Lecture 1 Slides -JJMC

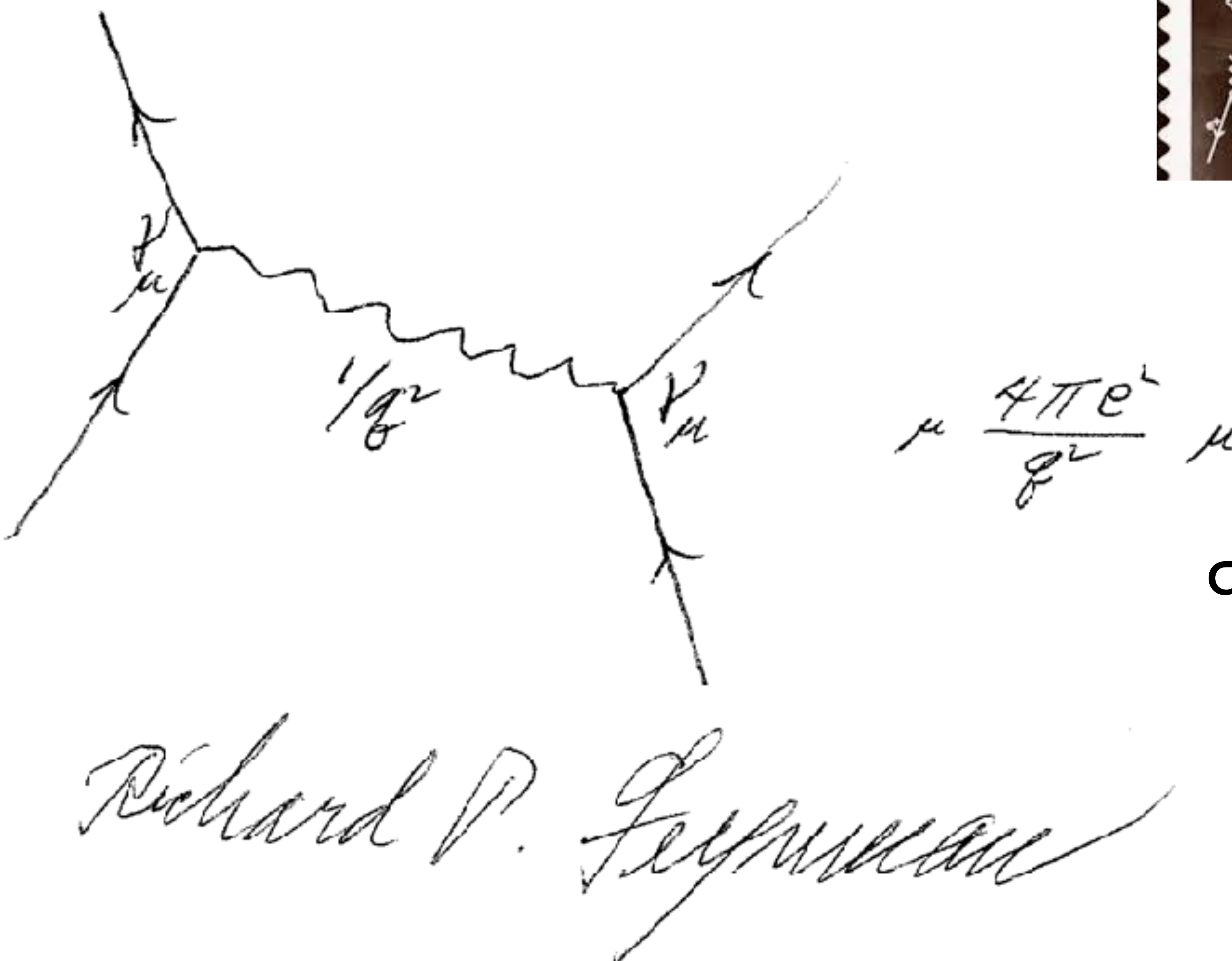
MOTIVATION

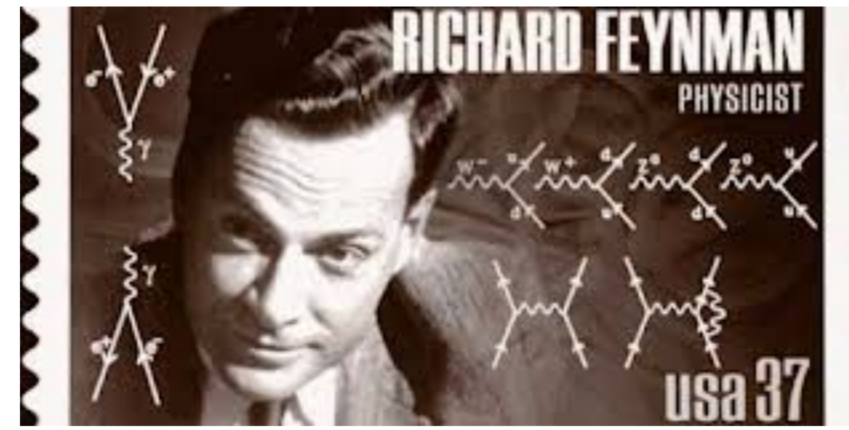
Need for Technical Simplicity

(the perils of infinite employment prospects)

Complexity of Carrying Unphysical Information

"Do Feynman rules represent a useful solution??"





trees: semi-classical

loops: increasing quantum corrections

Off-shell three-graviton vertex:

$$\begin{aligned}
\frac{\delta S^3}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma\tau}\delta\varphi_{\rho\lambda}} \rightarrow\; & 2\eta^{\mu\tau}\eta^{\nu\sigma}k_1{}^\lambda k_1{}^\rho + 2\eta^{\mu\sigma}\eta^{\nu\tau}k_1{}^\lambda k_1{}^\rho - 2\eta^{\mu\nu}\eta^{\sigma\tau}k_1{}^\lambda k_1{}^\rho + \\
& 2\eta^{\lambda\tau}\eta^{\mu\nu}k_1{}^\sigma k_1{}^\rho + 2\eta^{\lambda\sigma}\eta^{\mu\nu}k_1{}^\tau k_1{}^\rho + \eta^{\mu\tau}\eta^{\nu\sigma}k_2{}^\lambda k_1{}^\rho + \eta^{\mu\sigma}\eta^{\nu\tau}k_2{}^\lambda k_1{}^\rho + \eta^{\lambda\tau}\eta^{\nu\sigma}k_2{}^\mu k_1{}^\rho + \\
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& 2\eta^{\lambda\rho}\eta^{\mu\nu}k_1{}^\sigma k_1{}^\tau + 2\eta^{\lambda\nu}\eta^{\mu\rho}k_1{}^\sigma k_1{}^\tau + 2\eta^{\lambda\mu}\eta^{\nu\rho}k_1{}^\sigma k_1{}^\tau + \eta^{\mu\tau}\eta^{\nu\rho}k_1{}^\sigma k_2{}^\lambda + \eta^{\mu\rho}\eta^{\nu\tau}k_1{}^\sigma k_2{}^\lambda + \\
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& 2\eta^{\nu\rho}\eta^{\sigma\tau}k_2{}^\lambda k_2{}^\mu + \eta^{\mu\tau}\eta^{\rho\sigma}k_1{}^\lambda k_2{}^\nu + \eta^{\mu\sigma}\eta^{\rho\tau}k_1{}^\lambda k_2{}^\nu + \eta^{\lambda\tau}\eta^{\mu\rho}k_1{}^\sigma k_2{}^\nu - \eta^{\lambda\rho}\eta^{\mu\tau}k_1{}^\sigma k_2{}^\nu + \\
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& \eta^{\mu\rho}\eta^{\nu\sigma}k_1{}^\lambda k_3{}^\tau + \eta^{\lambda\nu}\eta^{\mu\rho}k_1{}^\sigma k_3{}^\tau + \eta^{\lambda\mu}\eta^{\nu\rho}k_1{}^\sigma k_3{}^\tau + \eta^{\mu\sigma}\eta^{\nu\rho}k_2{}^\lambda k_3{}^\tau + \eta^{\mu\rho}\eta^{\nu\sigma}k_2{}^\lambda k_3{}^\tau - \\
& \eta^{\mu\nu}\eta^{\rho\sigma}k_2{}^\lambda k_3{}^\tau + \eta^{\lambda\sigma}\eta^{\nu\rho}k_2{}^\mu k_3{}^\tau + \eta^{\lambda\nu}\eta^{\rho\sigma}k_2{}^\mu k_3{}^\tau + \eta^{\lambda\sigma}\eta^{\mu\rho}k_2{}^\nu k_3{}^\tau + \eta^{\lambda\mu}\eta^{\rho\sigma}k_2{}^\nu k_3{}^\tau - \\
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& 2\eta^{\lambda\rho}\eta^{\mu\nu}k_3{}^\sigma k_3{}^\tau + 2\eta^{\lambda\nu}\eta^{\mu\rho}k_3{}^\sigma k_3{}^\tau + 2\eta^{\lambda\mu}\eta^{\nu\rho}k_3{}^\sigma k_3{}^\tau - \eta^{\lambda\tau}\eta^{\mu\sigma}\eta^{\nu\rho}k_1\cdot k_2 - \eta^{\lambda\sigma}\eta^{\mu\tau}\eta^{\nu\rho}k_1\cdot \\
& k_2 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_1\cdot k_2 + \eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_1\cdot k_2 - \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_1\cdot k_2 + \eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_1\cdot k_2 + \\
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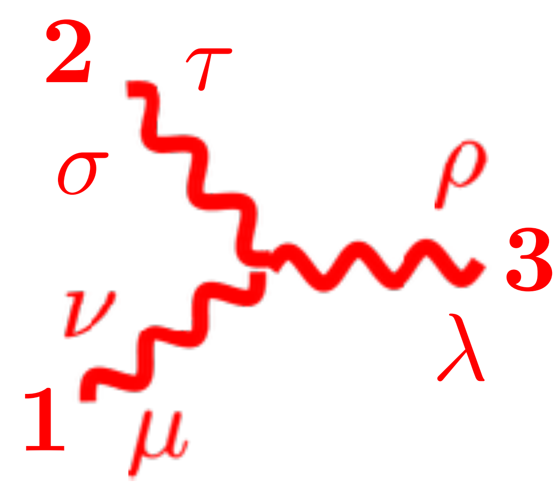
& 2\eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_1\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_1\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_1\cdot k_3 + 2\eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_1\cdot k_3 - \\
& \eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_1\cdot k_3 - \eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_1\cdot k_3 + 2\eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_1\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_1\cdot k_3 - \\
& \eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_1\cdot k_3 - 2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_1\cdot k_3 + \eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_1\cdot k_3 + \eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_1\cdot k_3 - \\
& \eta^{\lambda\tau}\eta^{\mu\sigma}\eta^{\nu\rho}k_2\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\tau}\eta^{\nu\rho}k_2\cdot k_3 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_2\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_2\cdot k_3 - \\
& \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_2\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_2\cdot k_3 + \eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_2\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_2\cdot k_3 - \\
& \eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_2\cdot k_3 + \eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_2\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_2\cdot k_3 - \eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_2\cdot k_3 - \\
& 2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_2\cdot k_3 + 2\eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_2\cdot k_3 + 2\eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_2\cdot k_3
\end{aligned}$$

171 terms

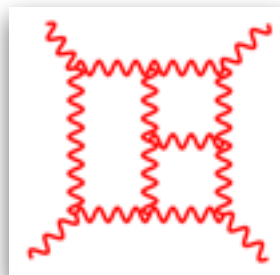


[DeWitt, 1967]

Textbook approach crumbles:

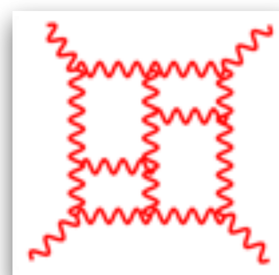
Feynman rules for a graviton: 171 terms per vertex
3 terms per edge

A single 3 loop diagram:



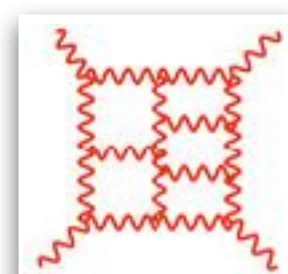
$\sim 10^{20}$ TERMS

4 loop diagram:



$\sim 10^{26}$ TERMS

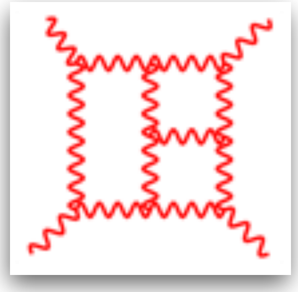
5 loop diagram:



$\sim 10^{31}$ TERMS

BUT FINAL EXPRESSIONS ARE TRACTABLE

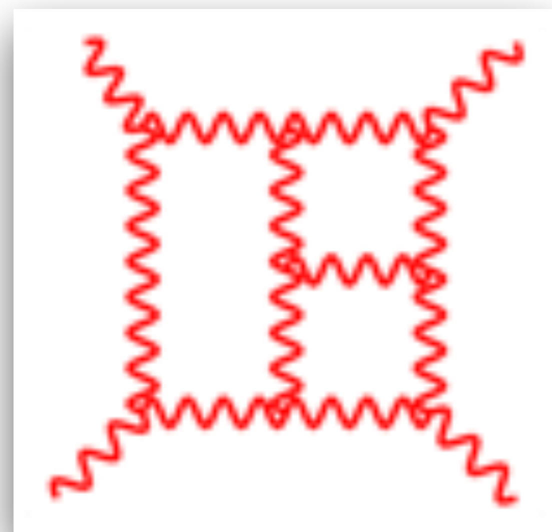
Vast majority of terms: unphysical freedom that must cancel



~10^{20}
TERMS

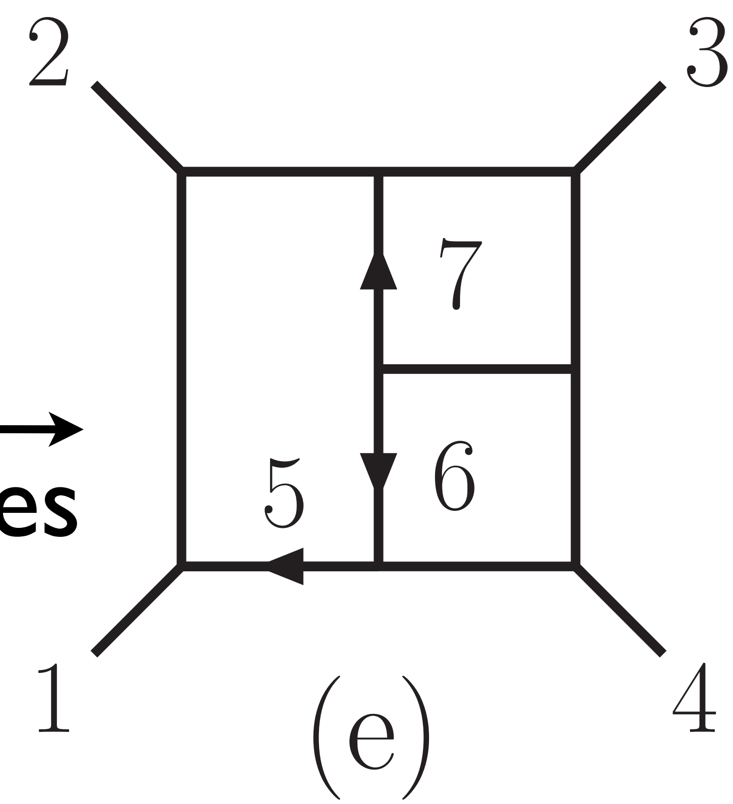
Bern, JJMC, Dixon, Kosower, Roiban '07

MOST SYMMETRIC 4D THEORY, N=8 SUGRA



~10^{20}
TERMS

add all other particles



$$\propto \int stu\mathcal{M}_4^{(0)} \frac{\left(s\,(k_4+l_5)^2\right)^2}{d\circ(e) \equiv (l_5^2 l_6^2 l_7^2 (k_1-l_5)^2 \cdots)}$$

Some secrets obscured in the Lagrangian

Calculate with physical (on-shell) quantities: $k_i^2 = 0$

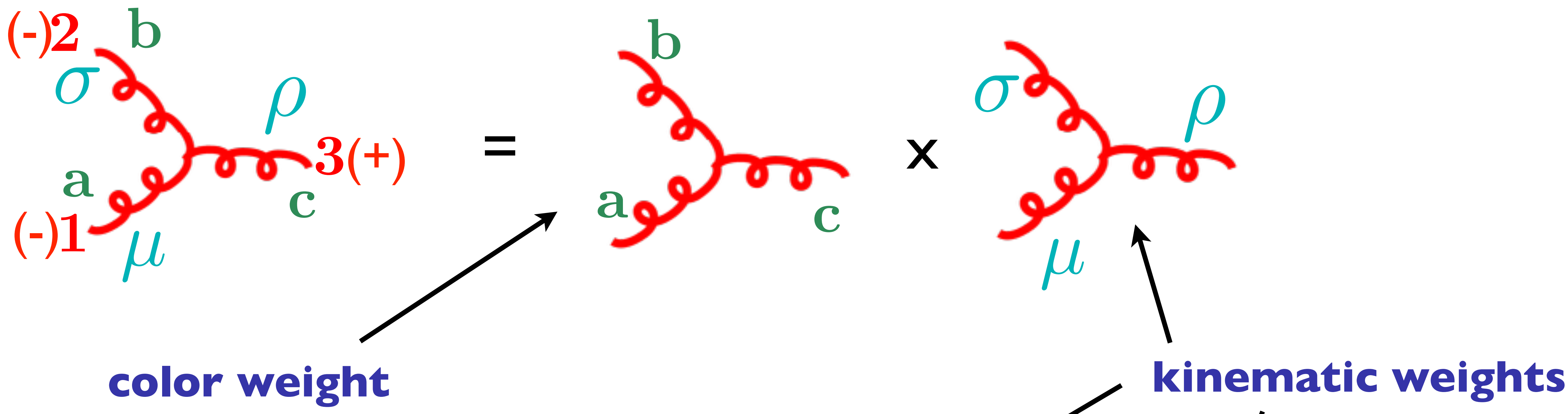
Physical (on-shell) **tree-level amplitudes contain all the information necessary to build *all* loop-level amplitudes** Bern, Dixon, Dunbar, and Kosower (‘94,’95)

Bern, Dixon, and Kosower (‘96)

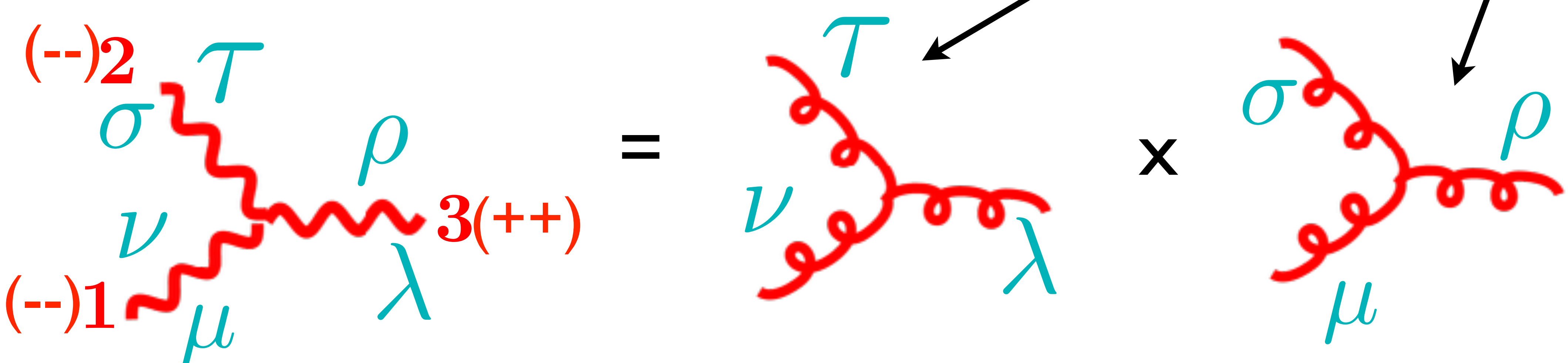
Physical (on-shell) **three-vertices contain all the information necessary to build *all* tree-level amplitudes** Britto, Cachazo, Feng, and Witten (’05)

$k_i^2 = 0$

Physical gluon 3-vertex: $\mathbf{f}^{\mathbf{abc}} \left(k_1{}^\sigma \eta^{\mu\rho} - k_2{}^\mu \eta^{\rho\sigma}\right)$



Physical graviton 3-vertex:

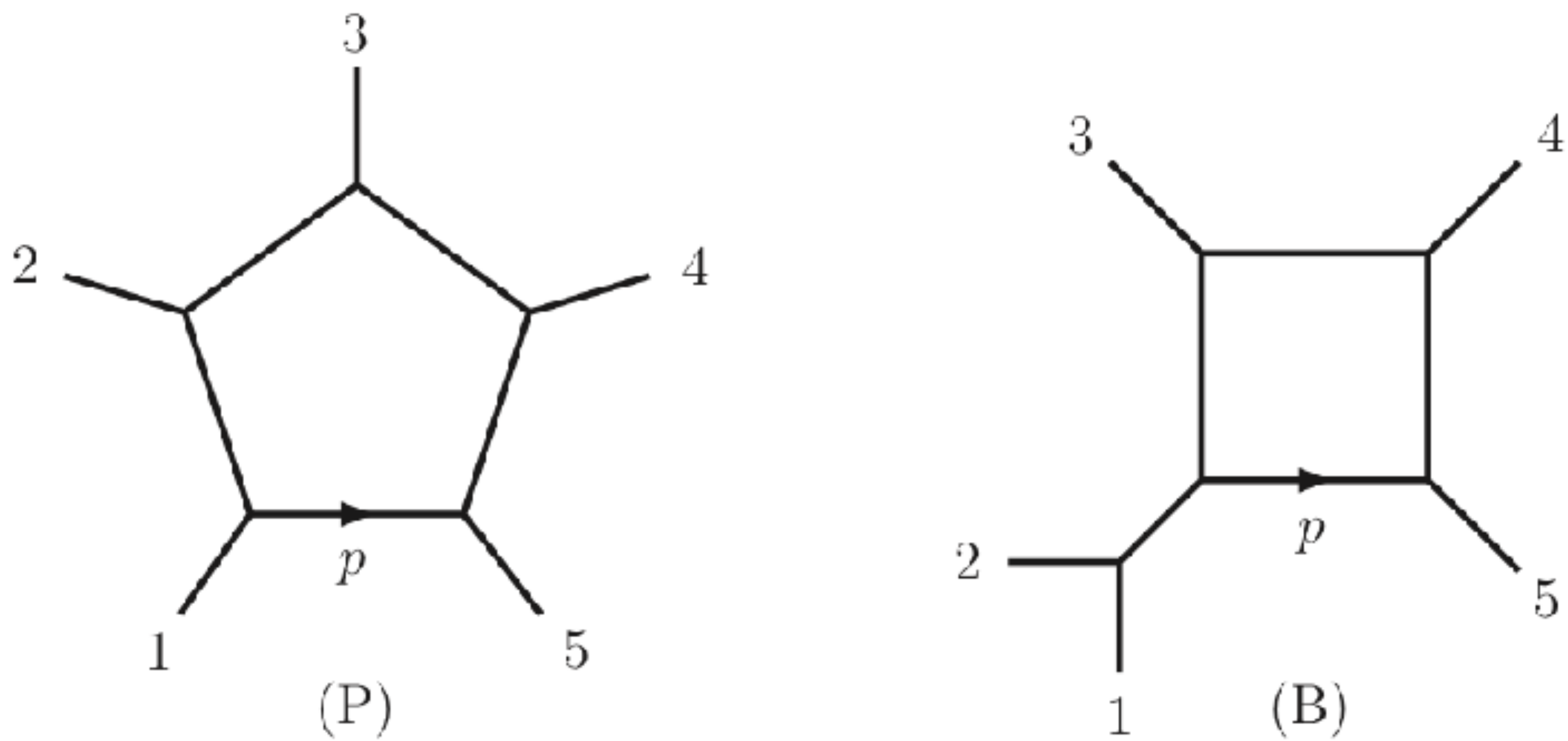


$$\left(\mathbf{k_1}{}^\sigma \eta^{\mu\rho} - \mathbf{k_2}{}^\mu \eta^{\rho\sigma}\right)\left(\mathbf{k_1}{}^\tau \eta^{\nu\lambda} - \mathbf{k_2}{}^\nu \eta^{\lambda\tau}\right)$$

Complexity of Insisting on Local Representations

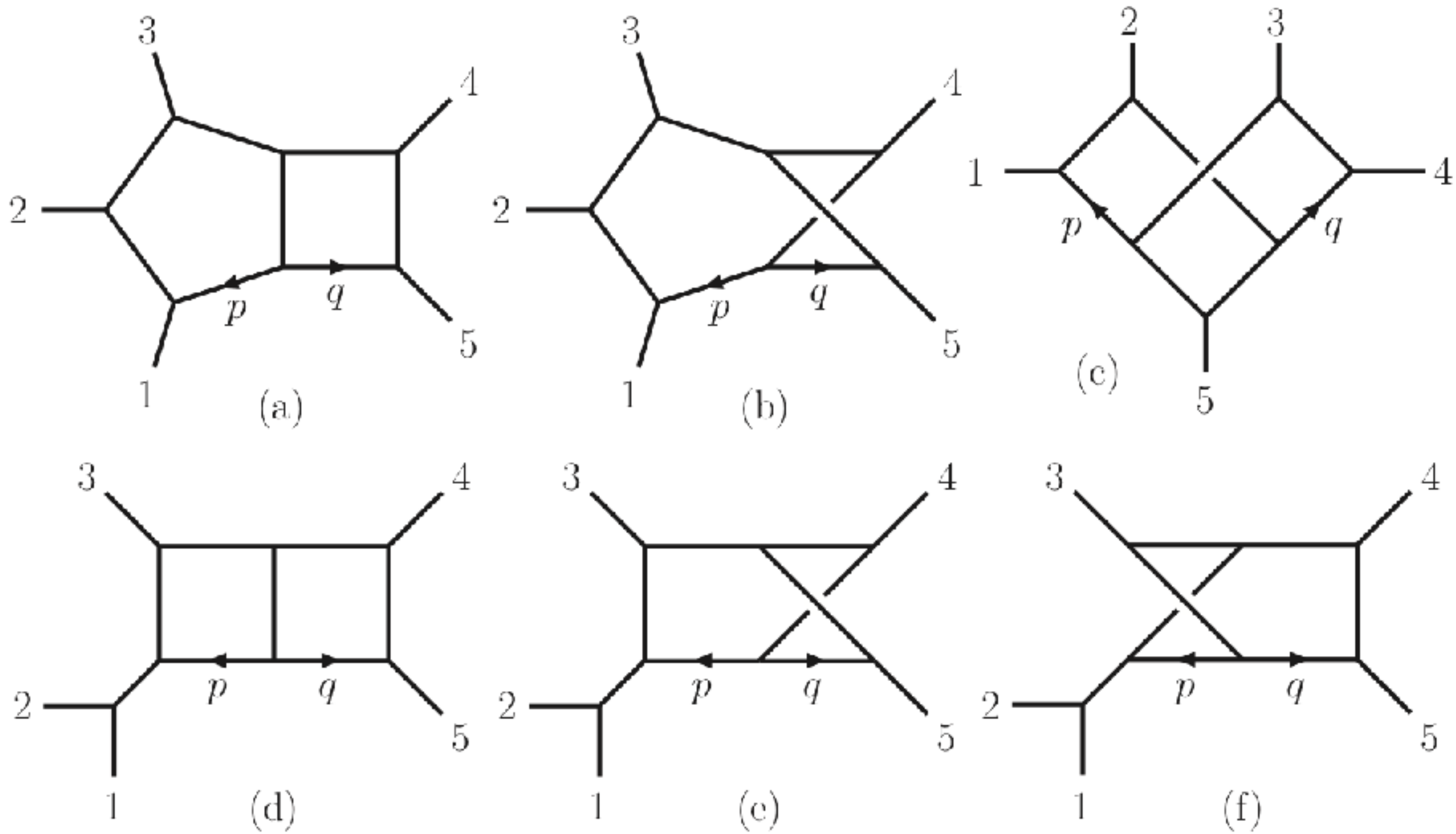
JJMC, Johansson (2011)

Five point 1-loop (no triangles, no bubbles)



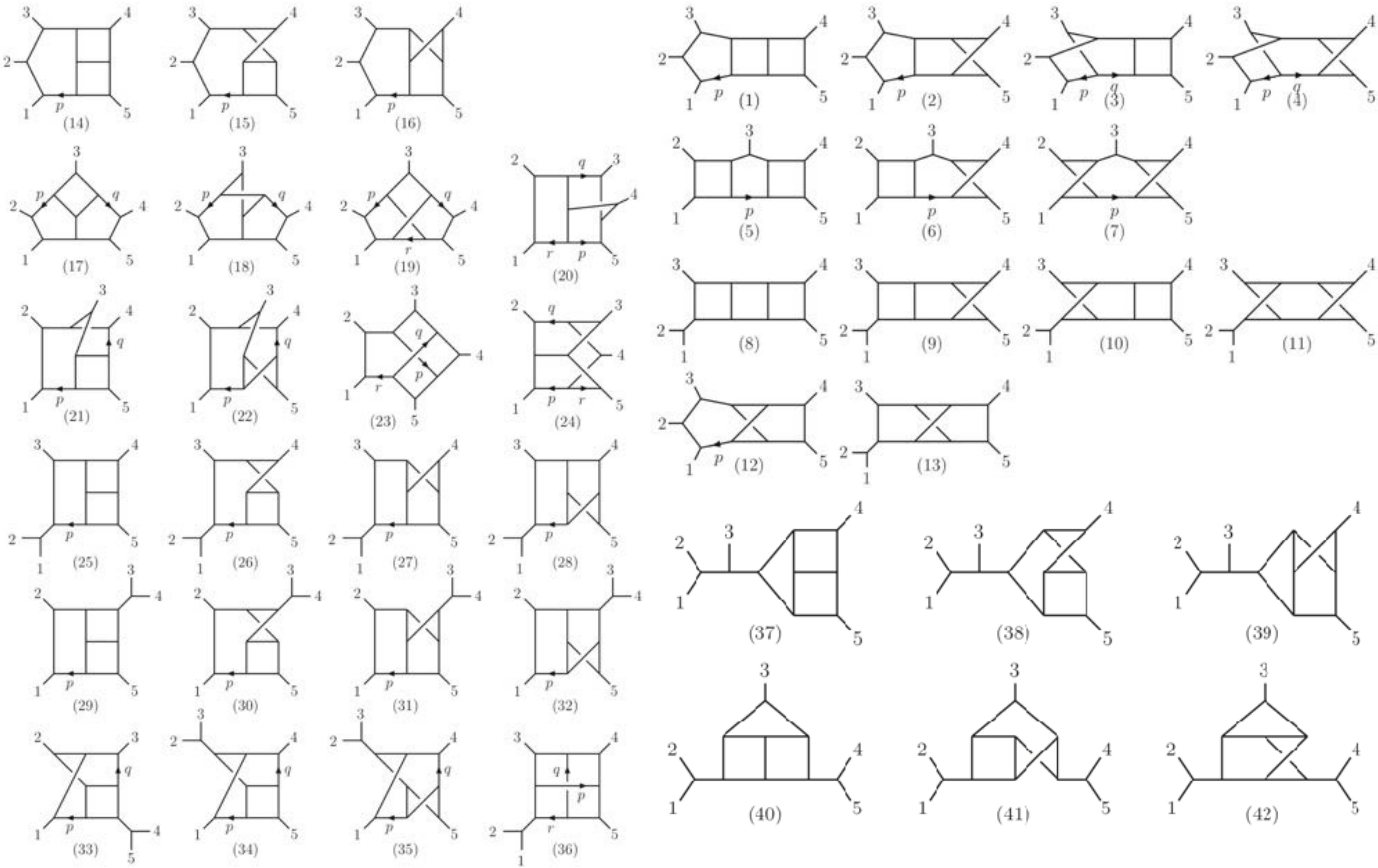
JJMC, Johansson (2011)

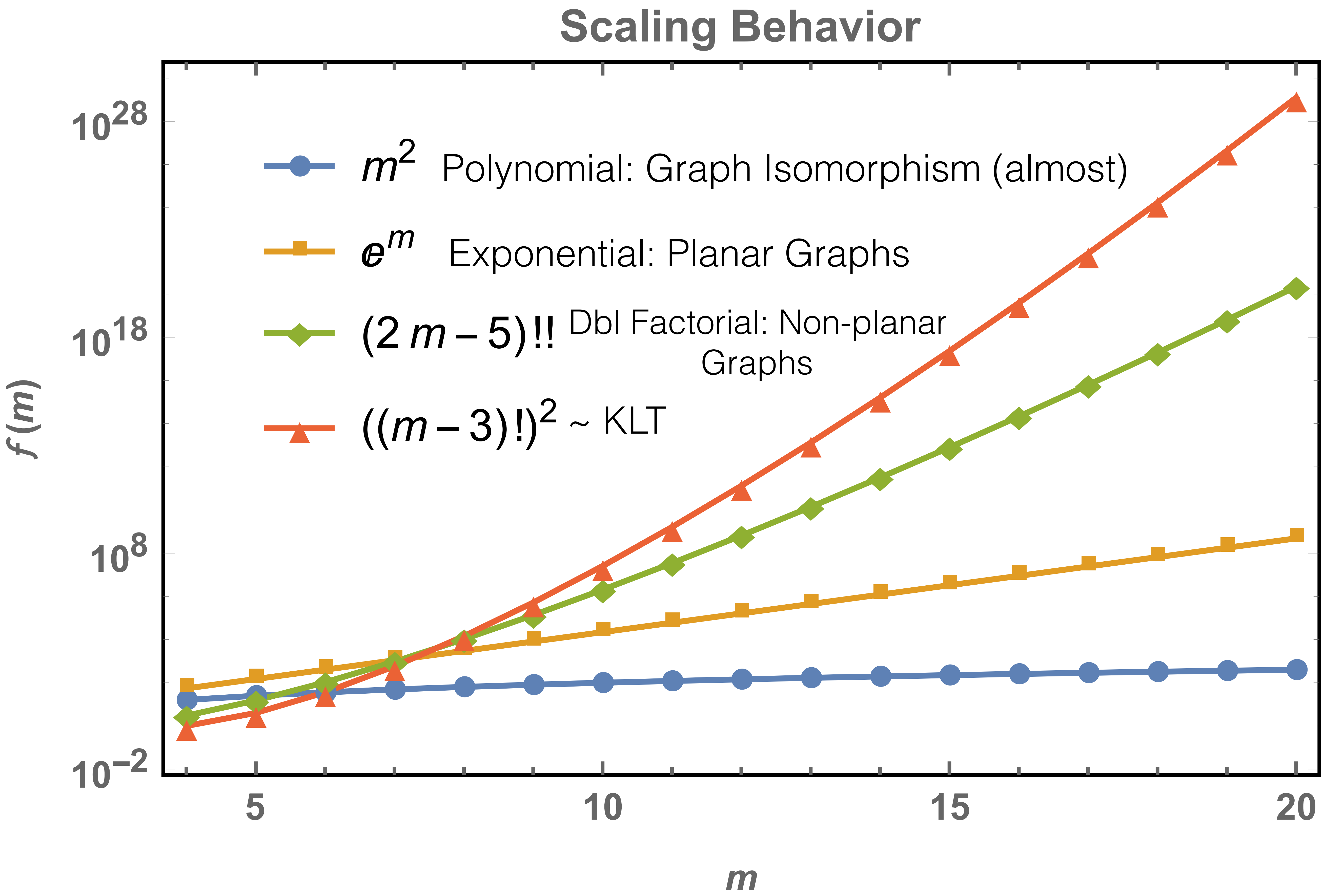
Five point 2-loop (no triangles, no bubbles)



Five point 3-loop (no bubbles, no triangles)

JJMC, Johansson (to appear)




Scaling Behavior
m^2 Polynomial: Graph Isomorphism (almost)
e^m Exponential: Planar Graphs
$(2m-5)!!$ Dbl Factorial: Non-planar Graphs
$((m-3)!)^2$ ~ KLT
10^{28}
10^{18}
10^{8}
10^{-2}
5
10
15
20
m
$f(m)$

VERIFIABILITY

Unitarity

$$= \mathcal{U}_c \circ \sum_g \frac{n \circ g}{d \circ g}$$

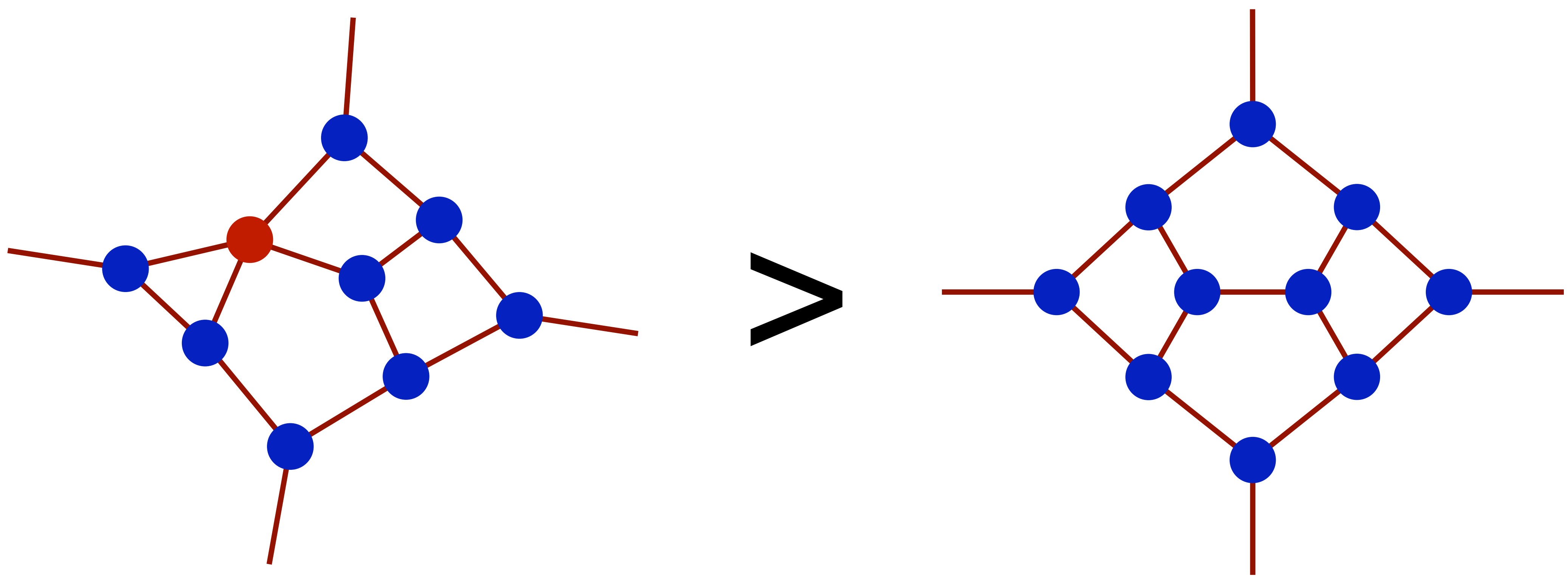
$$\forall\, \mathcal{U}_c \in \text{unitarity cuts}$$

Bern, Dixon, Dunbar, and Kosower ('94,'95)

Bern, Dixon, and Kosower ('96)

Britto, Cachazo, and Feng ('04)

SPANNING CUTS



leads to notion of a **Minimal Spanning Set**

EASY VERIFICATION

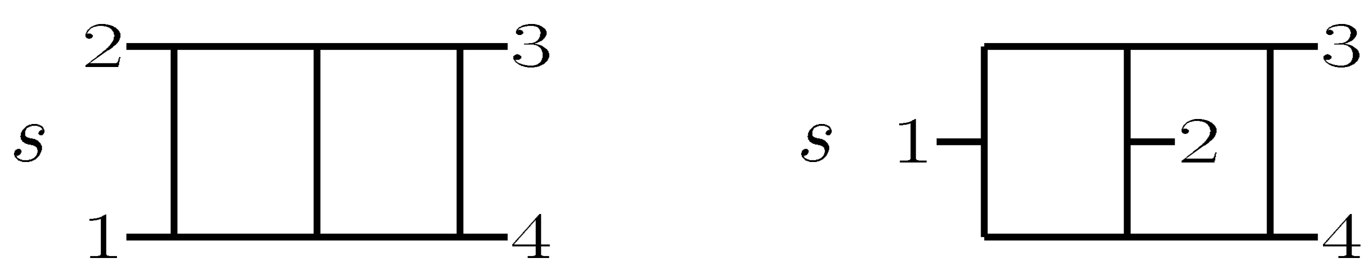
hep-th/9802162

SLAC–PUB–7751
UCLA/98/TEP/03
SWAT-98-183
February, 1998

On the Relationship between Yang-Mills Theory and Gravity and its Implication for Ultraviolet Divergences

Z. Bern[⋆,1], L. Dixon[†,2], D.C. Dunbar[♯,3], M. Perelstein[†,2] and J.S. Rozowsky[⋆,1]

N=4 Yang Mills



Satisfies planar 3-particle color-ordered cut:

$$\sum_{N=4 \text{ states}} A_5^{\text{tree}}(\ell_1, 1, 2, \ell_3, \ell_2) A_5^{\text{tree}}(-\ell_3, 3, 4, -\ell_1, -\ell_2) = -i\, st\, A_4^{\text{tree}}(1,2,3,4)$$

$$\times \Big[\frac{s}{(\ell_1 - k_4)^2 (\ell_3 + k_2)^2 (\ell_1 + \ell_2)^2 (\ell_2 + \ell_3)^2} + \frac{s}{(\ell_3 - k_3)^2 (\ell_1 + k_1)^2 (\ell_1 + \ell_2)^2 (\ell_2 + \ell_3)^2}$$

$$+ \frac{t}{(\ell_3 - k_3)^2 (\ell_1 - k_4)^2 (\ell_3 + k_2)^2 (\ell_1 + k_1)^2} \Big] . \tag{5.13}$$

N=4 Yang Mills

Satisfies non-planar 3-particle color-ordered cut:

$$
\begin{aligned}
\sum_{N=4 \text{ states}} & A_5^{\text{tree}}(1,\ell_1,\ell_3,2,\ell_2)A_5^{\text{tree}}(3,-\ell_3,-\ell_1,4,-\ell_2) = -i\, st A_4^{\text{tree}}(1,2,3,4) \\
& \times \Big[-\frac{s}{(\ell_1+\ell_3)^2(\ell_3+k_2)^2(\ell_1+k_1)^2(\ell_2-k_3)^2} + \frac{t}{(\ell_1+k_1)^2(\ell_2+k_2)^2(\ell_2-k_3)^2(\ell_1-k_4)^2} \\
& + \frac{t}{(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_1-k_4)^2(\ell_2-k_3)^2} - \frac{s}{(\ell_1+\ell_3)^2(\ell_2+k_2)^2(\ell_3-k_3)^2(\ell_1-k_4)^2} \\
& + \frac{t}{(\ell_1+k_1)^2(\ell_2+k_2)^2(\ell_1-k_4)^2(\ell_3-k_3)^2} + \frac{t}{(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_1-k_4)^2(\ell_3-k_3)^2} \\
& - \frac{u}{(\ell_2+k_1)^2(\ell_3+k_2)^2(\ell_1-k_4)^2(\ell_2-k_3)^2} - \frac{s}{(\ell_1+\ell_3)^2(\ell_2+k_1)^2(\ell_3-k_3)^2(\ell_1-k_4)^2} \\
& + \frac{t}{(\ell_2+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_1-k_4)^2} - \frac{s}{(\ell_1+\ell_3)^2(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_2-k_4)^2} \\
& - \frac{u}{(\ell_1+k_1)^2(\ell_2+k_2)^2(\ell_3-k_3)^2(\ell_2-k_4)^2} + \frac{t}{(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_2-k_4)^2} \\
& + \frac{t}{(\ell_2+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_2-k_4)^2}\Big] .
\end{aligned} \tag{5.14}
$$

Lecture 4 Slides -JJMC

Color-Kinematics and Double Copy Construction

Consider a Villanelle

Do Not Go Gentle Into That Good Night

Do not go gentle into that good night,
Old age should burn and rave at close of day;
Rage, rage against the dying of the light.

Though wise men at their end know dark is right,
Because their words had forked no lightning they
Do not go gentle into that good night.

Good men, the last wave by, crying how bright
Their frail deeds might have danced in a green bay,
Rage, rage against the dying of the light.

Wild men who caught and sang the sun in flight,
And learn, too late, they grieved it on its way,
Do not go gentle into that good night.

Grave men, near death, who see with blinding sight
Blind eyes could blaze like meteors and be gay,
Rage, rage against the dying of the light.

And you, my father, there on that sad height,
Curse, bless, me now with your fierce tears, I pray.
Do not go gentle into that good night.
Rage, rage against the dying of the light.

-Dylan Thomas

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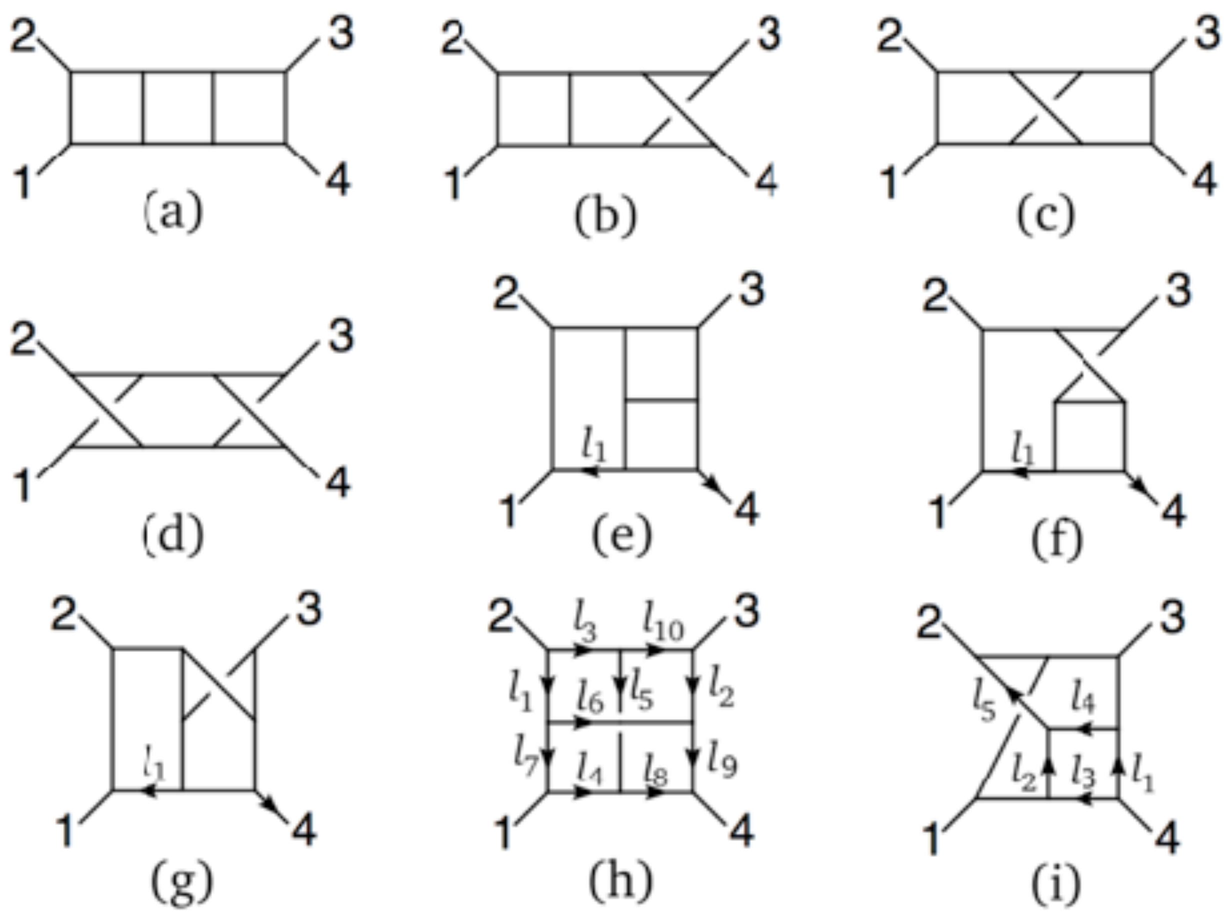
What’s going on?

- Minimal information in.
- Relations propagate this information to a full solution.

Consider an Amplitude

Bern, JJMC, Dixon, Kosower, Johansson, Roiban '07



Original solution of three-loop four-point N=4 sYM and N=8 sugra

Integral	$\mathcal{N}=4$ Yang-Mills	$\mathcal{N}=8$ Supergravity
(a)–(d)	s^2	$[s^2]^2$
(e)–(g)	$s(l_1+k_4)^2$	$[s\,(l_1+k_4)^2]^2$
(h)	$s(l_1+l_2)^2+t(l_3+l_4)^2 - sl_5^2 - tl_6^2 - st$	$(s(l_1+l_2)^2+t(l_3+l_4)^2-st)^2 - s^2(2((l_1+l_2)^2-t)+l_5^2)l_5^2 - t^2(2((l_3+l_4)^2-s)+l_6^2)l_6^2 - s^2(2l_7^2l_2^2+2l_1^2l_9^2+l_2^2l_9^2+l_1^2l_7^2) - t^2(2l_3^2l_8^2+2l_{10}^2l_4^2+l_8^2l_4^2+l_3^2l_{10}^2)+2stl_5^2l_6^2$
(i)	$s(l_1+l_2)^2-t(l_3+l_4)^2 -\frac{1}{3}(s-t)l_5^2$	$(s(l_1+l_2)^2-t(l_3+l_4)^2)^2 - (s^2(l_1+l_2)^2+t^2(l_3+l_4)^2+\frac{1}{3}stu)l_5^2$

Bern, JJMC, Dixon, Kosower, Johansson, Roiban '07

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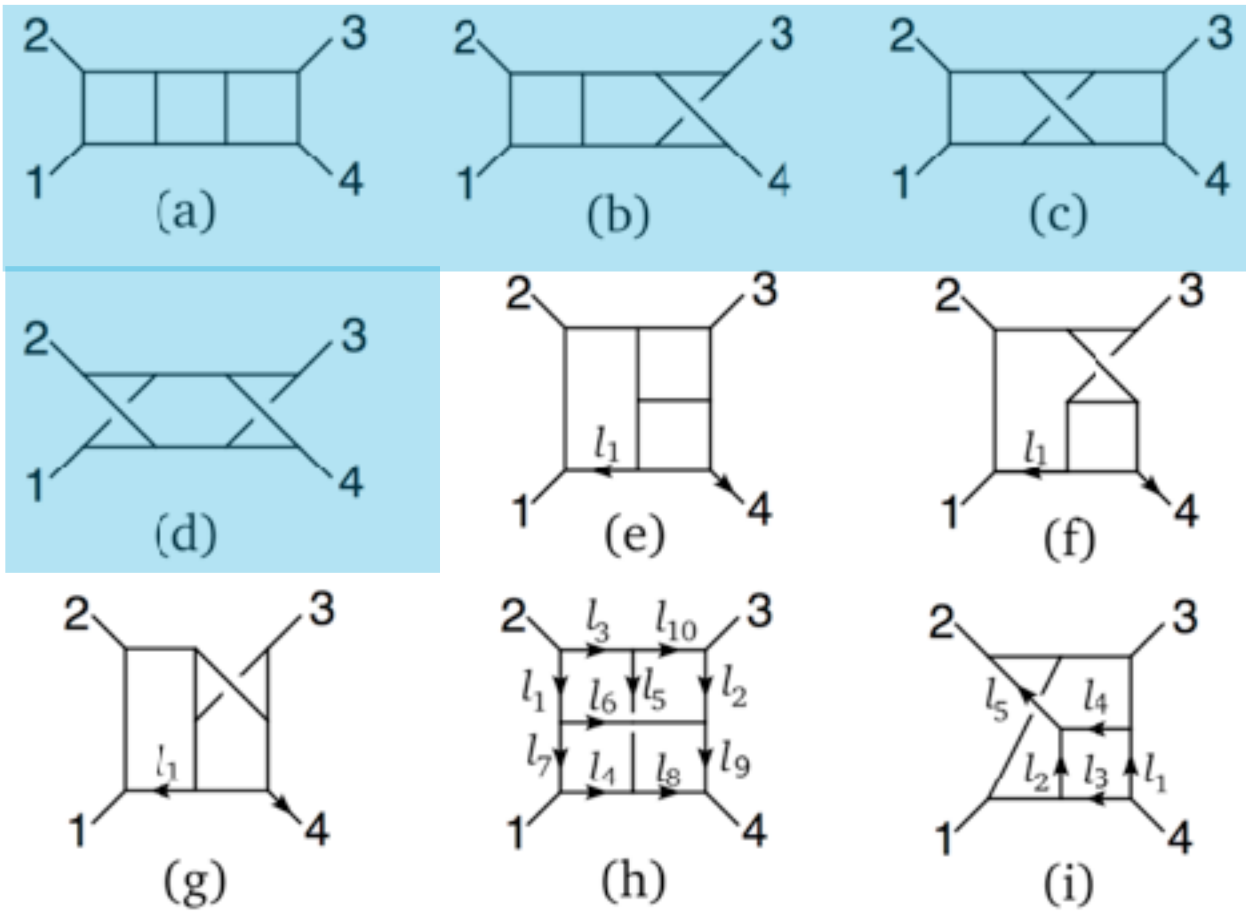

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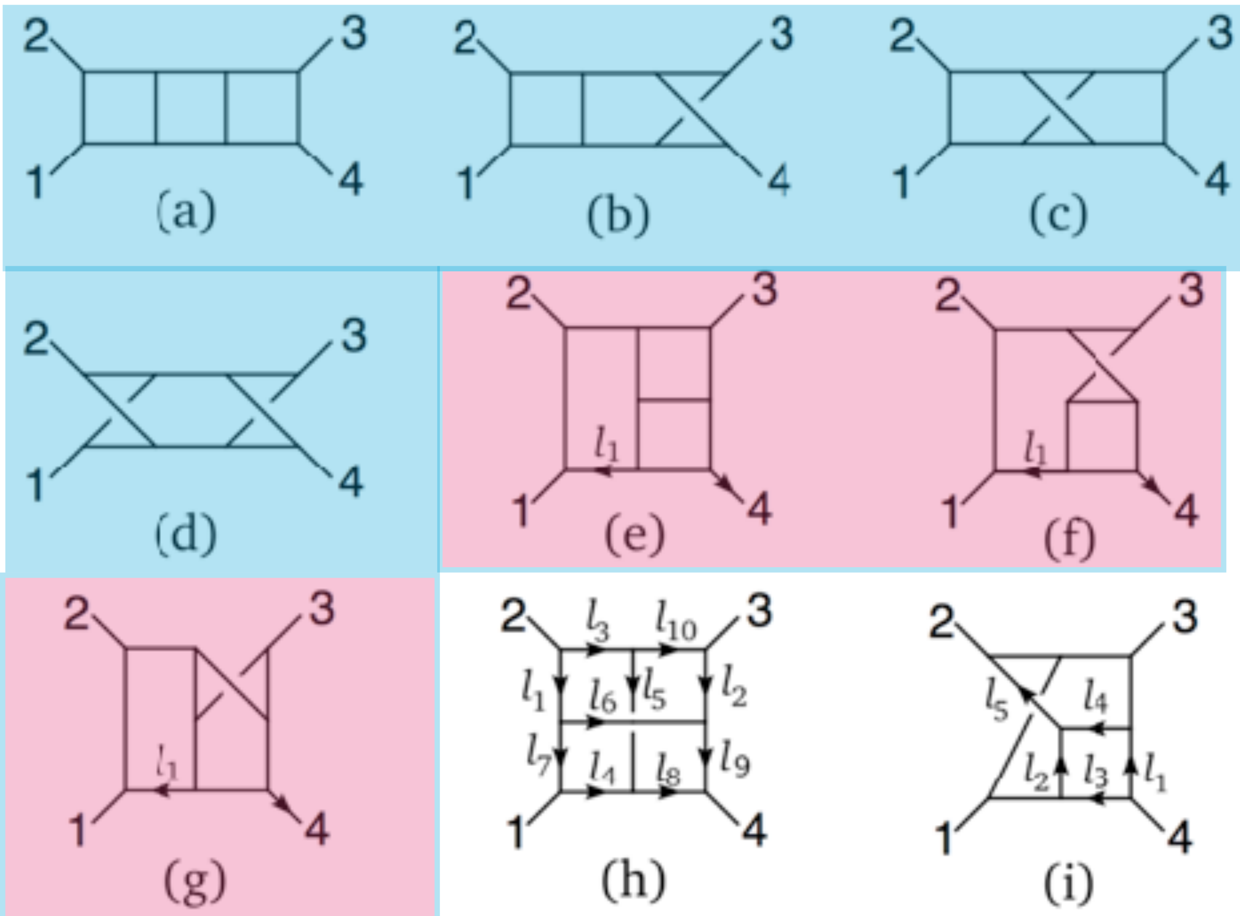

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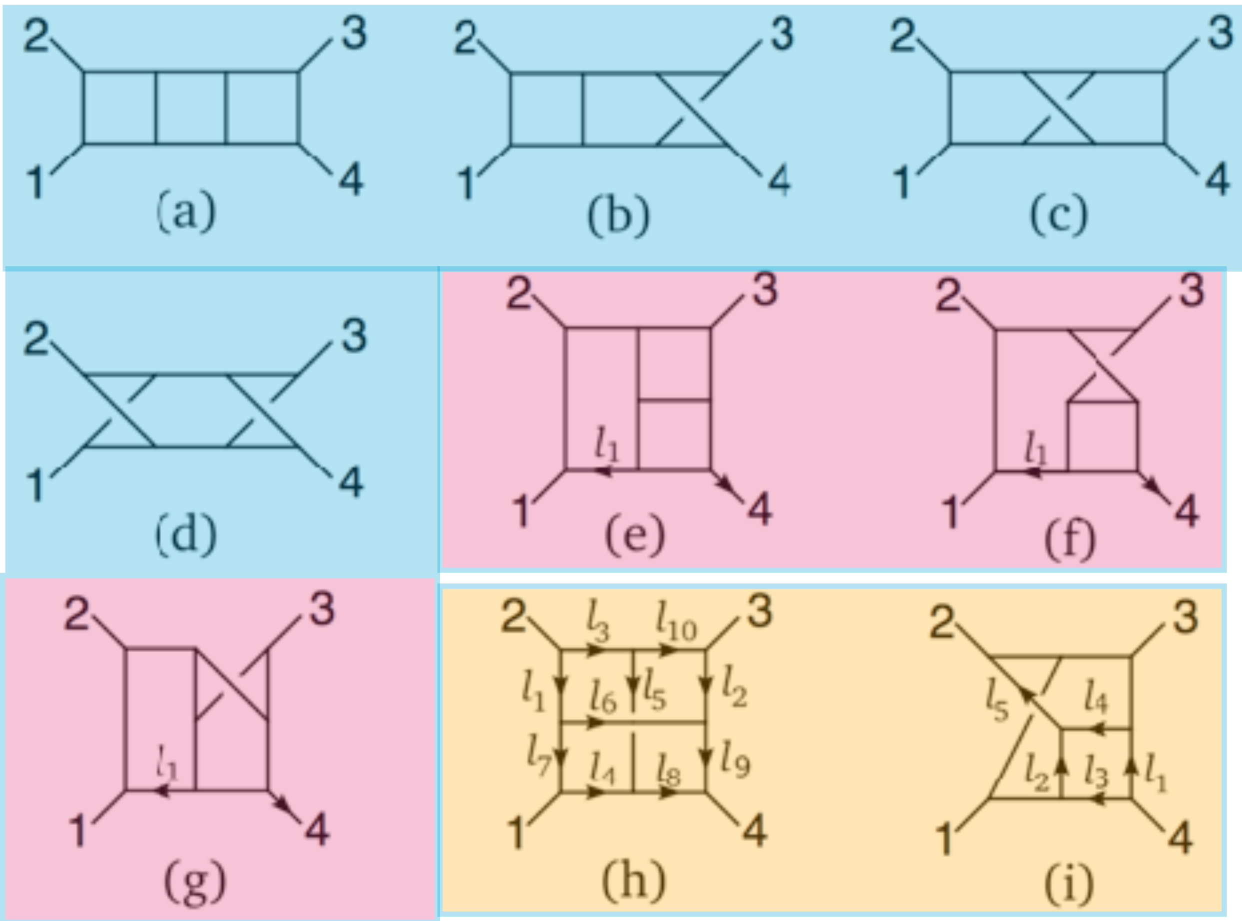

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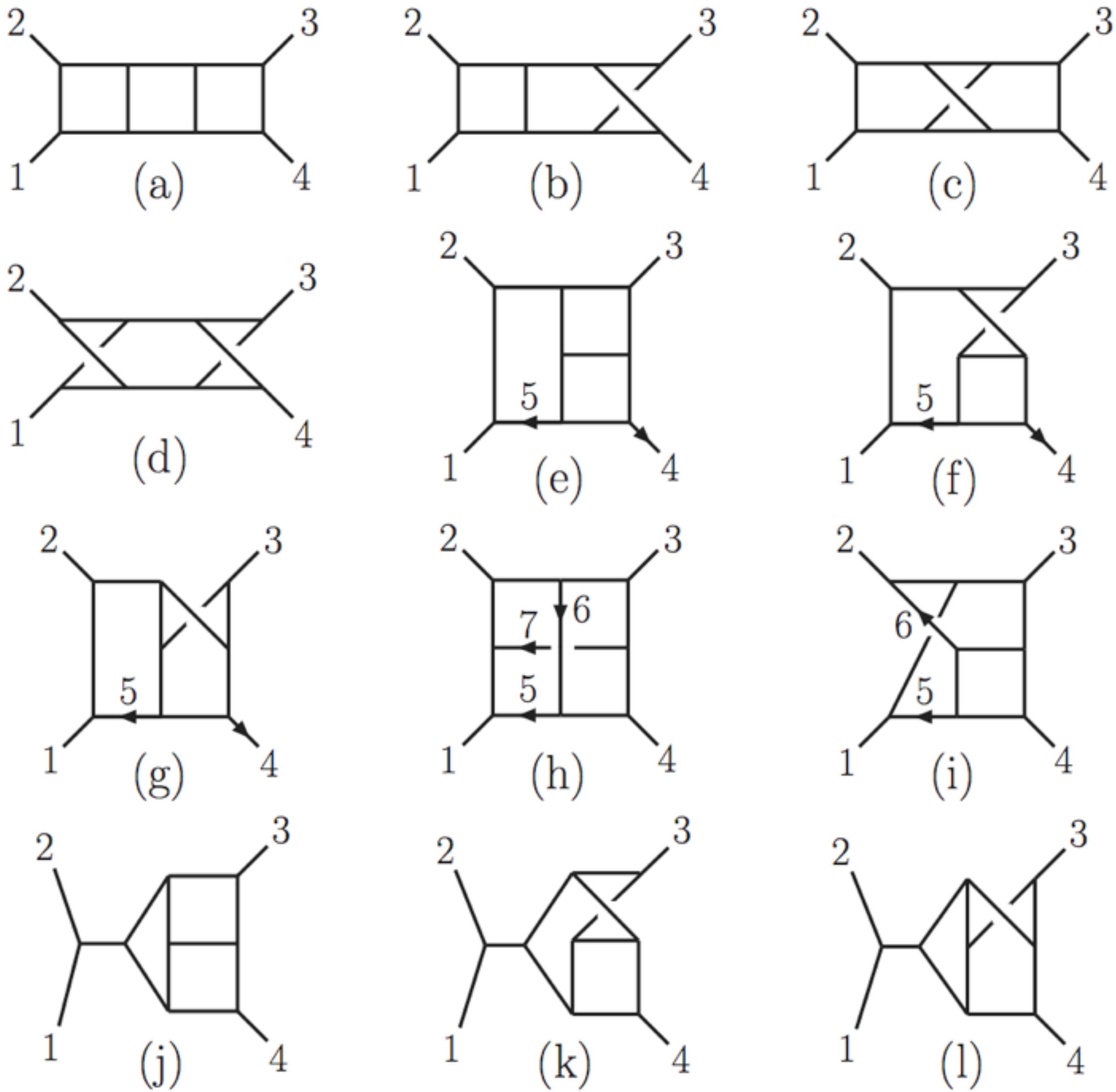
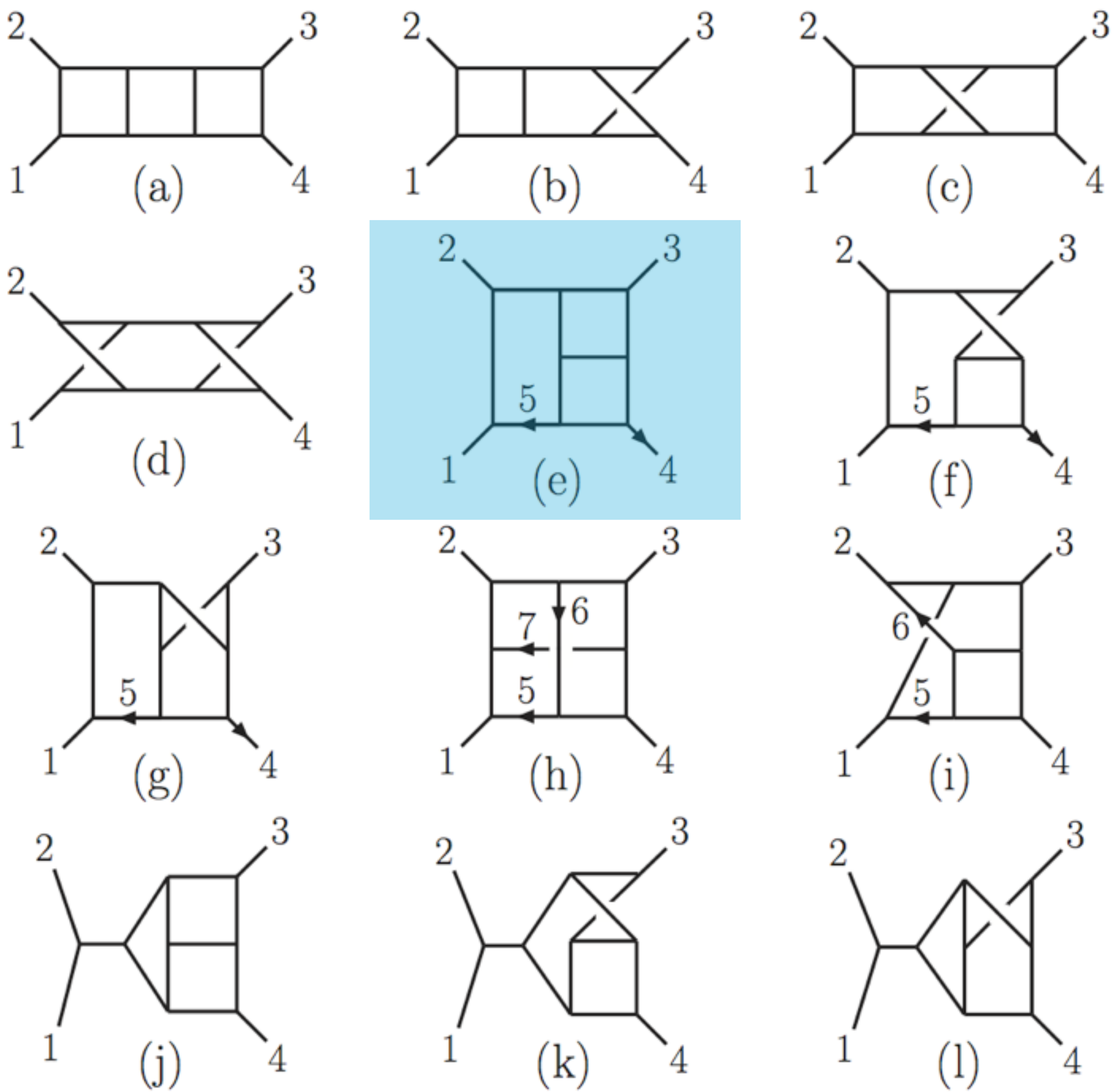
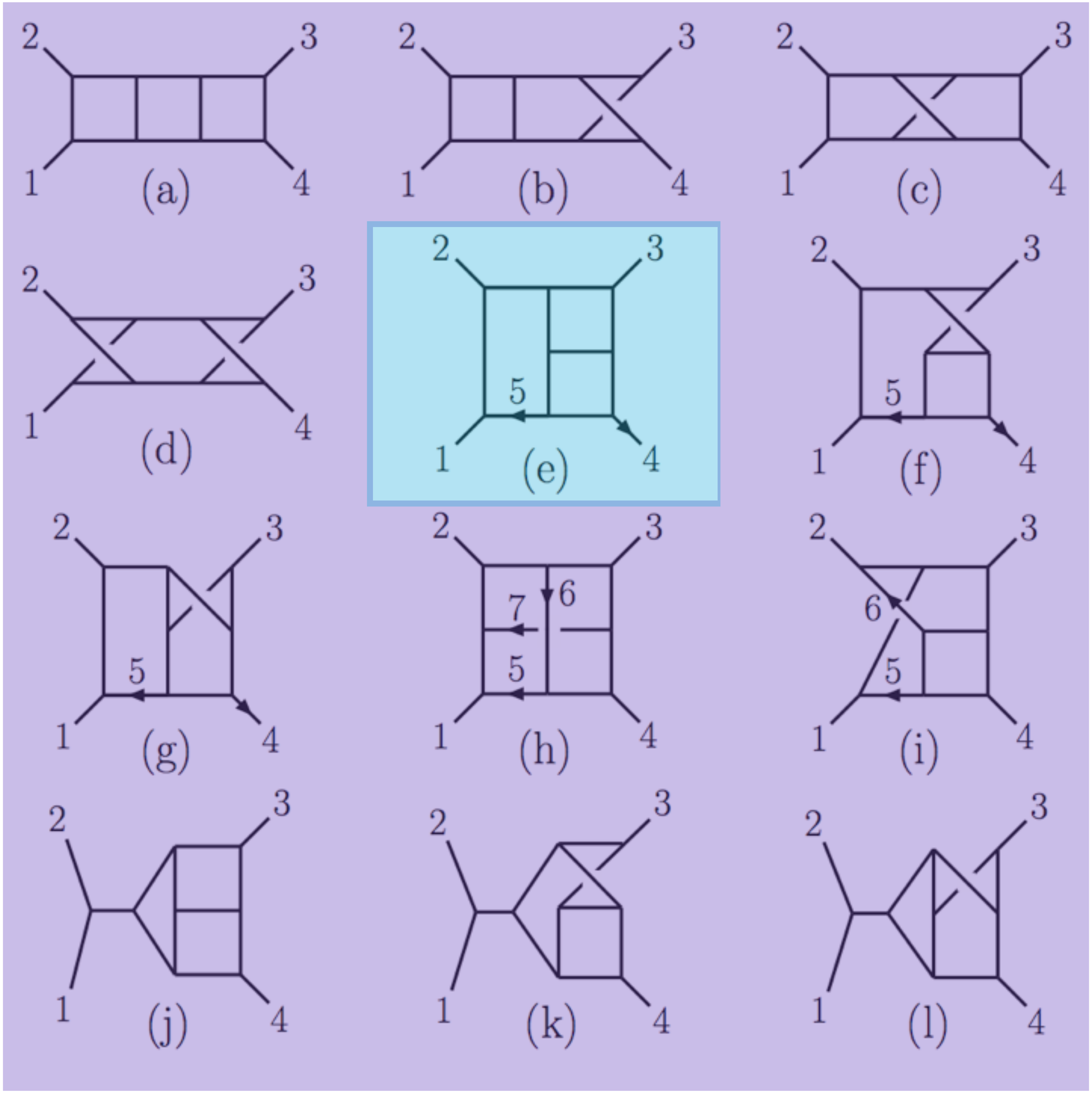
Bern, JJMC, Johansson, ‘10



Bern, JJMC, Johansson, '10

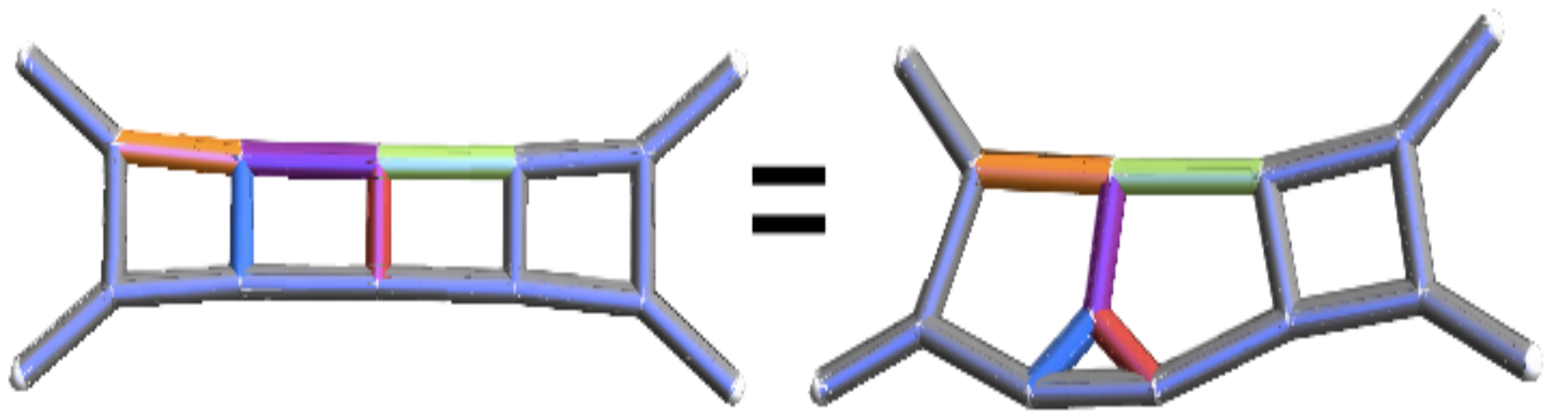


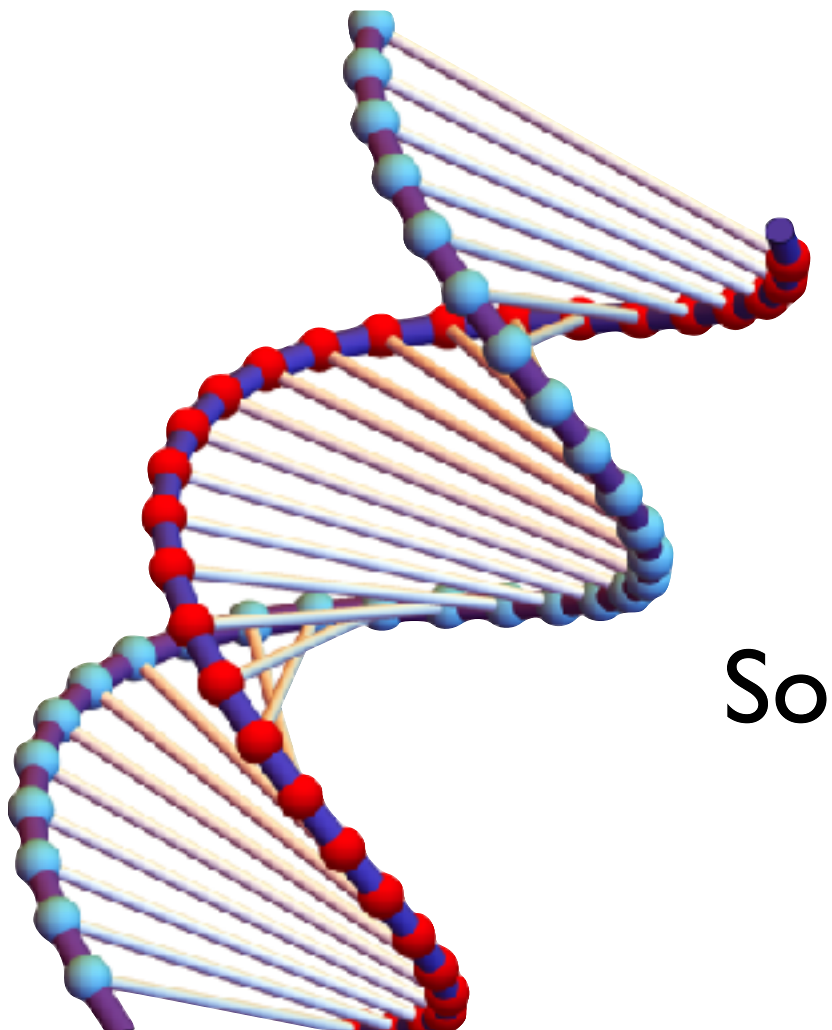


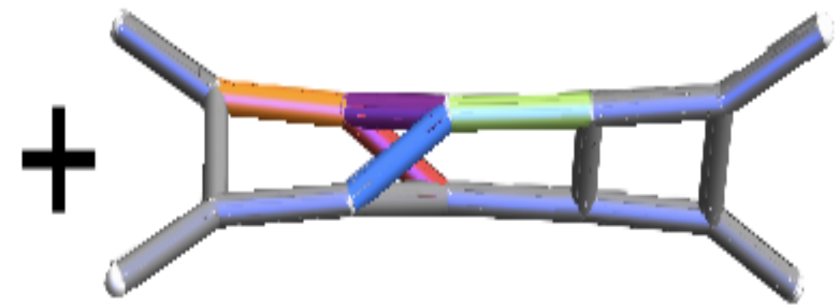
Bern, JJMC, Johansson, '10

Bern, JJMC, Johansson ('08,'10)

Color and Kinematics dance together.





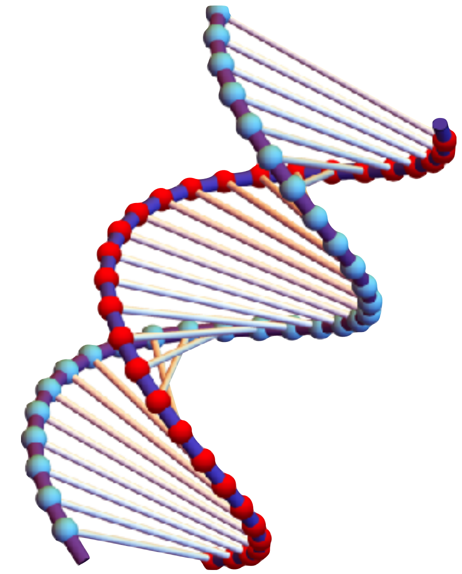


Solving Yang-Mills theories means solving Gravity theories.

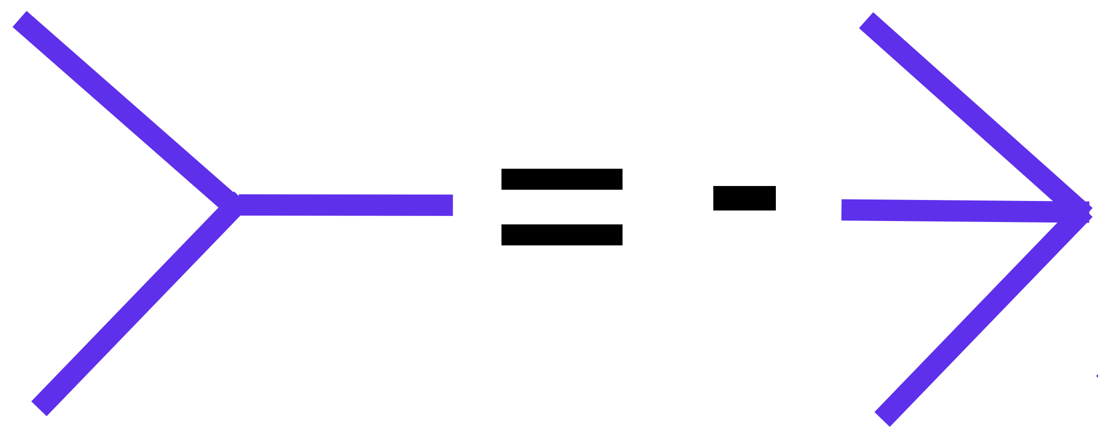
Bern, JJMC, Johansson ('08,'10)

Generic D-dimensional YM theories have a fascinating structure at tree-level

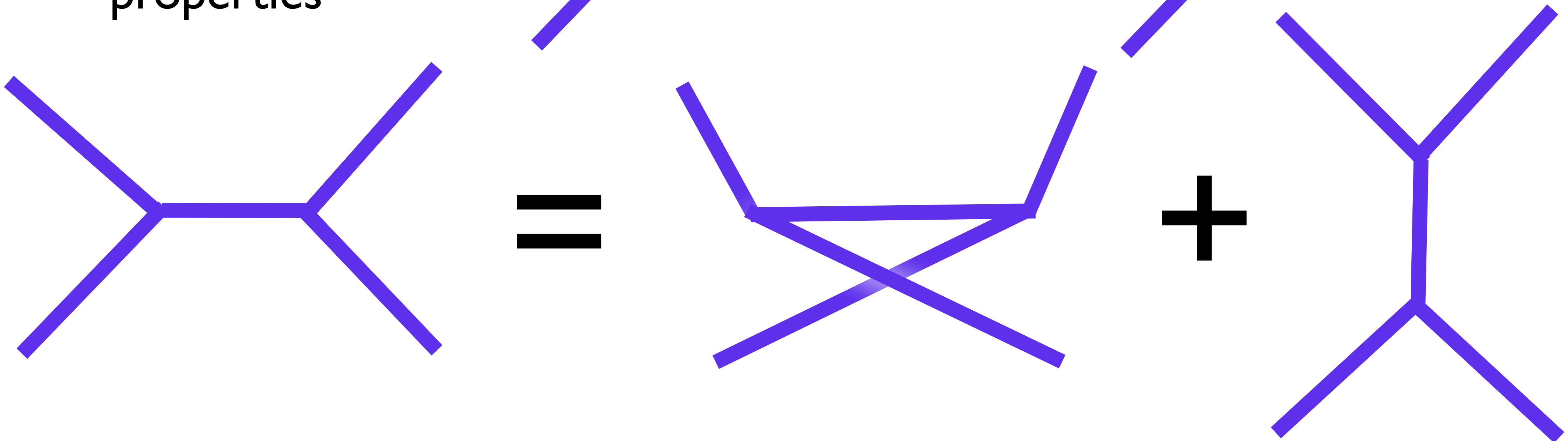
$$\mathcal{A}_m^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{c(\mathcal{G})n(\mathcal{G})}{D(\mathcal{G})}$$



Color factors and numerator factors satisfy similar lie algebra properties



Vertex Antisymmetry



Jacobi

Color-Kinematic Duality!

Bern, JJMC, Johansson ('08,'10)

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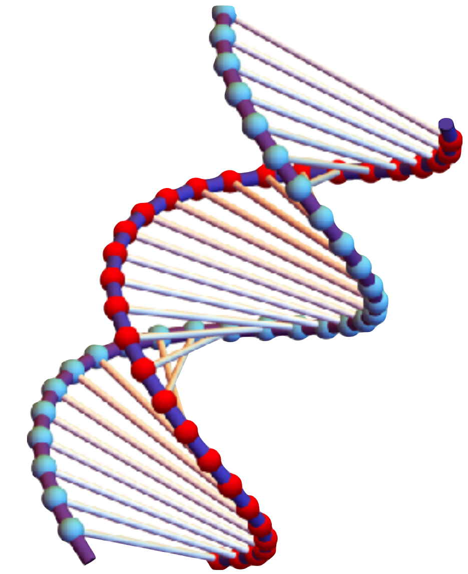
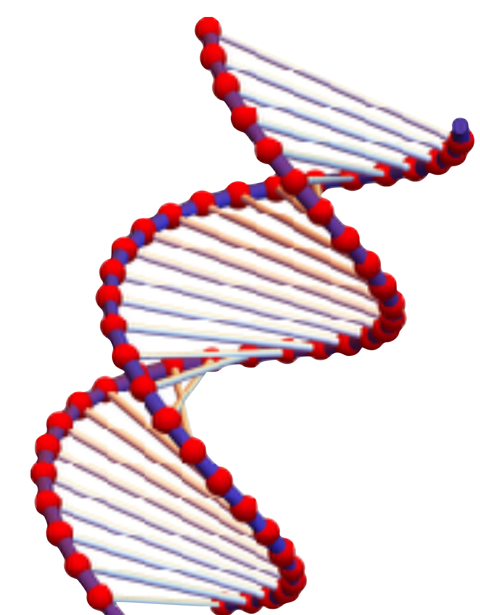
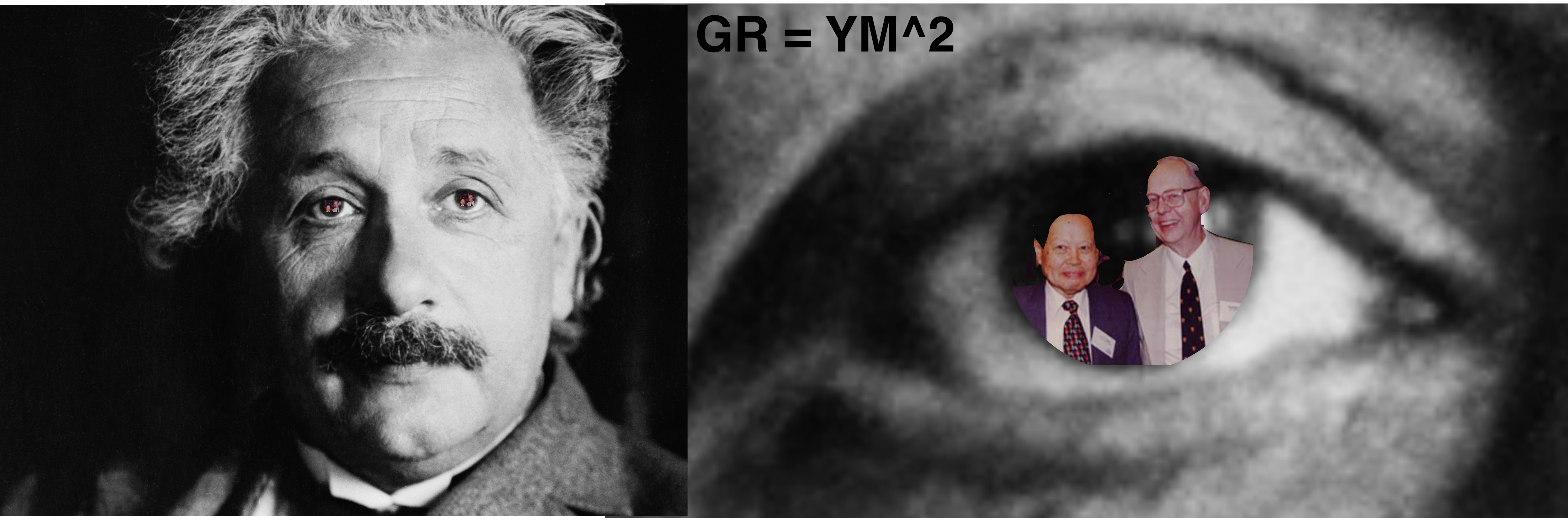


YM's Color-Kinematic duality makes manifest gravitational double copy structure:

$$-iM_n^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

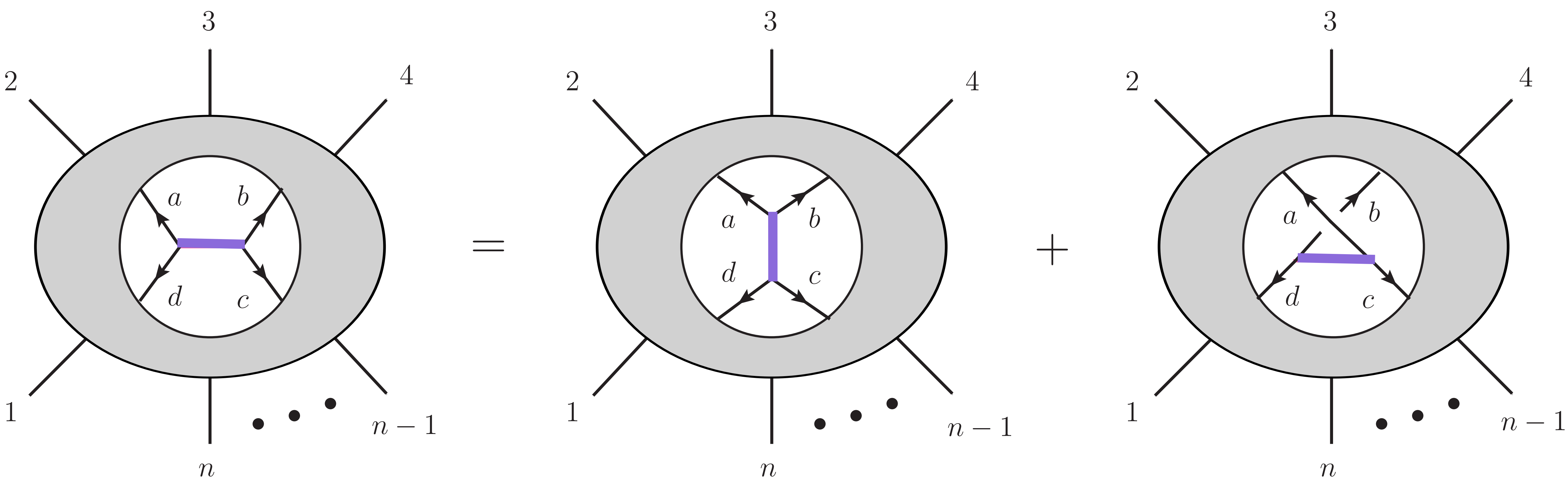



GR = YM^2

Valid multi-loop generalization?

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})}$$

CONJECTURE: for all graphs, can impose CK on every edge:



Consequence of unitarity: double copy structure holds.

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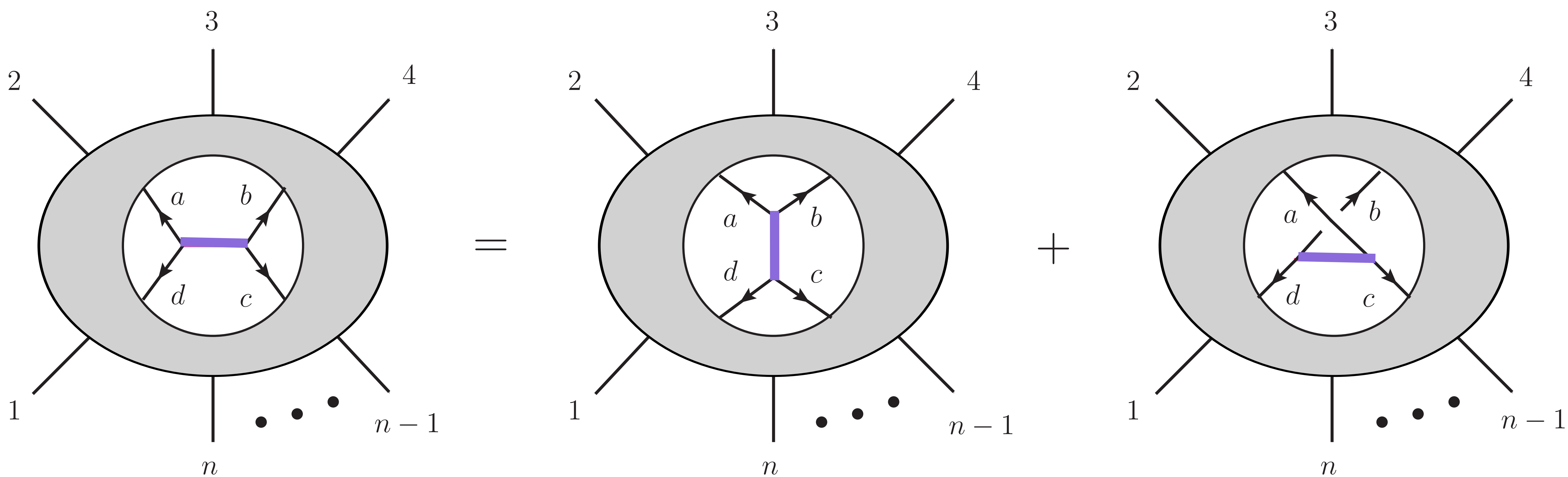


Consequence of unitarity: double copy structure holds.

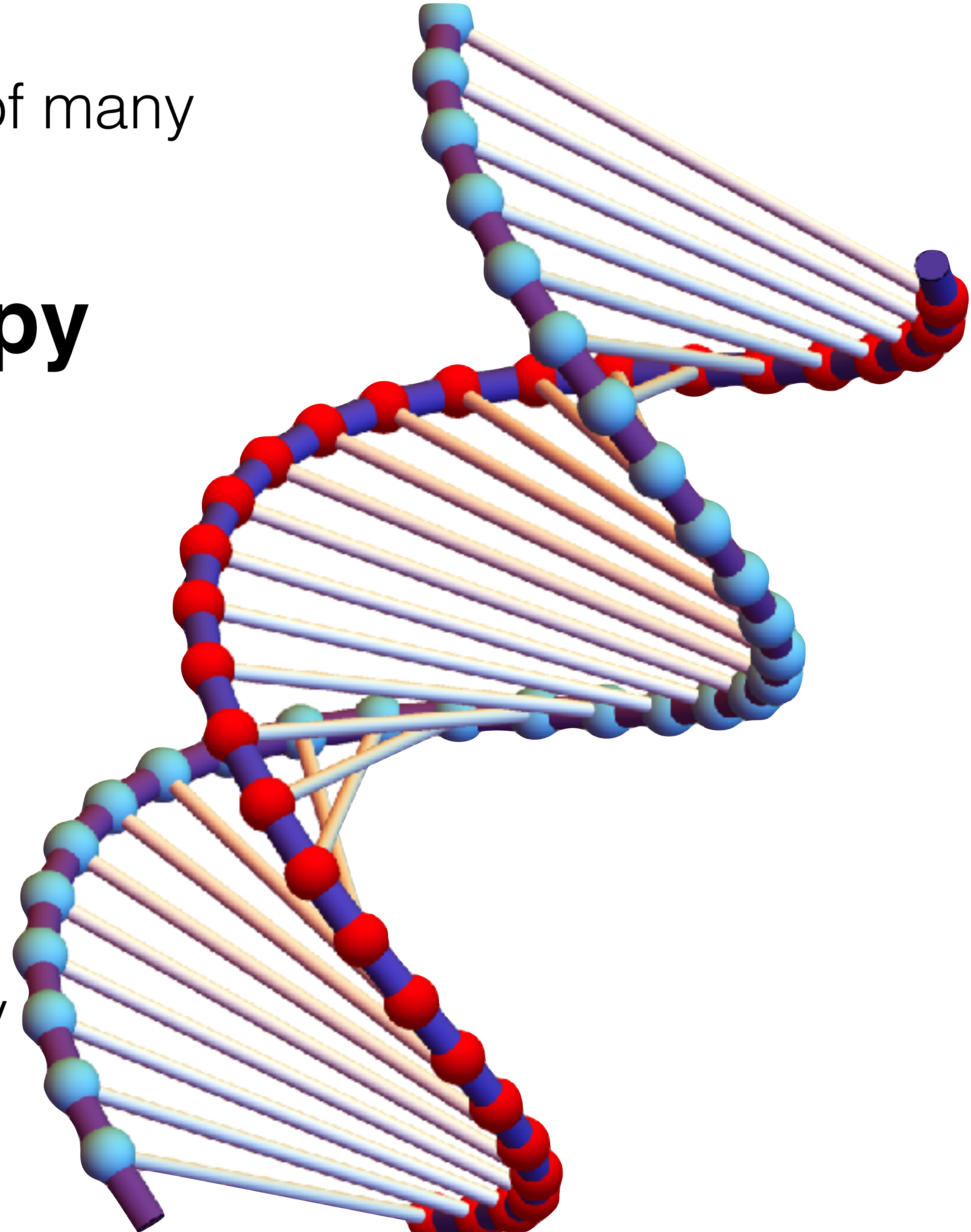
$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

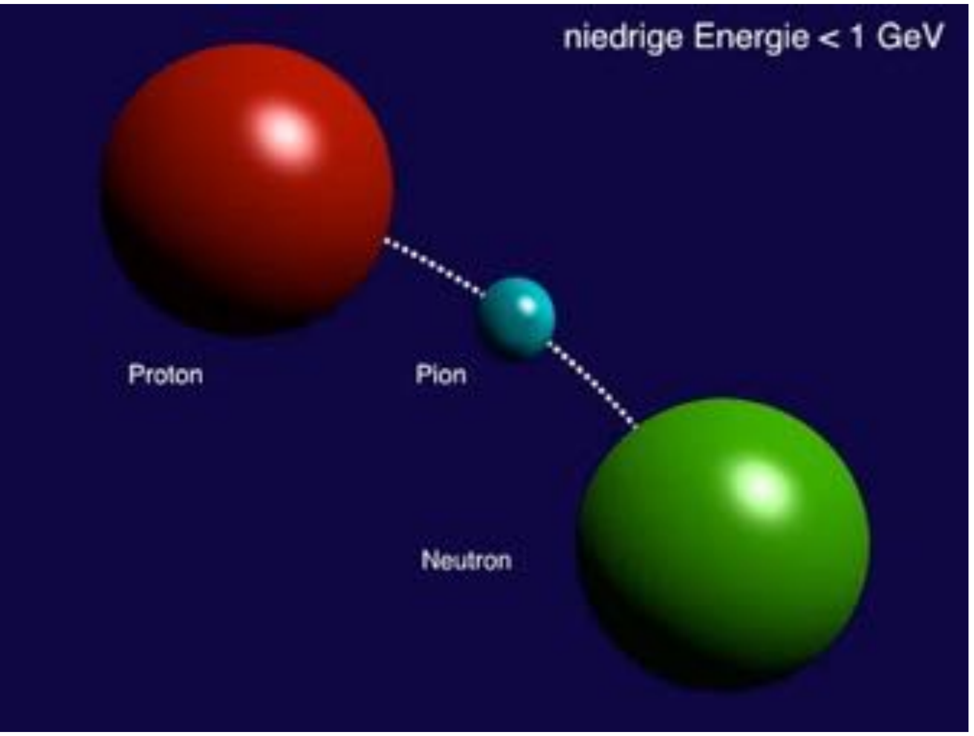
The scattering amplitudes of many relativistic theories admit a:

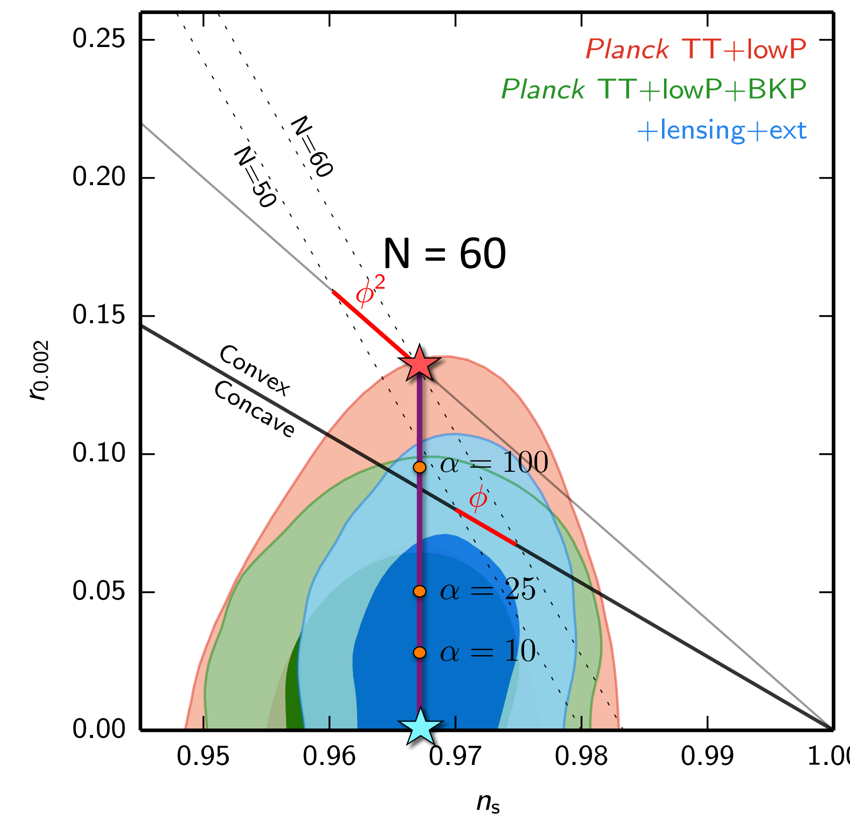
Double-copy
Numerator
Algebra

This points to previously hidden structure in many theories.

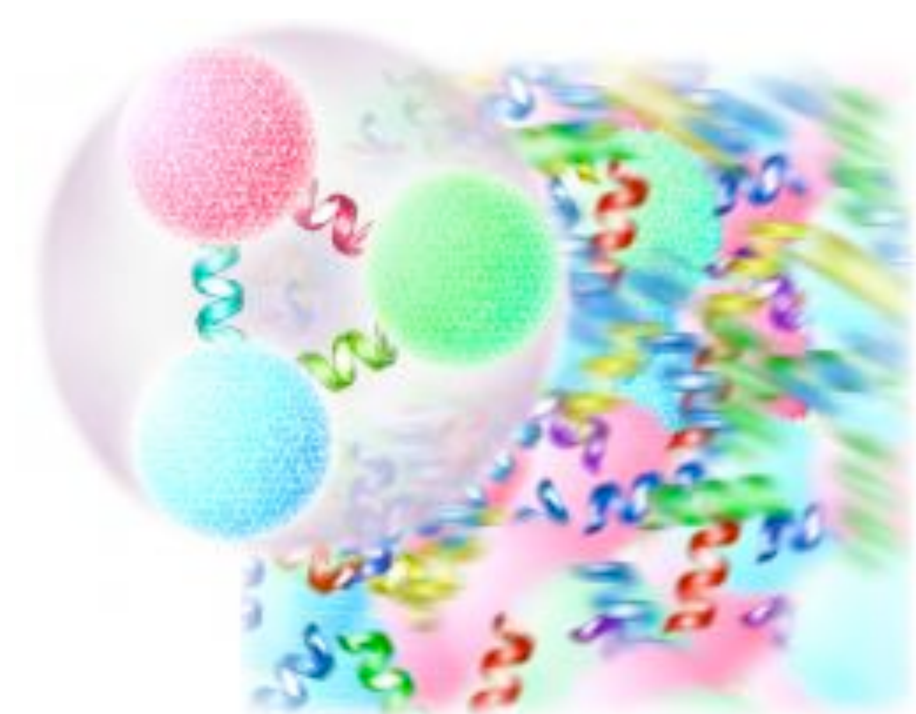
Structure yet to be generally understood at the level of the action.

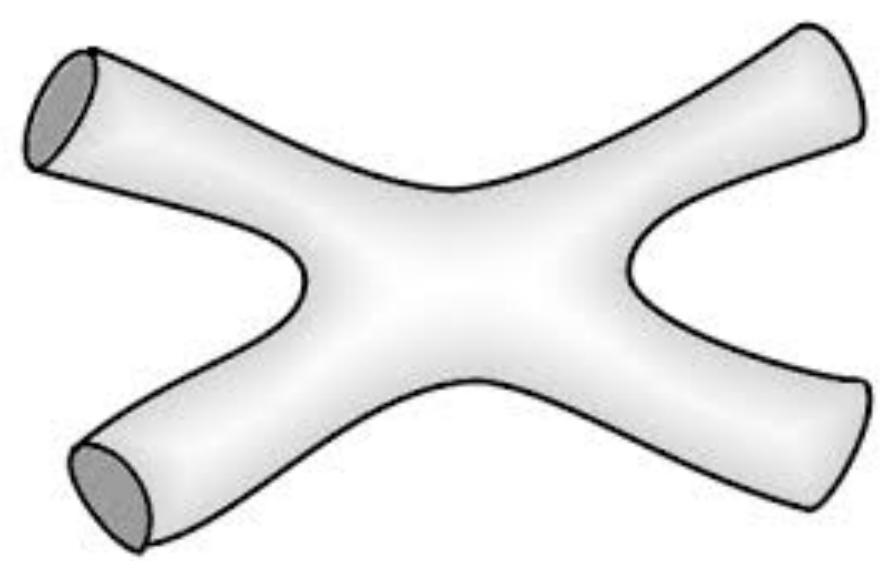






Many theories amplitudes are double copy!





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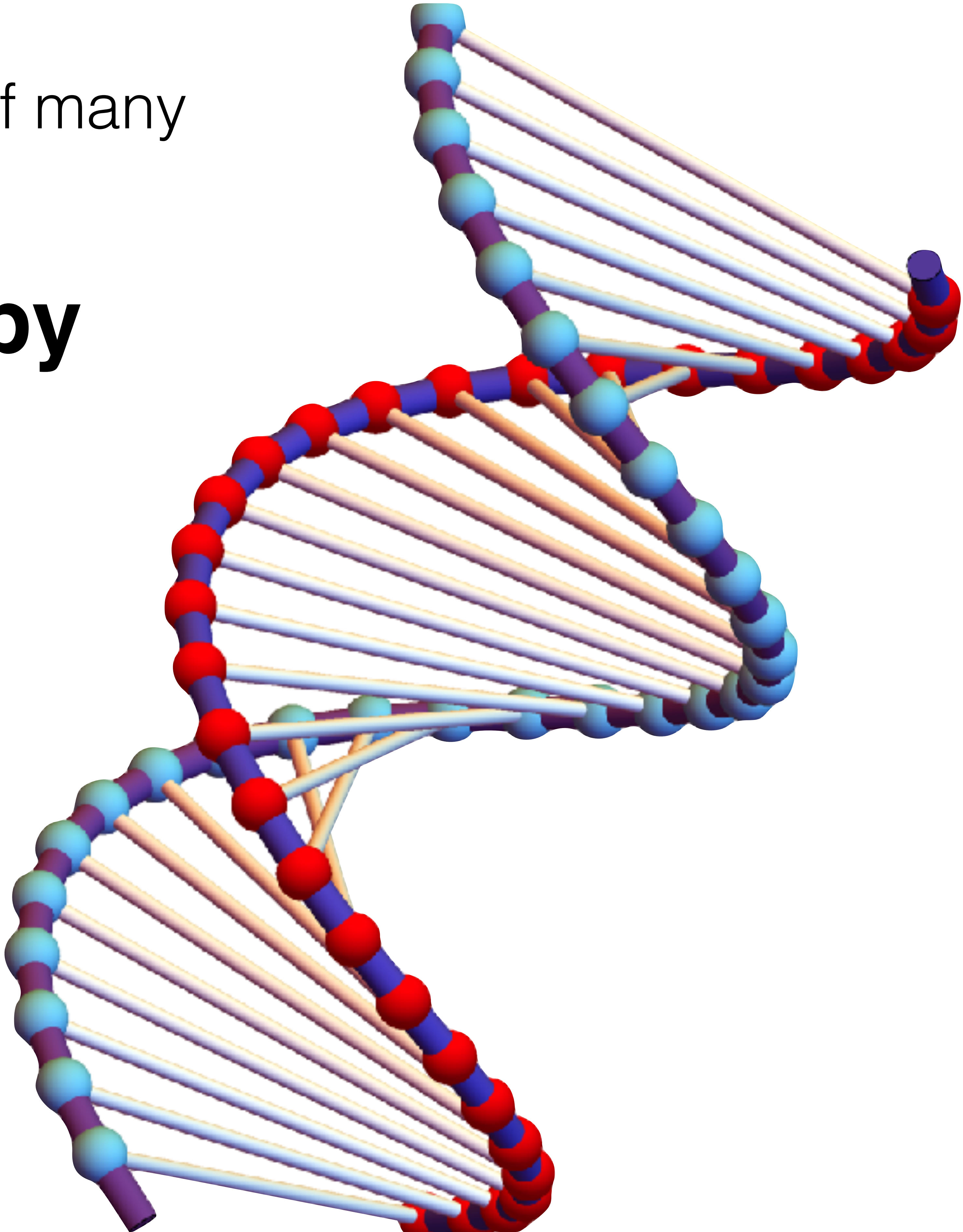

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Key Point: **MANY Theories are Double Copies**

Bi-Adjoint Scalar: color $\otimes$ color

Bern, de Freitas, Wong ('99); Bern, Dennen, Huang; Du, Feng, Fu; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell

(S)YM (…(S)QCD…): color $\otimes$ spin-1

BCJ ('08) Bjerrum-Bohr, Damgaard, Vanhove; Steiberger; Feng et al; Mafra, Schlotterer, ('08-'11); Johansson, Ochirov

(S)Gr (…(S)Einstein-YM…): spin-1 $\otimes$ spin-1

KLT('86); BCJ ('08); Chiodaroli, Gunaydin, Johansson, Roiban; Johansson, Ochirov; Johansson, Kälin, Mogull

NLSM / Chiral Lagrangian: "color" $\otimes$ even-spin-0

Chen, Du '13 Cachazo, He, Yuan '14 Cheung, Shen '16

(S)Born-Infeld: spin-1 $\otimes$ even-spin-0

Cachazo, He, Yuan '14

Special Galileon: even-spin-0 $\otimes$ even-spin-0

Cachazo, He, Yuan '14 Cheung, Shen '16

Open String: α' $\otimes$ spin-1

Broedel, Schlotterer, Stieberger

Closed String: spin-1 $\otimes$ α' corrected spin-1

Broedel, Schlotterer, Stieberger;

Z-theory: α' $\otimes$ "color"

Broedel, Schlotterer, Stieberger; JJMC, Mafra, Schlotterer

Key Point: **MANY Theories are Double Copies**

Ingredients:

α'

color

spin 0,1/2,1

For all these theories:

(S)YM
(...(S)QCD...)

Bi-Adjoint Scalar

(S)Gr
(...(S)Einstein-YM...)

NLSM

(S)Born-Infeld

Special Galileon

Z-theory

Open String

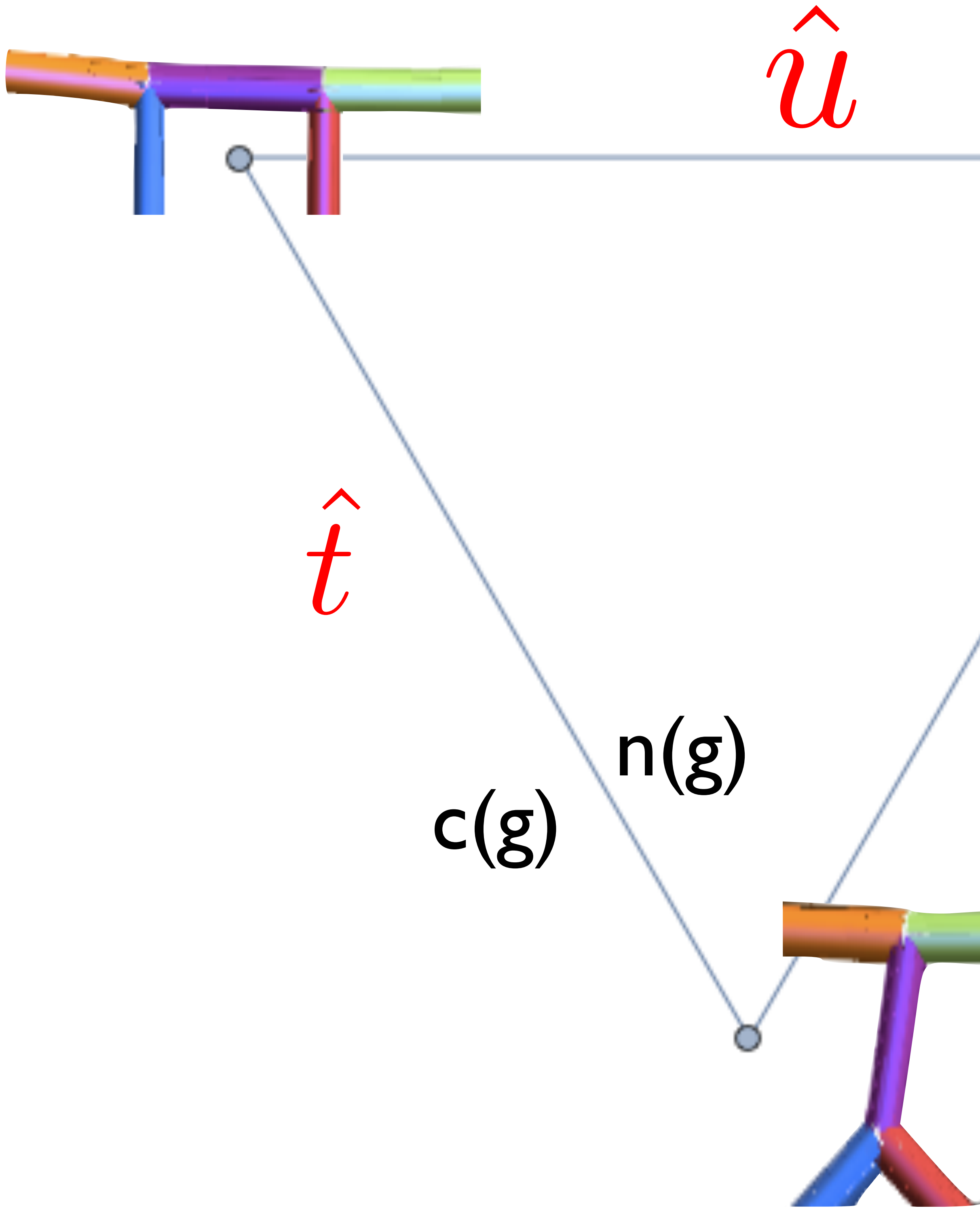
Closed String

a geometric guide to color-kinematics

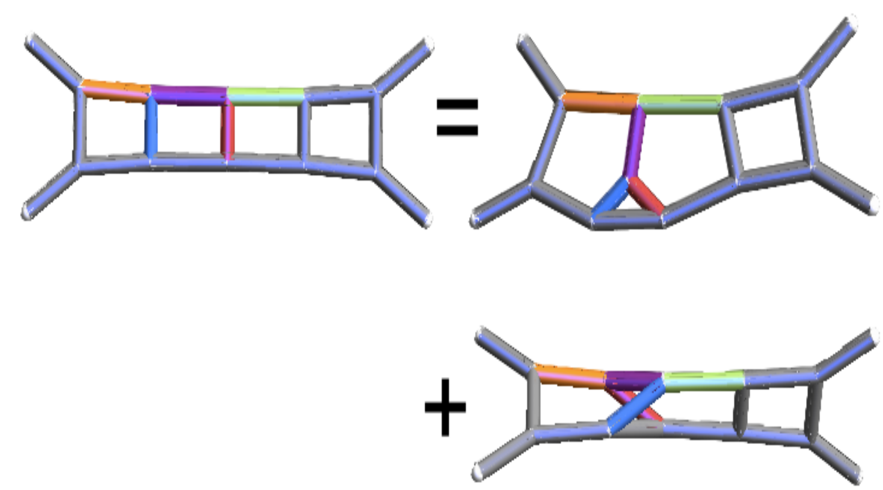
Physics = Geometry

(the best polytopes are graphs of graphs!)

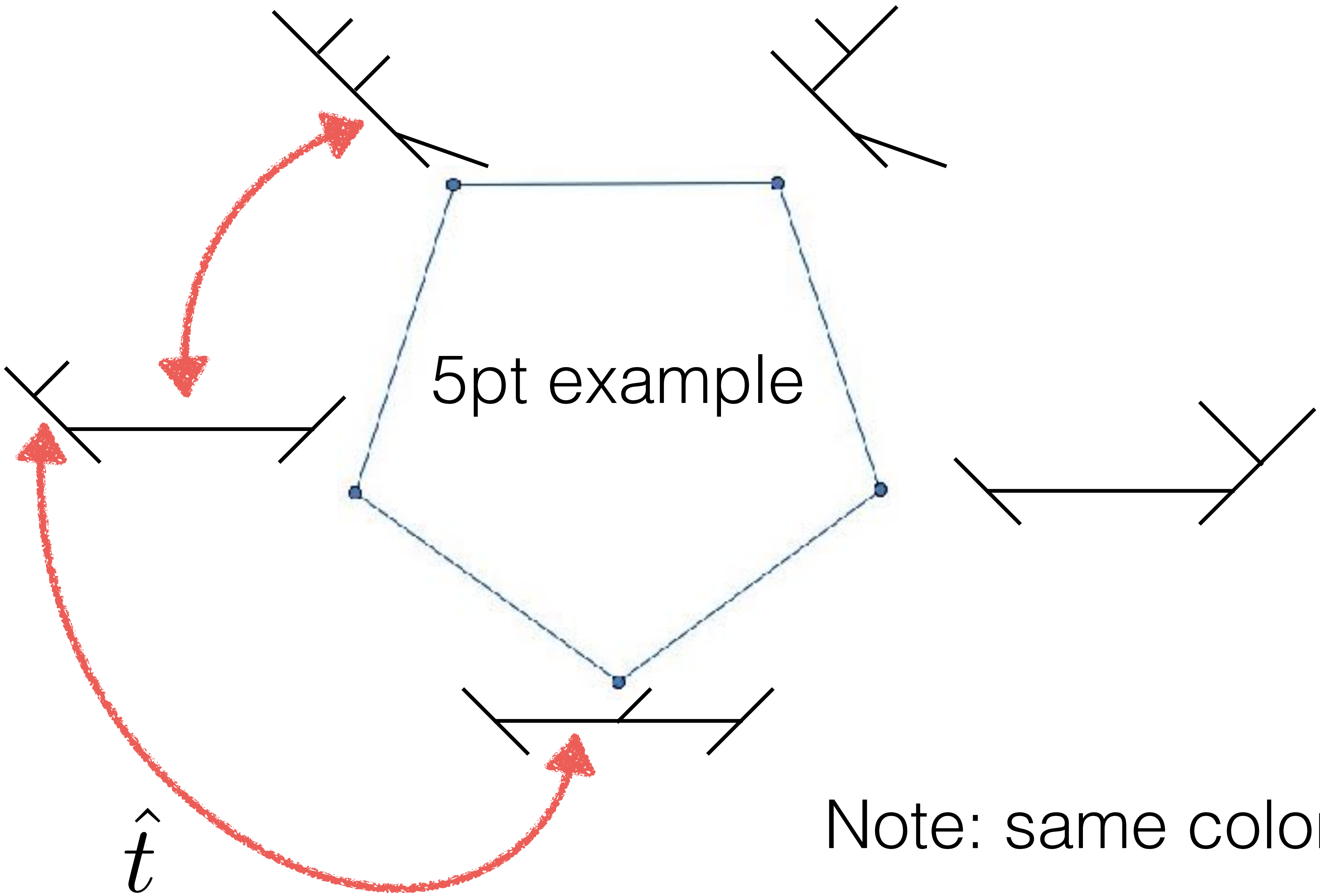
Convenient language: graphs of graphs



Color-Kinematics



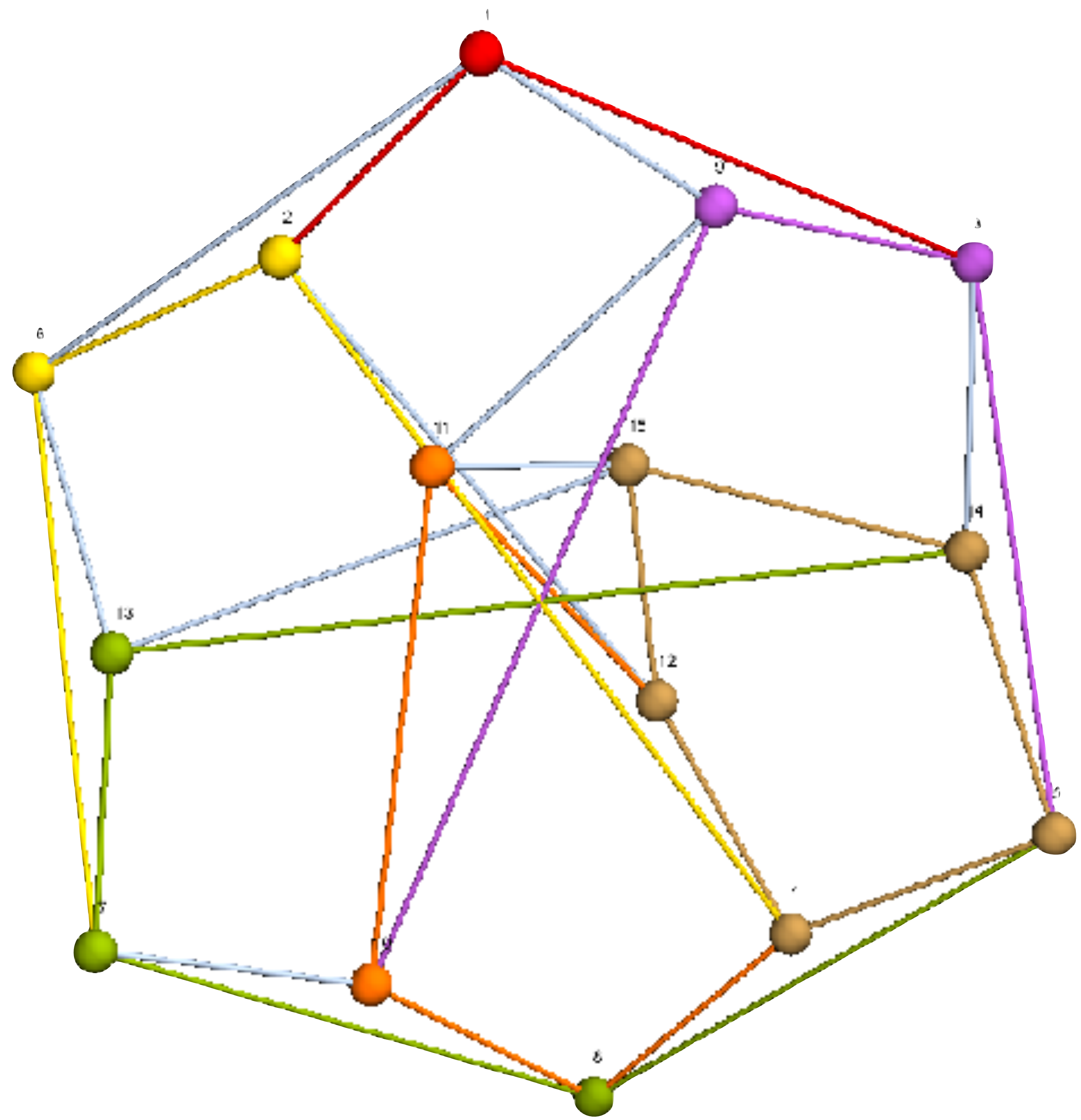
Graphs contributing to an ordered tree (color-stripped), generate the 1-skeleton of **Stasheff polytopes** joined only by $\hat{t}$



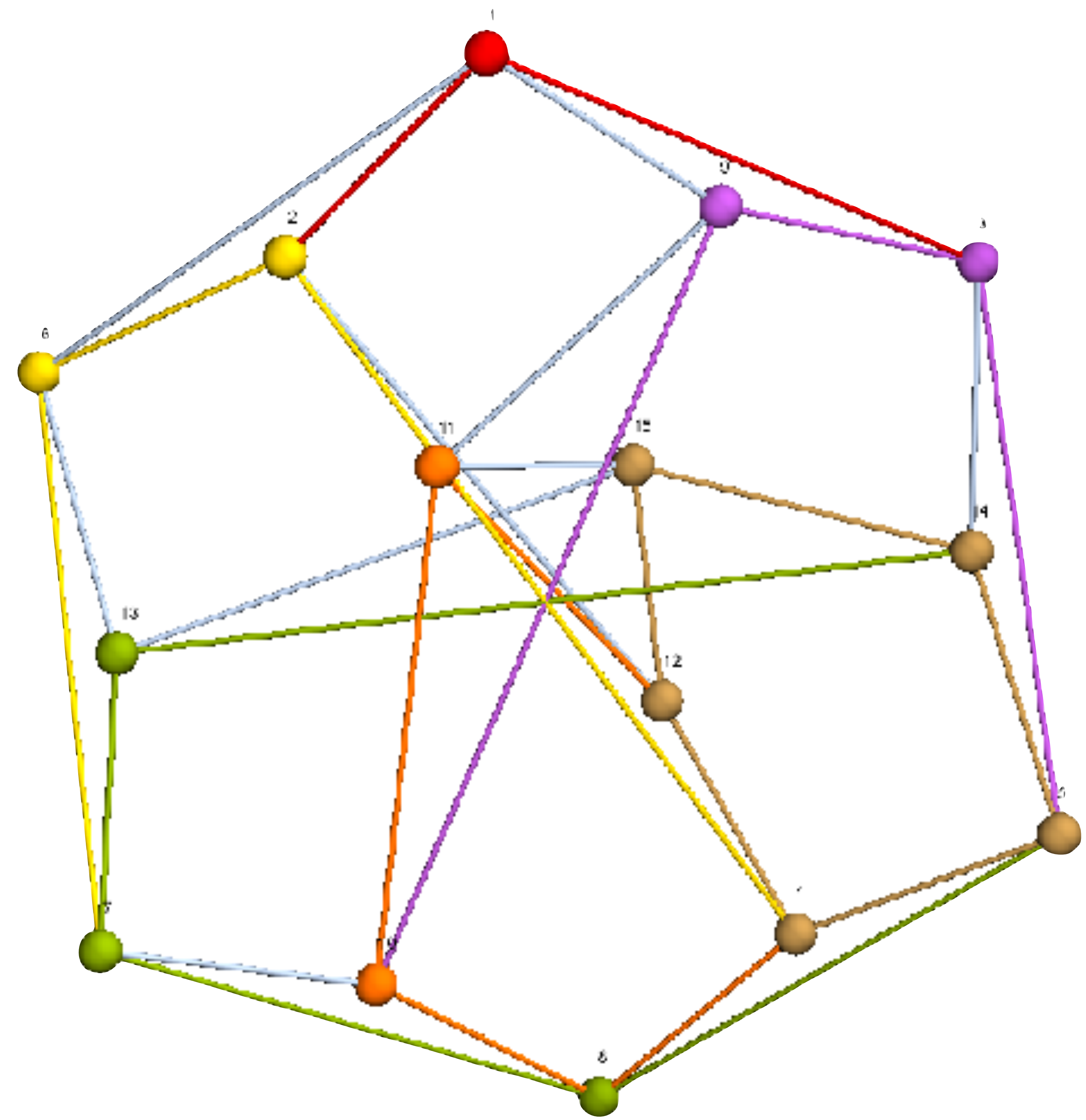
Note: same color-order!

(these polytopes are also called **associahedra**)

You might think you need (m-2)! of these color-ordered amplitudes to capture everything because this is what is required to touch every vertex at least once:



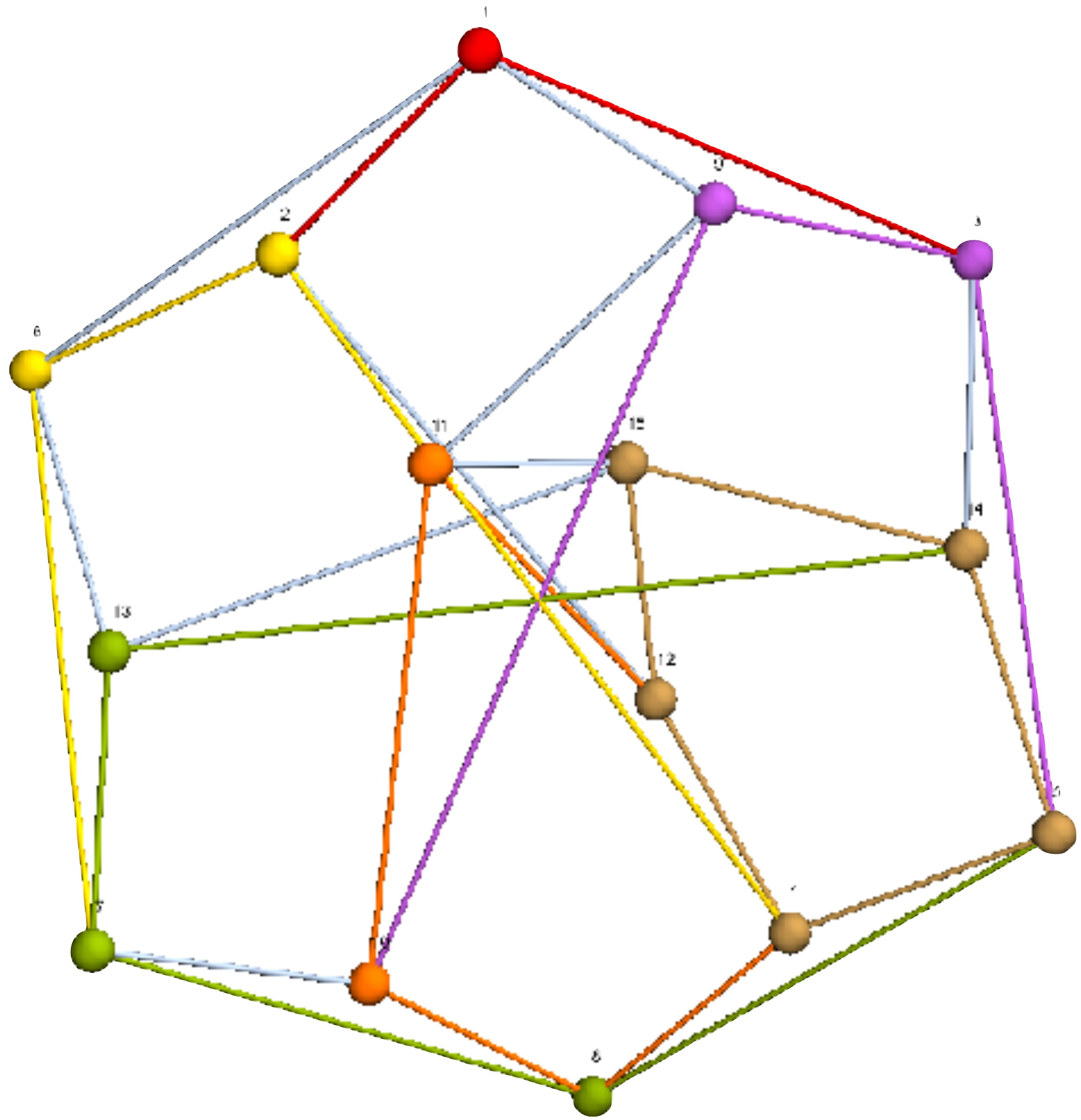
You might think you need (m-2)! of these color-ordered amplitudes to capture everything because this is what is required to touch every vertex at least once:



In fact, such a choice is the KK-basis, proven sufficient by Del Duca, Dixon, and Maltoni

JJMC

But notice, because of color-kinematics, only (m-2)! nodes are needed to specify both the color factors and numerator factors of everyone



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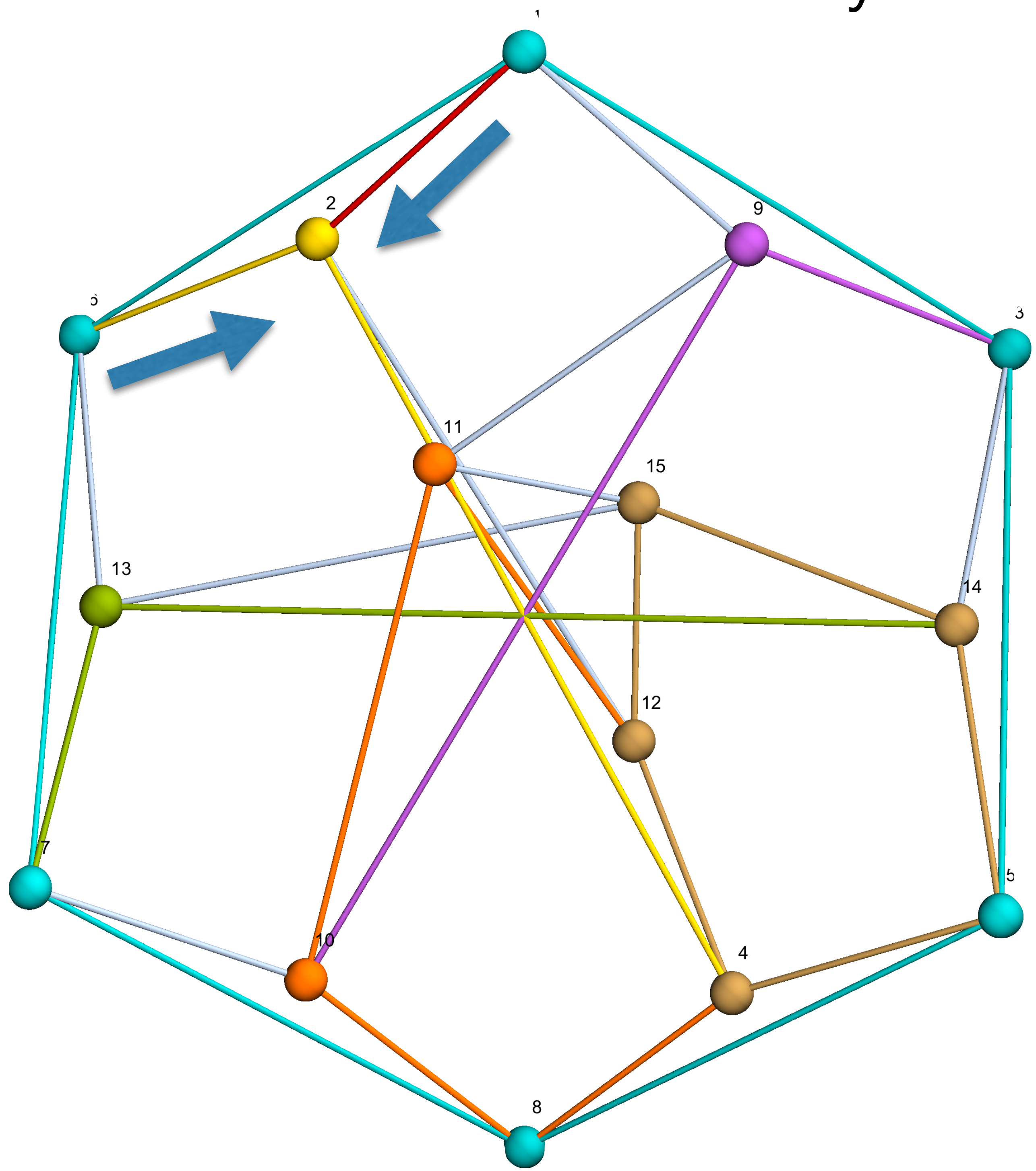

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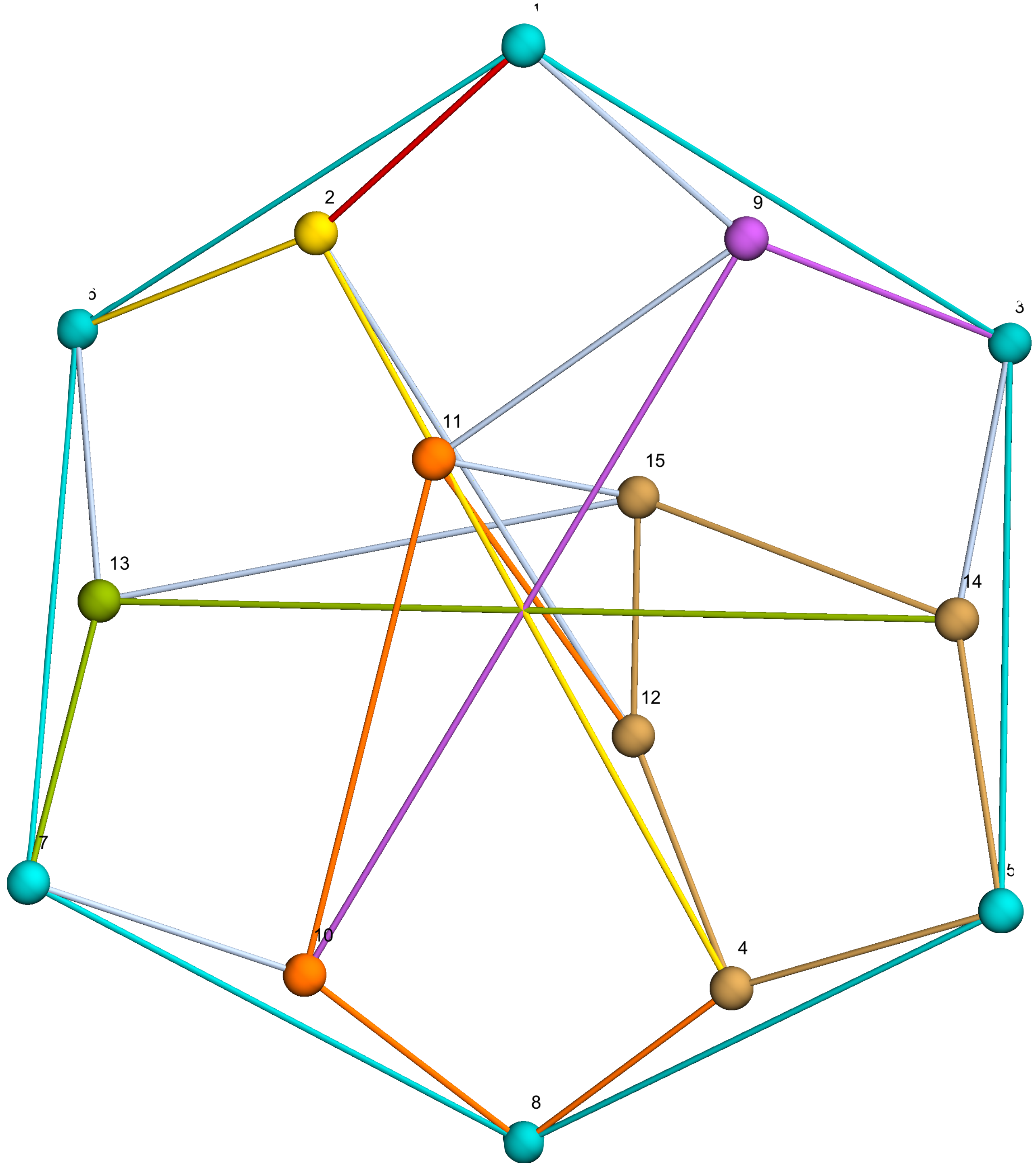
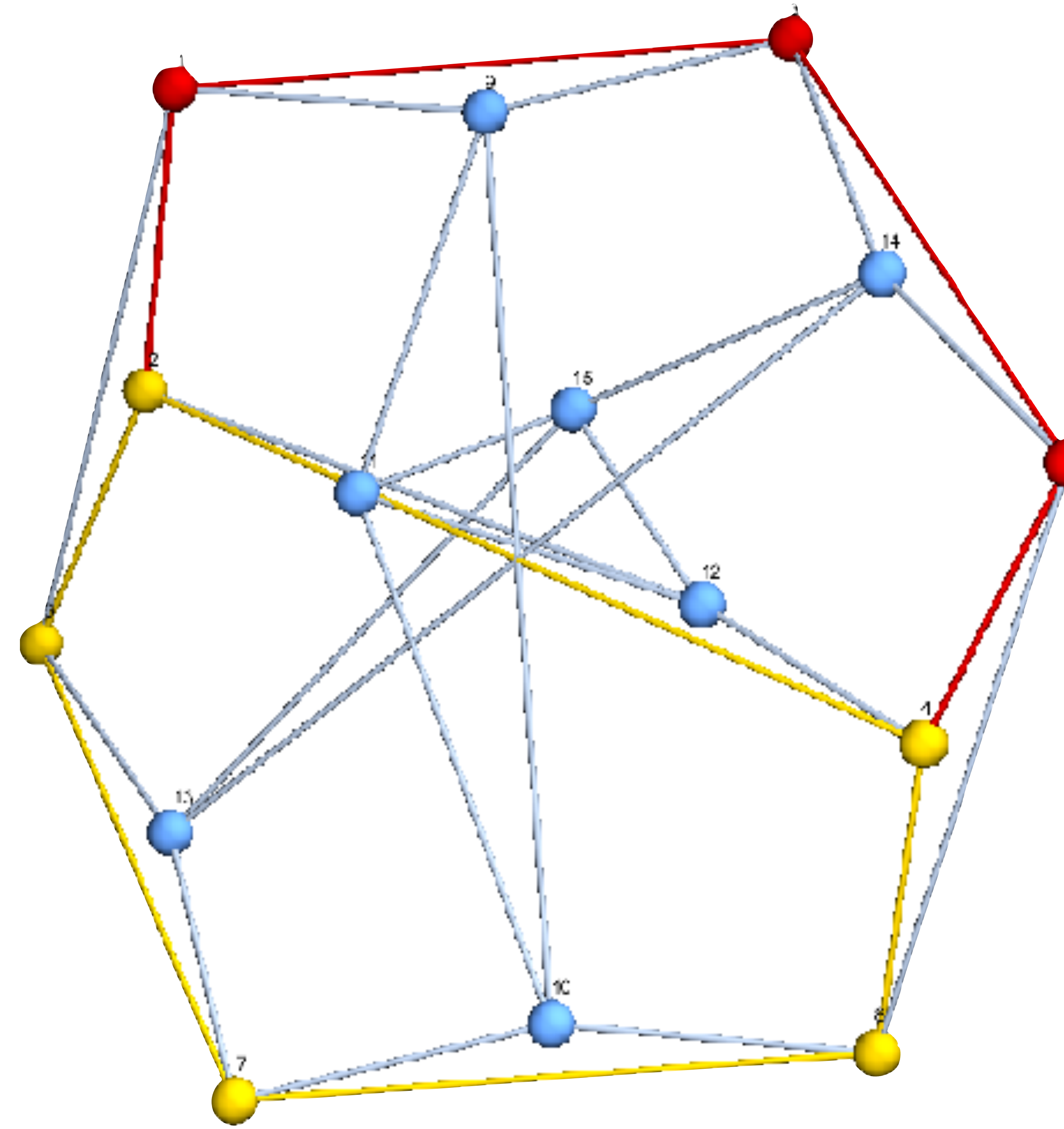
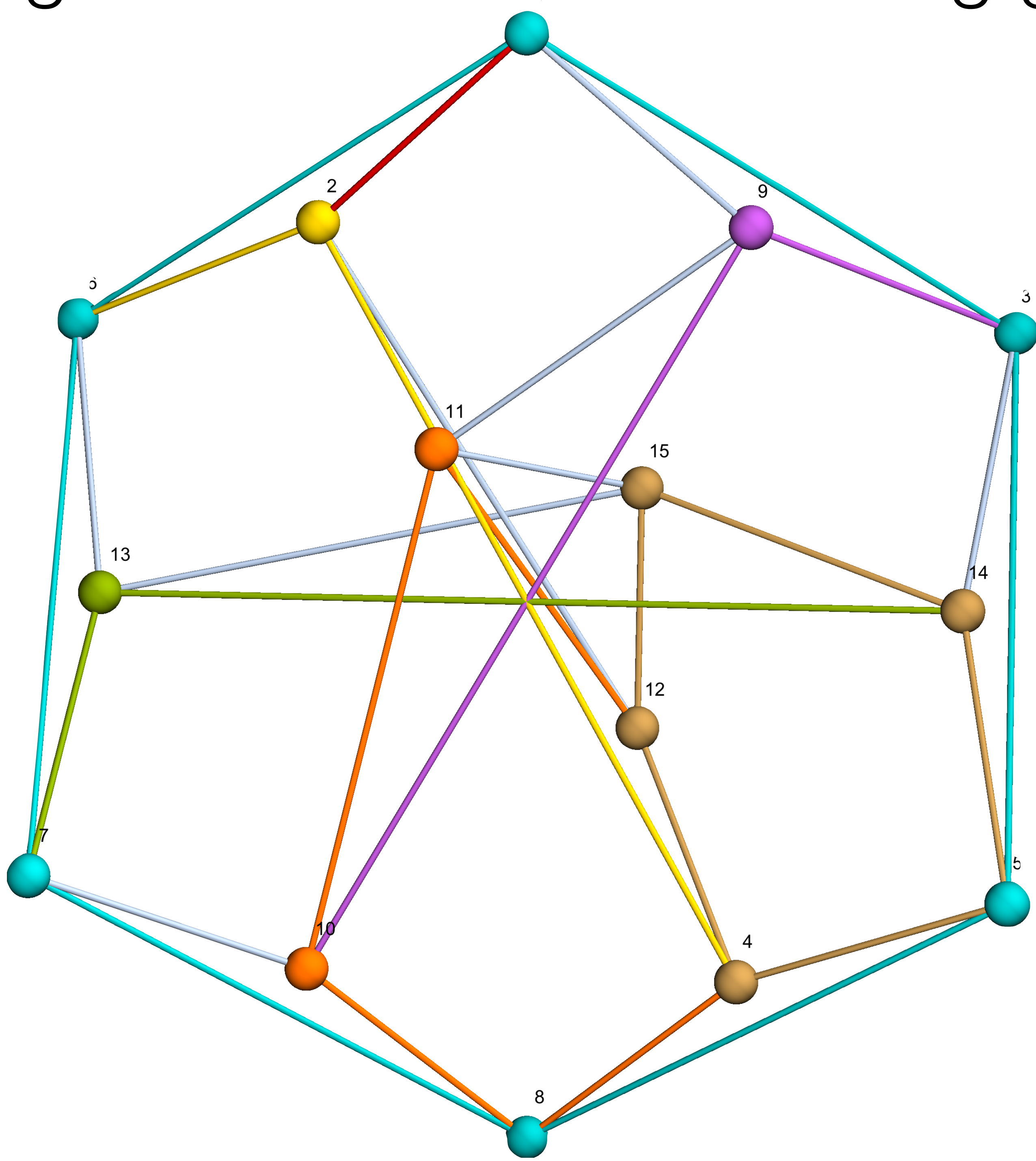
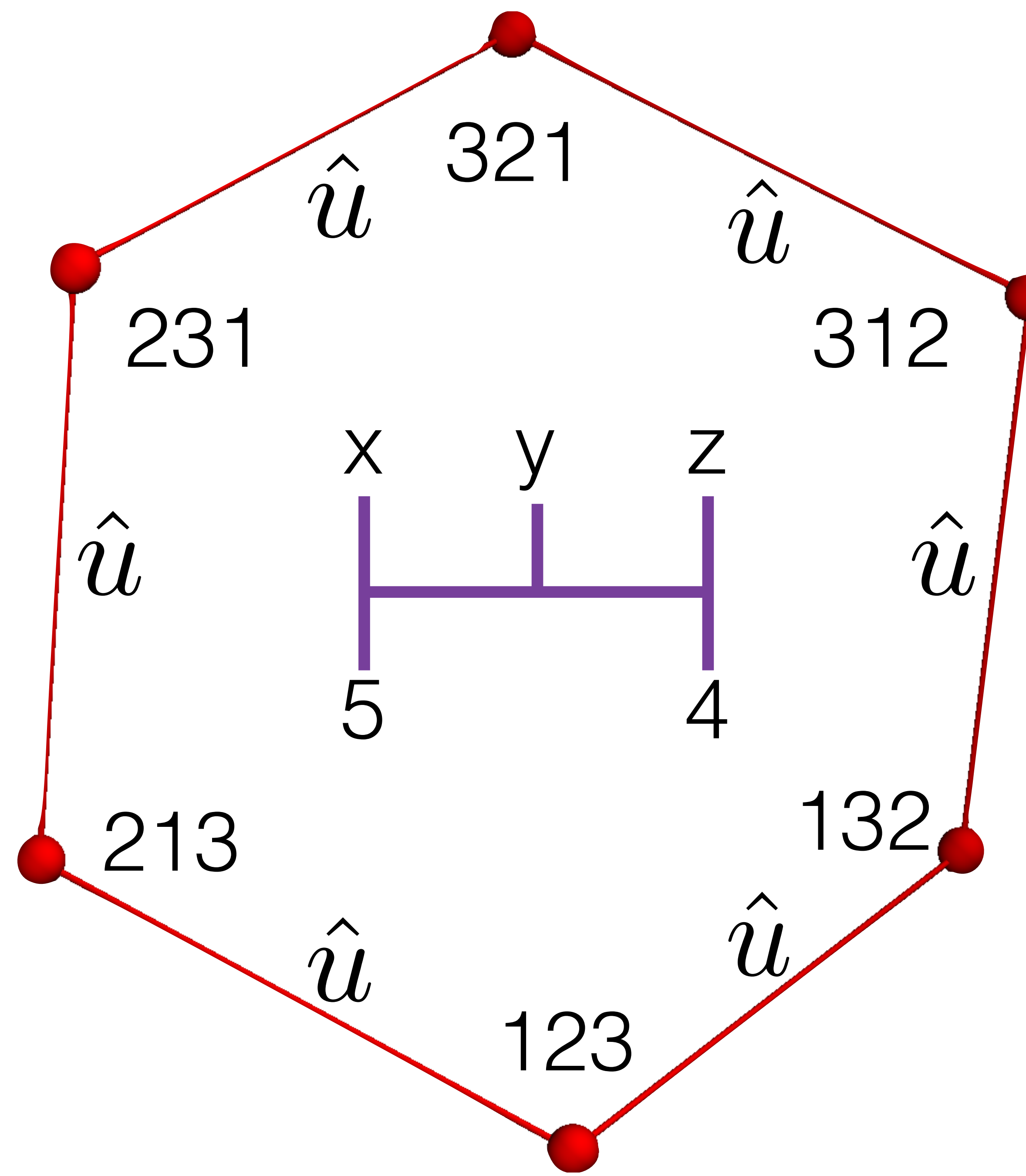




This reduces the set of necessary color-ordered amplitudes (associahedra) to (m-3)! : “BCJ” relations

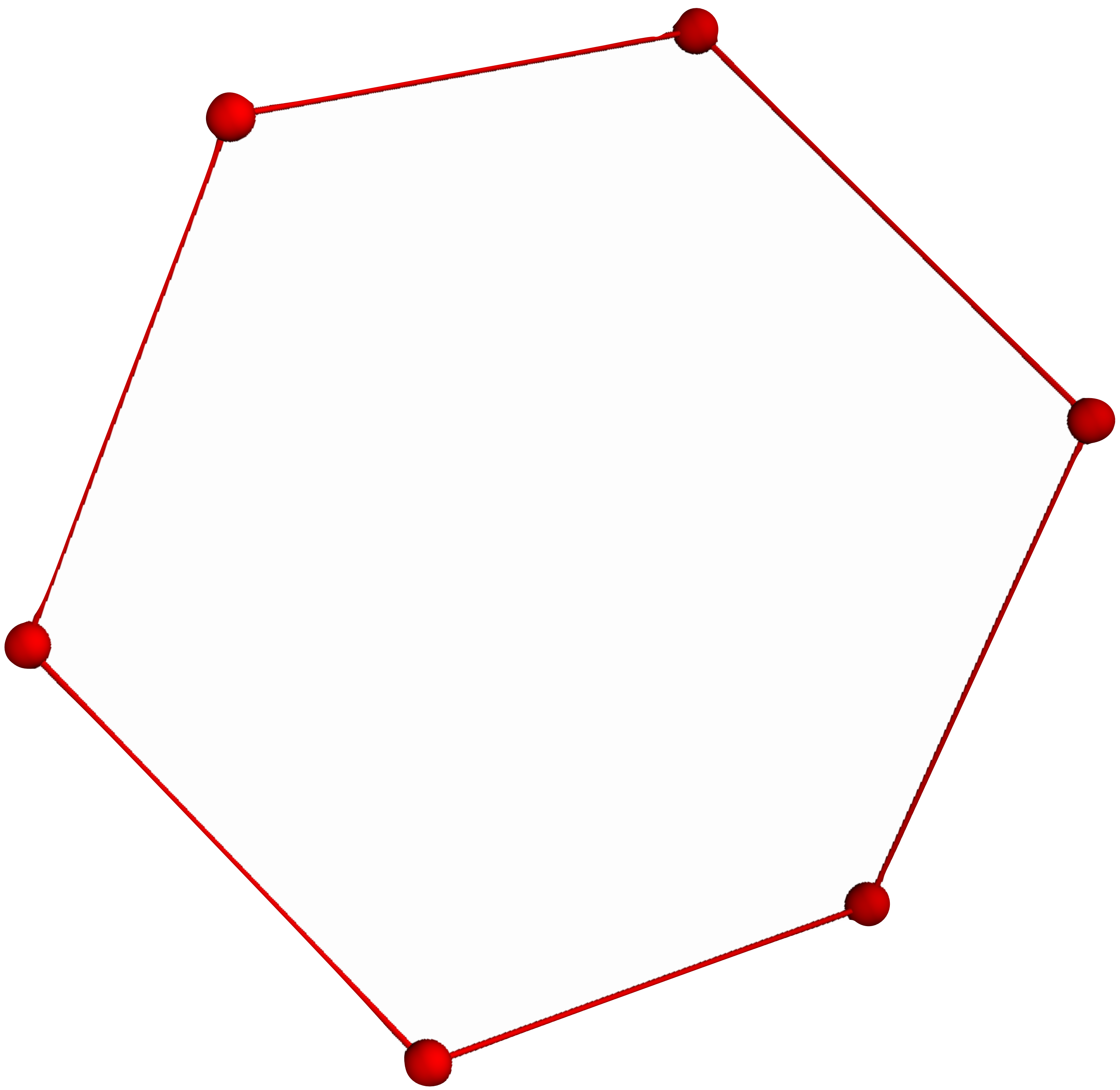
At every multiplicity the masters can be chosen to form the 1-skeleton of a polytope related by $\hat{u}$ on every internal edge of the relevant scattering graphs

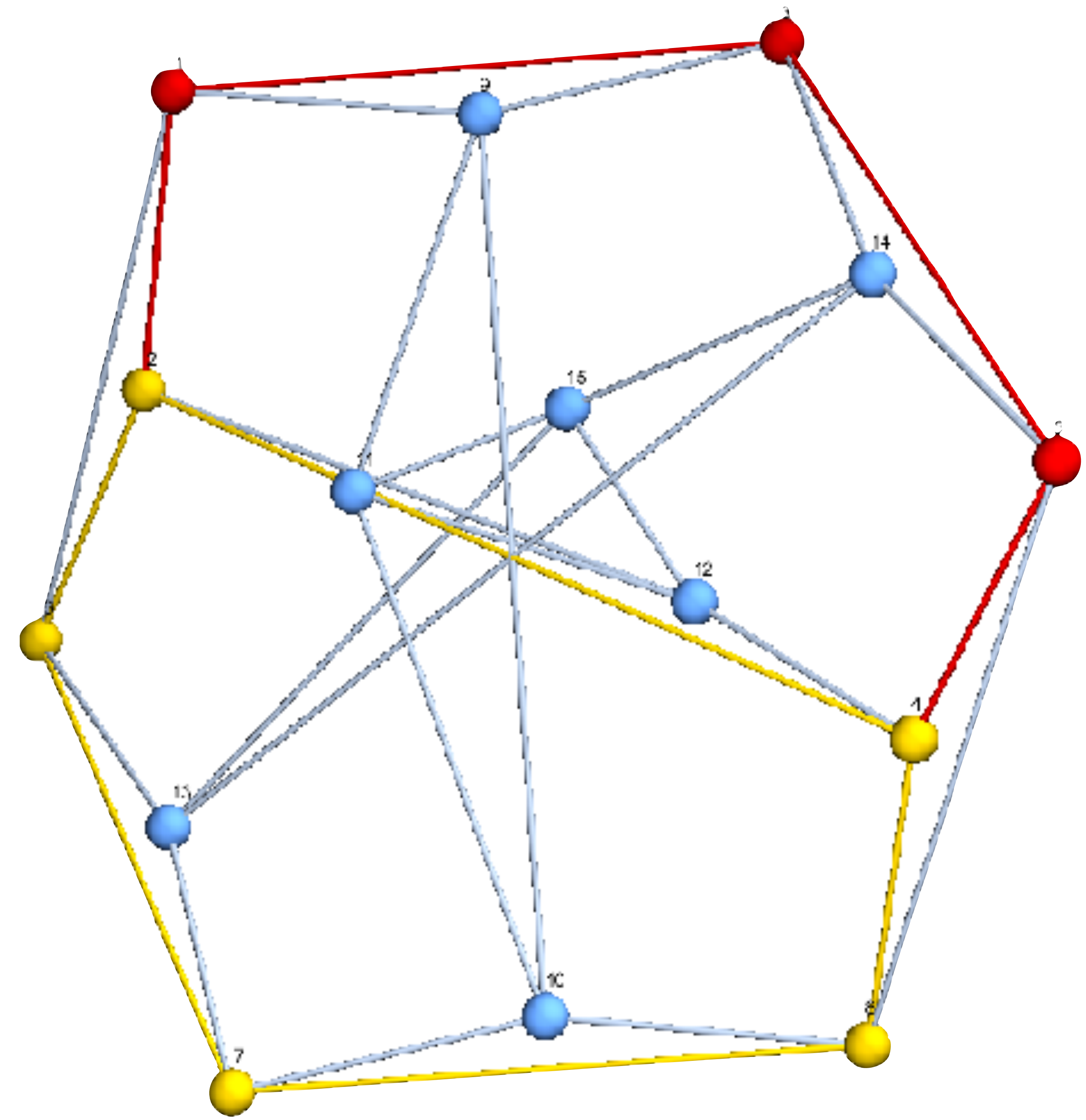




(these polytopes are called **permutahedra**)

Can linearly solve for the (m-2)! numerators of the masters in terms of the (m-3)! “BCJ” independent color-ordered amplitudes. In fact you get (m-3)! numerators in terms of the color-ordered amplitudes and (m-3)(m-3)! free functions.



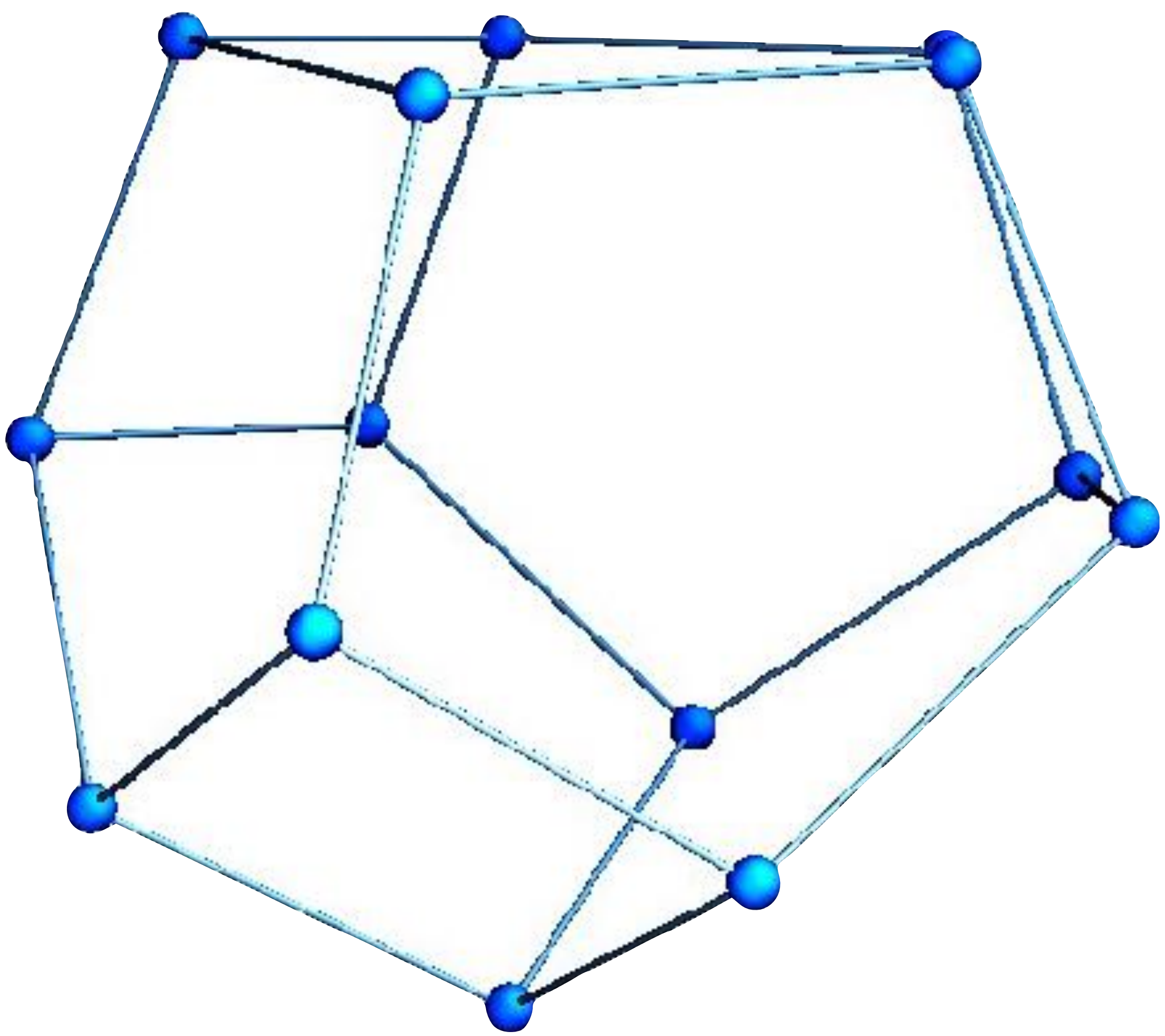


(generalized gauge freedom)

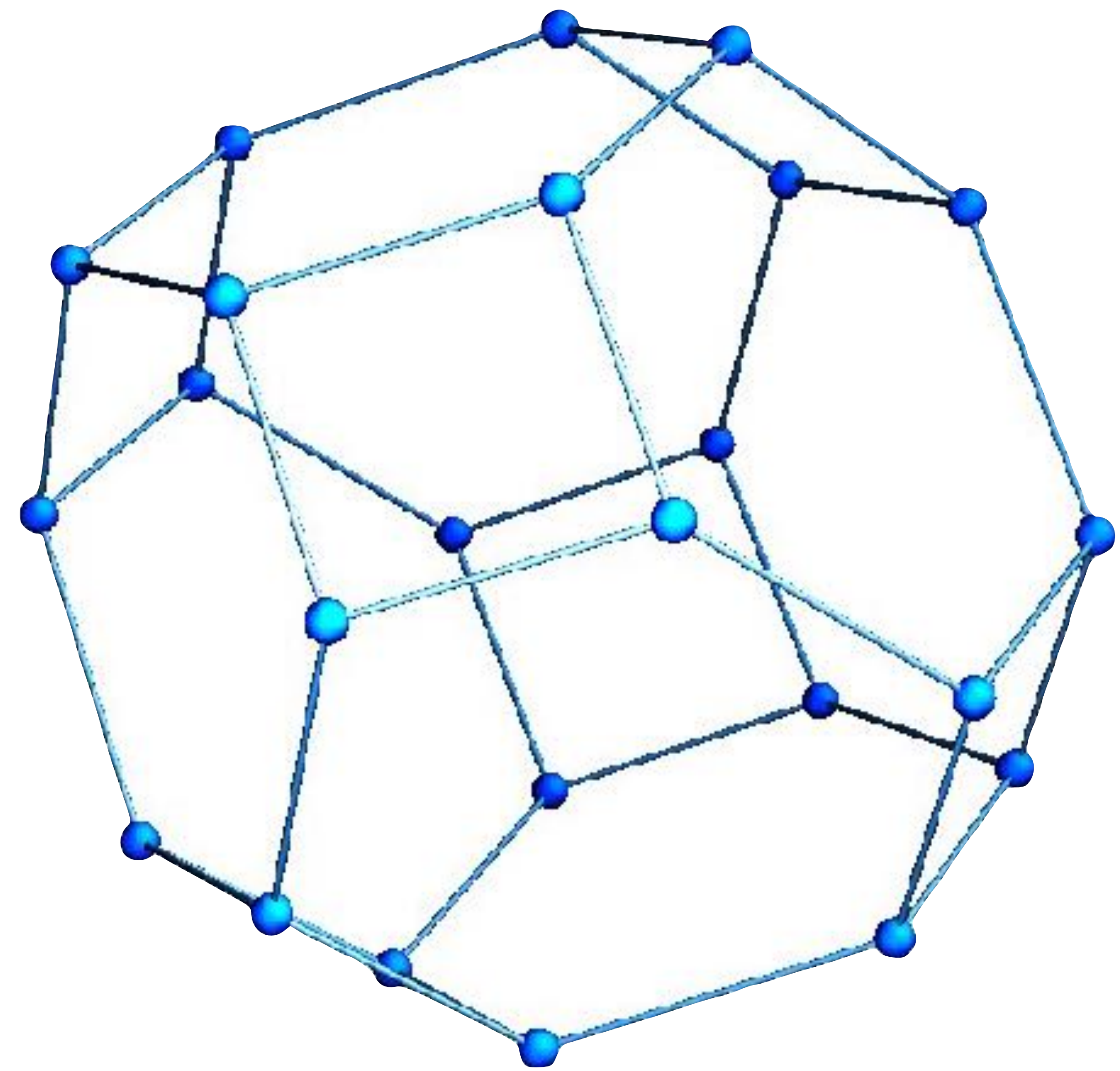
Building blocks at 6-points:

color-ordered amplitude

set of masters



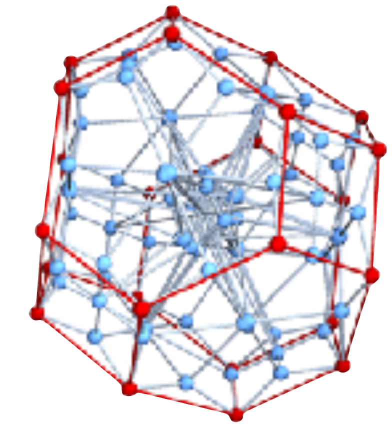
associahedron

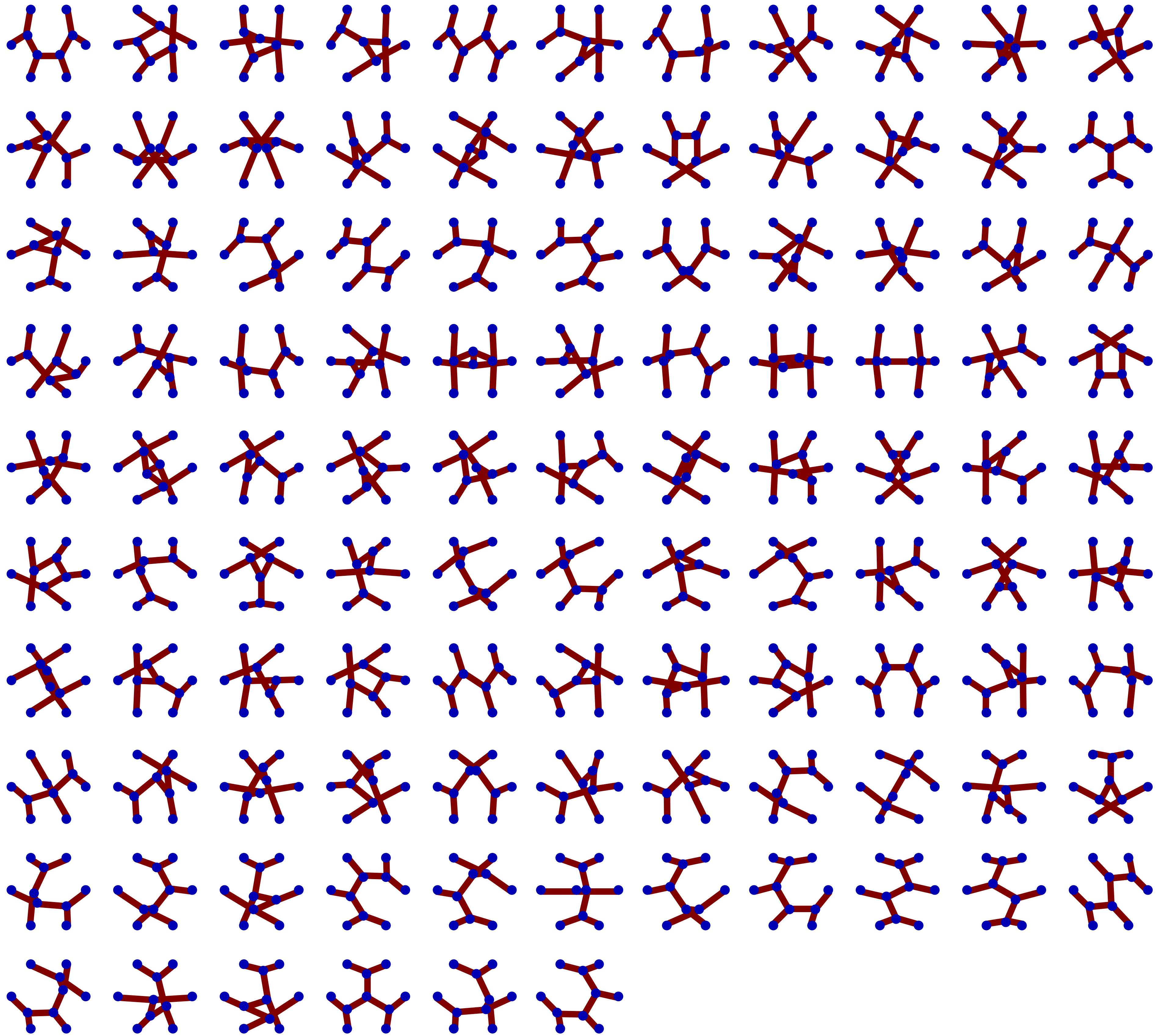


permutahedron

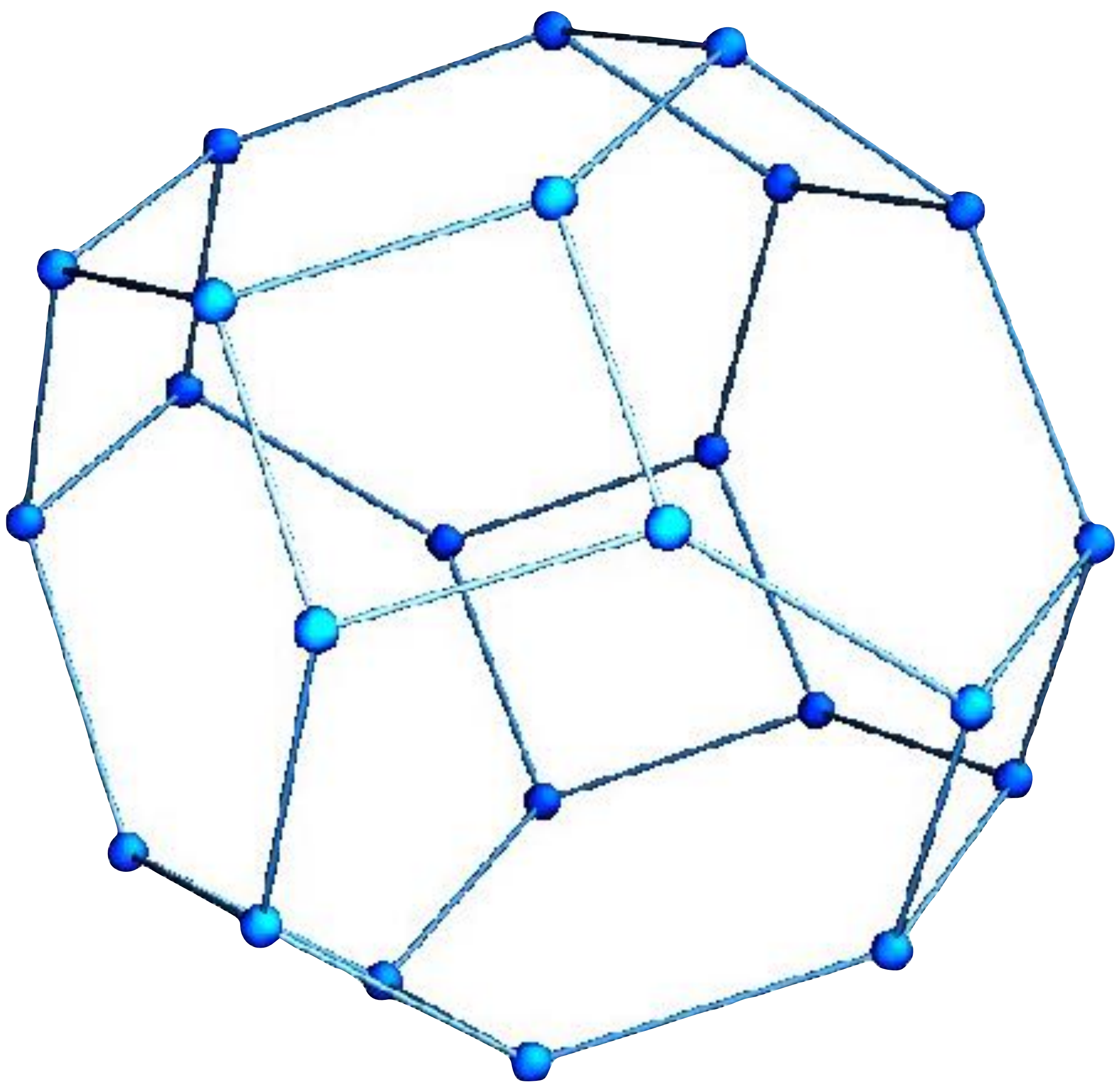
JJMC

105
cubic graphs at 6 pt

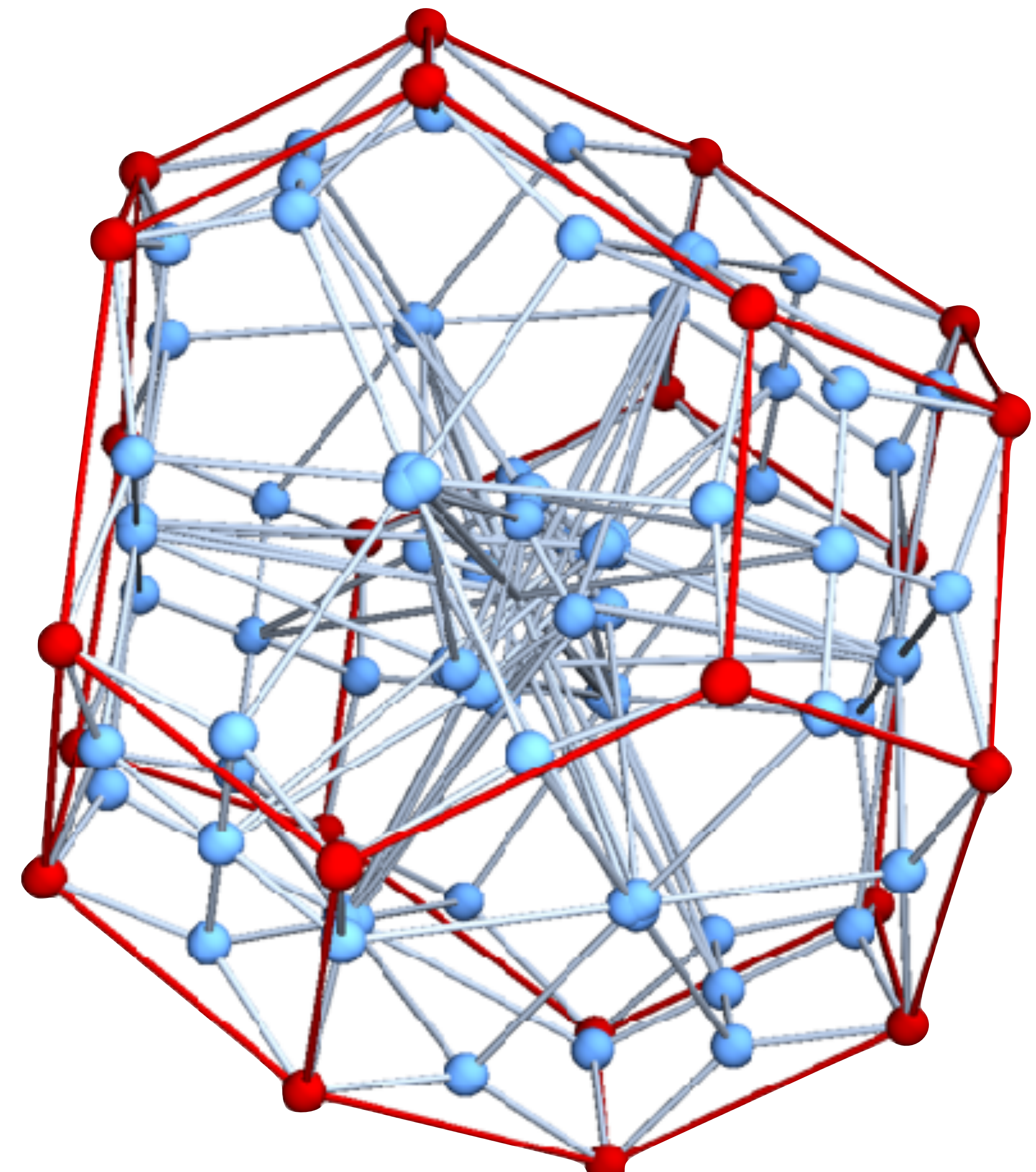


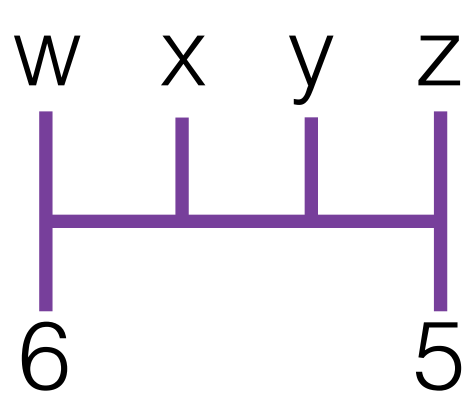


set of masters

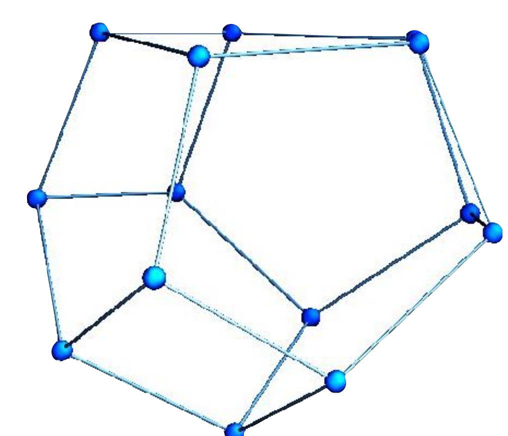


full amplitude



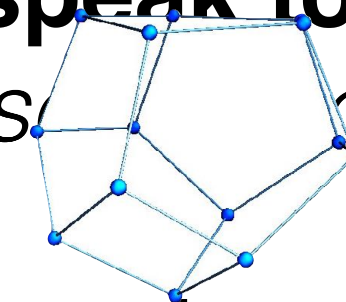


masters fixed by 6

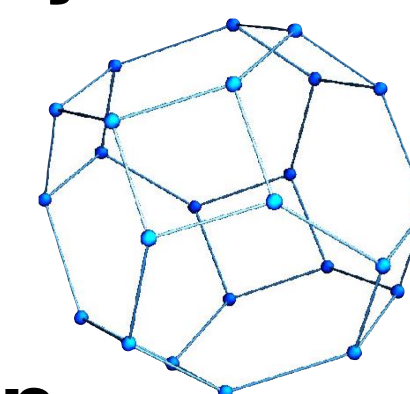


TREE-LEVEL SUMMARY

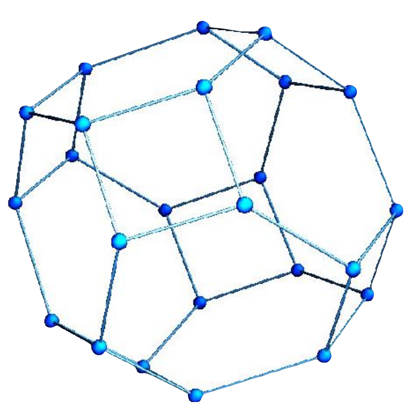
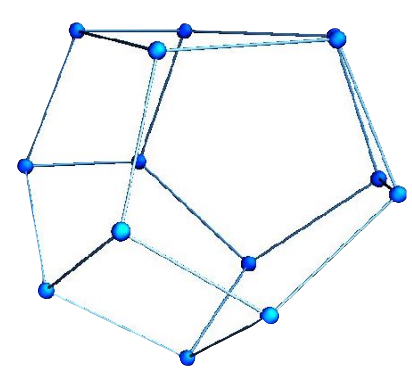
1. **Gauge invariant building blocks that speak to the theory:** color-ordered amplitudes, *associahedra*



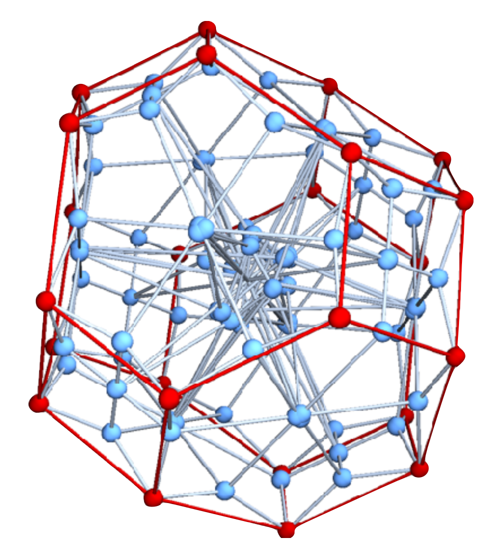
2. **CK means only need to specify the boundary data**: the master graphs, given by the relevant *permutahedron*



3. **Can solve for the *full amplitude efficiently* in terms of the (n-3)! independent *associahedra***

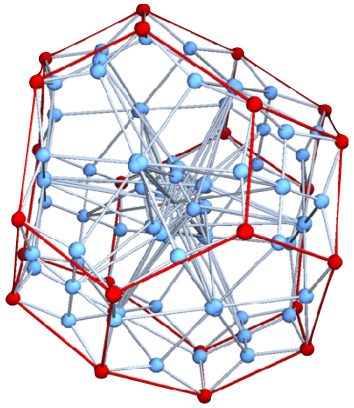
 = f()

(linear)



physics <—> geometry

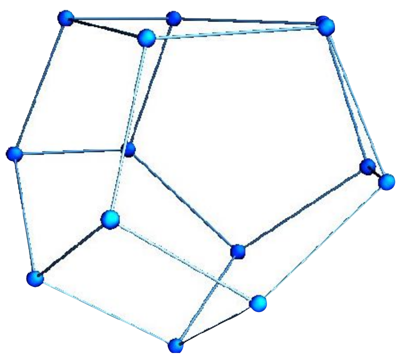
Full YM:



color $\otimes$ spin-1

$$\mathcal{A}_m^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{c(\mathcal{G})\,n(\mathcal{G})}{D(\mathcal{G})}$$

color-stripped YM

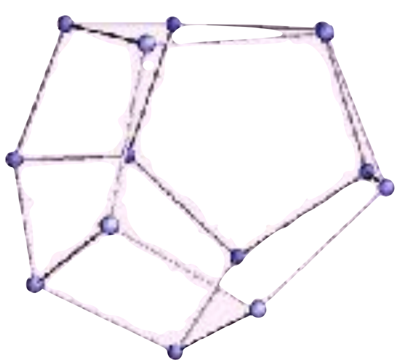


$$\mathbf{A}_{\mathbf{m}}^{\text{tree}}(\rho) = \sum_{\mathcal{G}\in\rho} \frac{\mathbf{n}(\mathcal{G})}{\mathbf{D}(\mathcal{G})}$$

(*same* as kinematic-stripped gravity

$$-iM_n^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$
)

kinematic-stripped YM



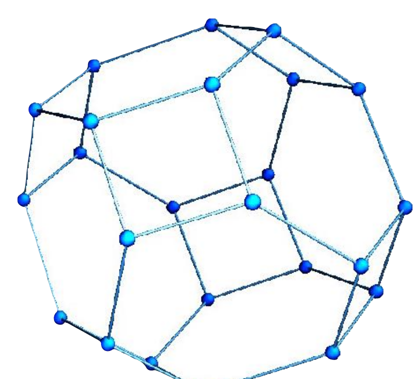
$$\mathbf{C}_{\mathbf{m}}^{\text{tree}}(\rho) = \sum_{\mathcal{G}\in\rho} \frac{\mathbf{c}(\mathcal{G})}{\mathbf{D}(\mathcal{G})}$$

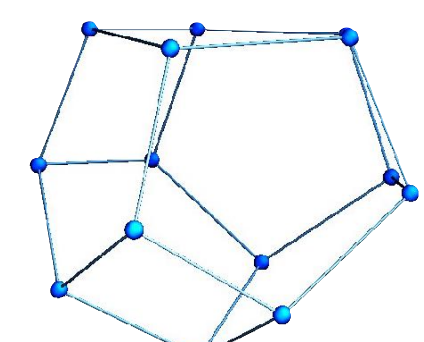
(*same* as color-stripped Bi-Adjoint Scalar

$$\mathcal{C}_{\mathbf{m}}^{\text{tree}}(\rho) = \sum_{\mathcal{G}\in\text{cubic}} \frac{\mathbf{c}(\mathcal{G})\tilde{\mathbf{c}}(\mathcal{G})}{\mathbf{D}(\mathcal{G})}$$
)

Can (pseudo) invert:

$$\mathbf{n}(\mathcal{G}) = \sum_{\rho} \mathbf{D}(\mathcal{G}|\rho)\,\mathbf{A}(\rho)$$

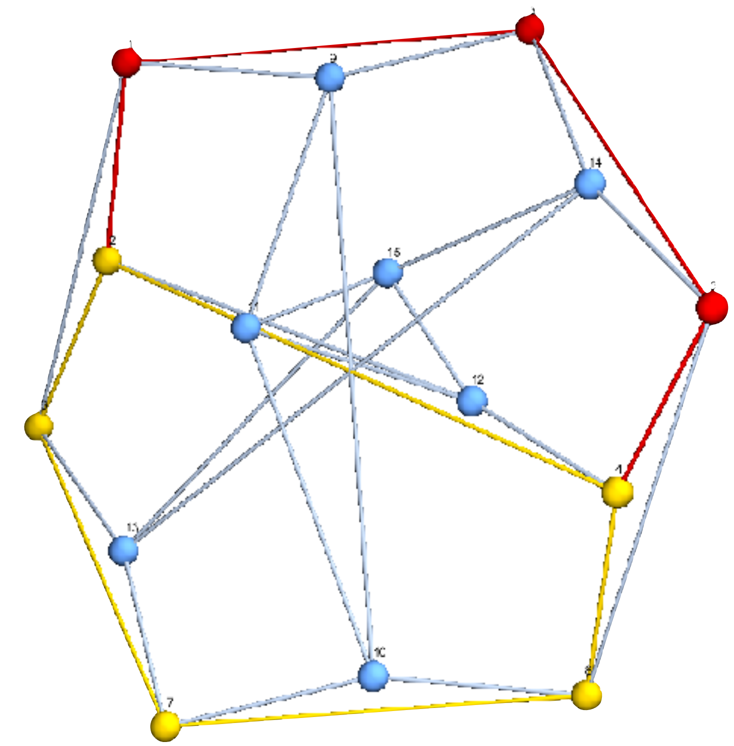


= f()

(linear)

Can only (pseudo) invert iff A(1,2, σ) aren't independent

$$\mathbf{n}(\mathcal{G}) = \sum_{\rho} \mathbf{D}(\mathcal{G}|\rho)\mathbf{A}(\rho)$$



This means additional relations giving (n-3)! BCJ relations:

$$A(1,2,\sigma) = \sum_{\sigma,\rho} f_{\sigma,\rho} A(1,2,\rho,n)$$

If assume A's proportional to gen. Park-Taylor factors can derive the scattering equations.

$$E_a := \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{a\,b}}{\sigma_a - \sigma_b} = 0, \quad \forall a \in \{1,2,\ldots,n\}.$$

Foundation of the powerful and elegant CHY formalism.

High energy strings: Gross, Mende

4D connected prescription twistor strings:
Witten ; Roiban, Spradlin, Volovich

D-dimensions YM+Grav+....
Cachazo, He, Yuan

Bern, JJMC, Johansson (2008)

color-kinematics ⟶ KLT-type relations

$$\mathcal{M}_m^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

$$= \sum_{\mathbf{g}\in\text{cubic},\rho,\tau} \frac{(\mathbf{D}(\mathbf{g},\rho)\mathbf{A}(\rho))(\mathbf{D}(\mathbf{g},\tau)\tilde{\mathbf{A}}(\tau))}{\mathbf{D}(\mathbf{g})}$$

$$= \sum_{\rho,\tau} \mathbf{A}(\rho) \left(\sum_{\mathbf{g}\in\text{cubic}} \frac{\mathbf{D}(\mathbf{g},\rho)\mathbf{D}(\mathbf{g},\tau)}{\mathbf{D}(\mathbf{g})} \right) \tilde{\mathbf{A}}(\tau))$$

$$= \sum_{\rho,\tau} \mathbf{A}(\rho)\mathbf{S_0}(\rho|\tau)\tilde{\mathbf{A}}(\tau)$$

Field theory KLT-type matrix
/ momentum kernel

Bern, Dixon, Perelstein, Rozowsky (1999)

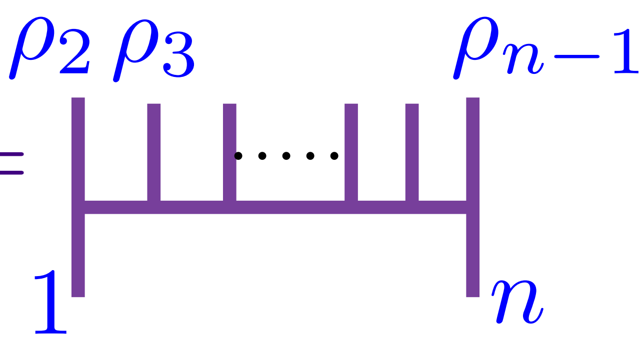
Bjerrum-Bohr, Damgaard, Feng, Sondergaard (2010)

Bjerrum-Bohr, Damgaard,Sondergaard,Vanhove (2011)

KLT-type relations ⟶ color-kinematics

$$\mathcal{A}^{\text{tree}}_{\mathbf{m}}(\rho) = \sum_{\mathcal{G}\in\text{cubic}} \frac{\mathbf{n}(\mathcal{G})\mathbf{c}(\mathcal{G})}{\mathbf{D}(\mathcal{G})}$$

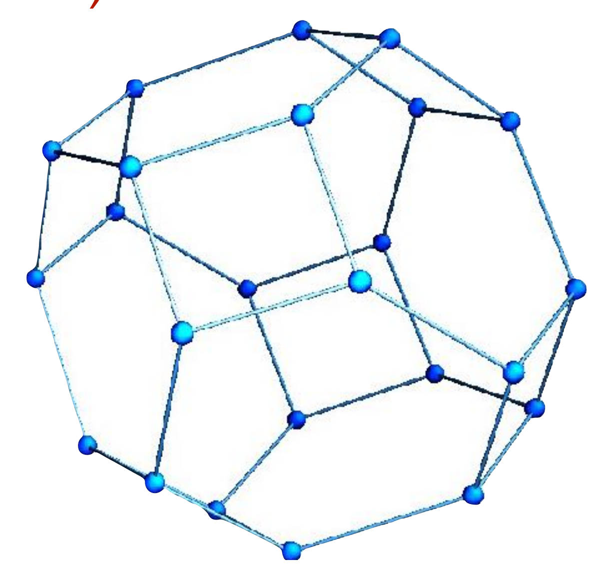
$$= \sum_{\rho,\tau} \mathbf{A}(\rho)\mathbf{S_0}(\rho|\tau)\mathbf{C}(\tau)$$

$c(\rho) =$ 

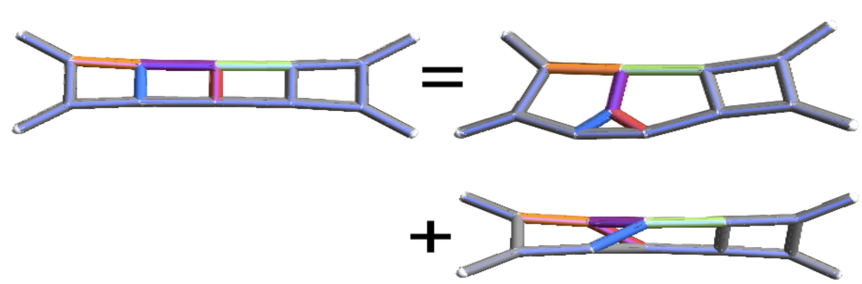
$$= \sum_{\rho} A(\rho)c(\rho)$$

Del Duca, Dixon, Maltoni (1999)

color weights of permutahedron:



relies only on color-Jacobi satisfaction



$$c(\rho) = \sum_{\tau} S_0(\rho|\tau)C(\tau)$$

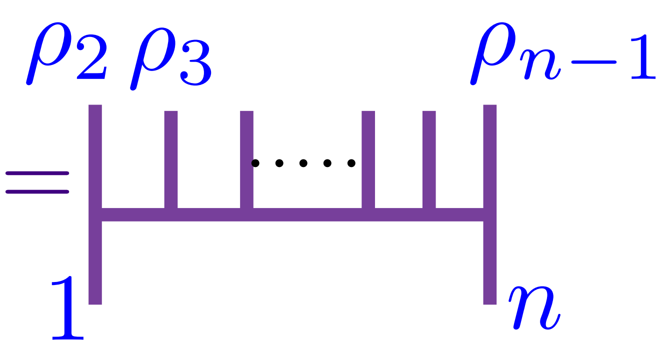
$$D(g(\rho)|\tau) = S_0(\rho|\tau)$$

KLT-type relations → color-kinematics

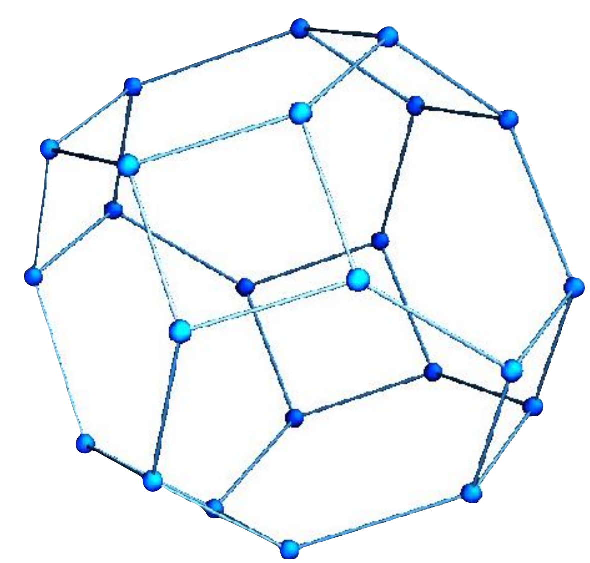
$$\mathcal{M}^{\text{tree}}_{\mathbf{m}}(\rho) = \sum_{\mathcal{G}\in\text{cubic}} \frac{\mathbf{n}(\mathcal{G})\tilde{\mathbf{n}}(\mathcal{G})}{\mathbf{D}(\mathcal{G})}$$

$$= \sum_{\rho,\tau} \mathbf{A}(\rho)\mathbf{S_0}(\rho|\tau)\tilde{\mathbf{A}}(\tau)$$

$$= \sum_{\rho} A(\rho)\tilde{n}(\rho) \qquad \tilde{n}(\rho) =$$



DDM basis for Gravity!



kinematic weights of permutahedron: relies only on kinematic-Jacobi satisfaction

Closed form (non-local) color-dual numerators:

$$\tilde{n}(\rho) = \sum_{\tau} S_0(\rho|\tau)\tilde{A}(\tau)$$

Kiermier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove (2010)

Can generalize c/k numerators to off-shell multi-loop:

By introducing ansatze. **BCJ; BCDJR; CJ; Bern, Davies, Dennen, Huang, Nohle; Johansson, Ochirov; Mogull, O'Connell; Johanson, Kälin, Mogull; . . .**

Yang **(FIRST 5-loop N=8 SG Calc: Form Factor!!!!)**

By introducing massive over-redundancy in graphs: **JJMC**

By exploiting BRST invariance of pure-spinor superstrings: **Mafra, Schlotterer**

By recycling forward limits & CHY formalism: **He, Schlotterer, Zhang**

Can generalize BCJ amp relns at loops: **Vanhove, Tourkine; Hohenegger, Stieberger; He, Schlotterer; Boels, Isermann**

Can take CHY tree-rep to loop integrand via ambitwistor string:

Adamo, Casali, Skinner; Geyer, Monteiro, Mason, Tourkine; He, Yuan; Baadsgaard, Bjerrum-Bohr, Bourjaily, Damgaard, Feng