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COMBINATORICS *and* TOPOLOGY *of* KAWAI—LEWELLEN—TYE RELATIONS¹

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¹Based on [\[1610.04230\]](#) and [\[1706.08527\]](#).

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- *Where do KLT relations come from?* In trying to answer this question, one finds a surprising connection between string theory amplitudes and the mathematics of *twisted cycles*.

**A RELATION BETWEEN TREE AMPLITUDES
OF CLOSED AND OPEN STRINGS***

H. KAWAI, D.C. LEWELLEN and S.-H.H. TYE

Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853, USA

Received 11 October 1985

**INTERSECTION THEORY FOR TWISTED COHOMOLOGIES
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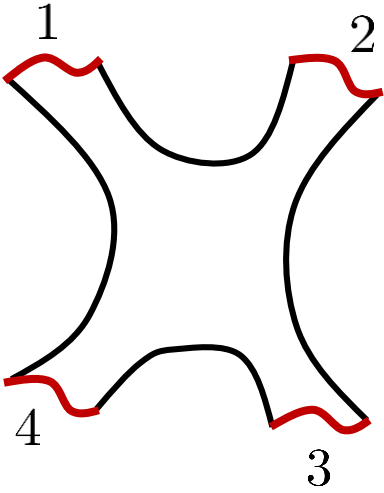
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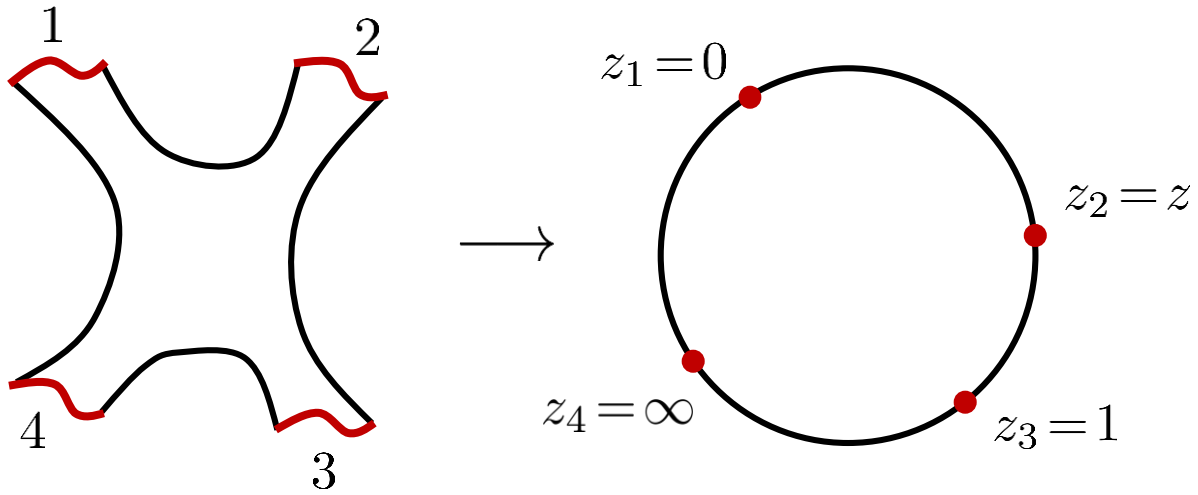
The goal for this talk:

How to understand KLT relations in terms of combinatorics and topology.

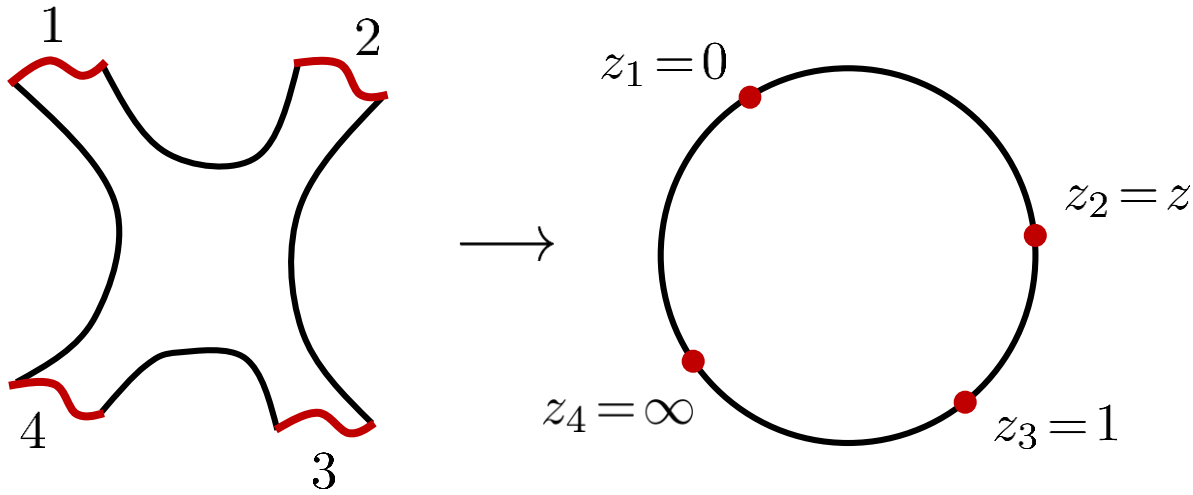
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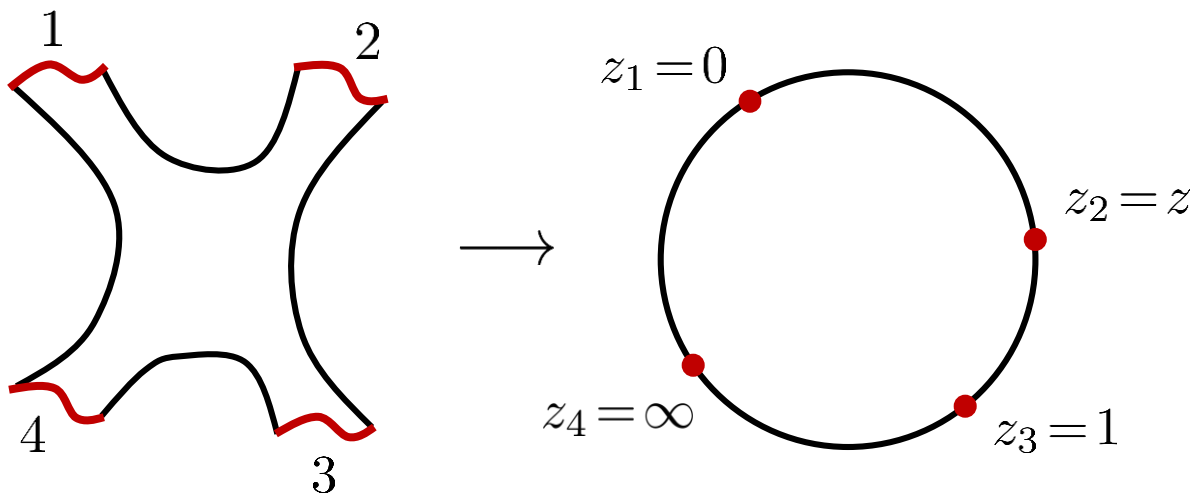


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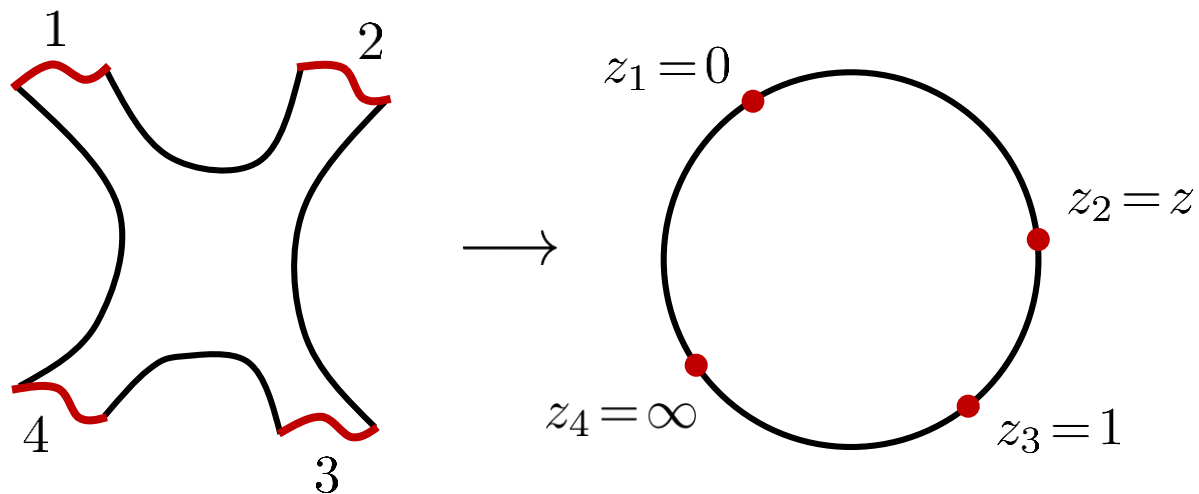


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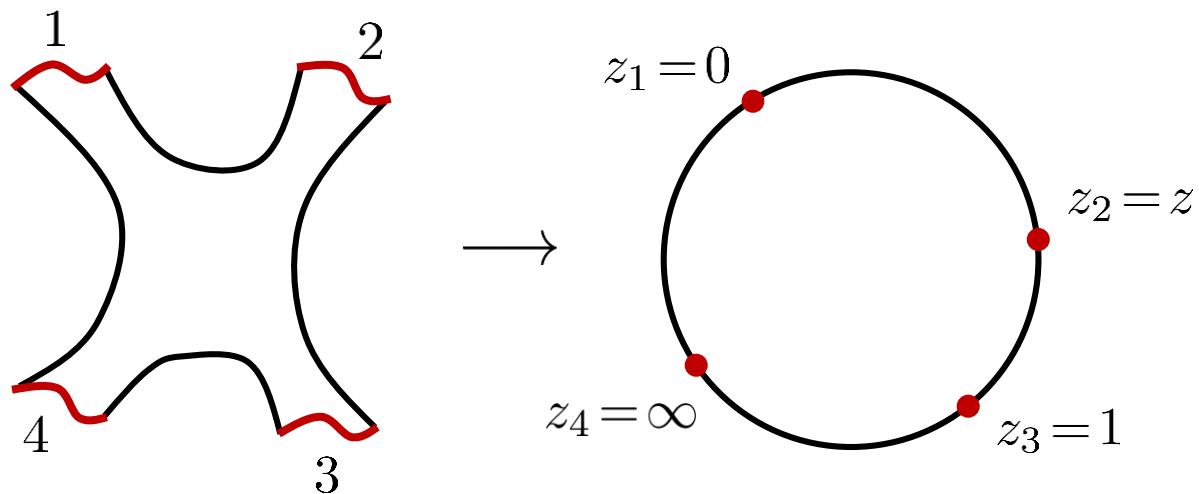
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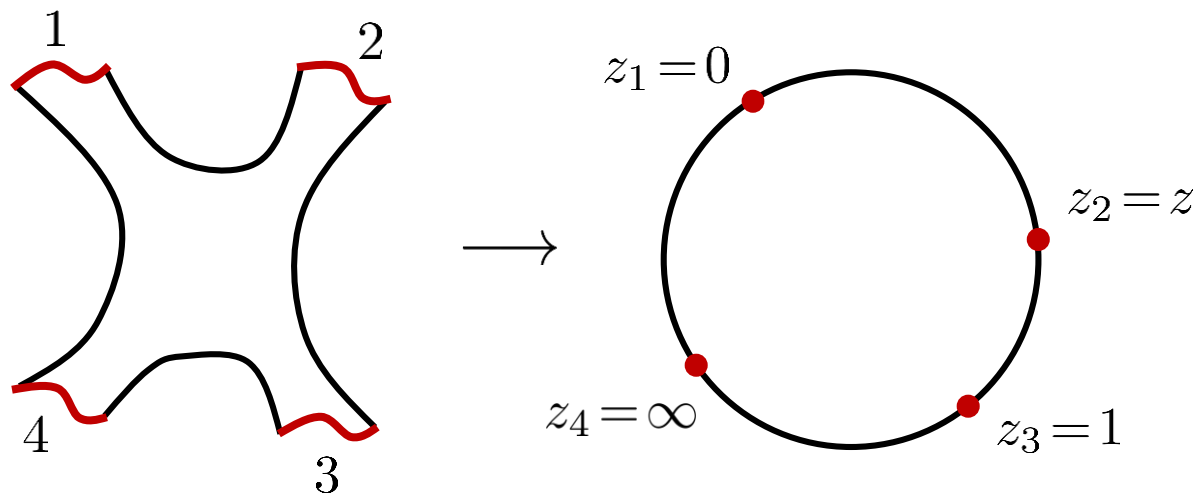
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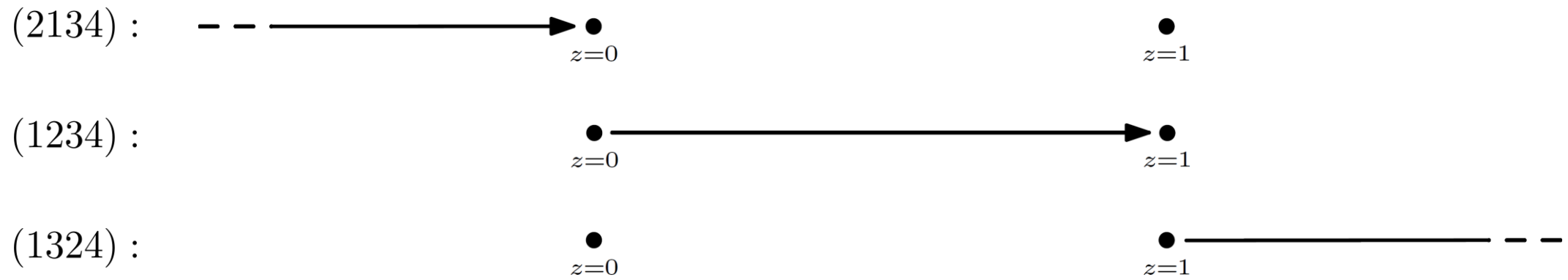
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- Topological cycle: $\overrightarrow{(0, 1)}$
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- Then an open string amplitude is given as a pairing between a twisted cycle $C(1234)$ and a twisted cocycle $\varphi(z)$, e.g.,

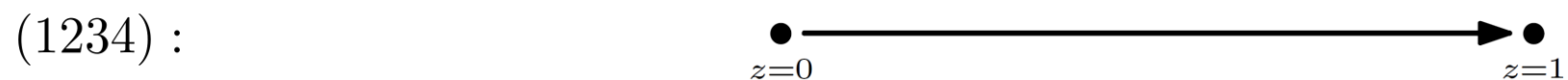
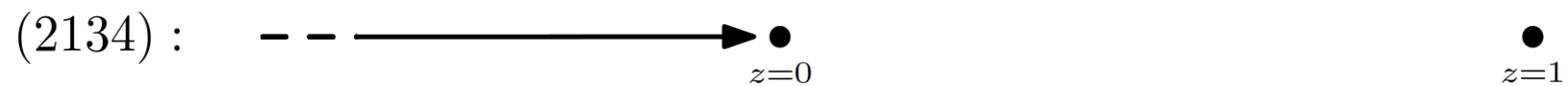
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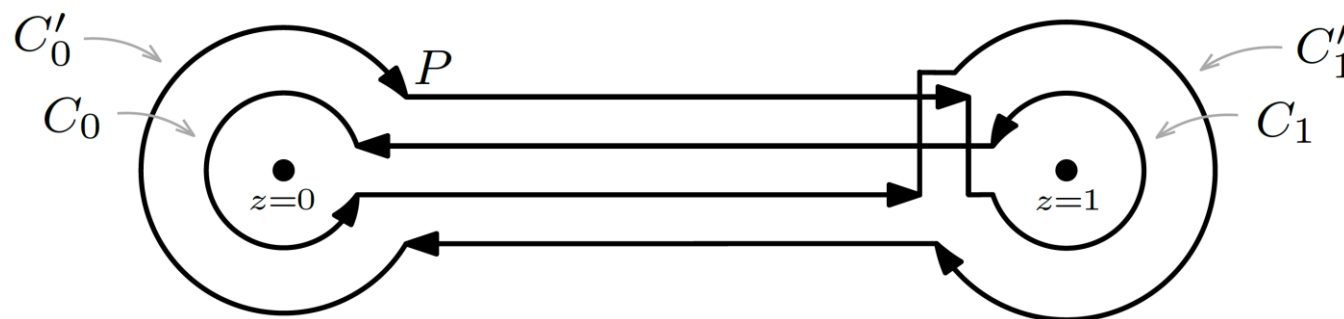
- Twisted cycles live in the real section of the moduli space $\mathcal{M}_{0,4} = \mathbb{C} \setminus \{0, 1\}$.

- It turns out we can compute invariants between two twisted cycles called *intersection numbers*.

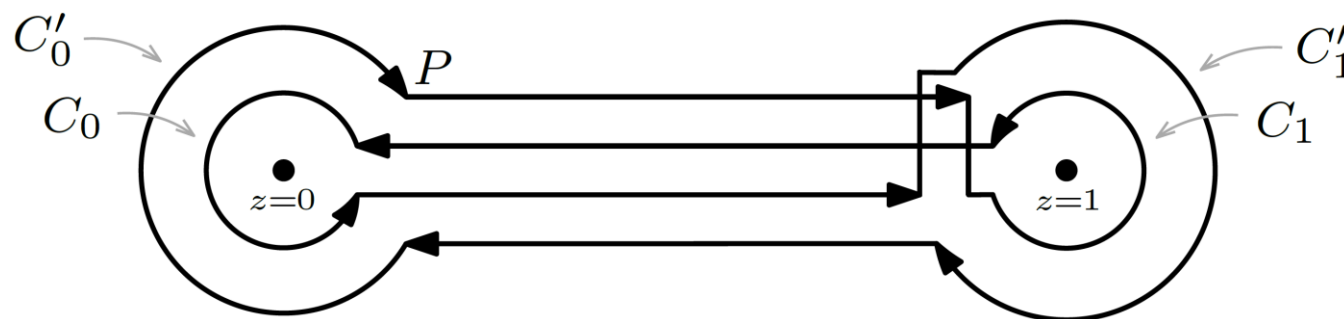
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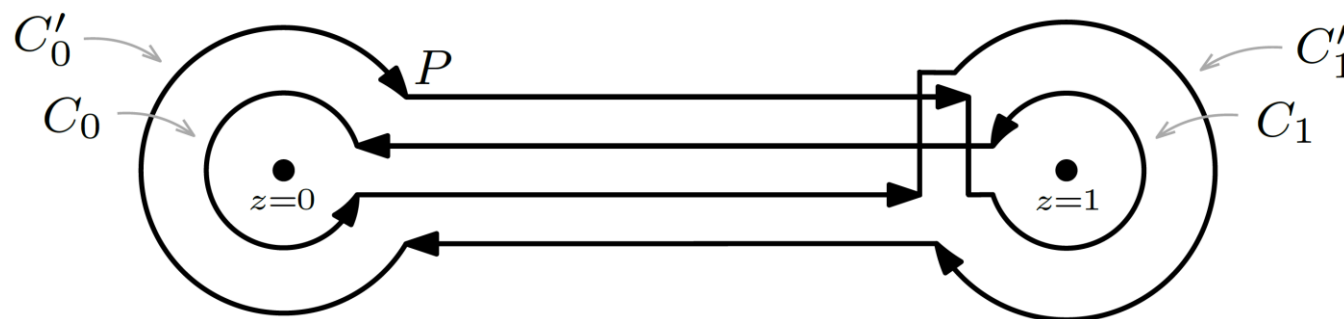


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$$\oint_{\Gamma} z^s (1-z)^t \varphi(z) = \left(1 - e^{2\pi i t} + e^{2\pi i (s+t)} - e^{2\pi i s} \right) \int_{0 < z < 1} z^s (1-z)^t \varphi(z),$$

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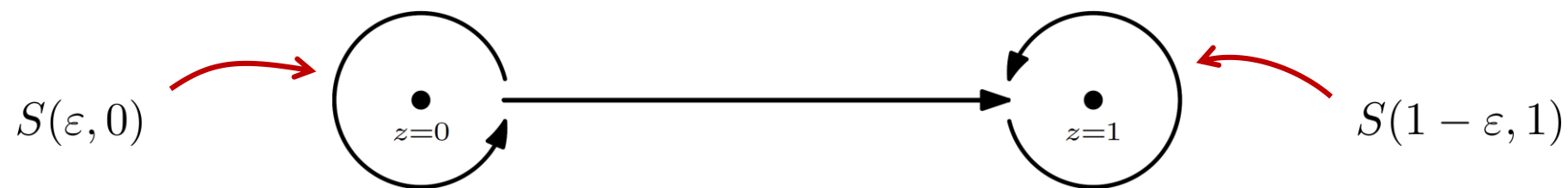
- Or, in other words: $\text{reg } \overrightarrow{(0, 1)} = \frac{\Gamma}{(e^{2\pi i s} - 1)(e^{2\pi i t} - 1)}$

- In fact, it is more convenient to split it into three contributions:

$$\text{reg } \overrightarrow{(0, 1)} = \frac{\Gamma}{(e^{2\pi i s} - 1)(e^{2\pi i t} - 1)} = \frac{S(\varepsilon, 0)}{e^{2\pi i s} - 1} + \overrightarrow{(\varepsilon, 1 - \varepsilon)} - \frac{S(1 - \varepsilon, 1)}{e^{2\pi i t} - 1}$$

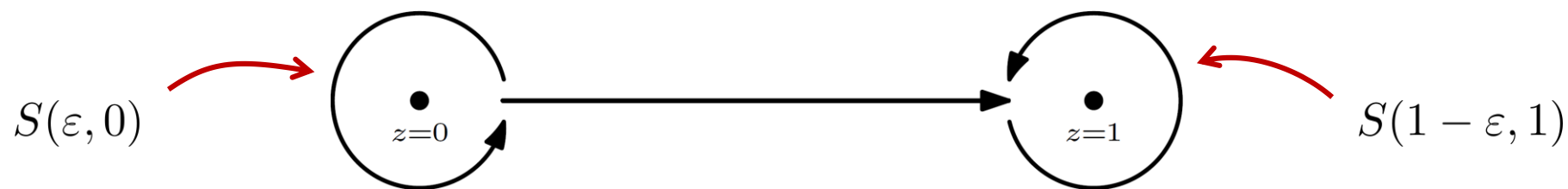
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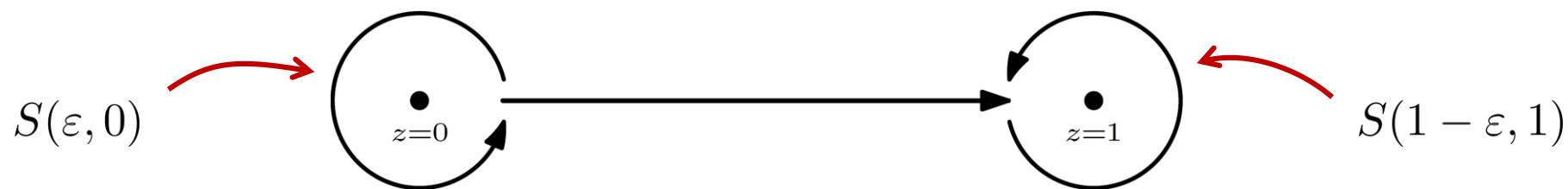
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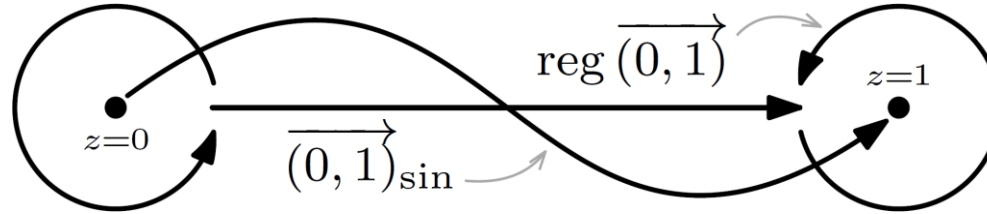


- The resulting regularized twisted cycle is compact. Let's call it $\text{reg } C(1234)$.
- We can now compute intersection numbers of twisted cycles with the rules:

$$\langle \omega_1, \omega_2 \rangle : \quad \begin{array}{c} \omega_2 \nearrow \nwarrow \omega_1 \\ \nwarrow \nearrow \end{array} = +1 \quad \text{or} \quad \begin{array}{c} \omega_1 \nwarrow \nearrow \omega_2 \\ \nearrow \nwarrow \end{array} = -1$$

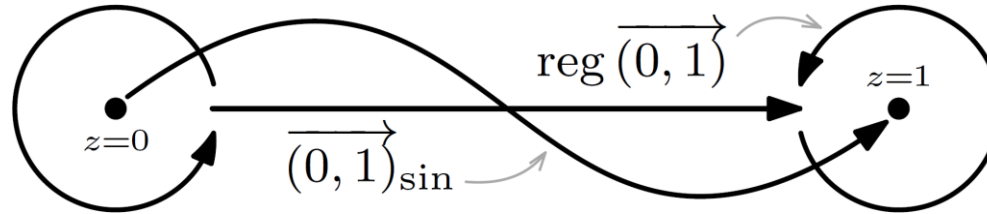
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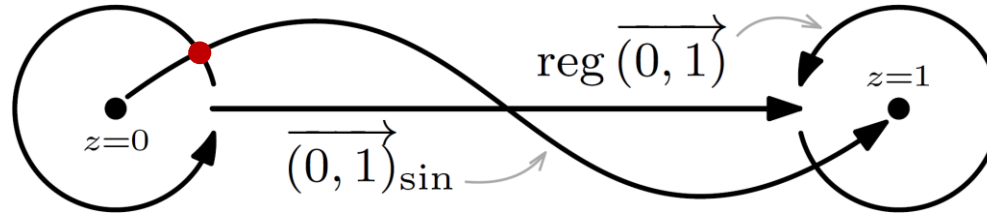
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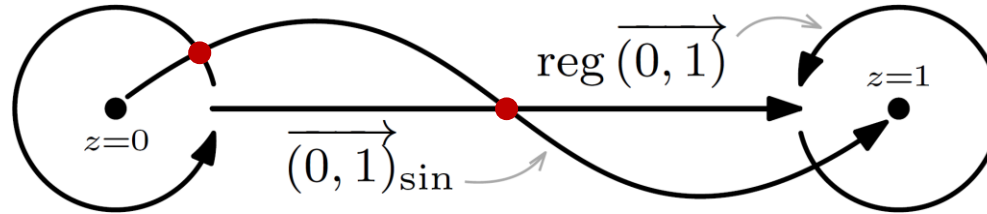


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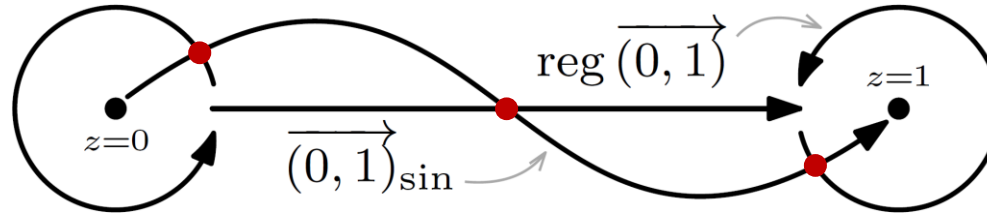


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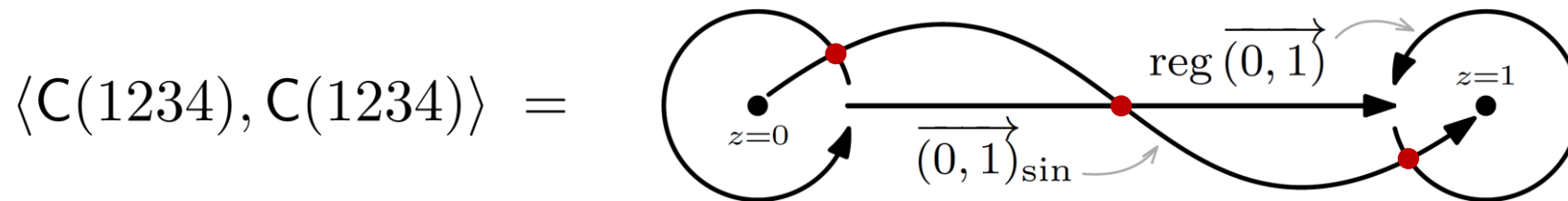
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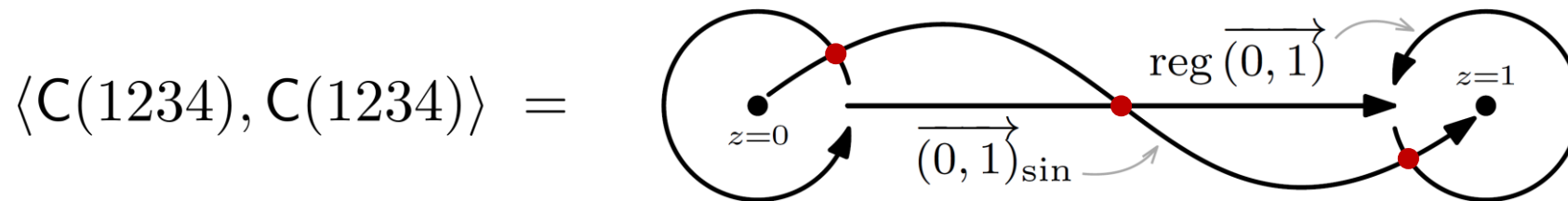
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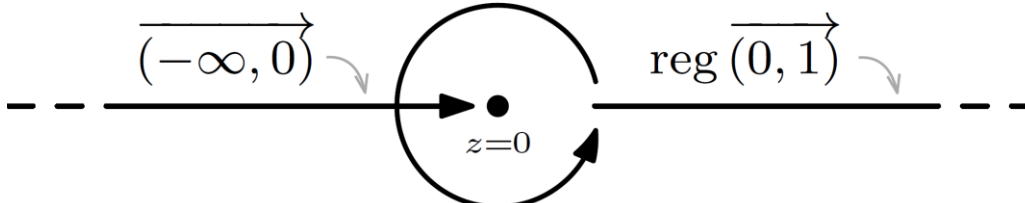


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$$\begin{aligned} \Rightarrow \langle C(1234), C(1234) \rangle &= - \left(\frac{1}{e^{2\pi i s} - 1} + 1 + \frac{1}{e^{2\pi i t} - 1} \right) \\ &= \frac{i}{2} \left(\frac{1}{\tan \pi s} + \frac{1}{\tan \pi t} \right) \end{aligned}$$

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 &= + \frac{e^{\pi i s}}{e^{2\pi i s} - 1} = \frac{i}{2} \left(-\frac{1}{\sin \pi s} \right)
 \end{aligned}$$

The diagram shows a horizontal real axis with a branch cut along the real axis starting from $z=0$. A red dot marks the origin $z=0$. A circular loop encircles the origin, with an arrow indicating a counter-clockwise direction. The region to the left of the origin is labeled $(-\infty, 0)$ with a right-pointing arrow above it. The region to the right of the origin is labeled $\text{reg } (0, 1)$ with a right-pointing arrow above it. Dashed lines extend from the ends of the horizontal axis.

- Similarly, intersection of $C(1234)$ with $C(2134)$:

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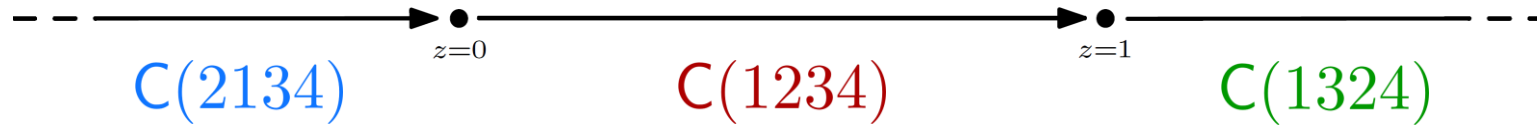
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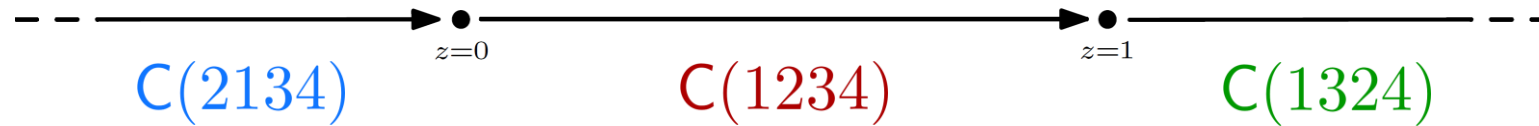
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- To summarize, $\mathcal{M}_{0,4}(\mathbb{R})$ is tiled by three twisted cycles:

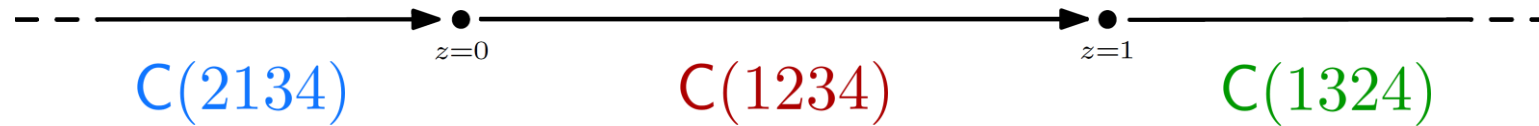


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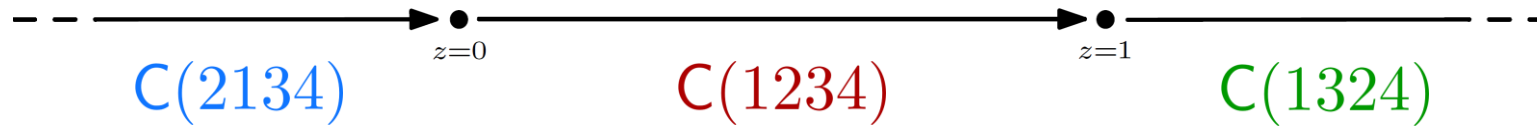
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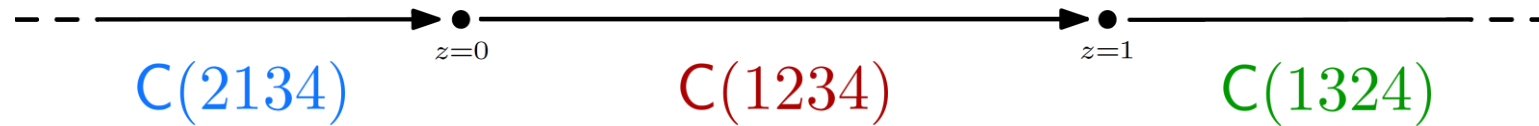


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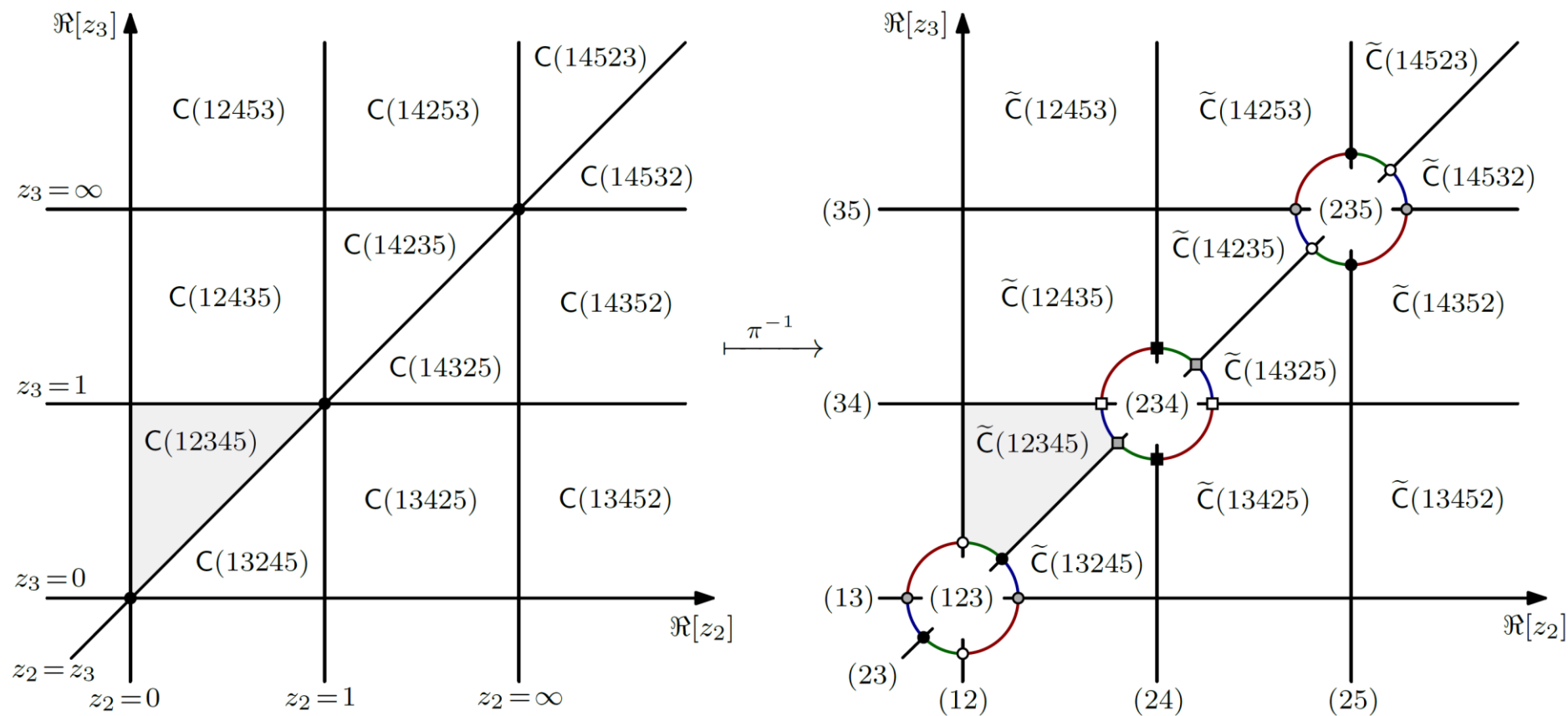
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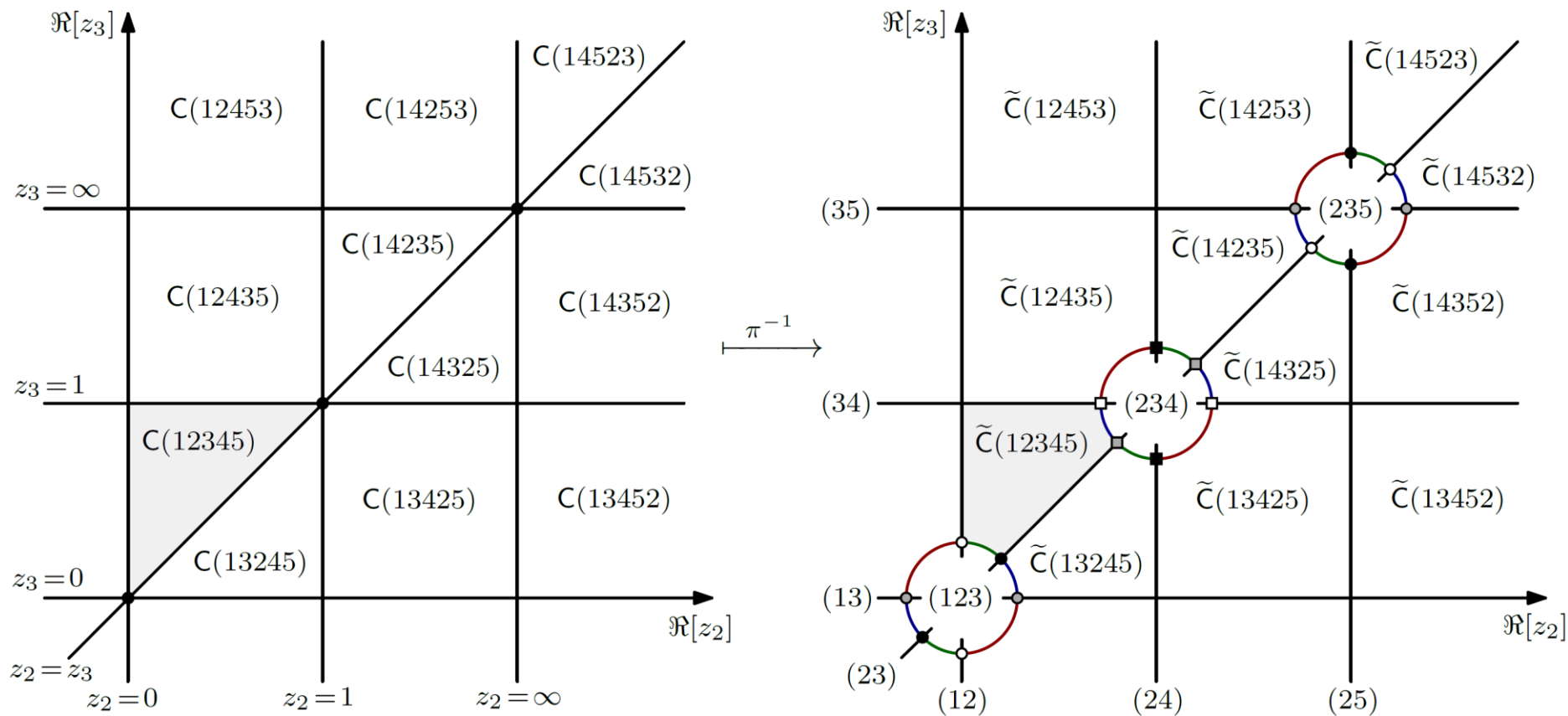
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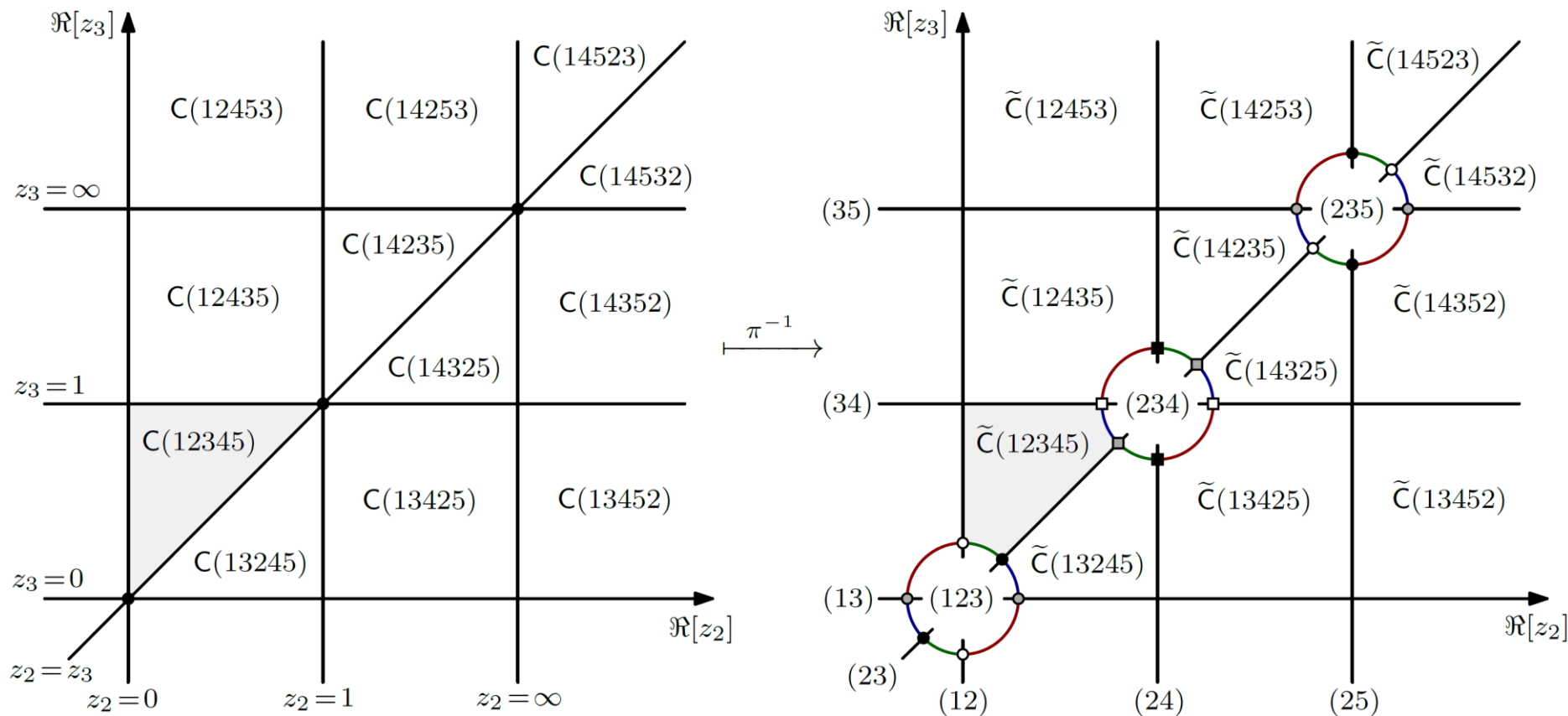


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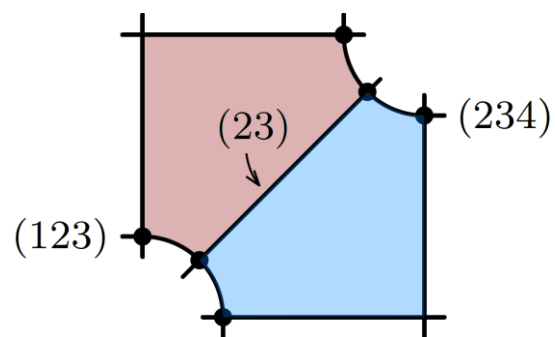


- Before blowup, e.g., $C(12345) = \{0 < z_2 < z_3 < 1\} \otimes \dots$
- After blowup each twisted cycle $\tilde{C}(\beta)$ is a pentagon.

- For higher-point twisted cycle we can calculate intersection numbers by projecting onto subspaces and reducing to previous cases, e.g.,

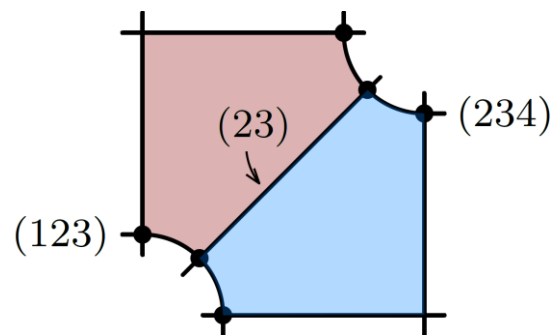
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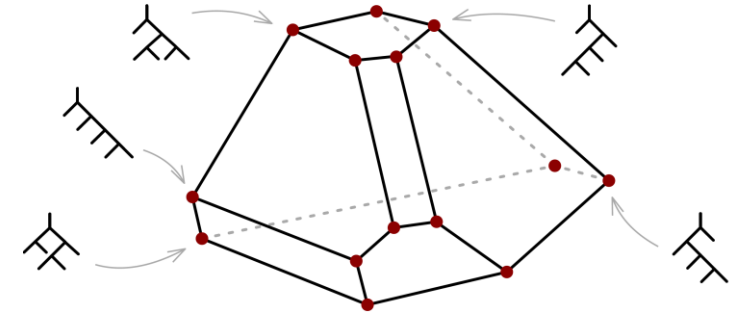
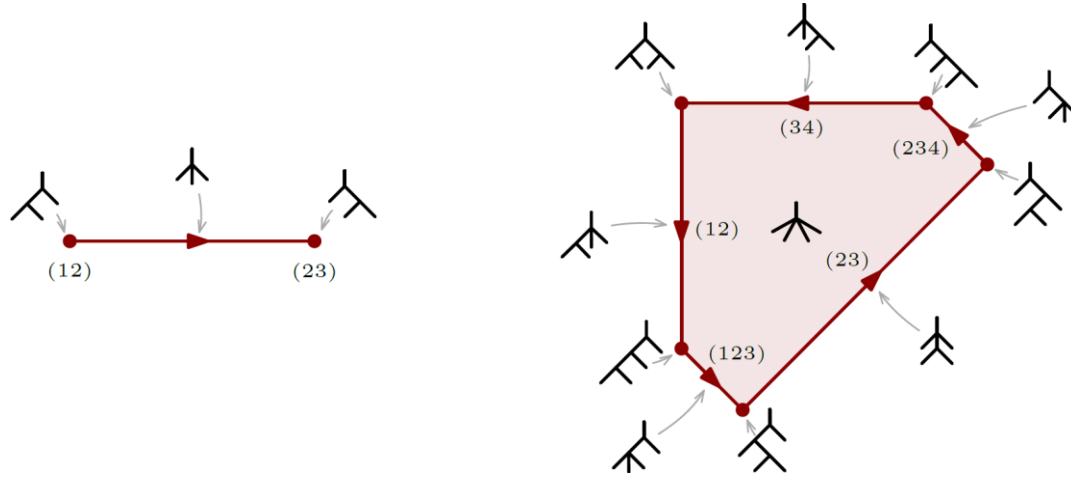
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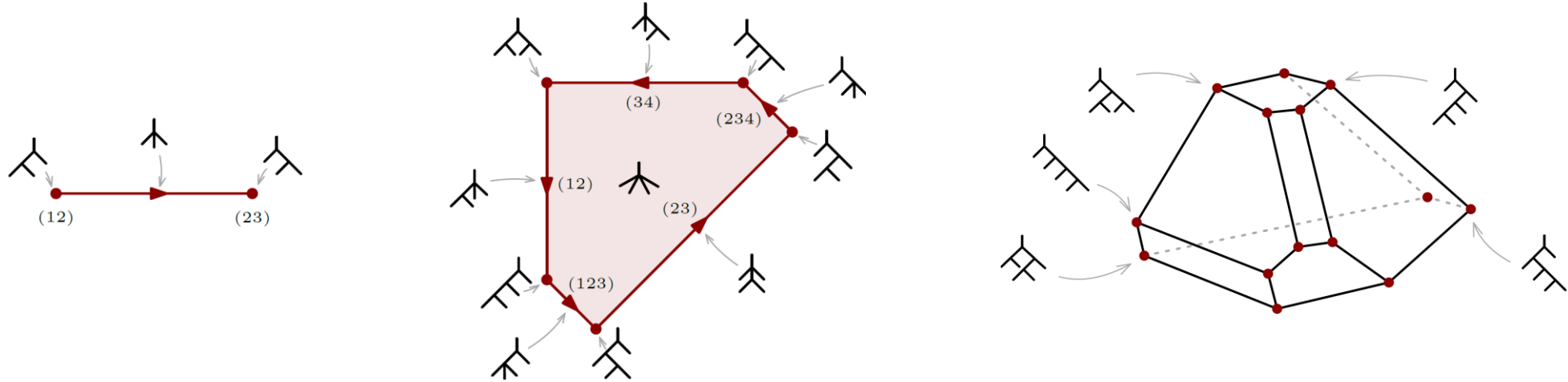
$$= \frac{e^{\pi i s_{23}}}{e^{2\pi i s_{23}} - 1} \left\langle \text{reg } \tilde{\mathbf{C}}(12345) \Big|_{(23)}, \tilde{\mathbf{C}}(13245) \Big|_{(23)} \right\rangle$$

- In general, the moduli space $\widetilde{\mathcal{M}}_{0,n}(\mathbb{R})$ is tiled by $(n-1)!/2$ copies of the so-called *associahedron*, K_{n-1} :
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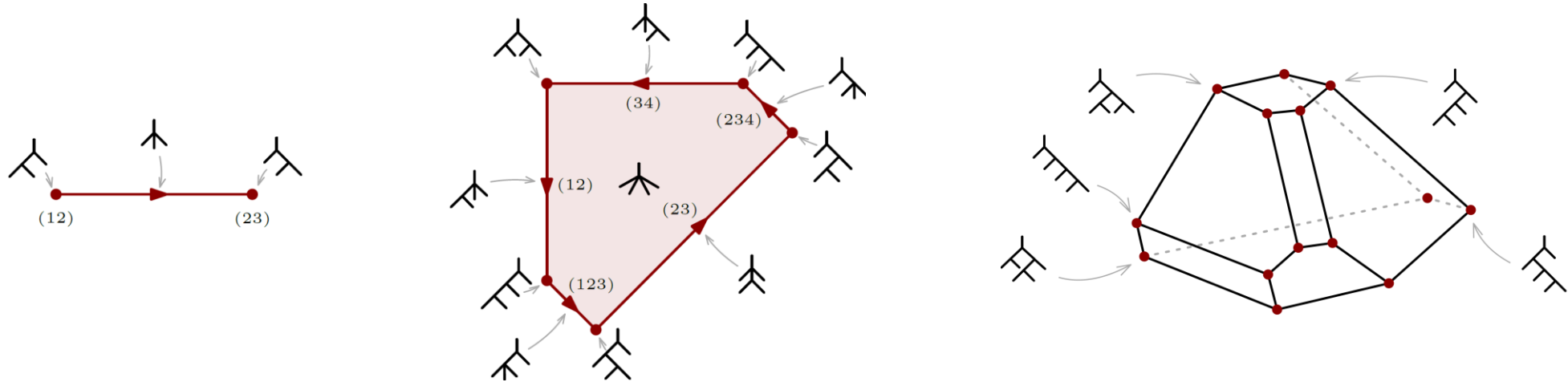


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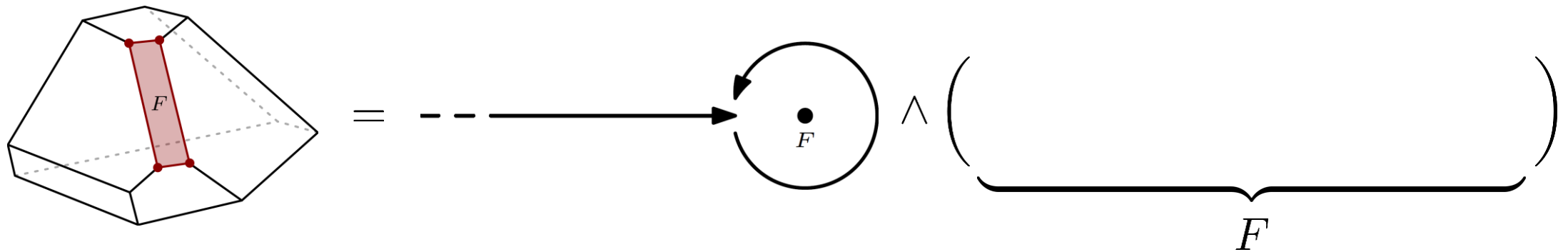


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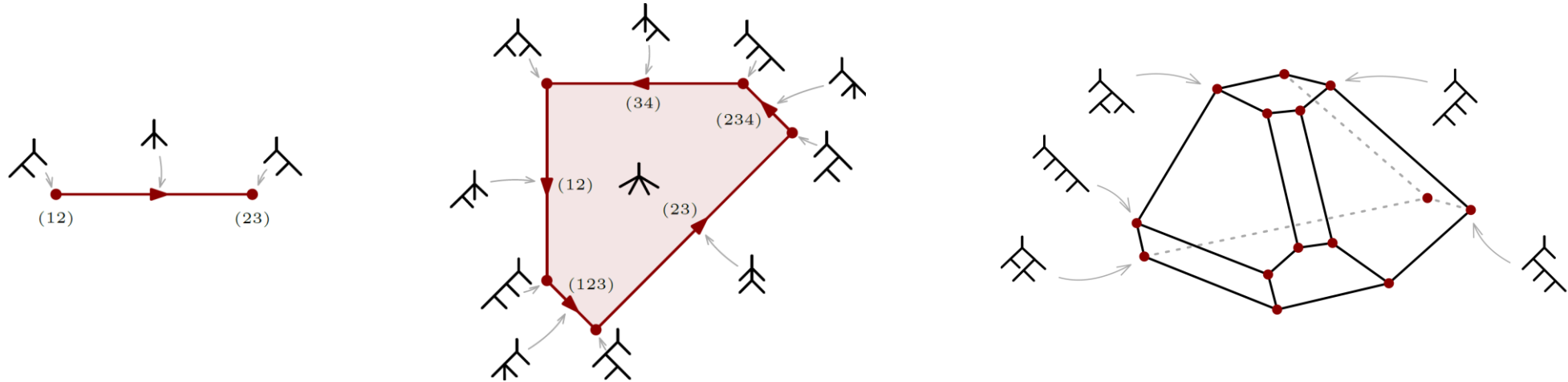
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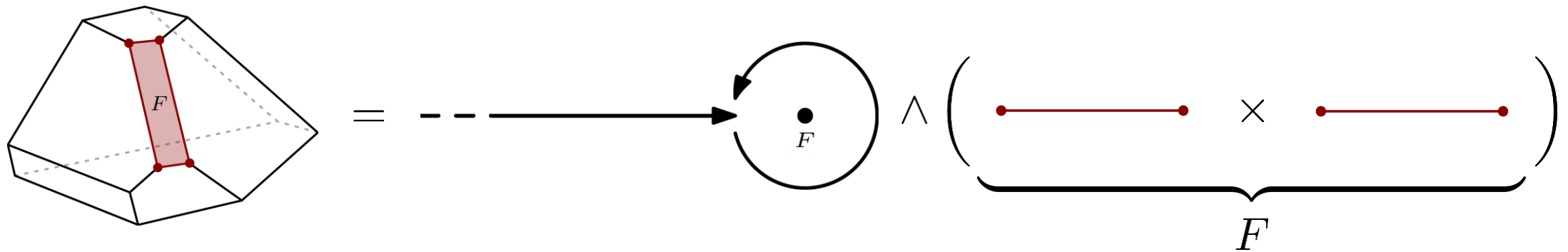
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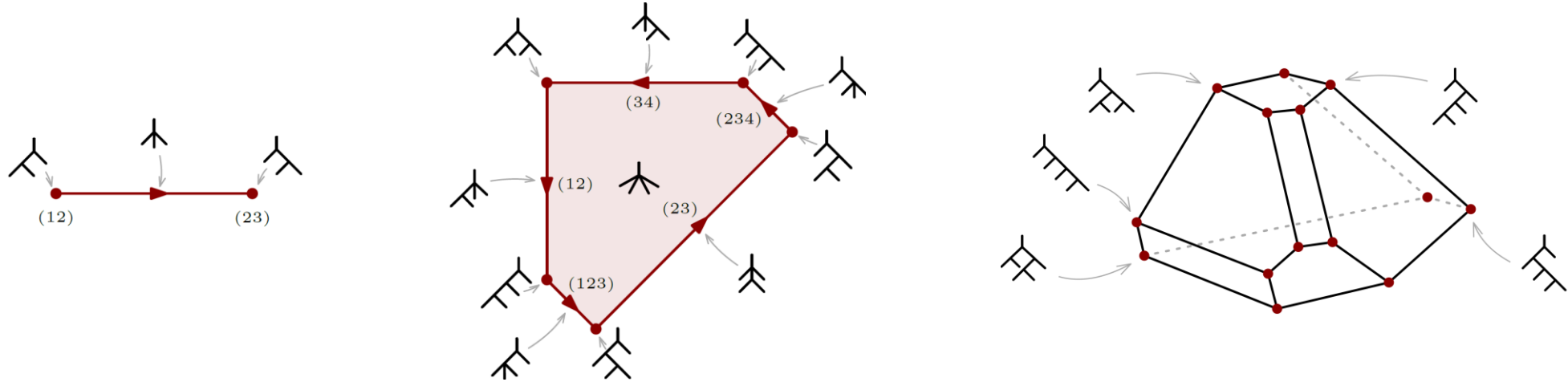
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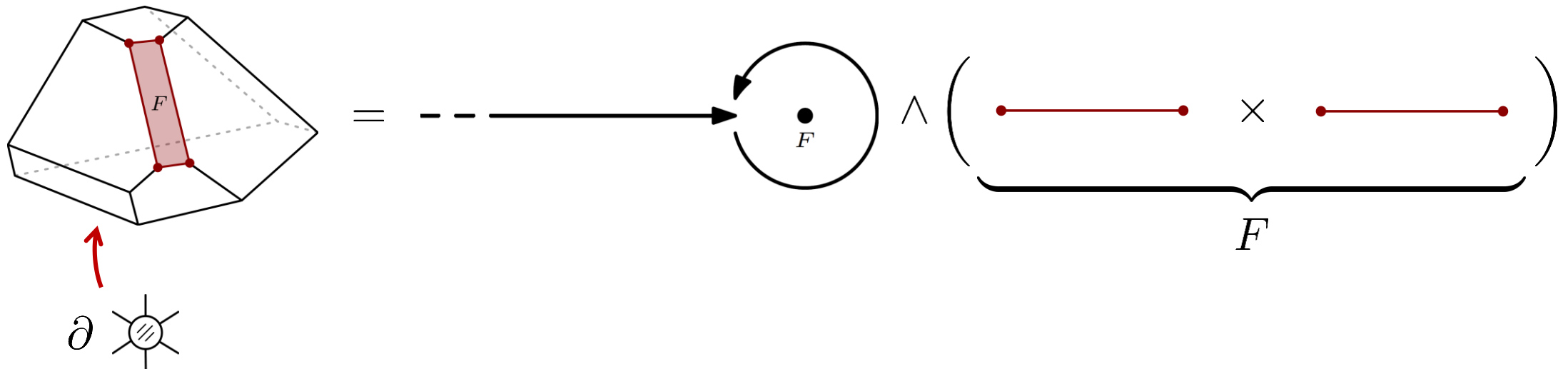
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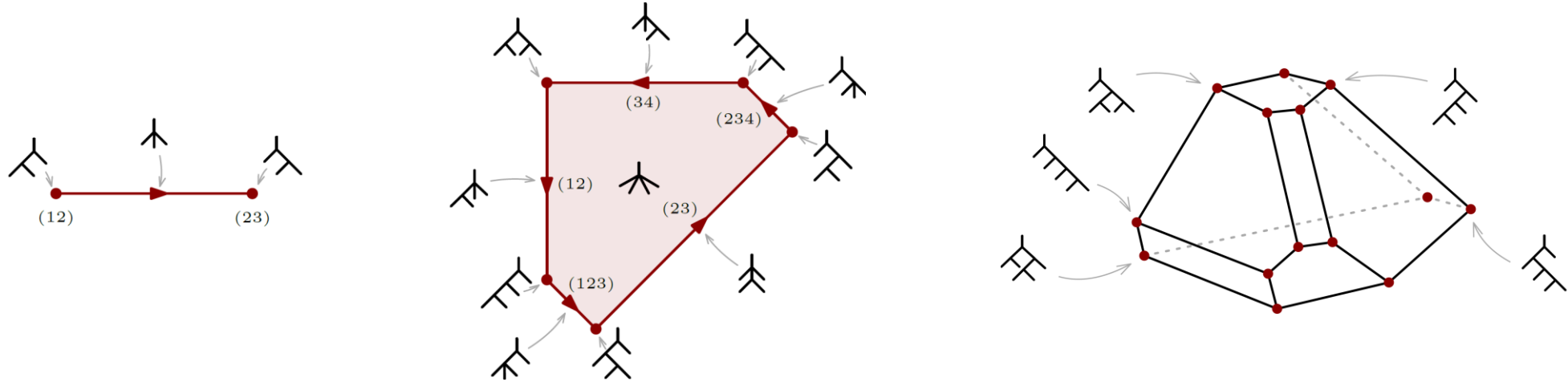
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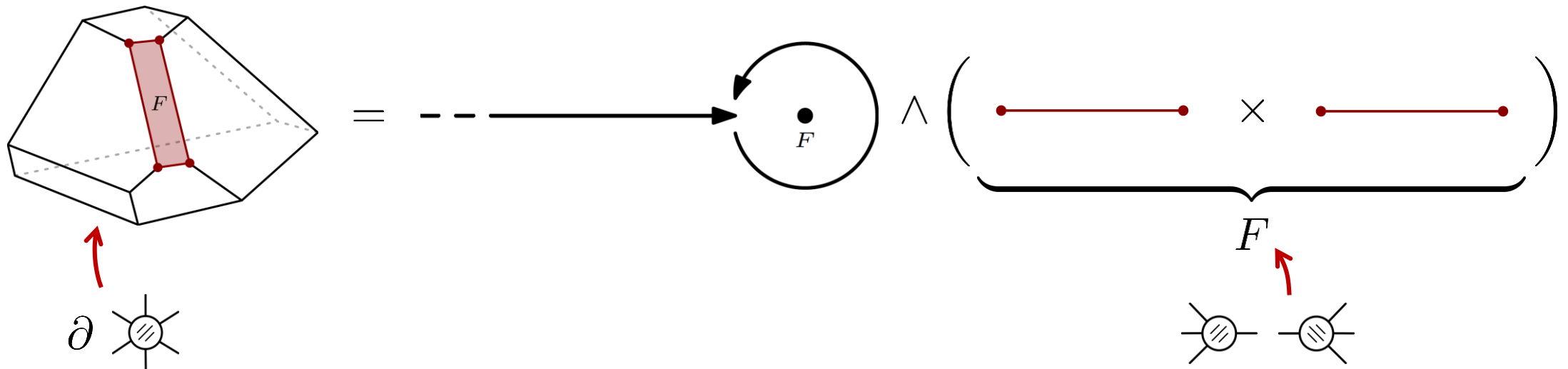
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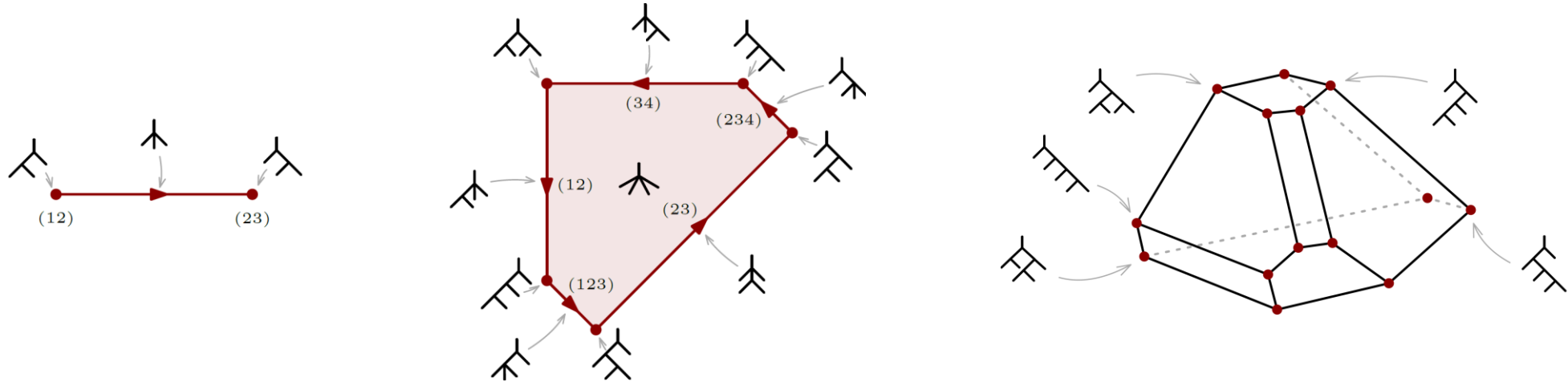
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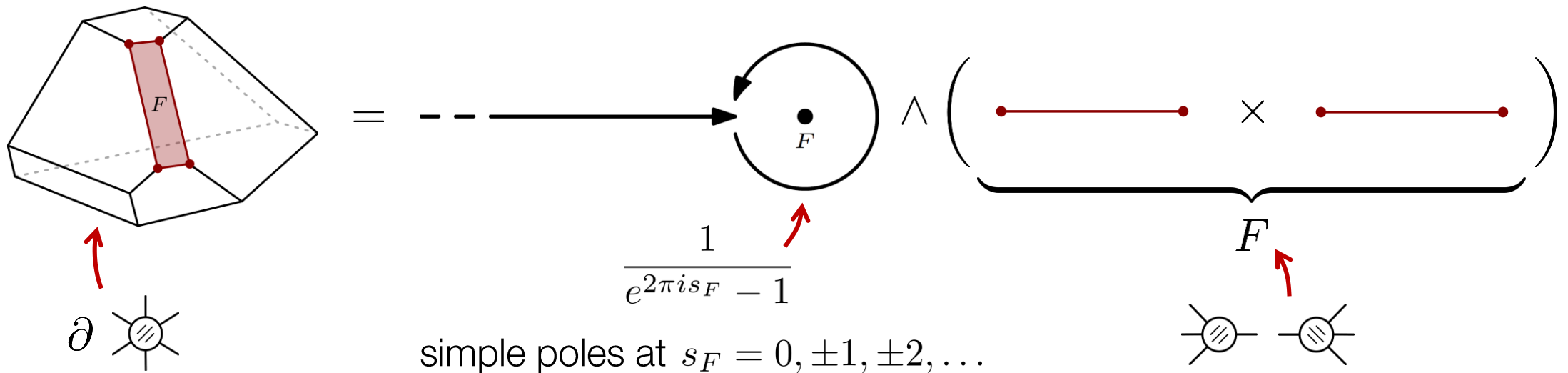
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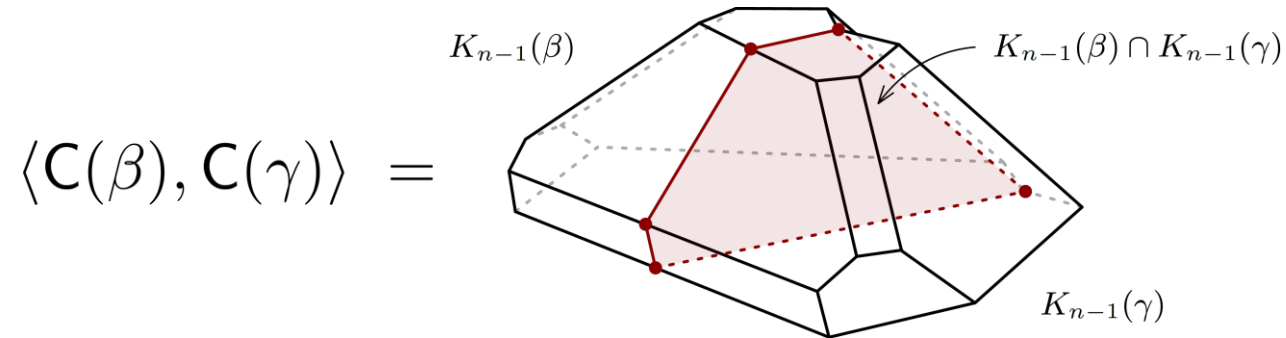


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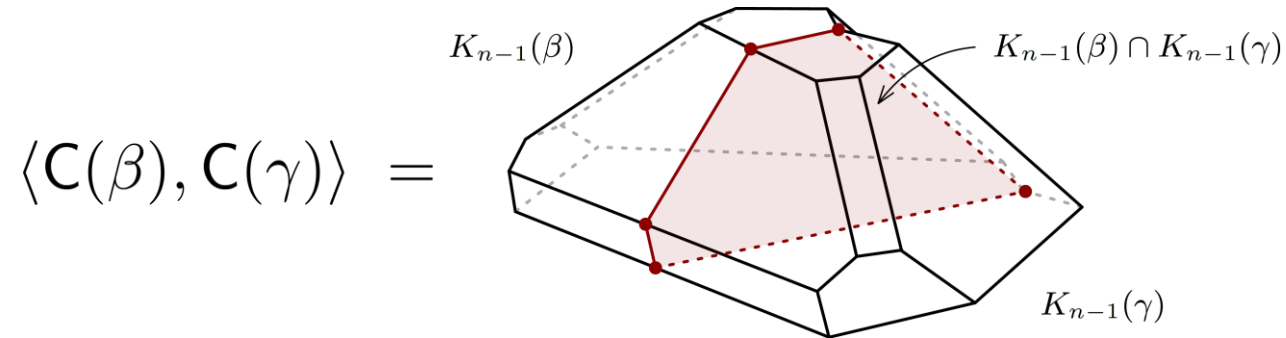


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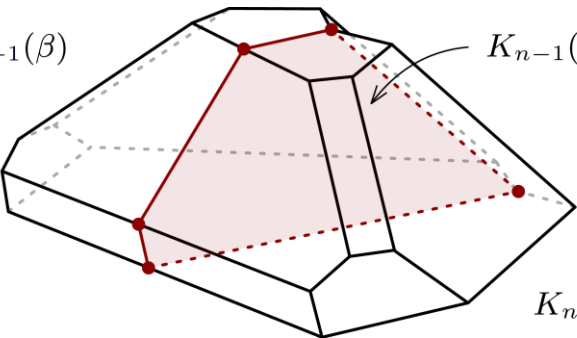
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- Always given as tree diagrams with “propagators” $1/\sin(\pi\alpha'p^2/2)$ and $1/\tan(\pi\alpha'p^2/2)$. But we don't need to actually draw associahedra. Use diagrams instead, e.g.,

[SM 16, 17]

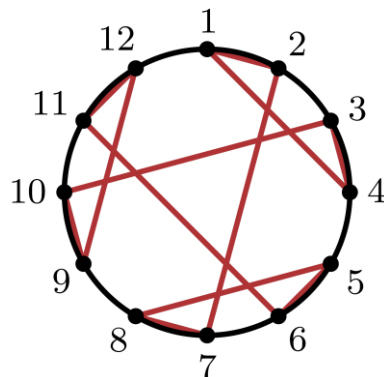
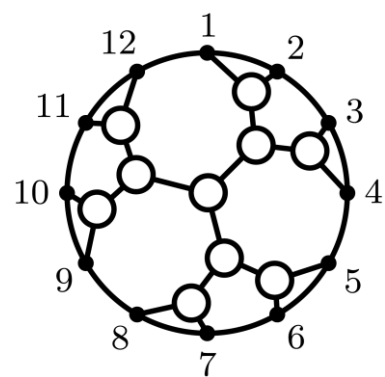
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The diagram illustrates the intersection of two twisted cycles, $K_{n-1}(\beta)$ and $K_{n-1}(\gamma)$, within a polyhedron. The intersection is highlighted in red, and the intersection number is denoted by $\langle C(\beta), C(\gamma) \rangle$.

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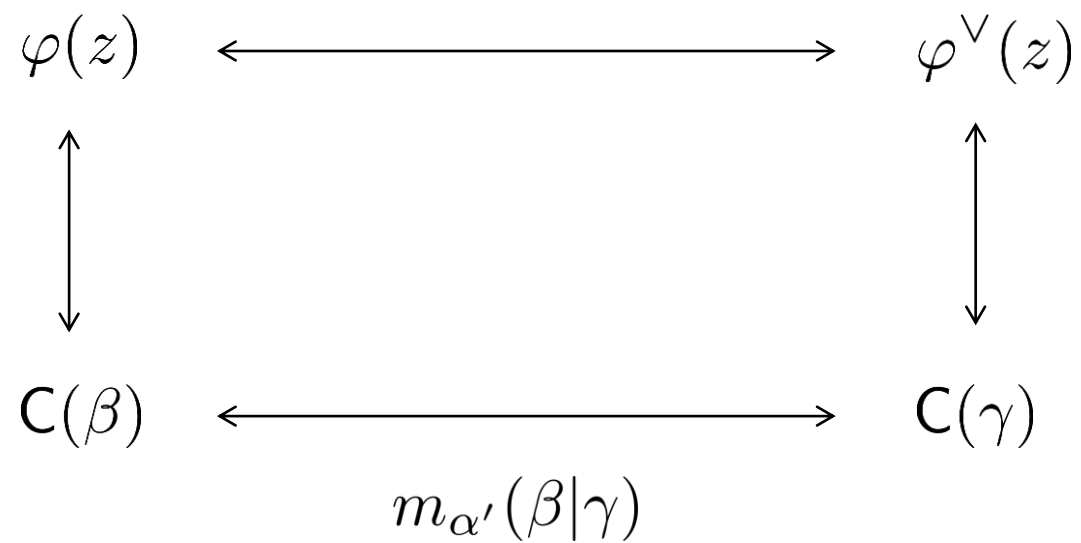
$$= - \frac{1}{\sin \pi s_{12}} \frac{1}{\sin \pi s_{34}} \frac{1}{\sin \pi s_{56}} \frac{1}{\sin \pi s_{78}} \dots$$

The diagram shows a tree diagram with 12 vertices and 11 edges, representing a propagator. The diagram is shown in two forms: a tree diagram and a tree diagram with internal vertices. The diagram is followed by an equals sign and a series of terms representing the propagator.

- There are three types of pairings we can define:

$$\begin{array}{ccc} \varphi(z) & \longleftrightarrow & \varphi^{\vee}(z) \\ \updownarrow & & \updownarrow \\ \mathbf{C}(\beta) & \longleftrightarrow & \mathbf{C}(\gamma) \end{array}$$

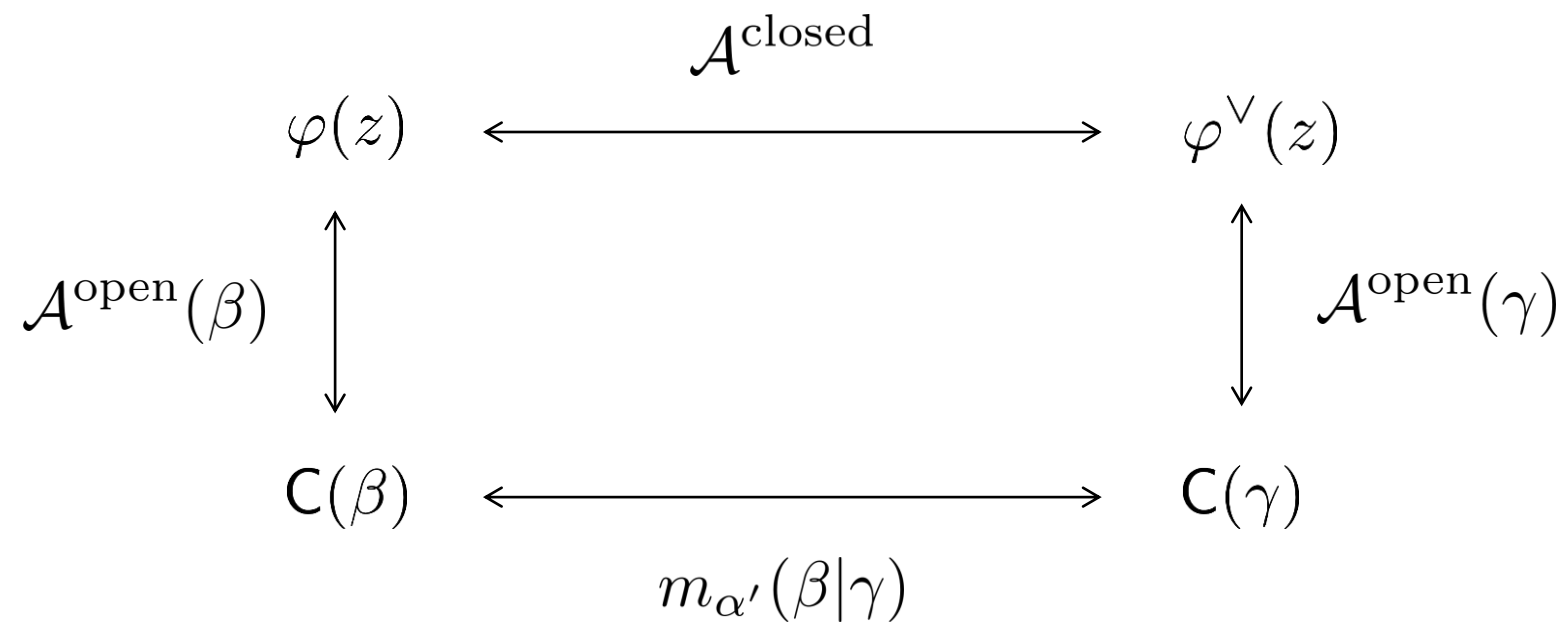
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- They are related by the *twisted period relations*: [Cho & Matsumoto 94, SM 16, 17]

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- These are the Kawai—Lewellen—Tye relations with KLT kernel given by $m_{\alpha'}^{-1}$!

- In the field theory limit, $\alpha' \rightarrow 0$, each facet contributes

$$\begin{array}{c} \text{---} \longrightarrow \end{array} \circlearrowright_F = \frac{1}{e^{2\pi i \alpha' s_F} - 1} \longrightarrow \frac{1}{2\pi i \alpha' s_F} + \dots$$

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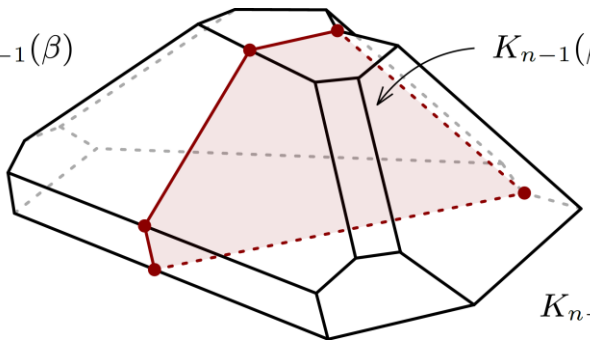
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 bi-adjoint scalar amplitudes

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to colour-kinematics duality [Carrasco later this week]

THANK YOU

REFERENCES

- S. Mizera, *Inverse of the String Theory KLT Kernel*, [[1610.04230](#)]
S. Mizera, *Combinatorics and Topology of Kawai—Lewellen—Tye Relations*, [[1706.08527](#)]