Amplitudes and Correlator Integrands to Ten Loops Using Graphical Bootstraps

Viet Tran
Supervisor: Paul Heslop

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Durham University
CPT - Department of Mathematical Sciences

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Outline

An overview of $\mathcal{N} = 4$ SYM. Finding the four-point stress-energy multiplet correlation function in planar $\mathcal{N} = 4$ SYM to **ten** loops.

Ideas: Integrand Basis Representation (Planar Graph Representation by Planarity), Dualities, Hidden Symmetry of the Correlator, Graphical Bootstraps.

- Tools: A coefficient is associated to each planar graph (f-graphs). Coefficients are fixed using various combinations of efficient graphical rules: Triangle, Square and Pentagon Rules.

Dicussion of novel results for $\ell \geq 8$ loops. Conclusion and future directions.

$\mathcal{N} = 4$ Super-Yang Mills (SYM)

$SU(N_c)$ 4D Gauge Theory in Planar Limit ($N_c \to \infty$ with 't Hooft coupling, $a = g^2 N_c$ fixed)

Field content (all in the adjoint rep.):

- One gauge field,
- Six massless scalar fields,
- Four massless fermions.

Motivation: the simplest and most symmetric $D = 4$ Quantum Field Theory.

Used as a toy model to understand QFT in 4D. The symmetries give rise to intriguing mathematical structures that encourage further study.

- Enjoys conformal symmetry ($\beta = 0$) and supersymmetry (4 generators) $\Rightarrow$ superconformal symmetry.
- AdS/CFT Correspondence.

Four-point correlators in $\mathcal{N} = 4$ SYM

- Simplest protected gauge invariant (half-BPS) operator $\mathcal{O}(x) \equiv \mathrm{Tr}(\phi(x)^2)$ (ϕ is one of the six scalars).
- Two- and three-point correlators formed from these operators are known to be **non-renormalised** - independent of *a* and fixed by the free value.
- Therefore, the simplest *non-trivial* correlation function is the four-point correlator (formed from the operators above):

$$\mathcal{G}_4(x_1, x_2, x_3, x_4) \equiv \langle \mathcal{O}(x_1)\overline{\mathcal{O}}(x_2)\mathcal{O}(x_3)\overline{\mathcal{O}}(x_4)\rangle$$

- $\mathcal{O}(x)$ belongs to the stress-energy tensor multiplet (which contains the Lagrangian and stress-tensor of the theory).
- This correlator is known to be **finite**.

Representing the Correlator Perturbatively

Define the $\mathcal{F}^{(\ell)}$ as the integrand of the ℓ loop correlator divided by its tree component:

$$\mathcal{F}^{(\ell)}(x_1, \ldots, x_4, x_5, \ldots, x_{4+\ell}) \equiv \frac{1}{2}\left(\frac{\mathcal{G}_4^{(\ell)}(x_1, x_2, x_3, x_4)}{\mathcal{G}_4^{(0)}(x_1, x_2, x_3, x_4)}\right)/\xi^{(4)},$$

where $\xi^{(4)} := x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2 (x_{13}^2 x_{24}^2)^2$.

It is useful to now regard $\mathcal{F}^{(\ell)}$ as the 'ℓ loop correlator'.

The integrands must have the following properties:

1. Locality and conformality $\Rightarrow$ rational conformally covariant functions of $x_{ij}^2 := (x_i - x_j)^2$ of weight 4 for each x_i.
2. OPE limits ensures only single poles.
3. Planar theory $\Rightarrow$ planar graphs.

Representing the Correlator Perturbatively - f-graphs

- We can therefore represent the integrands as planar graphs with $4+\ell$ vertices and signed degree 4 (by conformality) at each point.
- One can expand $\mathcal{F}^{(\ell)}$ in a basis with arbitrary coefficients

$$\mathcal{F}^{(\ell)} = \sum_{\alpha} c_{\alpha}^{(\ell)} f^{(\ell)}$$

- Enumerating the f-graphs is doable using their defining properties. The hard part is finding the coefficients.
- Superconformal symmetry $\Rightarrow$ (powerful) hidden symmetry:

Hidden Symmetry [Eden Heslop Korchemsky Sokatchev]:

$$f^{(\ell)}(x_1,\ldots,x_{4+\ell}) = f^{(\ell)}(x_{\sigma_1},\ldots,x_{\sigma_{4+\ell}}) \qquad \forall\,\sigma \in S_{4+\ell}$$

- This puts **external variables** $x_1,\ldots x_4$ and **integration variables** $x_5 \ldots x_{4+\ell}$ on the same footing!

f-graphs - Integrands as Graphs

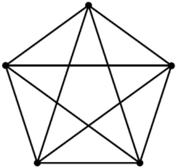
- The correlator is defined (perturbatively) via $f^{(\ell)}$
  - Conformal weight 4 at each point
  - Permutation invariant (hidden symmetry)
  - No double poles.
- Naively equivalent to: degree (valency) 4 graphs on $4+\ell$ points where

Graph edge between vertices x_i and $x_j = \dfrac{1}{x_{ij}^2} = \dfrac{1}{(x_i - x_j)^2}$

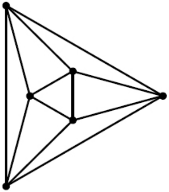
- but we also allow for numerator lines $\Rightarrow$ degree $\geq$ 4 graphs. So that the signed degree (no. x_i in denominator subtract no. x_i in numerator equals 4).
- Don't need to label graph thanks to hidden symmetry, sum over permutations $\Rightarrow$ sum over all labellings of a given f-graph.
- f graphs: (equivalent to the edges and vertices of 3D polytopes).

$\ell \leq 3$ correlator

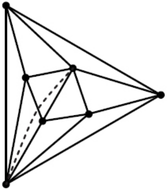
$$f^{(1)} = \frac{1}{\prod_{1\leq i<j\leq 5} x_{ij}^2}$$



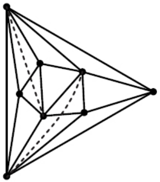
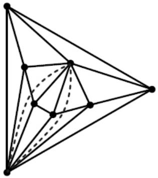
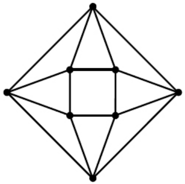
$$f^{(2)} = \frac{x_{12}^2 x_{34}^2 x_{56}^2 \ + \ S_6 \text{ perms}}{\prod_{1\leq i<j\leq 6} x_{ij}^2}$$



$$f^{(3)} = \frac{(x_{12}^4)(x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) \ + \ S_7 \text{ perms}}{\prod_{1\leq i<j\leq 7} x_{ij}^2}$$



$\ell = 4$ correlator

$f_1^{(4)} =$  $f_2^{(4)} =$  $f_3^{(4)} =$ 

- $\mathcal{F}^{(1)} = f^{(1)}$, $\mathcal{F}^{(2)} = f^{(2)}$, $\mathcal{F}^{(3)} = f^{(3)}$, $\mathcal{F}^{(4)} = f_1^{(4)} + f_2^{(4)} - f_3^{(4)}$
- The f-graph (integrand) expressions above are implicitly summed over permutations. Upon multiplication by $\xi^{(4)}$, these are integrated to define the perturbative correlator.
- Hidden symmetry uniquely fixes the four-point planar correlator to 3 loops (using the so-called “square” rule).

Graph Statistics

plane graphs, f-graphs and DCI integrands

ℓ	number of plane graphs	number of graphs admitting decoration	number of decorated plane graphs (f-graphs)	number of planar DCI integrands
1	0	0	0	1
2	1	1	1	1
3	1	1	1	2
4	4	3	3	8
5	14	7	7	34
6	69	31	36	284
7	446	164	220	3,239
8	3,763	1,432	2,709	52,033
9	34,662	13,972	43,017	1,025,970
10	342,832	153,252	900,145	24,081,425
11	3,483,075	1,727,655	22,097,035	651,278,237

The One-Loop Four-Point Gluon Amplitude

Consider the usual one-loop box integral in D-dimensional regularisation $D = 4 - 2\epsilon$, $\epsilon < 0$ and loop momenta k:

$$I^{(1)}(p_1, p_2, p_3, p_4) = \int \frac{d^D k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2}$$

one can rewrite this as (for $p_i^2 \neq 0$ (off-shell regularisation):

$$I^{(1)}(x_1, x_2, x_3, x_4) = \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

for $p_i := x_i - x_{i+1}$ (cyclicity over external points assumed) and $k = x_{15}$.

- x_i are called ‘dual momenta’, they **not** configuration space variables.
- We’ve taken $D = 4$ for a covariant conformal transformation under $x_i^\mu \to x_i^\mu / x_i^2$.

The One-Loop Four-Point Gluon Amplitude

Relation to the Correlator

$$I^{(1)}(x_1, x_2, x_3, x_4) = \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

Comparing this to integrand of the one loop correlator, we find they coincide:

$$\xi^{(1)}\mathcal{F}^{(1)}/(x_{13}^2 x_{24}^2) = \frac{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2 x_{13}^2 x_{24}^2}{x_{12}^2 x_{13}^2 x_{14}^2 x_{15}^2 x_{23}^2 x_{24}^2 x_{25}^2 x_{34}^2 x_{35}^2 x_{45}^2} = \frac{1}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

This motivates a duality between scattering amplitudes and correlators in the planar limit.

Scattering Amplitude/ Correlator Duality

Not LSZ reduction!

- Planar correlators in a polygonal light-like limit is the amplitude squared:

The (four-point) Scattering Amplitude/ Correlator Duality:

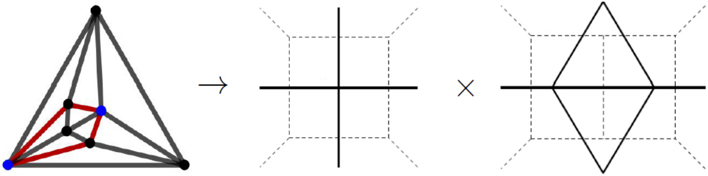
$$\lim_{x_{12}^2, x_{23}^2, x_{34}^2, x_{41}^2 \to 0} \left(\frac{\mathcal{G}_4(x_1, x_2, x_3, x_4)}{\mathcal{G}_4^{(0)}(x_1, x_2, x_3, x_4)} \right) = \mathcal{A}_4(x_1, x_2, x_3, x_4)^2,$$

- Momenta p_i have been reparametrised in terms of “dual momenta” x_i.
- Correlator is **finite** for generic x_i but diverge in the lightlike limit.
- Amplitudes are **IR divergent**.
- However, the integrands of both sides are well-defined.

Extracting four-point amplitudes from f-graphs

- Multiplying $\mathcal{F}^{(\ell)}$ by $\xi^{(4)} := x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2 (x_{13}^2 x_{24}^2)^2$ and taking the relevant limit gives a graphical procedure for extracting amplitudes.
- This corresponds to picking inequivalent four-cycles.
- Some four-cycles are on the "surface" - a **face** of the graph.
- Some four-cycles are on the "inside" - splitting into lower loop amplitude graphs.

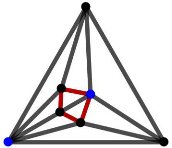
An example of an "inside" four-cycle, where one extracts $\mathcal{A}_4^{(1)} \times \mathcal{A}_4^{(2)}$ from $f^{(3)}$:



Extracting four-point amplitudes from f-graphs

- The 3-loop four-point lightlike limit in fact gives: $2\mathcal{A}_4^{(3)} + 2\mathcal{A}_4^{(1)}\mathcal{A}_4^{(2)}$.
  - Two 3-loop graphs from two inequivalent faces.
  - A lower loop amplitude product from an “inside” 4-cycle.
- We’ve ommitted external numerators $x_{12}^2 x_{34}^2$ factors.

The other two four-cycles of $f^{(3)}$:



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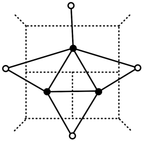

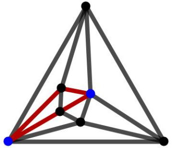

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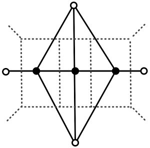


Higher point amplitudes from the four-point correlator

- Higher point lightlike limits of the four-point correlator gives higher point amplitudes (k = Grassmann odd expansion index):

$$\lim_{\substack{\text{n-point}\\ \text{light-like}}} \left(\xi^{(n)}\mathcal{F}\right) = \frac{1}{2}\sum_{k=0}^{n-4} \mathcal{A}_n^k \mathcal{A}_n^{n-4-k} / (\mathcal{A}_n^{n-4,(0)}).$$

- $\xi^{(n)} \equiv \prod_{a=1}^{n} x_{aa+1}^2 x_{aa+2}^2,$
- Pentagonal lightlike limit with $\mathcal{M}_5 \equiv \mathcal{A}_5^0/\mathcal{A}_5^{0,(0)}$ and $\overline{\mathcal{M}}_5 \equiv \mathcal{A}_5^1/\mathcal{A}_5^{1,(0)}$ (divided by tree).

$$\lim_{x_{12}^2,x_{23}^2,x_{34}^2,x_{45}^2,x_{51}^2 \to 0} \left(\xi^{(5)}\mathcal{F}^{(\ell+1)}\right) = \sum_{m=0}^{\ell} \mathcal{M}_5^{(m)} \overline{\mathcal{M}}_5^{(\ell-m)}.$$

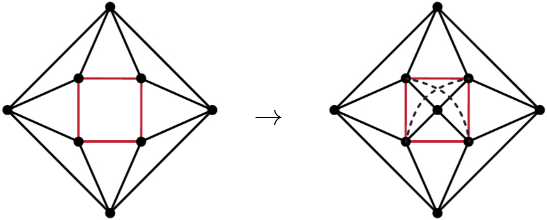
- At 5-points, one can disentangle $\mathcal{M}_5$ from $\mathcal{M}_5\bar{\mathcal{M}}_5$, the even and odd parts at ℓ and $\ell-1$ loops, resp. [Ambrosio Eden Goddard Heslop Taylor].

Fixing Coefficients using Graphical Rules

- Previous methods for fixing coefficients up to eight loops were algebraic. Due to symmetrisation, these expressions grow factorially and become problematic for $\ell \geq 9$.
- Three graphical rules applied on f-graphs turn out to be vastly more efficient in determining coefficients:
  - The "triangle" rule originates from the Euclidean limit of the correlator. The original method was algebraic but can be reinterpreted to be purely graphical (incorporating symmetrisation without many terms).
  - The "square" rule originates from four-point version of the scattering amplitude/ correlator duality.
  - Finally, the "pentagon" rule originates from understanding parts of the parity odd structure in the pentagonal lightlike limit (the five point version of the duality).
- These three rules completely fix the correlator to **ten** loops.

The Square rule [Eden Heslop Korchemsky Sokatchev]

- Gluing pyramids (preserving planarity) together to obtain higher loop f-graphs = "square rule" $\Rightarrow$ coefficient of ℓ loop f graph = coefficient of $\ell - 1$ loop f graph:



Derived from amplitude/correlator duality: [Eden Heslop Korchemsky Sokatchev]

Expanding the duality: $\lim_{x_{ij}^2 \to 0} \mathcal{F}^{(\ell)} \xi^{(4)} \to 2\mathcal{A}_4^{(\ell)} + 2\mathcal{A}_4^{(\ell-1)} \mathcal{A}_4^{(1)} + \ldots$ For $\mathcal{F}^{(\ell)}$ to correctly contain $\mathcal{A}_4^{(1)}$, coefficients must be inherited from lower loop f graph $\mathcal{F}^{(\ell-1)}$.

The Triangle Rule [Bourjaily Heslop Tran] [Eden Heslop Korchemsky Sokatchev]

Originates from projecting onto leading Konishi contribution of the log of the correlator in the single Euclidean limit, equivalent (by conformal invariance) to the double Euclidean limit in previous literature:

$\ell+1$ loop triangle shrinks = ℓ-loop edge shrinks

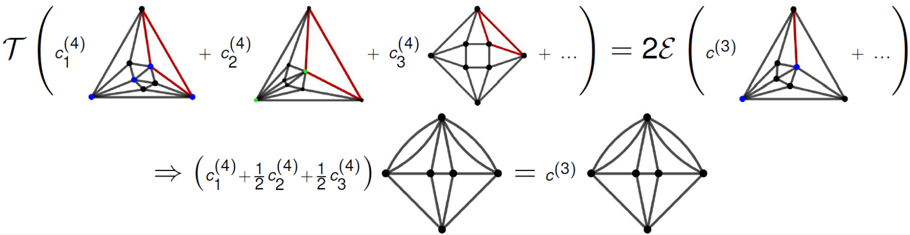
- Take the ℓ+1 loop correlator as a sum of f-graphs with arbitrary coefficients. Shrink all inequivalent triangular faces.
- This equals ($2\times$) the result of taking the (known) ℓ-loop result and shrinking all inequivalent edges.

1. It's important to include symmetry factors to "graphically" reproduce the algebra.
2. Each isomorphic shrunk graph induces a constraint on certain coefficients.
3. Proceed by solving the linear system of constraints.

The Triangle Rule Example

- Symmetry factors are taken to compensate for the difference between the *f*-graph and its reduced version. An example of a reduced graph equation:

3-Loop to 4-Loop Example



- In the example above, $\{c_1^{(4)}, c_2^{(4)}, c_3^{(4)}, c^{(3)}\} = \{+1, +1, -1, +1\}$.
- In practice, can one simply divide by the number of times an inequivalent edge or triangle is mapped to a permutation of itself under automorphisms of the chosen *f*-graph.

Combining the Triangle and Square Rules

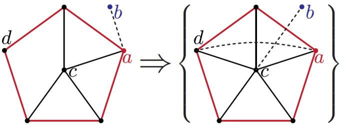
[Bourjaily Heslop Tran]

$\ell =$	2	3	4	5	6	7	8	9	10
number of f-graph coefficients:	1	1	3	7	36	220	2,709	43,017	900,145
unknowns remaining after square rule:	0	0	1	1	5	22	293	2,900	52,475
unknowns after square & triangle rules:	0	0	0	0	0	0	22	3	1,570

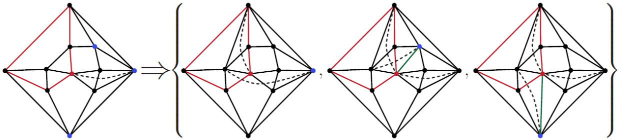
- Triangle and Square rules fix the correlator to seven loops.
- At eight loops, solving the linear system gives 22 free coefficients.
- Using the 22 parameter family solution at nine loops consistency fixes 20 of the eight loop coefficients (consistent with previous work on soft/collinear bootstrap [Bourjaily Heslop Tran]). There are 3 free nine loop coefficients giving 5 free coefficients overall.
- Consistency at ten loops fixes the remaining 5 free coefficients and therefore fixes the correlator to nine loops.
- Yet there are 1570 free ten loop coefficients that requiring further fixing. We can go to higher loops or find a new graphical rule.

Pentagon Rule

- Relates the following two topologies at the same loop order (with a minus sign) by the cross ratio $x_{ab}^2 x_{cd}^2/(x_{ad}^2 x_{bc}^2)$:



- An example:



- Rule implies the valid constraint $c_1^{(7)} + c_2^{(7)} + c_3^{(7)} + c_4^{(7)} = 0$.
- Arises from considering two seperate contributions of $\mathcal{M}_{odd}^{(\ell-1)}$ to $f^{(\ell+1)}$ (many contributions from multiplication of $\epsilon_{123456}\epsilon_{12345(m+6)}$).
- One finds a constraint equation for each highlighted "pentawheel with missing spoke" $\Rightarrow$ another linear system of equations.

The Ten Loop Correlator

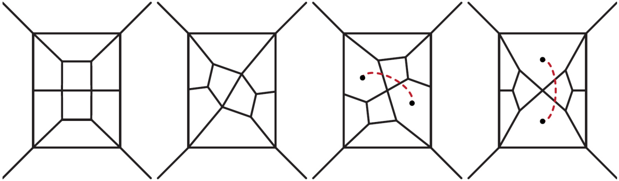
- A table on the strength of just the square and pentagon rules combined:

$\ell =$	2	3	4	5	6	7	8	9
number of f-graph coefficients:	1	1	3	7	36	220	2,709	43,017
unknowns remaining after square rule:	0	0	1	1	5	22	293	2,900
unknowns after square & pentagon rules:	0	0	0	0	1	0	17	64

- Square, triangle and pentagon rules in combination determine everything to ten loops (the pentagon rule can be used to fix the remaining 1570 free coefficients from the triangle rule).

Novel Features for $\ell \geq 8$: Finite Graphs

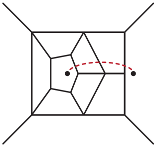
- Whilst the amplitude is IR divergent, there are finite (elliptic?) integrals contributing to the amplitude from 8 loops:



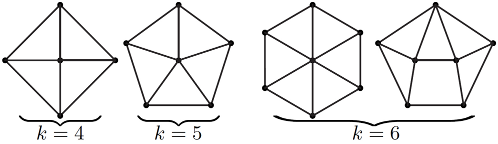
- Signals potential non-polylog contribution to 4-point amplitude?
  - Consistent with BDS (elliptic contribution to the constant part) but potential breakdown of maximal transcendentality?
  - Contradiction may occur in [Arkani-Hamed Bourjaily Cachazo Goncharov Postnikov Trnka] - MHV and NMHV amplitudes are purely polylogarithmic.
- On the other hand, the non-polylog contributions could cancel (coefficients above are $\{-1, 1/2, 1/2, 1\}$), preventing any contradiction.

Novel Features for $\ell \geq 8$: Divergent Graphs

- Conversely: divergent integrals contributing to the **finite** correlator occur from eight loops (off shell).
- E.g.



- Previously assumed to have vanishing coefficient.
- Divergences are logarithmic by a power counting. The divergence can be removed by adding numerator terms.
- The contributing divergent subgraphs cancel in the sum, keeping the correlator finite.



Rule Comparison

$\ell =$	3	4	5	6	7	8	9	10	11
number of f-graph coefficients:	1	3	7	36	220	2,709	43,017	900,145	22,097,035
number unfixed by square rule:	0	1	1	5	22	293	2,900	52,475	1,017,869
percent fixed by square rule (%):	100	67	86	86	90	89	93	94	95

$\ell =$	2	3	4	5	6	7	8	9	10
number of f-graph coefficients:	1	1	3	7	36	220	2,709	43,017	900,145
unknowns remaining after square rule:	0	0	1	1	5	22	293	2,900	52,475
unknowns after square & triangle rules:	0	0	0	0	0	0	22	3	1,570

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unknowns remaining after square rule:	0	0	1	1	5	22	293	2,900
unknowns after square & pentagon rules:	0	0	0	0	1	0	17	64

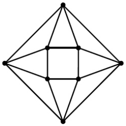
New Features at High Loops

- Appearance of half- and quarter-integer coefficients.
- New integers appear only at even loops (up to signs).
- Appearance of special coefficients for “anti-prism” graphs.

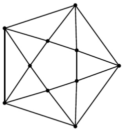
ℓ	±1	0	±2	$\pm1/2$	$\pm3/2$	±5	$\pm1/4$	$\pm3/4$	$\pm5/4$	$+7/4$	$\pm9/4$	$\pm5/2$	$+4$	$+14$
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4	3	0	0	0	0	0	0	0	0	0	0	0	0	0
5	7	0	0	0	0	0	0	0	0	0	0	0	0	0
6	25	10	1	0	0	0	0	0	0	0	0	0	0	0
7	126	93	1	0	0	0	0	0	0	0	0	0	0	0
8	906	1,649	9	141	3	1	0	0	0	0	0	0	0	0
9	7,919	32,492	54	2,529	22	1	0	0	0	0	0	0	0	0
10	78,949	763,712	490	50,633	329	9	5,431	559	18	5	4	4	1	1

Anti-prism Coefficients

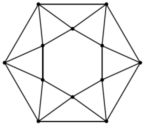
- First new coefficient (-1) is the four-loop “4-sided anti-prism graph” (first with a square face):



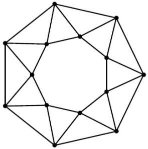
- Next new coefficient (+2) is the six-loop “5-sided anti-prism graph” (first with a pentagonal face):



- Next new coefficient (-5) is the eight-loop “6-sided anti-prism graph” (first with a hexagonal face):



- $+3/2$ appears for the first time at nine loops ($-3/2$ was previously seen at eight loops).
- At ten loops, a new graph with unique coefficient $+14$, the "7-sided anti-prism graph" (first seven-sided face):



- Whilst the octahedron (three-sided antiprism) and four-side antiprism can be accessed by the square and pentagon rule, resp. Clearly, the coefficients of all higher loop antiprisms cannot be found using these. The triangle rule is required to determine them.
- **Conjecture**: Catalan numbers (up to a sign). $\Rightarrow$ twelve loop octagonal anti-prism with coefficient -42
- where $C_n = \dfrac{1}{n+1}\dbinom{2n}{n} = \dfrac{(2n)!}{(n+1)!\,n!}$ is the n^{th} Catalan number, $n = 0, 1, 2, \ldots$

Conclusions and Further Directions

Finding the ten loop correlator essentially gives the following:

1. Four-point (*MHV*) amplitude to ten loops.
2. Five-point (*MHV* and $\overline{MHV}$) parity (even) amplitude to nine loops
3. Five-point (*MHV* and $\overline{MHV}$) amplitude to eight loops.

- Different rules give a consistent overlapping result.
- How far can we take these rules? Higher loop correlator eleven-, twelve-loops?
- Extraction of higher point amplitudes from four-point correlator.
- Six point amplitudes have been successfully extracted to one loop and "even" part at two loops. Six point graphical rule?
- n-point extraction from the four-point correlator?

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