Maximal Cuts in Arbitrary Dimensions

Amplitudes Summer School 2017, Edinburgh
Based on [1704.04255] with Mads Søgaard & Yang Zhang

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ETH Zurich

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SwissMAP
The Mathematics of Physics
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Motivation

Want to calculate loop amplitudes,

$$A_n^{L-loop} = \sum_{\text{diagrams}} \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}l_i}{(2\pi)^D} \right) \frac{N_j}{\prod_{\alpha_j} D_{\alpha_j}},$$

where

- $N_j = N_j(l, k, \epsilon)$ are numerators,
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Using *maximal cuts*, progress can be made.

Table of Contents

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(Generalized) Unitarity Method

[Cutkosky, . . . , Bern, Dixon, Dunbar, Kosower, . . .]

See reviews [Bern & Huang '11; Britto '11] and the textbook [Elvang & Huang '15]

Main idea: use *unitarity* to determine the *analytic structure* of loop amplitudes.

Write $S = \mathbb{1} + i\mathcal{T}$ and insert in $S^\dagger S = \mathbb{1}$:

$$2\mathrm{Im}(\mathcal{T}) = \mathcal{T}^\dagger \mathcal{T}.$$

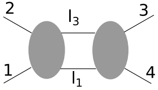
Perturbatively this means Im(L-loop) $\propto \prod$(Lower loop).

(Generalized) Unitarity Method

Unitarity cut: put propagators corresponding to a physical momentum channel on-shell.

Example: s-Channel unitarity cut of $A_4^{\text{1-Loop}}$:

$$\Delta_s A_4^{\text{1-Loop}} =$$



$$= \sum_{\text{states}} A_4^{\text{tree}}[-l_1, 1, 2, l_3] A_4^{\text{tree}}[-l_3, 3, 4, l_1]$$

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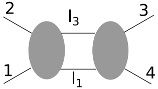

$$= \sum_{\text{states}} A_4^{\text{tree}}[-l_1, 1, 2, l_3] A_4^{\text{tree}}[-l_3, 3, 4, l_1]$$

Generalized unitarity cut: put any combination of propagators on-shell, even if they don't correspond to a physical momentum channel.

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Two ways to use information from cuts:

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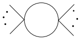
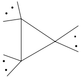
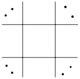
$A^{\text{1-Loop}} = c_2$  $+\, c_3$  $+\, c_4$  $+$ rational terms

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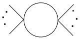
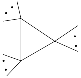
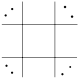
$A^{\text{1-Loop}} = c_2$  $+ c_3$  $+ c_4$  $+$ rational terms

Rational terms:

- Come from the -2ϵ-dimensional part;
- Can be captured by working in arbitrary D dimensions.

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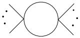
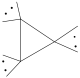
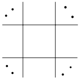
$A^{\text{1-Loop}} = c_2$  $+\, c_3$  $+\, c_4$  $+$ rational terms

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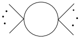
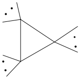
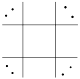
$A^{\text{1-Loop}} = c_2$  $+ c_3$  $+ c_4$  $+$ rational terms

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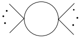
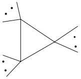
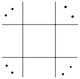
$\Delta A^{\text{1-Loop}} = c_2 \Delta \Big($  $\Big) + c_3 \Delta \Big($  $\Big) + c_4 \Delta \Big($  $\Big)$

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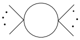
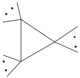
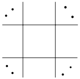
$A^{\text{1-Loop}} = c_2$  $+ c_3$  $+ c_4$  $+$ rational terms

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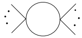
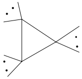
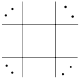
$\Delta A^{\text{1-Loop}} = c_2 \Delta \Big($  $\Big) + c_3 \Delta \Big($  $\Big) + c_4 \Delta \Big($  $\Big)$

Take different cuts to determine $c_i' s$.

1. Start with *maximal cuts*, which cut D propagators, to get c_D;
2. Then consider $D-1$ cuts to get c_{D-1};
3. ...

Maximal Cuts & Leading Singularities

Cut conditions on maximal cut generically have 2^L distinct solutions.
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All loop integrands in planar $\mathcal{N} = 4$ Super Yang-Mills are completely determined by their LSs [Arkani-Hamed, Bourjaily, Cachazo, Trnka '12].

Integration-by-parts (IBP) Reduction

[Chetyrkin & Tkachov '81]

What if cows are not spherical and we don't know a full basis of integrals?

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What if cows are not spherical and we don't know a full basis of integrals?

Use *integration-by-parts* identities to relate different integrals:

$$\int \left(\prod_{i=1}^{L} \mathrm{d}^D l_i \right) \sum_{j=1}^{L} \frac{\partial}{\partial l_j^\mu} \frac{v_j^\mu}{\prod_\alpha D_\alpha^{a_\alpha}} = 0,$$

where

- v_j^μ are polynomials in internal and external momenta;
- $a_\alpha \geq 0$ are integers.

Integration-by-parts (IBP) Reduction

More precisely, this relates integrals within an *integral family:*

Fix propagators $D_1, \ldots, D_k$ and irreducible scalar products $D_{k+1}, \ldots, D_m$. Their integral family is the set of integrals

$$I[a_1, \ldots, a_m] = \int \left(\prod_{i=1}^{L} \mathrm{d}^D l_i \right) \frac{D_{k+1}^{a_{k+1}} \ldots D_m^{a_m}}{D_1^{a_1} \ldots D_k^{a_k}},$$

with $a_i \geq 0$ integers.

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More hardcore / practical approach to loop amplitudes:

1. Generate diagrams;

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More hardcore / practical approach to loop amplitudes:

1. Generate diagrams;
2. Reduce to scalar integrals;
3. Reduce to master integrals via (scalar) IBPs;
4. Evaluate master integrals.

Differential Equations

[Kotikov '91, Henn '13]

See also the lecture notes by Johannes Henn

Evaluate master integrals by deriving and solving a DE system in external kinematic invariants for them:

1. Calculate $\partial_x \vec{I}$;

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4. Solve DE system $\rightarrow$ expressions for MIs.

Table of Contents

The Baikov Representation

We now discuss a method to compute maximal cuts of multi-loop Feynman integrals.

For the integral family, use propagators and irreducible scalar products as variables,

$$z_i = D_i.$$

Integrate out the (-2ϵ)-dim solid angle.

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$$\rightarrow \quad I[a_1, \ldots, a_m] = C(D, k, \epsilon) \int_A \mathrm{d}z_1 \cdots \mathrm{d}z_m \, F(z)^{\frac{D-L-m}{2}} \, \frac{z_{k+1}^{a_{k+1}} \cdots z_m^{a_m}}{z_1^{a_1} \cdots z_m^{a_m}},$$

where $F(z)$ is called the *Baikov polynomial* and A is defined by $F \geq 0$.

The Baikov Representation

One preferably avoids doubled propagators at any step, i.e. one only considers the integrals $I[1, \ldots, 1, a_{k+1}, \ldots, a_m]$.

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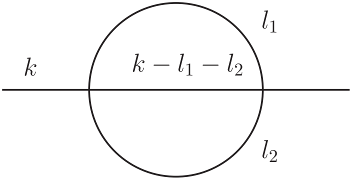
Cuts in the Baikov representation turn out to be very simple:

$$\Delta_i : \quad \int F(z) \frac{1}{z_i} \quad \mapsto \quad F(z)|_{z_i=0}.$$

See also the related works [Frellesvig & Papadopoulos '17; Harley, Moriello, Schabinger '17].

Example: Massless Sunset



$$D_1 = l_1^2, \qquad D_2 = l_2^2, \qquad D_3 = (l_1 + l_2 - k)^2,$$
$$D_4 = (l_1 + k)^2 - 2s, \qquad D_5 = (l_2 + k)^2 - 2s,$$
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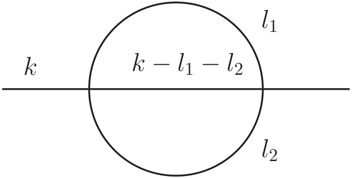

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In the Baikov representation:

$$I[1,1,1,a,b] = \frac{1}{s} \frac{2^{D-4}\pi^{D-2}}{\Gamma(D-2)} \int_A \mathrm{d}z_1 \cdots \mathrm{d}z_5 F(z)^{\frac{D-4}{2}} \frac{z_4^a z_5^b}{z_1 z_2 z_3}.$$

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On the maximal cut the Baikov polynomial is

$$F(z_4, z_5) = \frac{1}{s} z_4 z_5 \left(s + z_4 + z_5\right).$$

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Rescale the variables to $x = z_4/s$ and $y = z_5/s$. This yields

$$F(x, y) = xy(1 + x + y)$$

and

$$I[a, b] = C(D, s)\, s^{a+b} \int_{\Omega} \mathrm{d}x\mathrm{d}y\, F^{\frac{D-4}{2}} x^a y^b.$$

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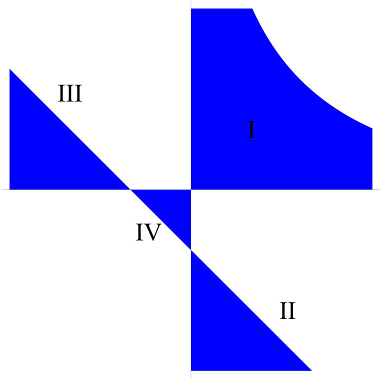
$$I[a,b] = C(D,s)\, s^{a+b} \int\limits_{\Omega} \mathrm{d}x\mathrm{d}y\, F^{\frac{D-4}{2}} x^a y^b .$$

The integration region Ω splits into four different regions:



Example: Massless Sunset

Consider the first region,

$$I_1[a,b] = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \, (xy(1+x+y))^{\frac{D-4}{2}} x^a y^b.$$

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The integration yields

$$J_1[a,b] = \frac{\Gamma(4-a-b-3D/2)\Gamma(-1+a+D/2)\Gamma(-1+b+D/2)}{\Gamma(2-D/2)}.$$

Example: Massless Sunset

Recall Γ-function identities,

$$\Gamma(z+1) = z\Gamma(z),$$
$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \quad (z \notin \mathbb{Z}),$$
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Applying these we find for the other regions,

$$J_1[a,b] = J_2[a,b] = J_3[a,b],$$
$$J_4[a,b] = (1+\cos D\pi)J_1[a,b],$$

so there is only one independent region.

Example: Massless Sunset

Moreover, Γ-identities also yield the known IBP relations,

$$J[a,b] = J[0,0]\, s^{a+b} \frac{(-1+D/2)_a\,(-1+D/2)_b}{(3-3D/2)^{\underline{(a+b)}}},$$

with the ascending factorial

$$(z)_a = \frac{\Gamma(z+a)}{\Gamma(z)} = z(z+1)\cdots(z+a-1)$$

and the descending factorial

$$(z)^{\underline{(a)}} = \frac{\Gamma(z+1)}{\Gamma(z-a+1)} = z(z-1)\cdots(z-a+1).$$

Example: Massless Sunset

In the same fashion one recovers dimension shift identities,

$$J[0,0](D+2) = -J[0,0](D)\, s^2\pi^2 \frac{(-1+D/2)^2}{2(1-3D/2)_3\,(D-1)},$$

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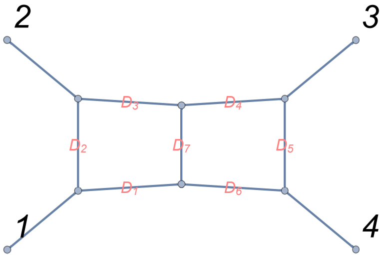
and the differential equation,

$$\frac{\partial}{\partial s} J[0,0] = \frac{D-2}{s} J[0,0],$$

which is immediately in ϵ-form in $D=2$.

Example: Massless Double Box



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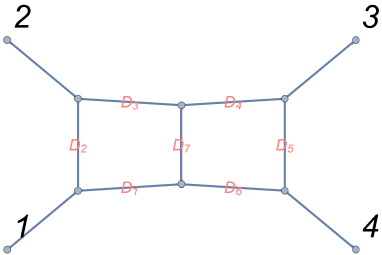

On the maximal cut we have

$$F(z) = \frac{z_8 z_9 (s^2 \chi - s z_8 - s z_9 - z_8 z_9)}{4 s^2 \chi (\chi + 1)}$$

and we want to calculate

$$J[a, b] = \int_\Omega \mathrm{d}z_8 \mathrm{d}z_9 \, F(z)^{\frac{D-6}{2}} z_8^a z_9^b .$$

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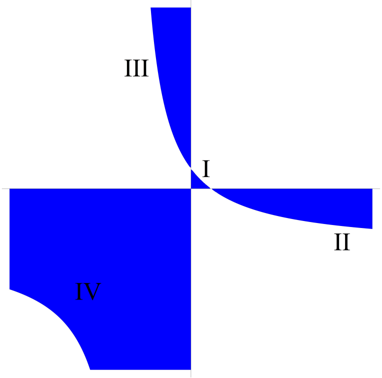
$$J[a,b] = \int_{\Omega} \mathrm{d}z_8 \mathrm{d}z_9 \, F(z)^{\frac{D-6}{2}} z_8^a z_9^b.$$

Ω, defined by $F \geq 0$, splits into four regions:



Example: Massless Double Box

$$J_1[a,b] = \frac{\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(a+\frac{D}{2}-2\right)\Gamma\left(b+\frac{D}{2}-2\right)s^{a+b+D-7}\chi^{a+b+D-5}}{16\pi^4\Gamma(D-4)}$$
$$\times\ {}_2\tilde{F}_1\left(a+D-4, b+D-4; a+b+\frac{3D}{2}-6; -\chi\right),$$

where ${}_2\tilde{F}_1$ is the regularized hypergeometric function,
${}_2\tilde{F}_1(\alpha,\beta,\gamma,z) = {}_2F_1(\alpha,\beta,\gamma,z)/\Gamma(\gamma)$.

Example: Massless Double Box

Similarly,

$$J_2[a,b] = -\frac{(-1)^{a+b}\chi^{2-\frac{D}{2}}\sin(\pi D)\Gamma\left(\frac{D}{2}-2\right)\Gamma(-a-D+5)\Gamma(-b-D+5)s^{a+b-}}{16\pi^4\Gamma(D-4)\sin\left(\frac{3\pi D}{2}\right)}$$

$$\times\ {}_2\tilde{F}_1\left(-a-\frac{D}{2}+3, -b-\frac{D}{2}+3; -a-b-\frac{3D}{2}+8; -\chi\right)$$

$$+\frac{\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(a+\frac{D}{2}-2\right)\Gamma\left(b+\frac{D}{2}-2\right)s^{a+b+D-7}\chi^{a+b+D-5}}{16\pi^4\Gamma(D-4)\big(1+2\cos(\pi D)\big)}$$

$$\times\ {}_2\tilde{F}_1\left(a+D-4, b+D-4; a+b+\frac{3D}{2}-6; -\chi\right),$$

and, after applying hypergeometric function identities,

$$J_3[a,b] = J_2[a,b],$$
$$J_4[a,b] = J_1[a,b] - 2\cos(\pi D)J_2[a,b].$$

Example: Massless Double Box

Hypergeometric function identities also

- give that $J_i[a, b]$ is generated by $J_i[0, 0]$ and $J_i[1, 0]$ for any region;
- provide all IBPs and dimension shift identities.

Example: Massless Double Box

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Put the independent functions from the independent regions in a matrix,

$$S = \begin{pmatrix} J_1[0,0] & J_2[0,0] \\ J_1[1,0] & J_2[1,0] \end{pmatrix}.$$

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For massive external legs one also gets ${}_2\tilde{F}_1$ functions or Appel F1 functions, depending on the configuration.

Table of Contents

Conclusion

- Developed a consistent and precise method to compute (maximal) cuts of Feynman integrals in D dimensions;
- This method also works for massive and nonplanar integrals;
- We found compact, analytic results in all examples considered and all integral relations correspond to relations of special functions;
- The number of independent regions equals the number of master integrals;
- The independent functions form the Wronskian of the DE system. (See also recent work [Primo & Tancredi '17; Zeng '17])

Outlook

- Extend our method to complex momenta and complex Baikov variables;
- Consider elliptic Feynman integrals;
- Compute non-maximal cuts.

Thank you