Azurite

Finding master Integrals using algebraic geometry

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Outline

IBPs and Master Integrals

- Any L-loop integral can be expressed in terms of a finite basis of Integrals (Smirnov and Petukhov 2011)

$$\mathrm{I}_n^L = \sum_k^N C_k \mathrm{I}_k$$

- In order to reduce the starting integral we can use IBPs identities (Chetyrkin and Tkachov 1981)

$$\int \prod_{j=1}^{L} \left(\frac{d^D l_j}{i\pi^{D/2}} \right) \sum_{i=1}^{L} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{a_1} \dots D_k^{a_k}} = 0$$

- Apply the Laporta algorithm to solve the linear system that arises from the IBPs identities (Laporta 2000)

Baikov Representation (Baikov 1996)

- A general L-loop Feynman Integral, can be written as

$$I[a_1,\ldots,a_k;N] = \int \prod_{j=1}^{L}\left(\frac{\mathrm{d}^D l_j}{i\pi^{D/2}}\right)\frac{N(l_1,\ldots,l_L)}{D_1^{a_1}\cdots D_k^{a_k}}$$

- We can now reparametrize the loop momenta as $l_i = l_i^d + l_i^\perp$ and define $\mu_{ij} = -l_i^\perp \cdot l_j^\perp$ (Neerven and Vermaseren 1984). We obtain $n_{sp} = (n-1)L + L(L+1)/2$ variables.
- We can now identify the inverse propagators and irreducible scalar products as new variables z_i, which express the starting integrals as

$$\langle 12\ldots k\rangle[N] \propto \int \mathrm{d}z_1\cdots\mathrm{d}z_{n_{sp}} F(z_1,\ldots,z_{n_{sp}})^{\frac{D-h}{2}}\frac{N(z_1,\ldots,z_{n_{sp}})}{z_1\cdots z_m}$$

IBPs in Baikov representation

The general algorithm for IBPs has been developed in (Larsen and Zhang 2016). In here we will focused on the particular case of a maximal cut (G and Zhang 2015).

- The integrand on a maximal cut takes the following form

$$\int \mathrm{d}z_{m+1}\cdots \mathrm{d}z_{n_{sp}} F(z_{m+1},\ldots,z_{n_{sp}})^{\frac{D-h}{2}} N(z_{m+1},\ldots,z_{n_{sp}})\,.$$

- we can now write a exact form of degree $n_{sp}-m-2$ to generate IBP relations

$$0=\int \mathrm{d}\left(\sum_{i=m+1}^{n_{sp}}(-1)^{i+1}a_i(z)F(z)^{\frac{D-h}{2}}\mathrm{d}z_{m+1}\cdots\widehat{\mathrm{d}z_j}\cdots\mathrm{d}z_{n_{sp}}\right)$$

$$0=\int\left(\sum_{i=m+1}^{n_{sp}}\frac{\partial a_i}{\partial z_i}+\frac{D-h}{2F}a_i\frac{\partial F}{\partial z_i}\right)F(z)^{\frac{D-h}{2}}\mathrm{d}z_{m+1}\cdots\mathrm{d}z_{n_{sp}}$$

Syzygy equations

In order to get the correct form the second term of the previous equation must be a polynomial in the z variables. Which give rise to the following constrain

$$\sum_{i=m+1}^{n_{sp}} a_i \frac{\partial F}{\partial z_i} + bF = 0$$

Which determines the parameters $\{b(z), a_i(z)\}$.
This kind of equations are known in the litterature as *syzygy* (Gluza, Kajda, and Kosower 2011) .

Tangent Algebra

$F = 0$ defines an affine variety V, the solution to the previous equation defines the algebra of tangent polynomial vector fields to V.
Depending on the properties of F we can distinguish two cases of solutions:

- F is smooth, the tangent algebra is generated by the principal syzygy.
- F has singular points, the tangent algebra is generated by the principal syzygy and weighted euler vectors around the singular points.

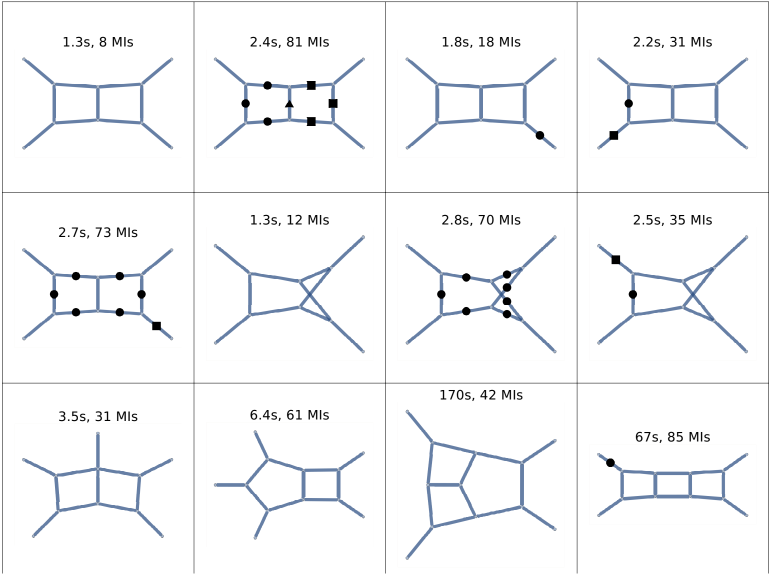
Algorithm

Given a Diagram G as input to the program:

- Determines the automorphism group of G and G_s, discarding all the graphs connected by discrete symmetries.
- Detect and discard scaleless integrals
- Generates the IBPs on the maximal cut of each diagram. This is done using finite field values for the kinematics.

After this steps the program finds a linear basis of integrals which contains the lowest possible numerator degrees.

Results


1.3s, 8 MIs
2.4s, 81 MIs
1.8s, 18 MIs
2.2s, 31 MIs
2.7s, 73 MIs
1.3s, 12 MIs
2.8s, 70 MIs
2.5s, 35 MIs
3.5s, 31 MIs
6.4s, 61 MIs
170s, 42 MIs
67s, 85 MIs

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Conclusions and Future Work

- A new algorithm for finding bases of MI.
- Implement Graph theory tools to reduce the number of diagrams to be analyzed.
- The implementation is general and works in different configurations
- Applications to integral evaluation (See Jorrit's talk tomorrow!)
- Implementing non maximal cuts efficiently.

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Mathematica

After this introduction I can show you how what I explained is implemented.
With an illustrative Mathematica Notebook