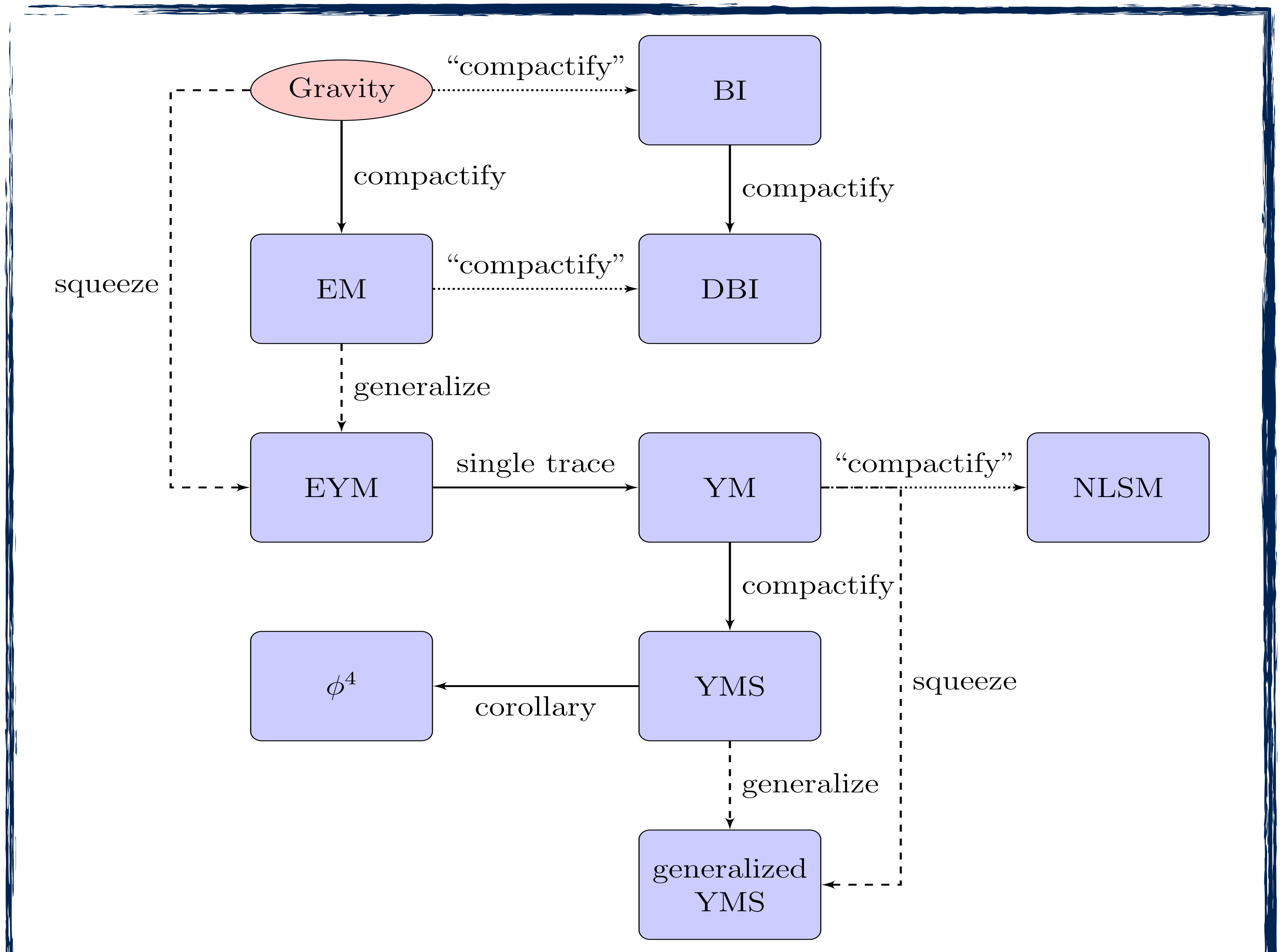
- YM and Gravity
  [Cachazo, Strominger; Casali; Schwab, Volovich; Afkhami-Jeddi; ….]
  - Weinberg soft theorem
    + Sub-(sub)-leading
- Effective Field Theories
  - "Exceptional" soft behavior fixes theories
  - Analogs of YM/GR
    [Cheung, Kampf, Novotny, CHS, Trnka; CHY]
    [Arkani-Hamed, Rodina, Trnka]



Cachazo-He-Yuan Formulae

- Structure only in CHY?

[CHY, 1412.3479]



Outline

- Transmutation
- Unified Web of Theories
- Proof
- Unifying Soft Theorems

Transmutation

Transmutation

- Goal: map amplitudes of one theory to another

- Inputs: amplitudes in terms of formal polarizations, *i.e.,* functions of Lorentz invariants

$$A(p_i p_j, p_i e_j, e_i e_j, p_i \tilde{e}_j, \tilde{e}_i \tilde{e}_j)$$

  - Agnostic about the representations (Feynman diagrams, recursion, CHY,….)

Toy Example: Dimensional Reduction

- Dimensional Reduction: $A_\mu \to \phi_I$

$$P = (\vec{p}, 0) \qquad \varepsilon = (\vec{0}, 1) \quad \Rightarrow \quad pe_{i,j} \to 0, \quad e_i e_j \to 1$$

- Differential operator:

$$\mathcal{T}_{ij} \equiv \partial_{e_i e_j}$$

- Multi-linearity of polarization kills pe effectively.

General Construction

- Operators of other invariants $\partial_{pe}, \partial_{pp}$?
- Consistency: maintain on-shell conditions
  - Momentum conservation: $\mathcal{P}_v \equiv \sum_i p_i v$
  - Gauge invariance: $\mathcal{W}_i \equiv \sum_v p_i v\, \partial_{v e_i}$

$$[\,\mathcal{T}\,,\,\mathcal{P}_v\,] = 0 \;\rightarrow\; \begin{cases} \mathcal{T}_{ij} \equiv \partial_{e_i e_j} \\ \mathcal{T}_{ijk} \equiv \partial_{p_i e_j} - \partial_{p_k e_j} \\ \mathcal{T}_{ijkl} \equiv \partial_{p_i p_j} - \partial_{p_k p_j} + \partial_{p_k p_l} - \partial_{p_i p_l} \end{cases}$$

Transmutation

- Trace operator: $\mathcal{T}_{ij} \equiv \partial_{e_i e_j}$
  - Equivalent to dimensional reduction
- Insertion operator: $\mathcal{T}_{ijk} \equiv \partial_{p_i e_j} - \partial_{p_k e_j}$
  - does NOT preserve gauge invariance of i and k
  - needs to be accompanied with a trace operator
  - Induce color ordering!

Example

- 3pt YM amplitude:

$$A(g_1, g_2, g_3) = e_3 e_1 (p_1 e_2 - p_3 e_2)/2 + \text{cyclic}$$

- YM to bi-adjoint scalars: $\mathcal{T}[123] = \mathcal{T}_{123}\mathcal{T}_{13}$

$$A(\phi_1\phi_2\phi_3) = \mathcal{T}[123] \cdot A(g_1, g_2, g_3) = 1$$
$$A(\phi_1\phi_3\phi_2) = \mathcal{T}[132] \cdot A(g_1, g_2, g_3) = -1$$

- All points gluon amplitudes:

$$A_{\text{YM}}(\beta) = \sum_\alpha N(\alpha) A_{\text{BS}}(\alpha|\beta)$$

- Transmutation as projection: $\mathcal{T}[\alpha] \cdot N(\beta) = \delta_{\alpha\beta}$

Transmutation

- Longitudinal operator:

$$\mathcal{L}_i \equiv \sum_j p_i p_j \partial_{p_j e_i} \quad \text{and} \quad \mathcal{L}_{ij} \equiv -p_i p_j \partial_{e_i e_j}$$

- Not gauge invariant! $[\,\mathcal{L}_i \,,\, \mathcal{W}_j\,] = -\mathcal{L}_{ij}$

- Have to transmute all particles (+ trace operator)

$$\mathcal{L} \equiv \prod_i \mathcal{L}_i \sim \sum_{\rho \in \mathrm{pair}} \prod_{i,j \in \rho} \mathcal{L}_{ij}$$

- Important consequence on color-kinematic duality
  [Cheung, Shen '16; +Congkao Wen, work in progress]

Transmutation Summary

- Trace & Insertion operator: new color trace

$$\mathcal{T}[i_1 i_2 \ldots i_n] \equiv \mathcal{T}_{i_1 i_n} \cdot \prod_{m=2}^{n-1} \mathcal{T}_{i_{m-1} i_m i_n}$$

  - gluon to bi-adjoint scalar; graviton to gluon

- Longitudinalization: $\mathcal{L} \equiv \prod_i \mathcal{L}_i \sim \sum_{\rho \in \mathrm{pair}} \prod_{i,j \in \rho} \mathcal{L}_{ij}$

  - YM to NLSM; gravity to Born-Infield theory (to special Galileon)

- Commute with KLT & BCJ relations

Unified Web of Theories

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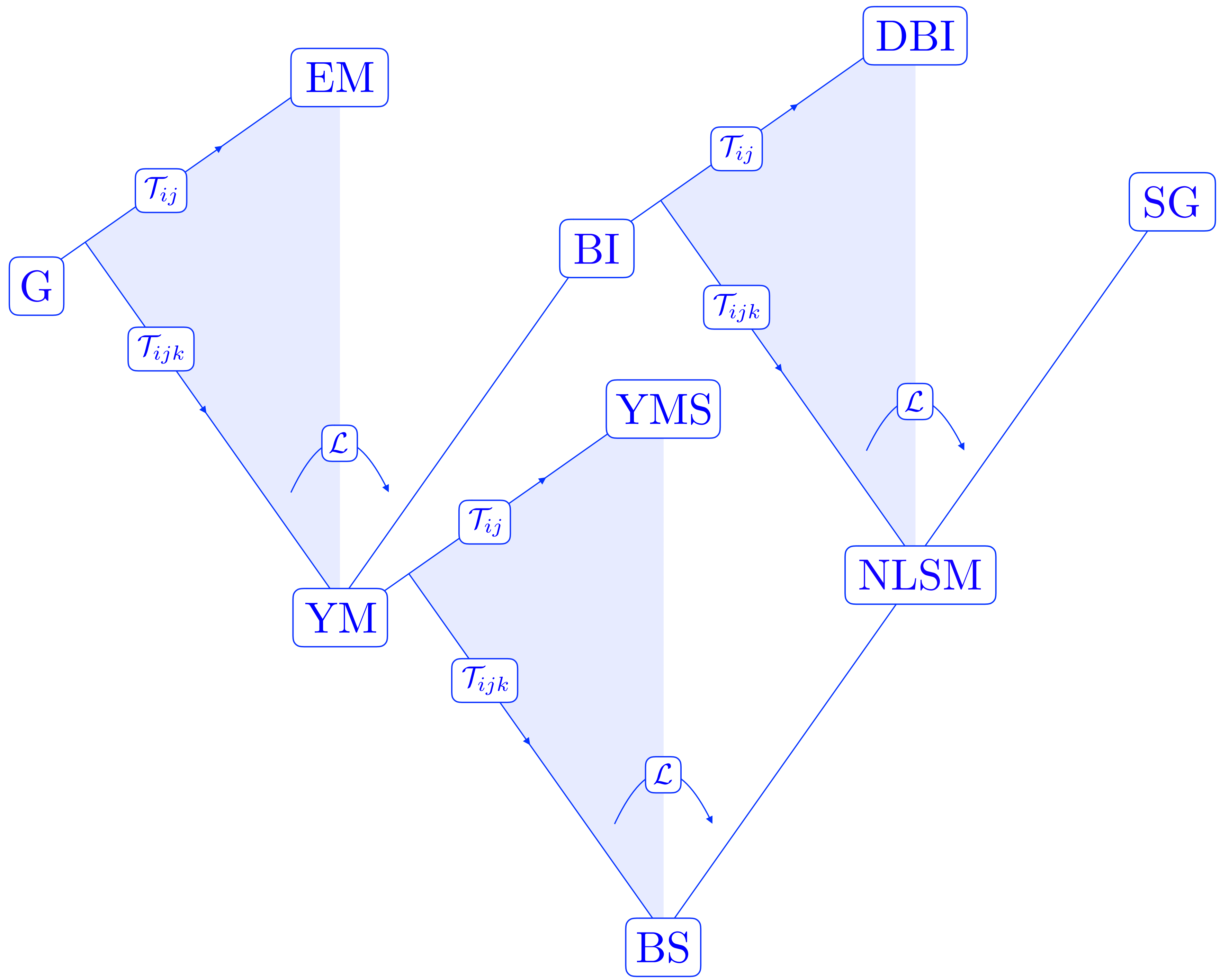
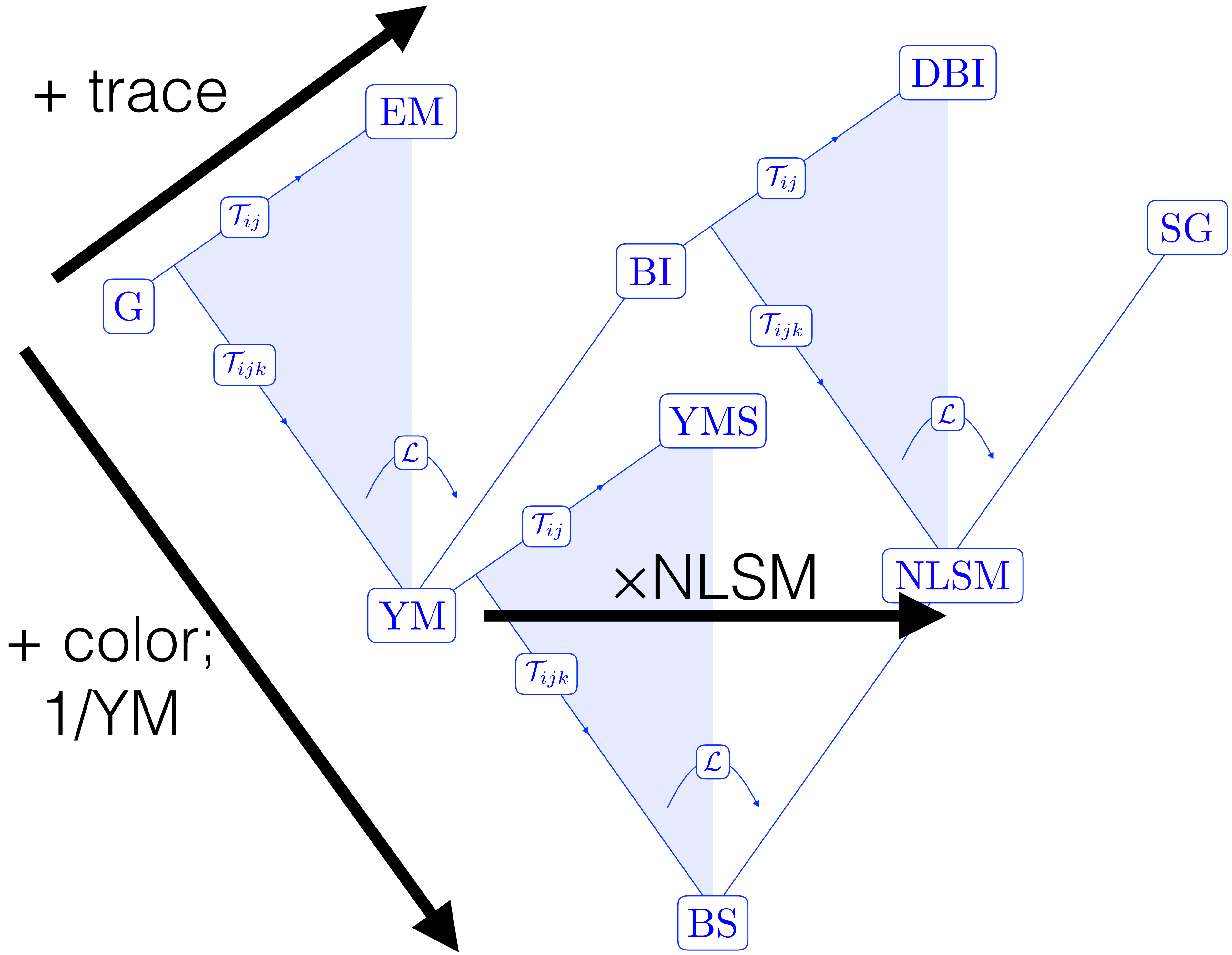


Unified Web of Theories



String Completion

- Transmutation also applies to string completion

[Broedel, Mafra, Schlotterer, and Stieberger]

$$A_{\text{open}} = A_{\text{YM}} \otimes A_{\text{Z}}$$

- Generate (abelian) Z-theory, a UV completion for bi-adjoint scalars or pions.

$$\mathcal{T}[i_1 \cdots i_n] \cdot A_{\text{open}}(g_{i_1}, \cdots, g_{i_n}) = A_{\text{Z}}(\phi_{i_1} \cdots \phi_{i_n})$$

$$\mathcal{T}[i_1 \cdots i_n] \cdot A_{\text{open}}(\gamma_{i_1}, \ldots, \gamma_{i_n}) = A_{\text{Z}}(\pi_{i_1} \cdots \pi_{i_n})$$

[Broedel, Schlotterer, and Stieberger; Carrasco, Mafra, Schlotterer]

- Or… $\mathcal{L} \cdot \mathcal{T}[i_1 i_n] \cdot A_{\text{open}} = A_{\text{Z}} \otimes A_{\text{NLSM}}$

Proof

Proof

- Proof via Recursion: factorizations (+ soft limits)

  [BCFW; Cohen, Elvang, Kiermaier ’10; Cheung, Kampf, Novotny, CHS, Trnka ‘15]

  - Transmutation is a boundary operation.

    Factorization & permutation are NOT manifest.

- Double Copy Construction

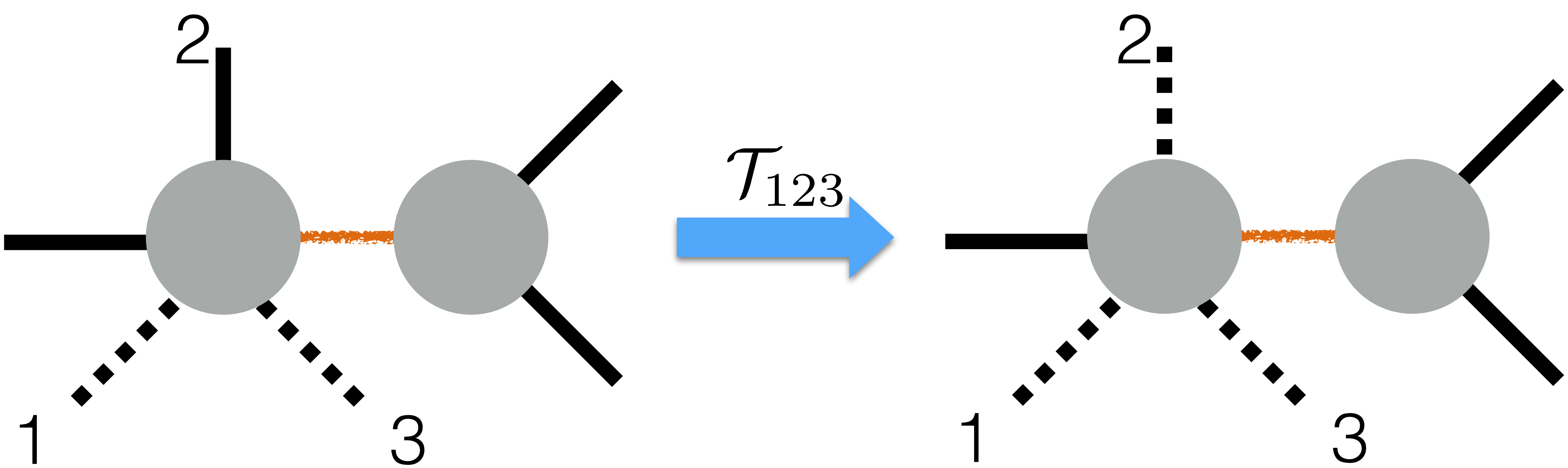
  [Bern, Carrasco, Johansson; +Dennen, Huang, Kiermaier; Cachazo, He, Yuan;…]

- Short cut for YM/NLSM: [Arkani-Hamed, Rodina, Trnka, ’16]

  gauge invariance/soft limits + locality

Factorization

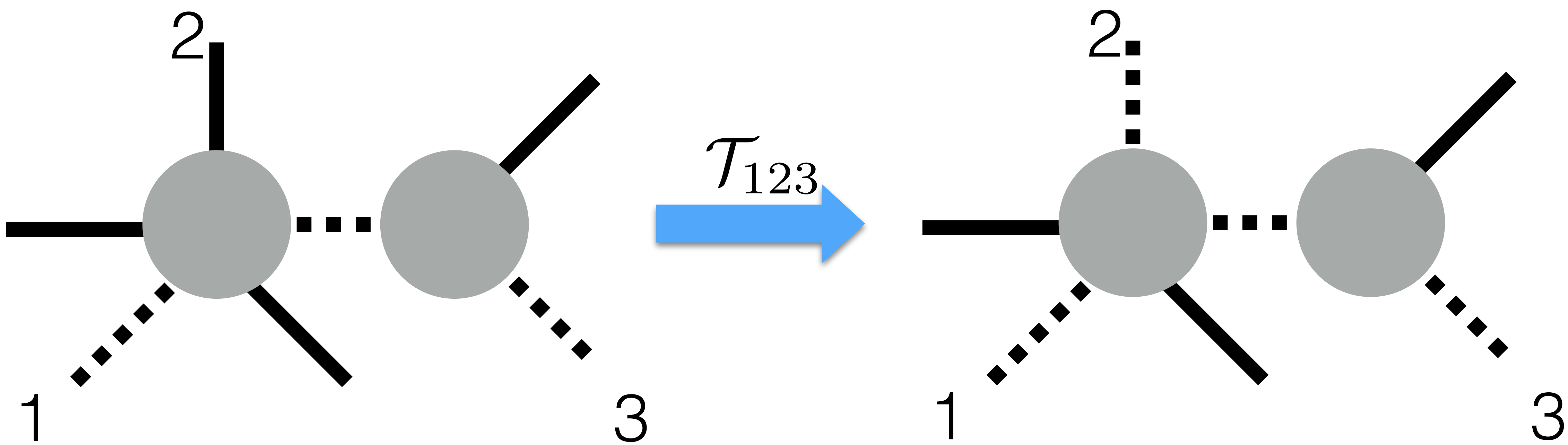
- Insertion operator: $\mathcal{T}_{123} = \partial_{p_1 e_2} - \partial_{p_3 e_2}$
- Case 1



Factorization

- Insertion operator: $\mathcal{T}_{123} = \partial_{p_1 e_2} - \partial_{p_3 e_2}$
- Case 2

$$\partial_{p_1 e_2} - \partial_{p_3 e_2} \rightarrow \partial_{p_1 e_2} - \partial_{p_L e_2}$$

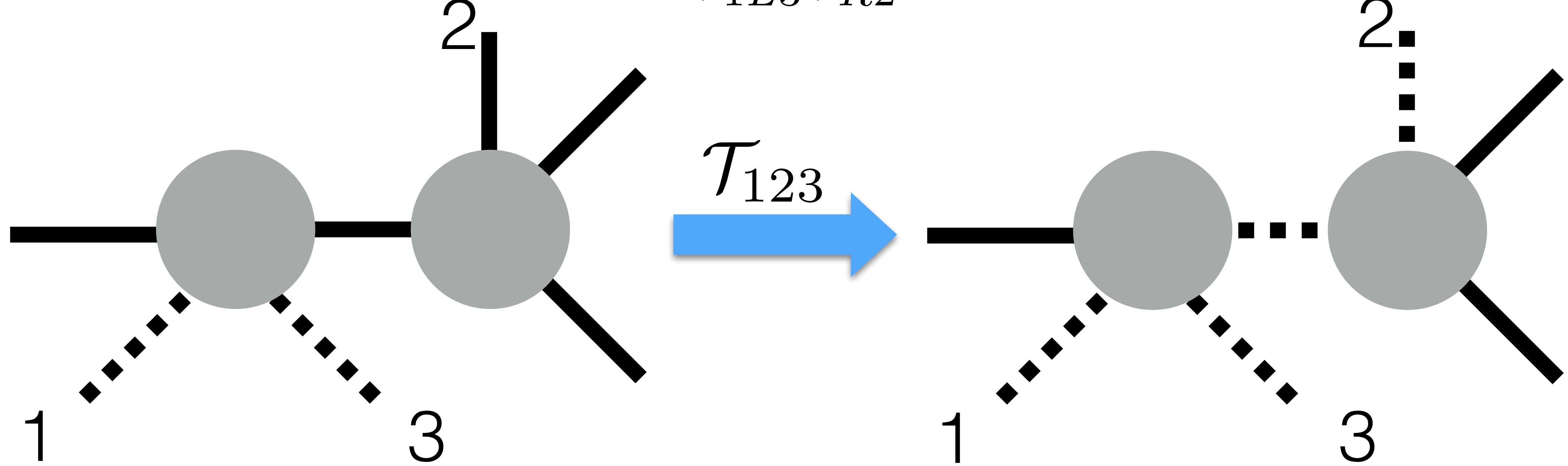


Factorization

- Insertion operator: $\mathcal{T}_{123} = \partial_{p_1 e_2} - \partial_{p_3 e_2}$

- Case 3

$$\partial_{p_1 e_2} - \partial_{p_3 e_2} \to \sum_I (\partial_{p_1 e_L} - \partial_{p_3 e_L})\partial_{e_R e_2}$$
$$\to \mathcal{T}_{1L3}\mathcal{T}_{R2}$$



Unifying Soft Theorems

Unified Soft Theorems

- Soft Theorems

$$A^{(n)} \xrightarrow{p_i \to 0} \mathcal{S}^{(i)} \cdot A^{(n-1)}$$

- Factor the transmutation into soft and hard parts

$$\mathcal{T}^{(n)} = \mathcal{T}^{(i)} \cdot \mathcal{T}^{(n-1)}$$

- Soft theorems for descendants

$$\underline{\mathcal{T}^{(n)} \cdot A^{(n)}} \xrightarrow{p_i \to 0} [\, \mathcal{T}^{(i)} \,,\, \mathcal{S}^{(i)} \,] \cdot \underline{\mathcal{T}^{(n-1)} \cdot A^{(n-1)}}$$

- Soft theorems from transmutation!

Leading Order

- Weinberg soft theorems in gravity

$$\mathcal{S}_{\mathrm{G}}^{(j)} = \sum_{j\neq i} \frac{p_i e_j \, p_i \tilde{e}_j}{p_i p_j}$$

- to Weinberg soft theorems in YM

$$\mathcal{S}_{\mathrm{YM}}^{(ijk)} = [\, \mathcal{T}_{ijk} \, , \, \mathcal{S}_{\mathrm{G}}^{(j)} \,] = \frac{p_i e_j}{p_i p_j} - \frac{p_k e_j}{p_k p_j}$$

- to bi-adjoint scalar

$$S_{\mathrm{BS}}^{(ijk|ljm)} = [\, \mathcal{T}_{ljm} \, , \, \mathcal{S}_{\mathrm{YM}}^{(ijk)} \,] = \frac{\Delta_{lim}}{p_i p_j} - \frac{\Delta_{lkm}}{p_k p_j}$$

- to Adler zero in NLSM/ BI [Cheung, CHS '16]

$$S_{\mathrm{BI}}^{(i)} = S_{\mathrm{NLSM}}^{(ijk)} = 0$$

Sub-leading Order

- Sub-leading soft theorems in gravity (+ dilaton, 2-form)

$$\mathcal{S}_{\mathrm{G}}^{(j)} = \sum_{i \neq j} \frac{e_j \mathcal{O}_i \tilde{e}_j}{2 p_i p_j}$$

[Di Vecchia, Marotta, Mojaza, 1706.02961]

- to sub-leading soft theorems in YM

$$\mathcal{S}_{\mathrm{YM}}^{(ijk)} = [\mathcal{T}_{ijk}, \mathcal{S}_{\mathrm{G}}^{j}] = \frac{p_i e_j - p_j J_i e_j}{p_i p_j} - (i \leftrightarrow k)$$

- to bi-adjoint scalar (*new!*) Symmetry perspective?

[Campiglia, Coito, Mizera '17]

- to extended theory in NLSM/BI

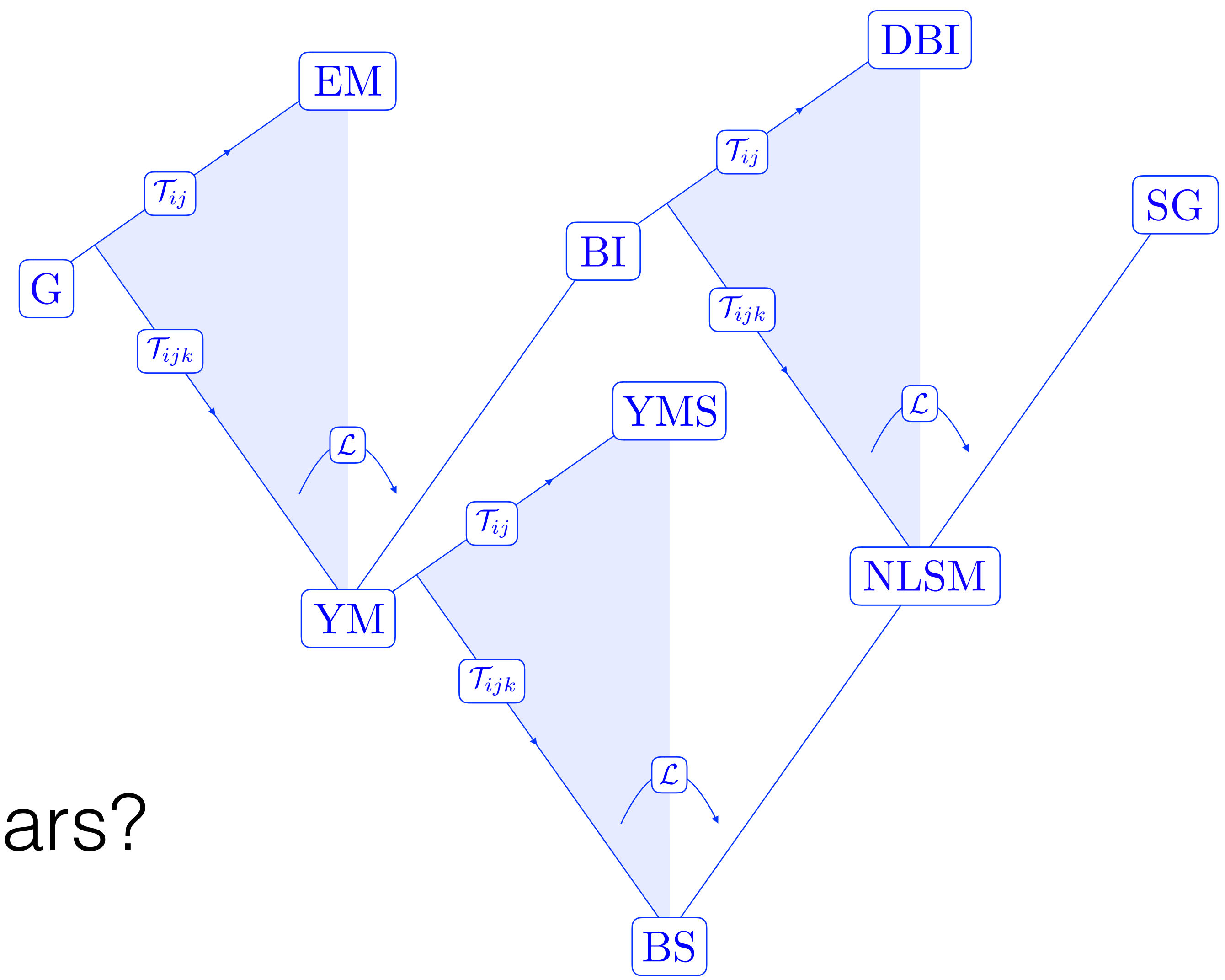
$$\mathcal{S}_{\mathrm{NLSM}}^{(ijk)} = \sum_{l \neq i,k} p_j p_l \mathcal{T}_{ilk}$$

[Cachazo, Cha, Mizera, '16]

Summary / Outlook

Summary / Outlook

- Unified many novel properties of different theories
  - KLT/BCJ
  - Soft Theorems
  - CHY
- *Physical* meaning?
- Defining property for scalars?
- Loop level?



Thank you

Example

- YM amplitudes expanded by bi-adjoint scalars

$$A_{\text{YM}}(\beta) = \sum_{\alpha} N(\alpha) A_{\text{BS}}(\alpha|\beta)$$

- Transmutation implies

$$\mathcal{T}[\alpha] \cdot N(\beta) = \delta_{\alpha\beta}$$

Why?

- $N(\beta) \sim (ee)(pe)^{n-2} + (ee)^2(pe)^{n-4}(pp) + \ldots$
- $N(\beta) = e_{\beta_1} e_{\beta_n} \prod_{i=2}^{n-1} \sum_{j=1}^{i-1} p_{\beta_j} e_{\beta_i} + \ldots$