Edinburgh, July 2017

Combinatorics *and* Topology *of* Kawai—Lewellen—Tye Relations[1]

Sebastian Mizera

Perimeter Institute for Theoretical Physics & University of Waterloo, Canada

[1]Based on [1610.04230] and [1706.08527].

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- *Where do KLT relations come from?* In trying to answer this question, one finds a surprising connection between string theory amplitudes and the mathematics of *twisted cycles*.

A RELATION BETWEEN TREE AMPLITUDES OF CLOSED AND OPEN STRINGS*

H. KAWAI, D.C. LEWELLEN and S.-H.H. TYE

Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853, USA

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INTERSECTION THEORY FOR TWISTED COHOMOLOGIES AND TWISTED RIEMANN'S PERIOD RELATIONS I

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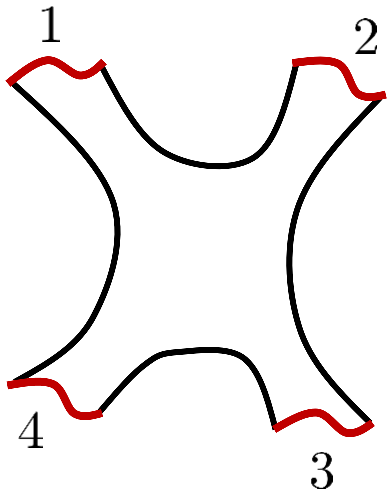
The goal for this talk:
How to understand KLT relations in terms of combinatorics and topology.

- Quick recap of tree-level open string amplitudes:



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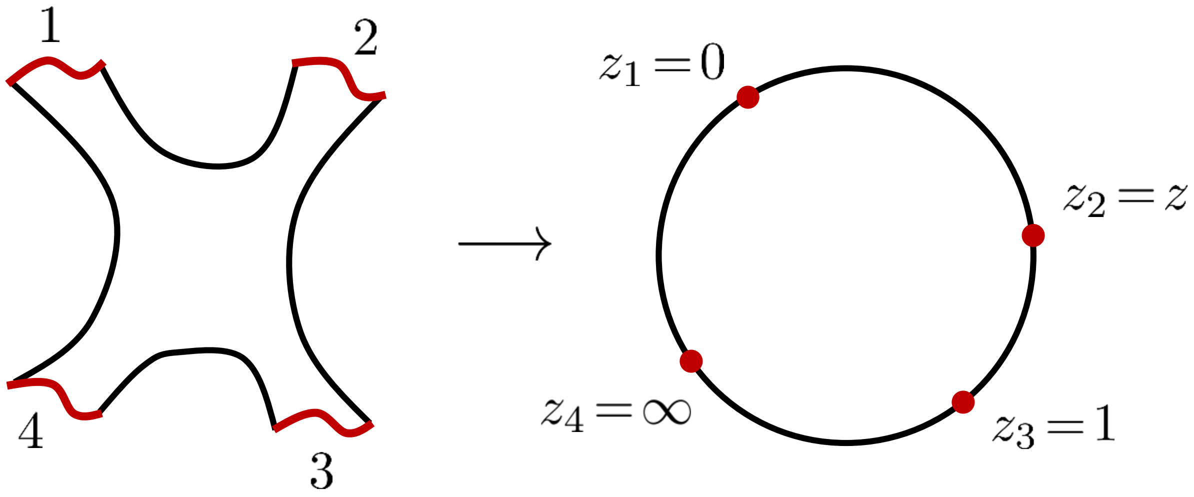

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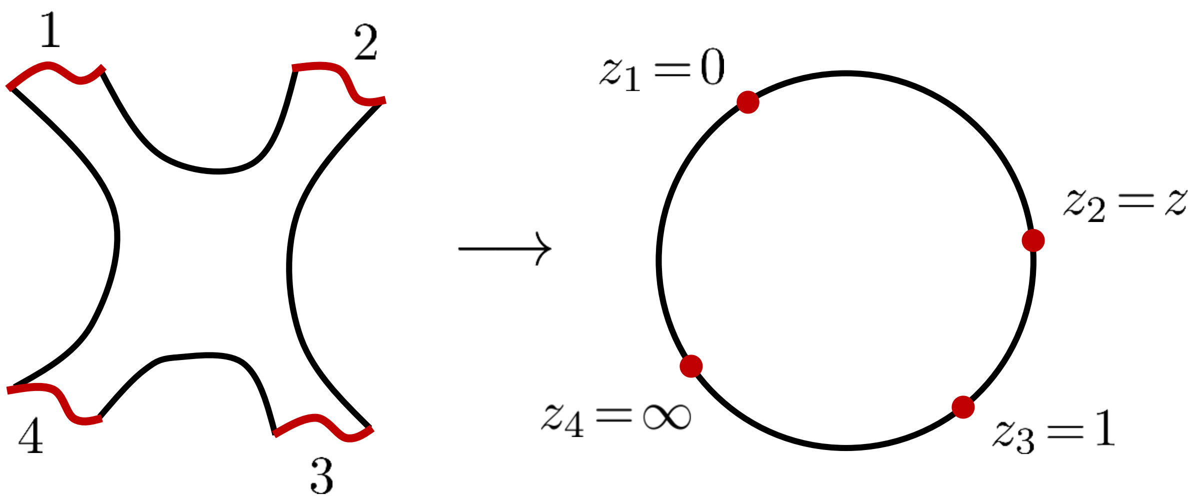

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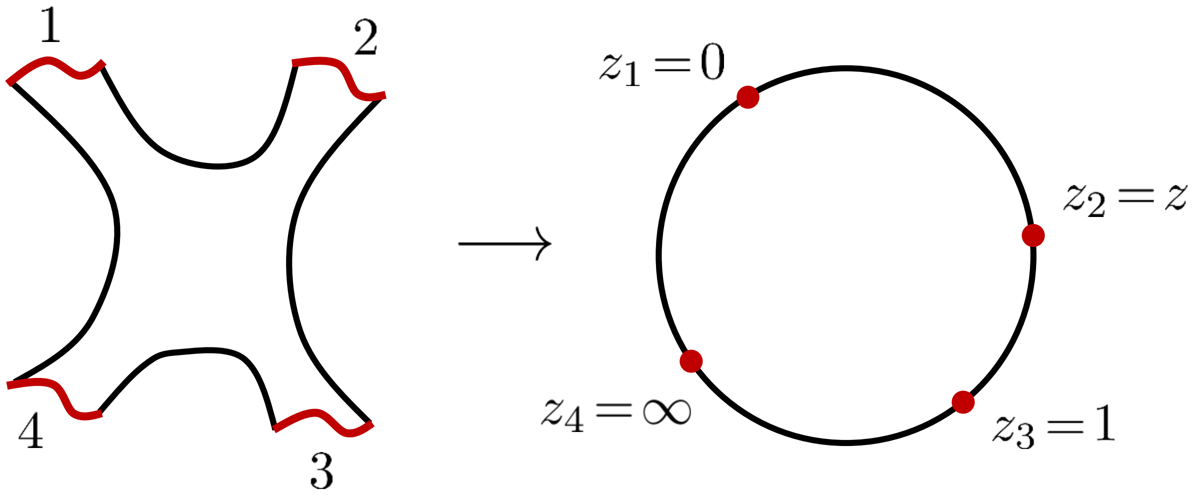

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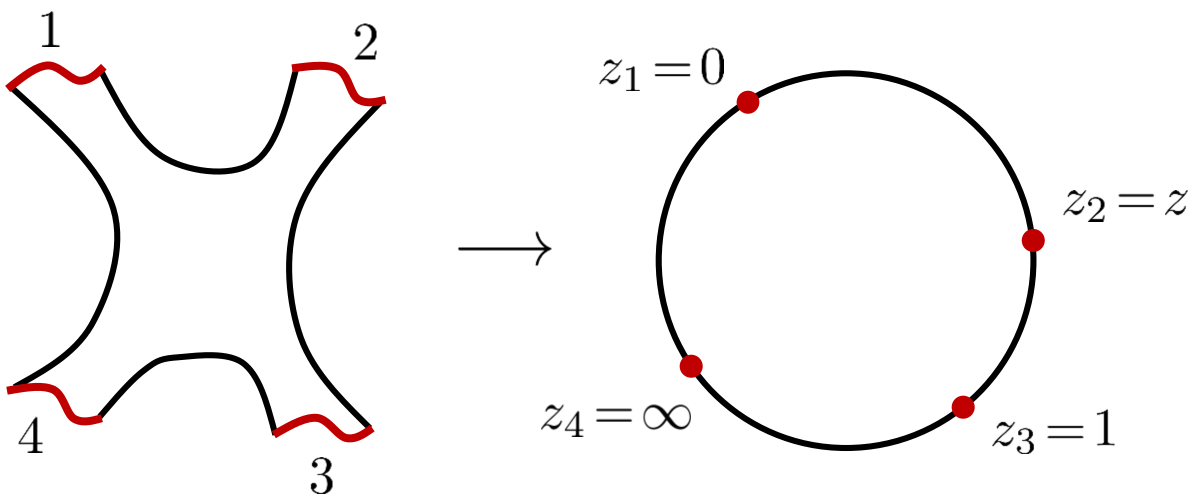

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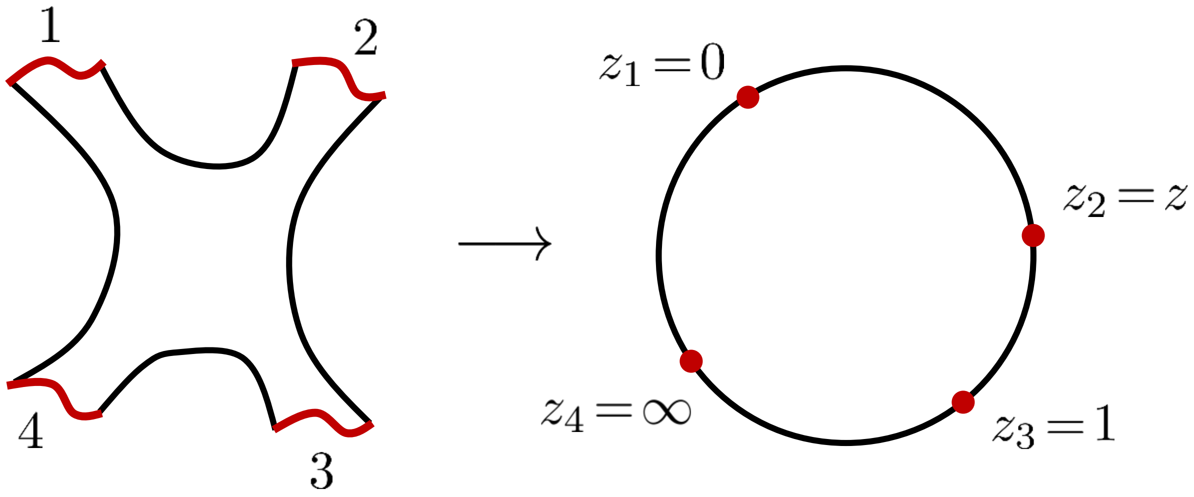

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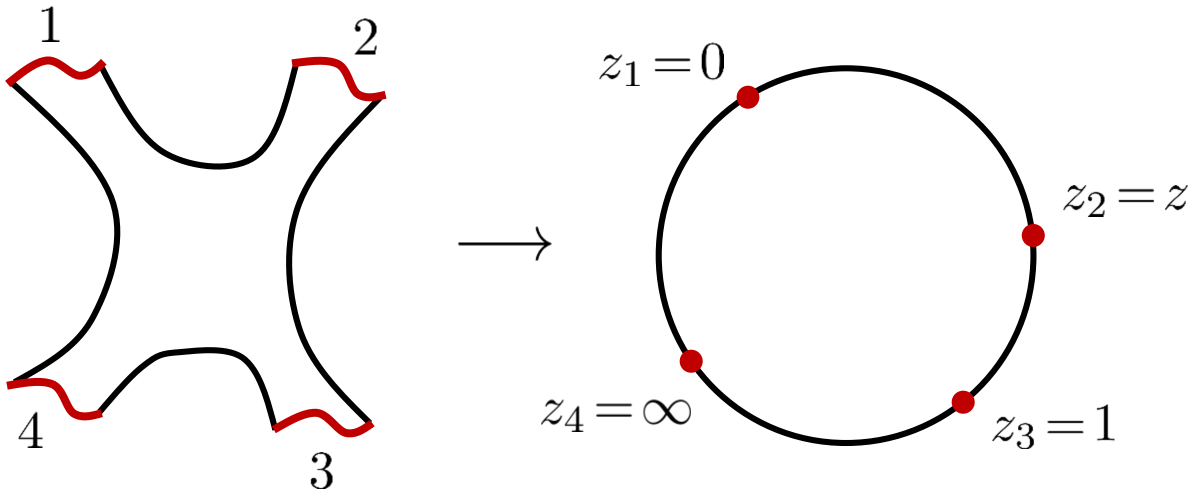

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- Twisted cycle: $\mathsf{C}(1234) = \overrightarrow{(0,1)} \otimes z^s (1-z)^t$
- Then an open string amplitude is given as a pairing between a twisted cycle $\mathsf{C}(1234)$ and a twisted cocycle $\varphi(z)$, e.g.,

$$\mathcal{A}^{\text{open}}(1234) = \langle \mathsf{C}(1234), \varphi \rangle$$

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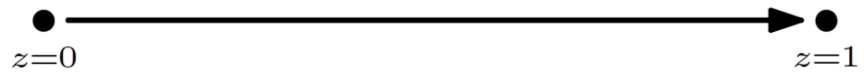

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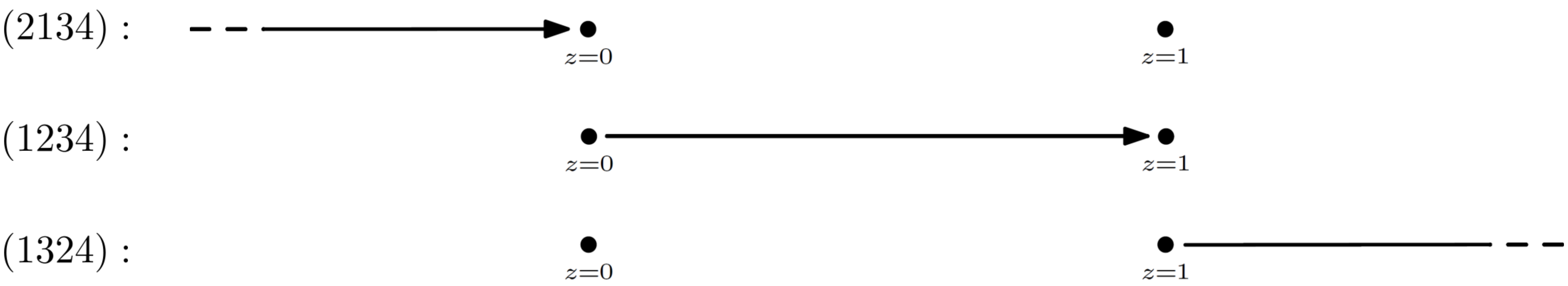

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$$\mathsf{C}(2134) = \overrightarrow{(-\infty, 0)} \otimes (-z)^s(1-z)^t$$

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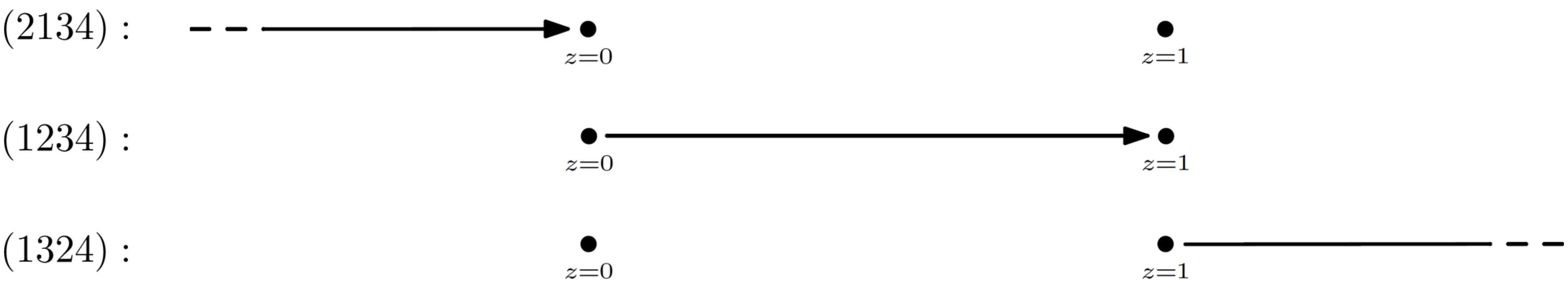

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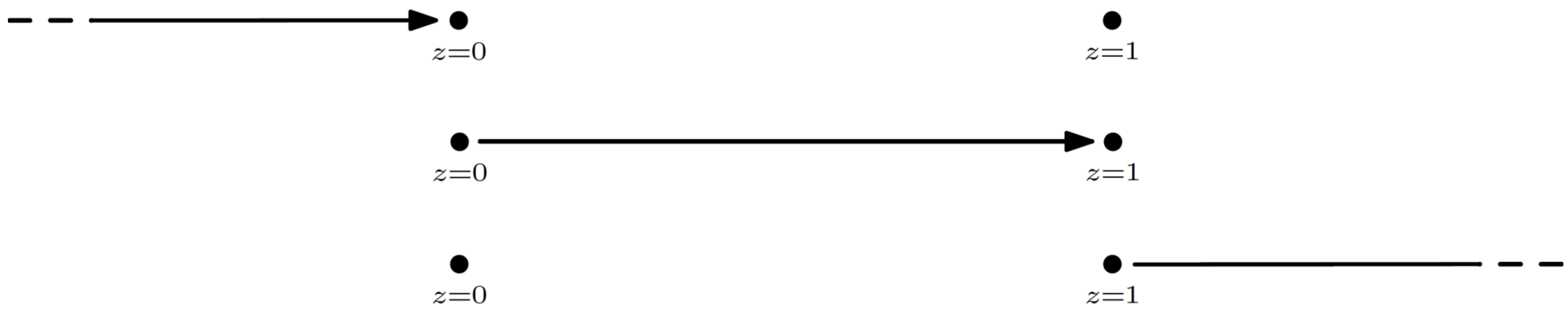

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- Twisted cycles live in the real section of the moduli space $\mathcal{M}_{0,4} = \mathbb{C} \setminus \{0, 1\}$.

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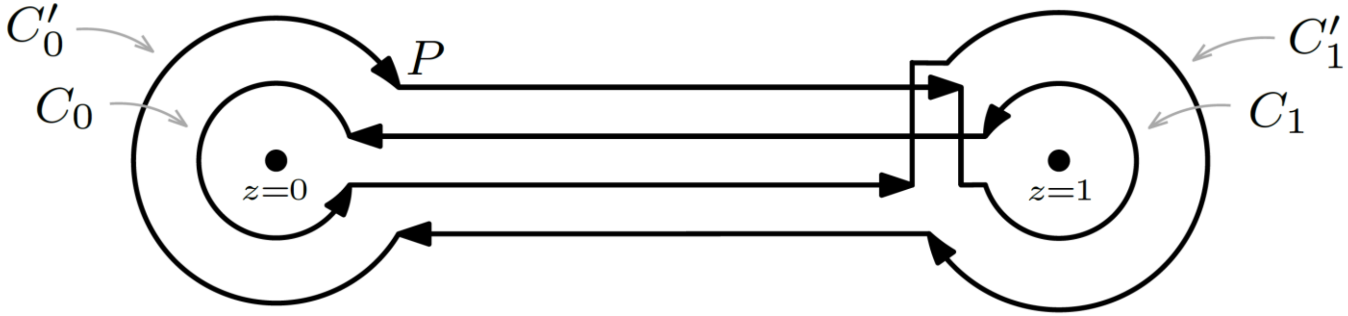

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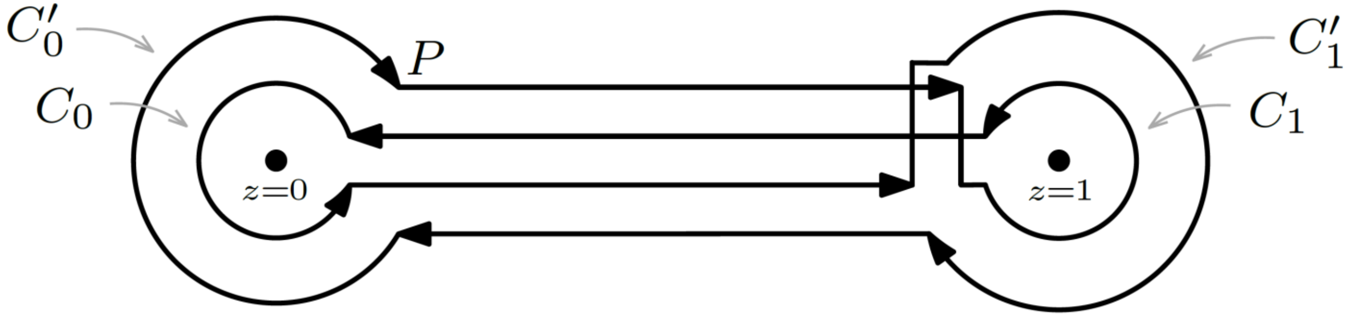

$$\oint_{\Gamma} z^s (1-z)^t \, \varphi(z) = \left(1 - e^{2\pi i t} + e^{2\pi i (s+t)} - e^{2\pi i s}\right) \int_{0<z<1} z^s (1-z)^t \, \varphi(z),$$

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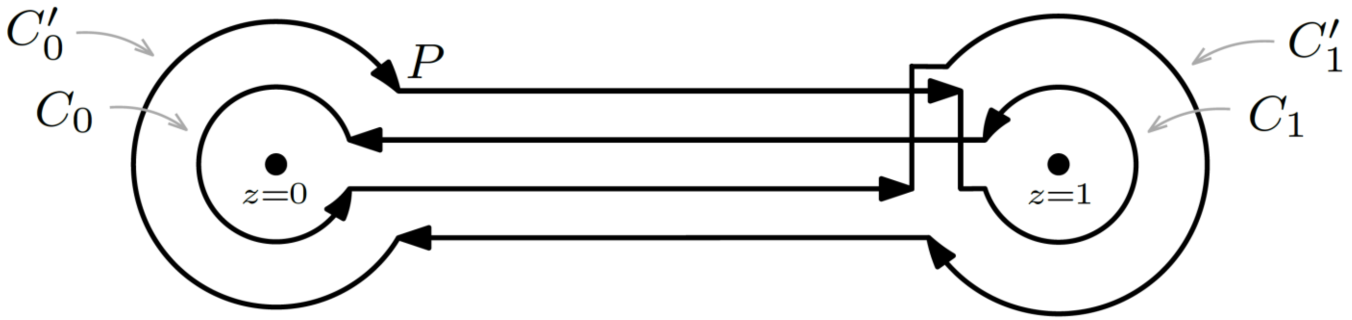

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- Or, in other words: $\operatorname{reg}\overrightarrow{(0,1)} = \dfrac{\Gamma}{(e^{2\pi i s}-1)(e^{2\pi i t}-1)}$

- In fact, it is more convenient to split it into three contributions:

$$\operatorname{reg}\overrightarrow{(0,1)} = \frac{\Gamma}{(e^{2\pi is}-1)(e^{2\pi it}-1)} = \frac{S(\varepsilon,0)}{e^{2\pi is}-1} + \overrightarrow{(\varepsilon,1-\varepsilon)} - \frac{S(1-\varepsilon,1)}{e^{2\pi it}-1}$$

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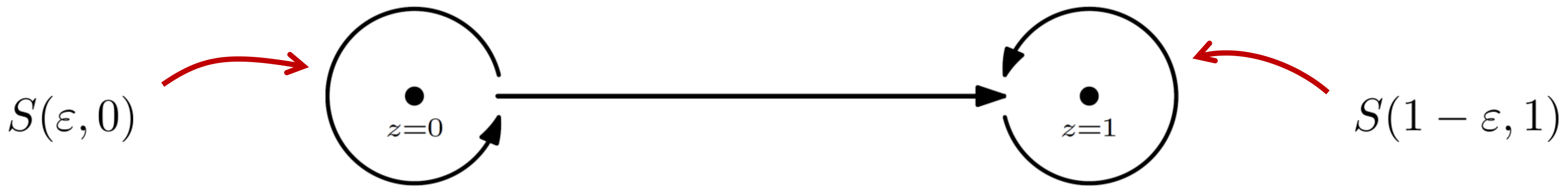

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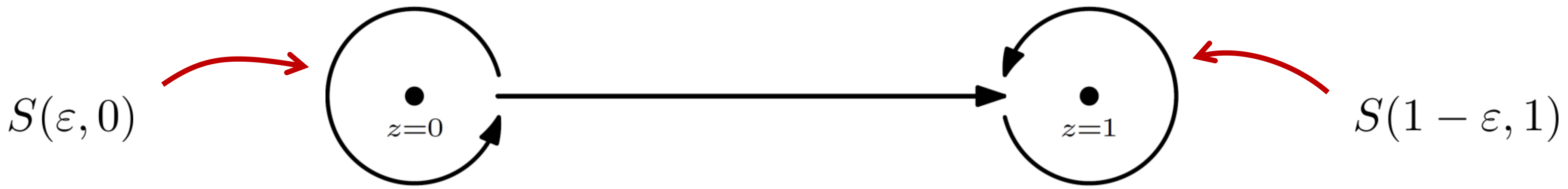

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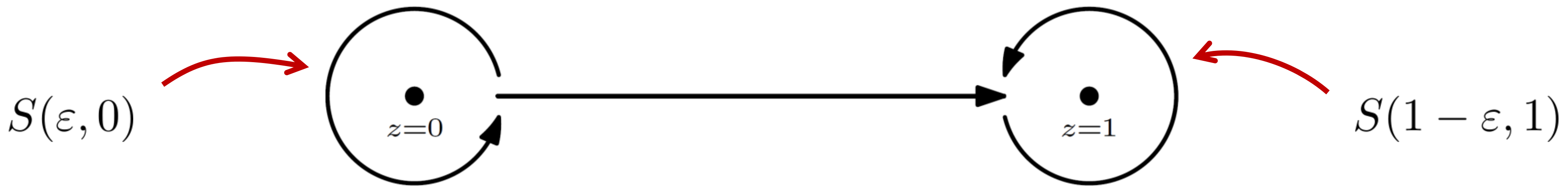

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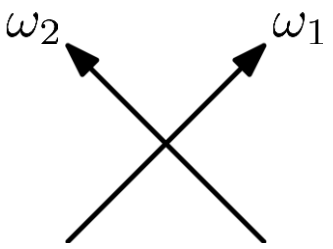
- We can now compute intersection numbers of twisted cycles with the rules:

$\langle \omega_1, \omega_2 \rangle$:



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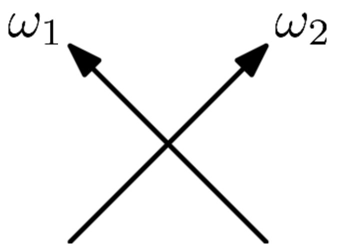

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- First calculate the self-intersection number of $\mathsf{C}(1234)$ with itself:

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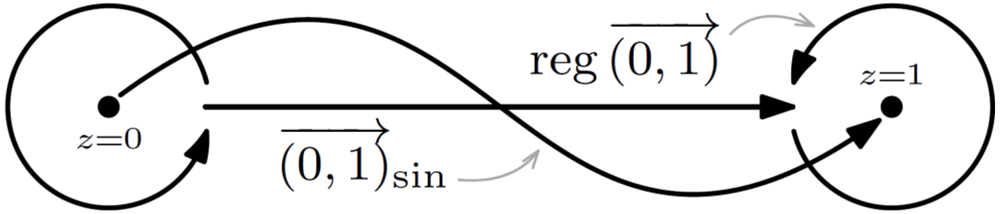

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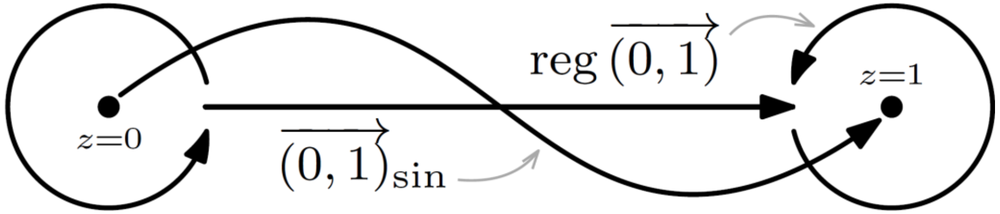

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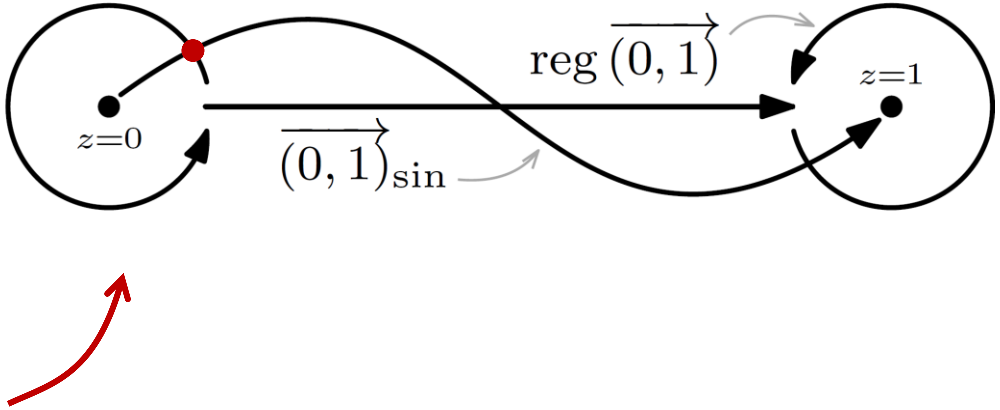

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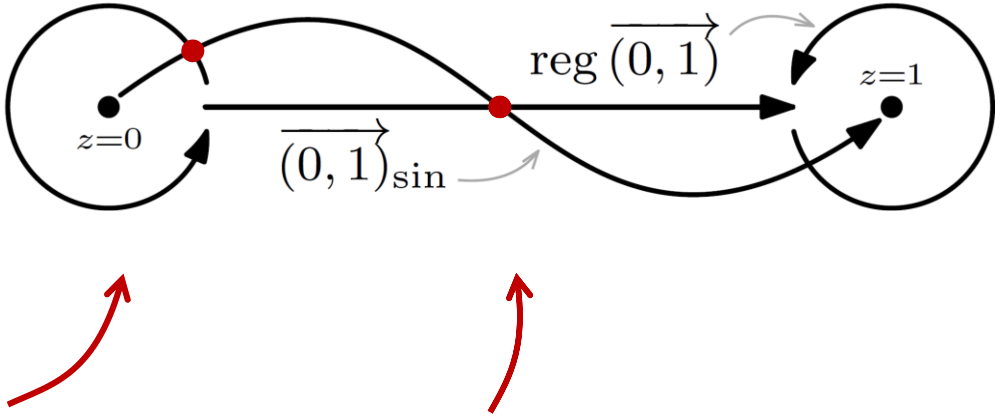

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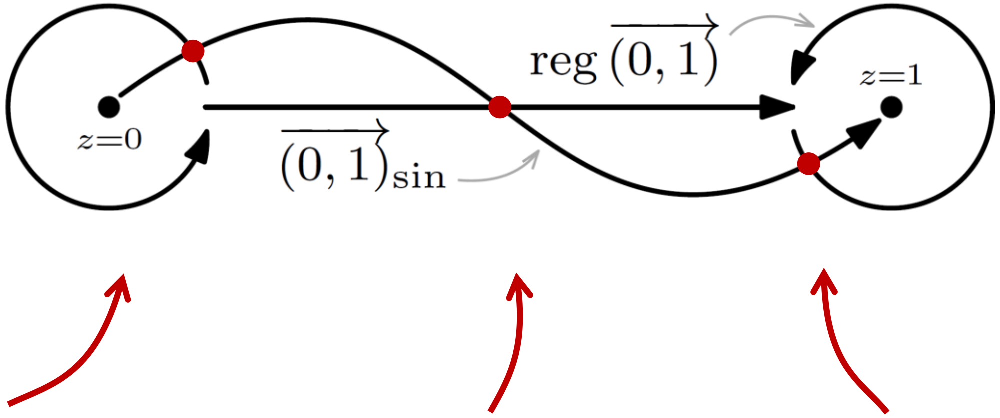

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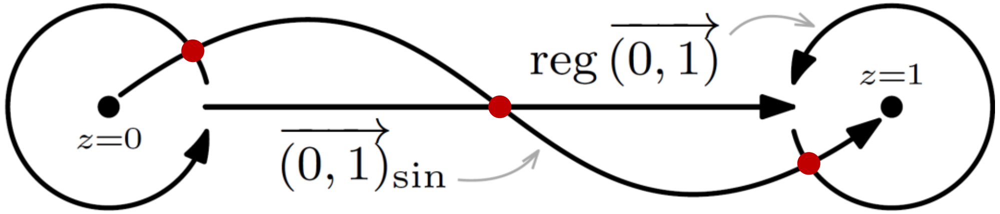

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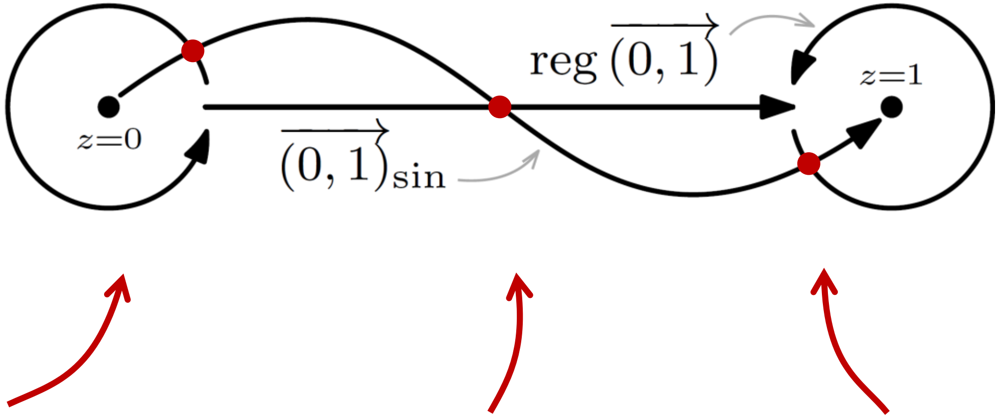

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$$= \frac{i}{2} \left(\frac{1}{\tan \pi s} + \frac{1}{\tan \pi t} \right)$$

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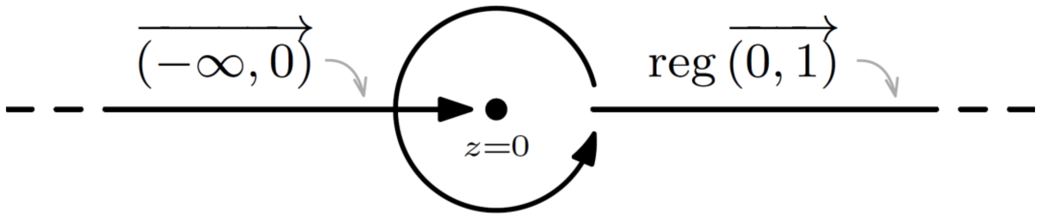

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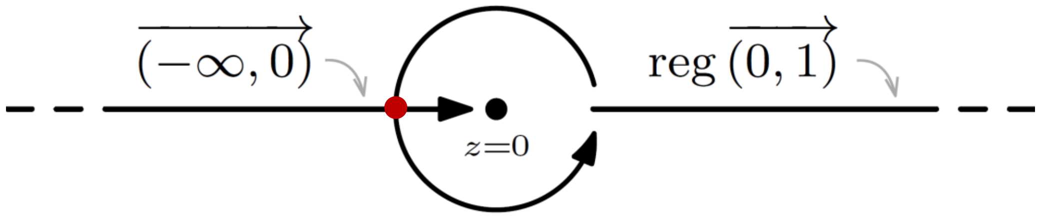

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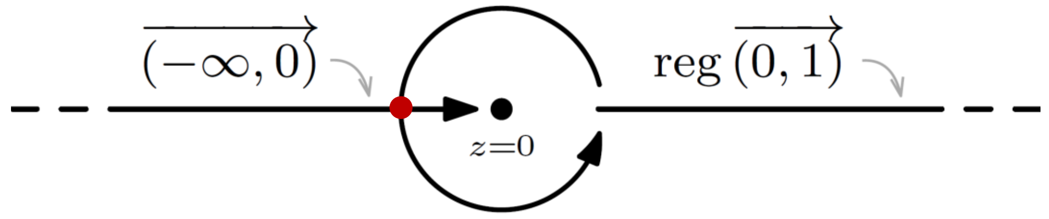

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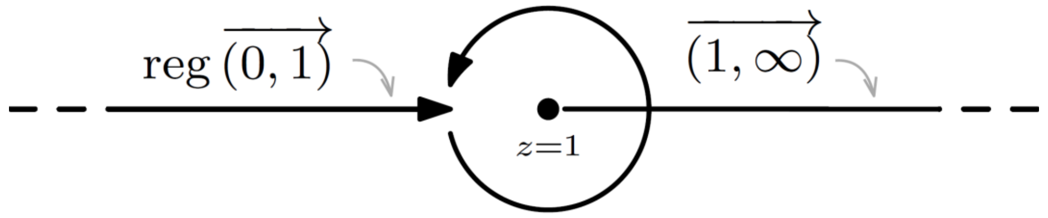

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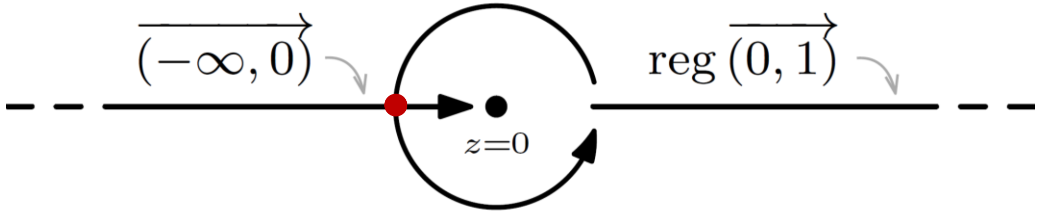

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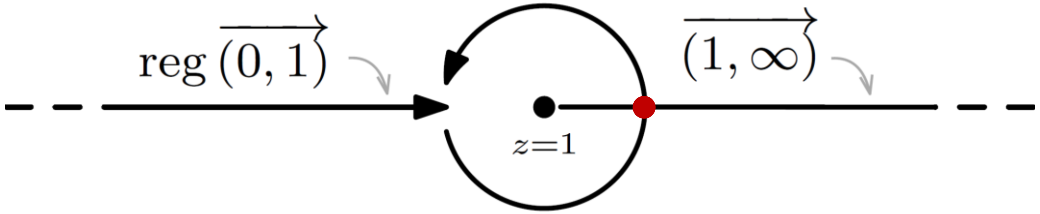

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$$\langle \mathsf{C}(1234), \mathsf{C}(1234)\rangle = -\left(\frac{1}{e^{2\pi i s} - 1} + 1 + \frac{1}{e^{2\pi i t} - 1}\right)$$ s,t-channel + contact

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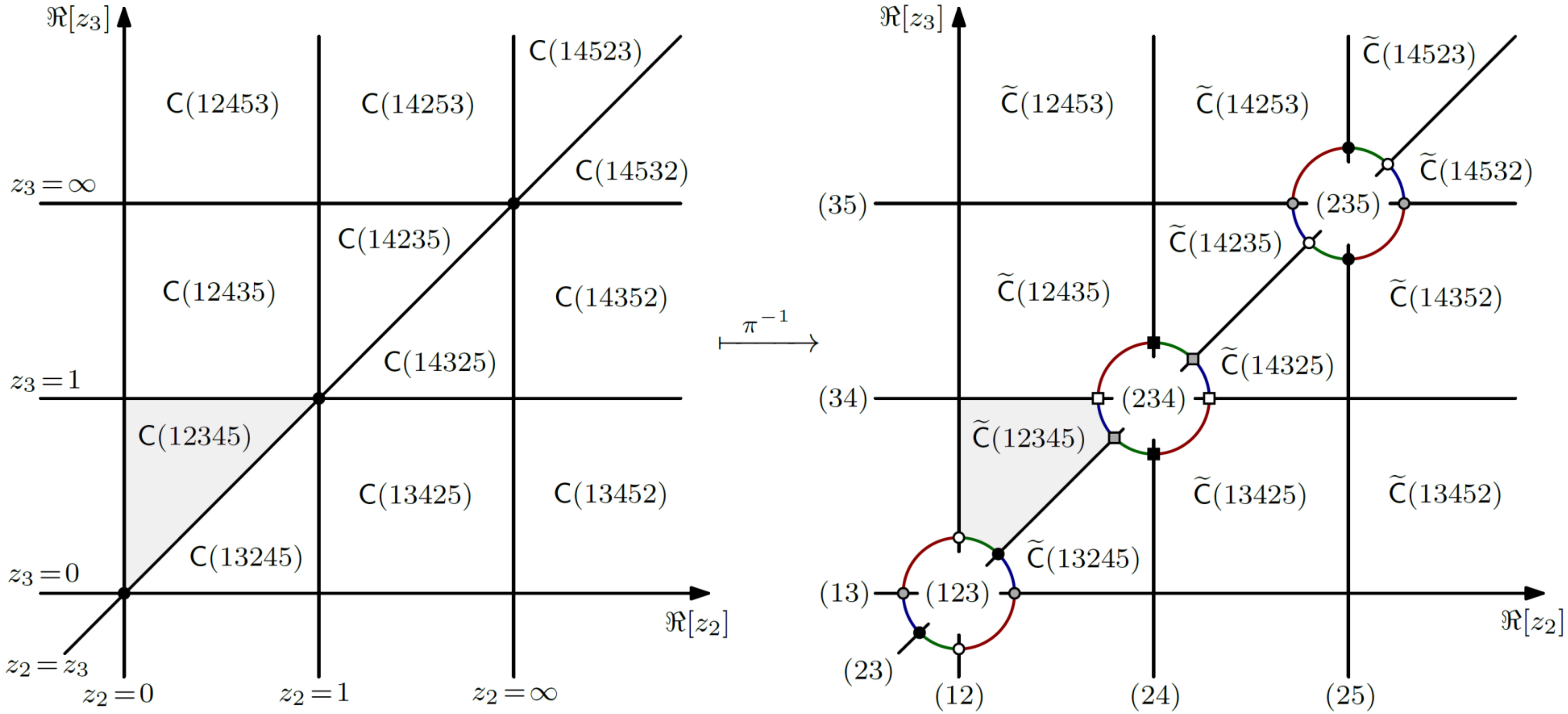

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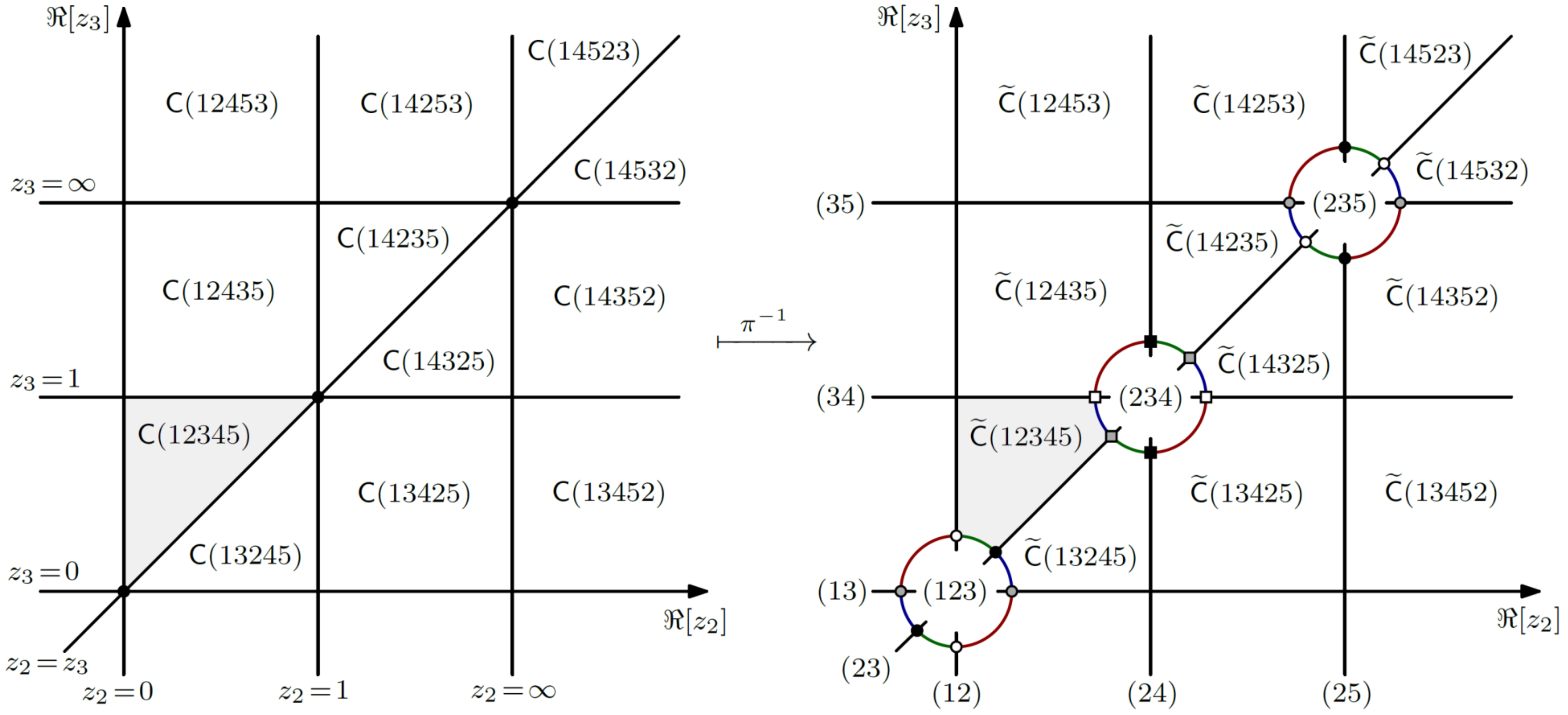

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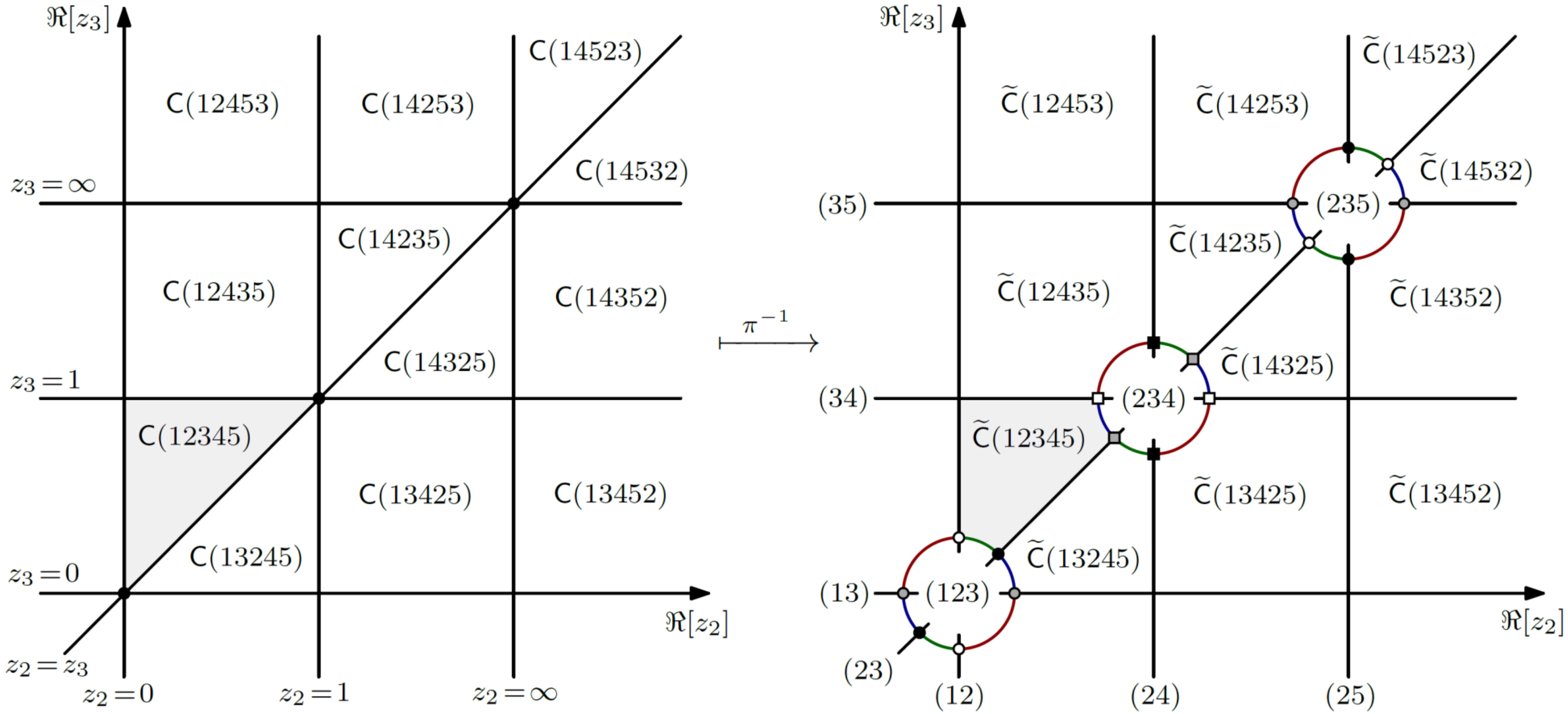

- Before blowup, e.g., $\mathsf{C}(12345) = \{0 < z_2 < z_3 < 1\} \otimes \ldots$
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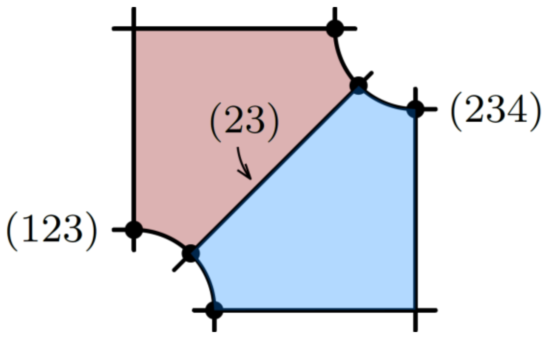

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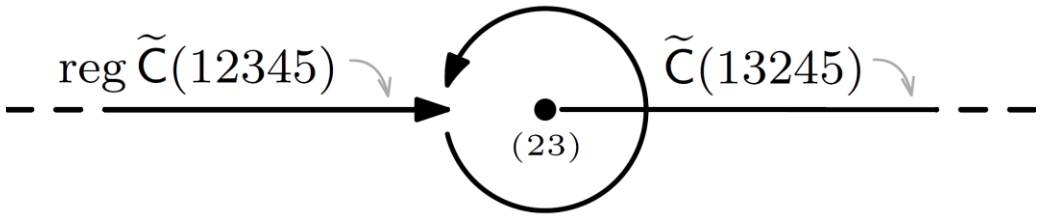

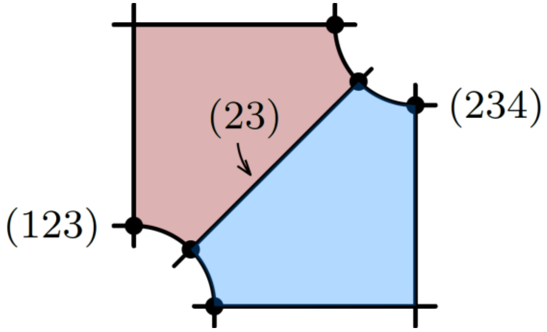

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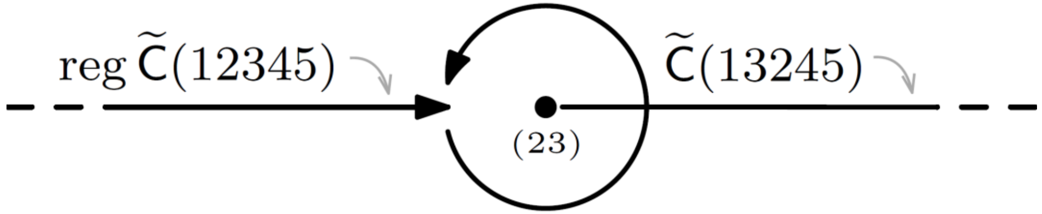

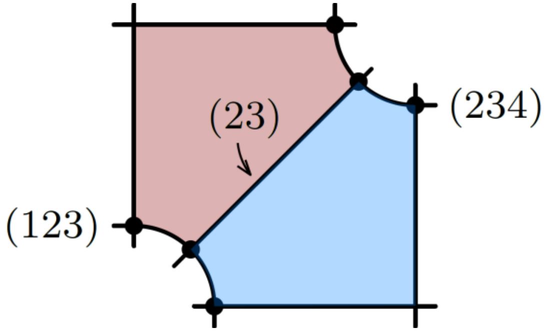

$$= \frac{e^{\pi i s_{23}}}{e^{2\pi i s_{23}} - 1} \left\langle \operatorname{reg} \widetilde{\mathsf{C}}(12345)\Big|_{(23)}, \widetilde{\mathsf{C}}(13245)\Big|_{(23)} \right\rangle$$

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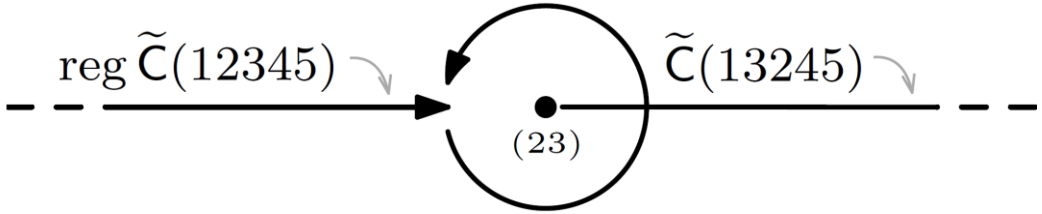

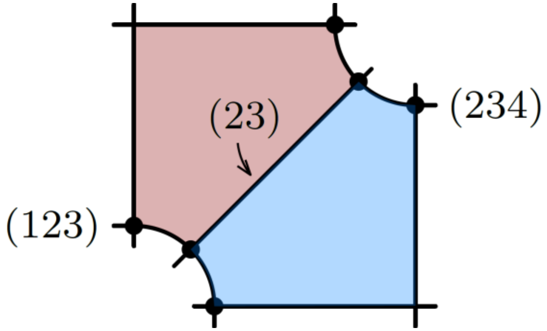

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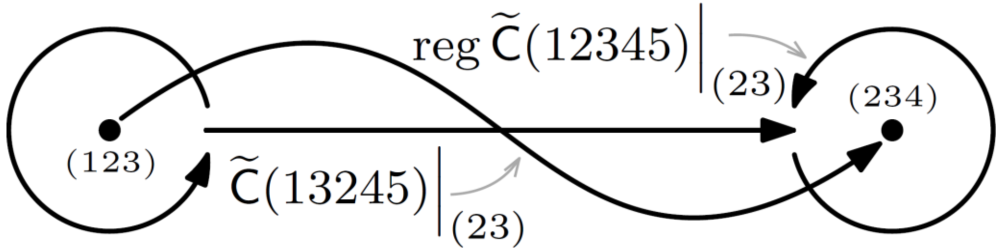

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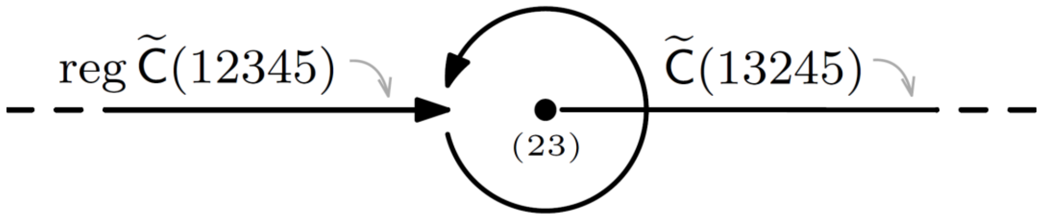

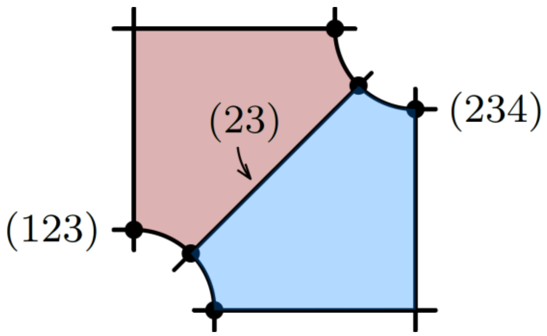

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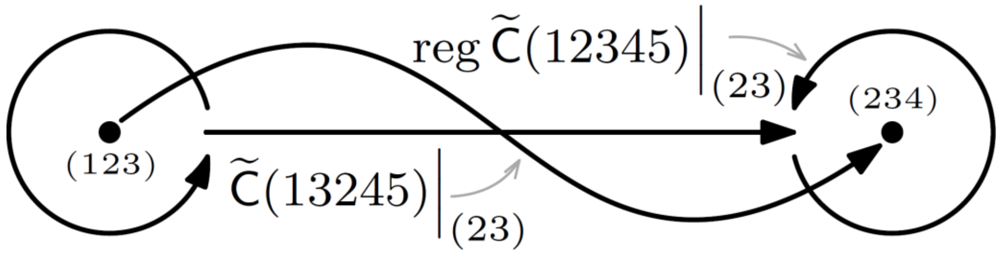

$$\Bigg)$$

$$= -\left(\frac{i}{2}\right)^2 \frac{1}{\sin \pi s_{23}} \left(\frac{1}{\tan \pi s_{45}} + \frac{1}{\tan \pi s_{51}} \right).$$

- In general, the moduli space $\widetilde{\mathcal{M}}_{0,n}(\mathbb{R})$ is tiled by $(n-1)!/2$ copies of the so-called *associahedron,* K_{n-1}: [Stasheff, Kapranov, Devadoss, …]

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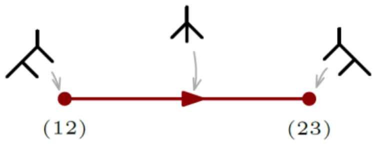

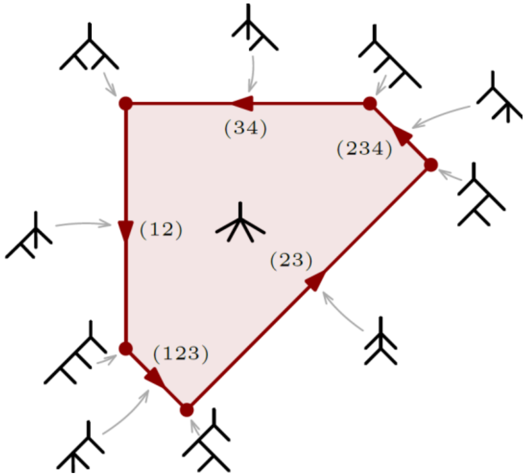

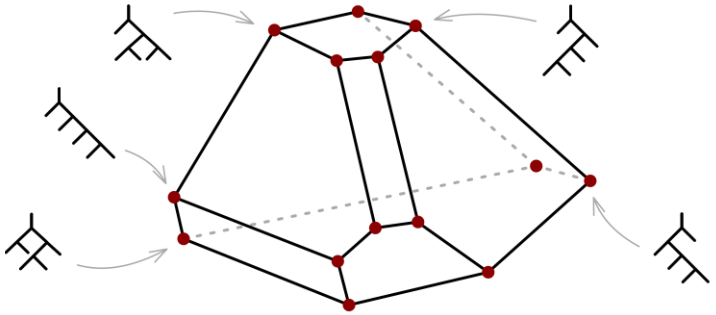

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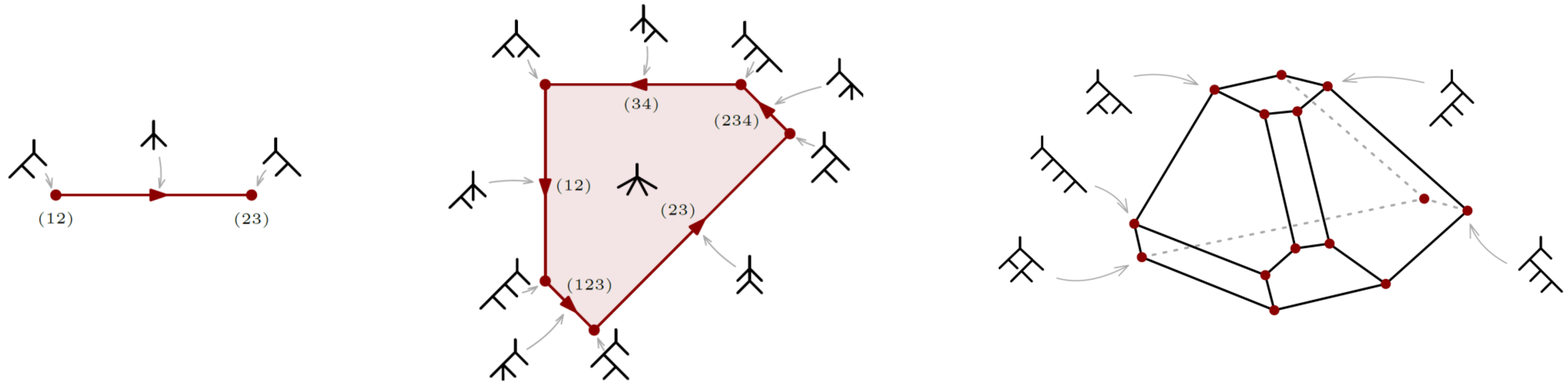

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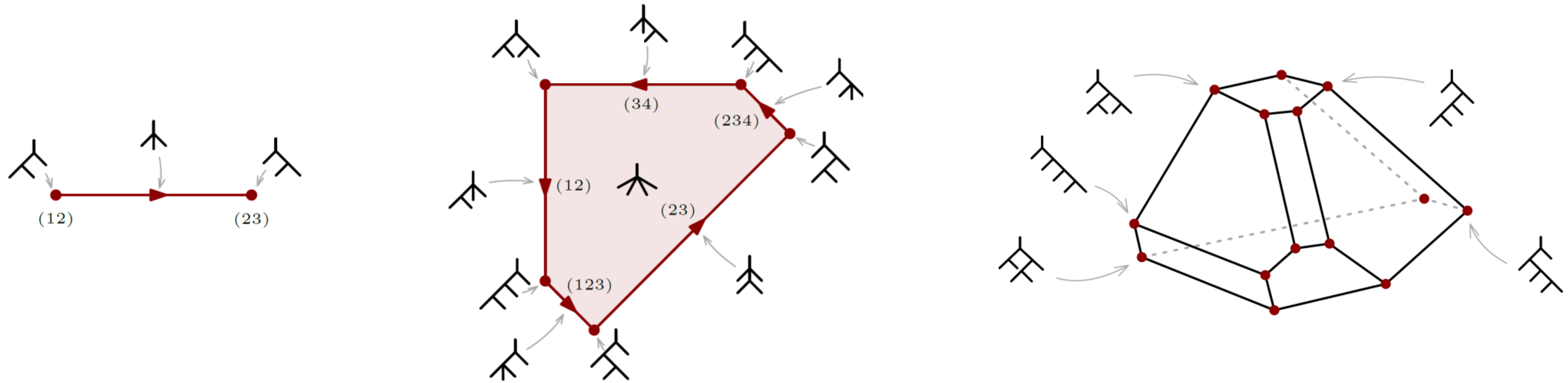

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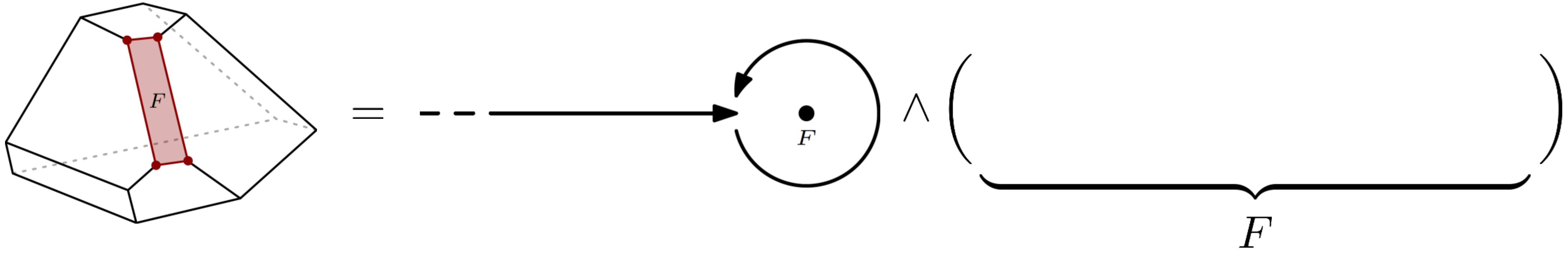

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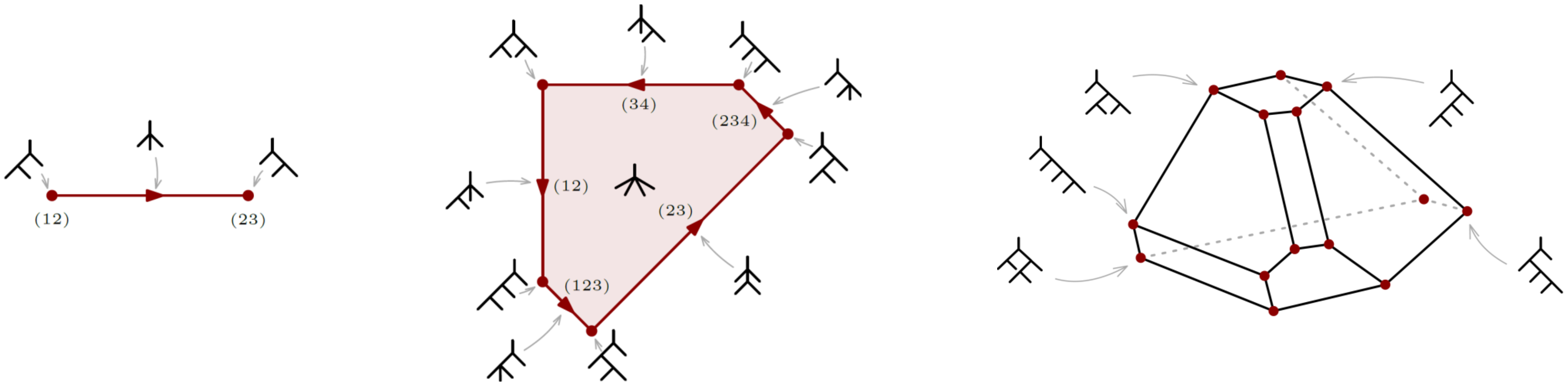

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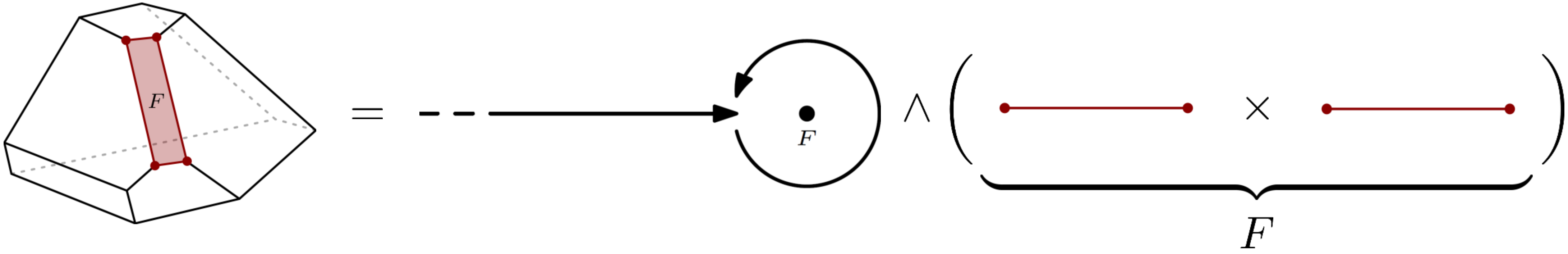

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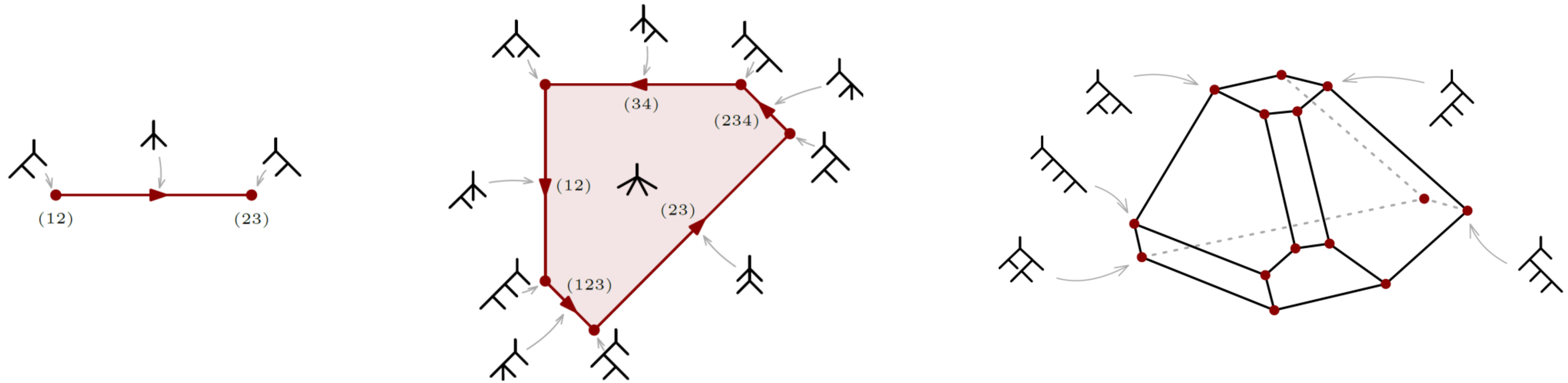

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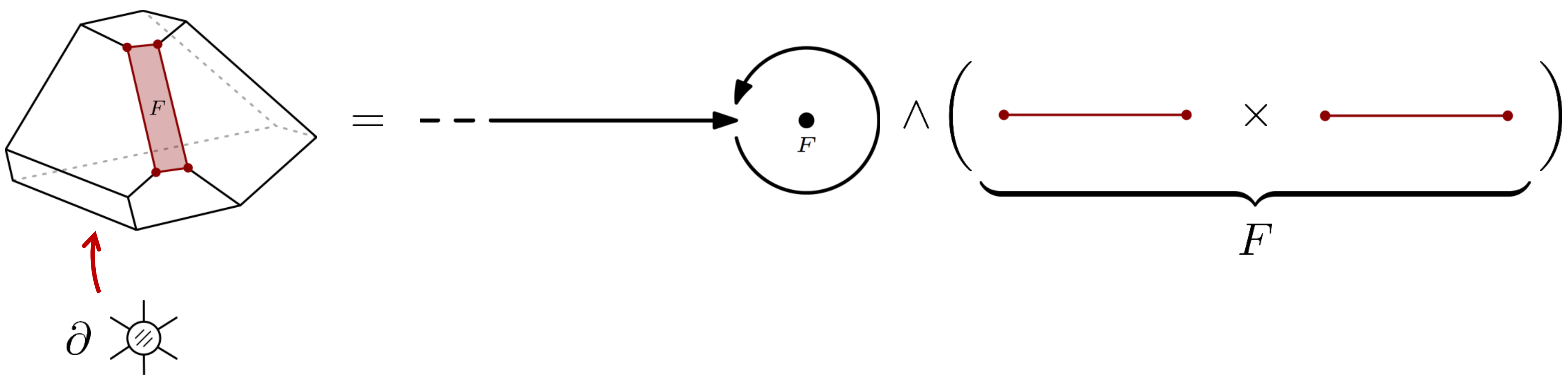

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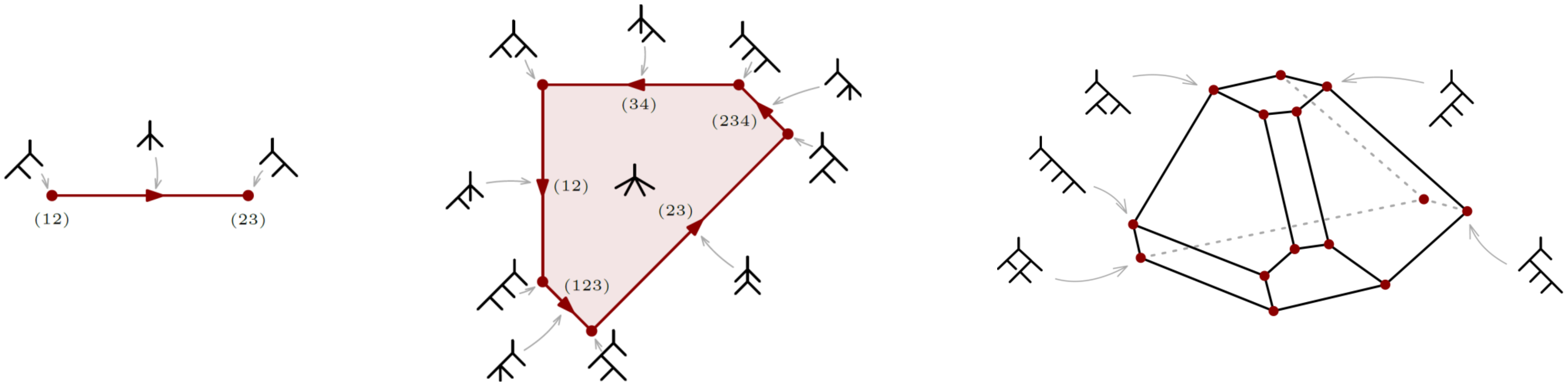

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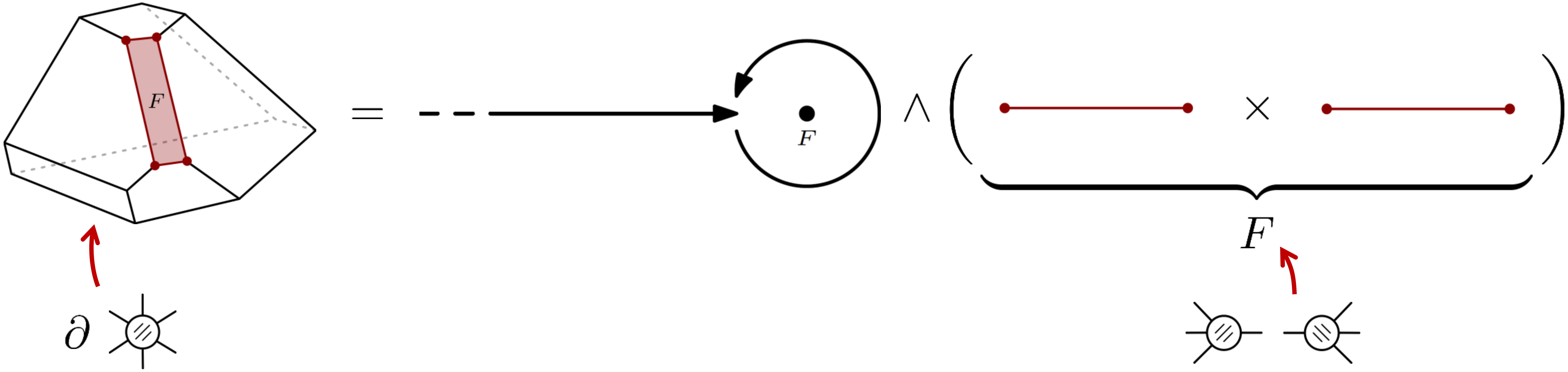

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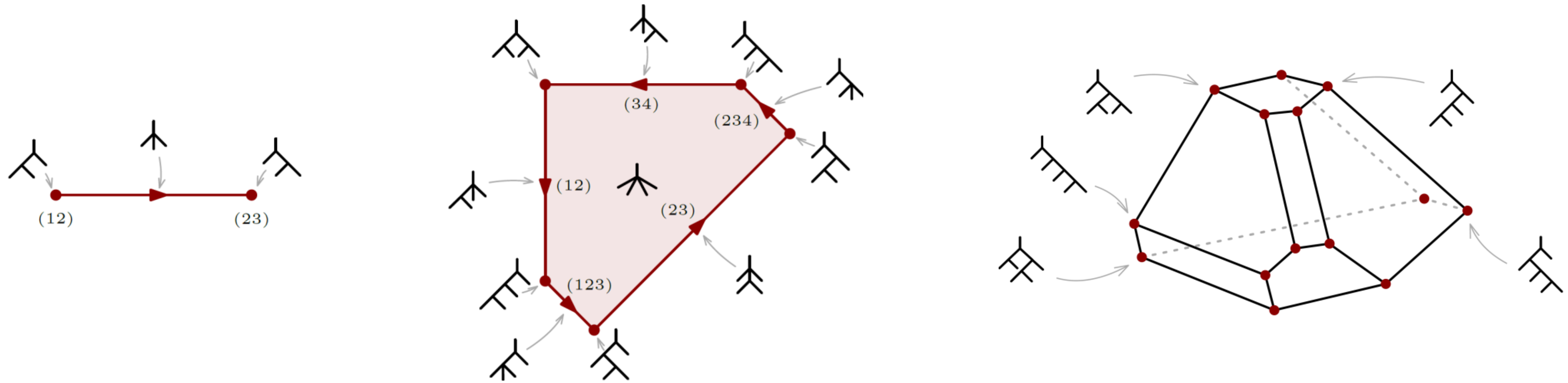

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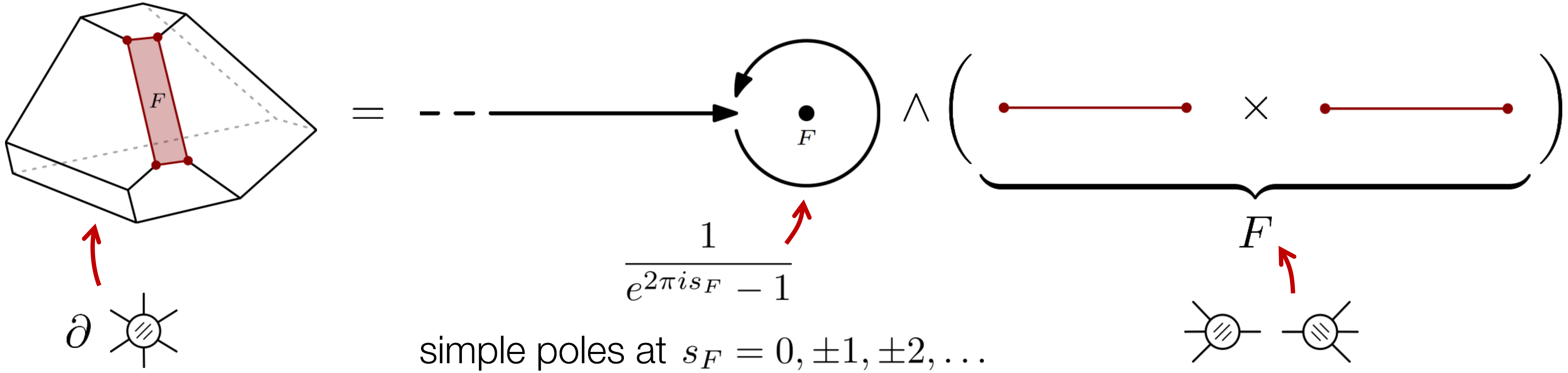
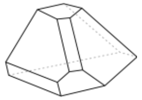
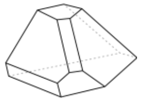


- Twisted cycles given by: $\mathsf{C}(\beta) =$  $_{\beta} \otimes \ \ldots$, where β is a colour ordering.

- Twisted cycles given by: $\mathsf{C}(\beta) =$  $_{\beta} \otimes \ldots$, where β is a colour ordering.
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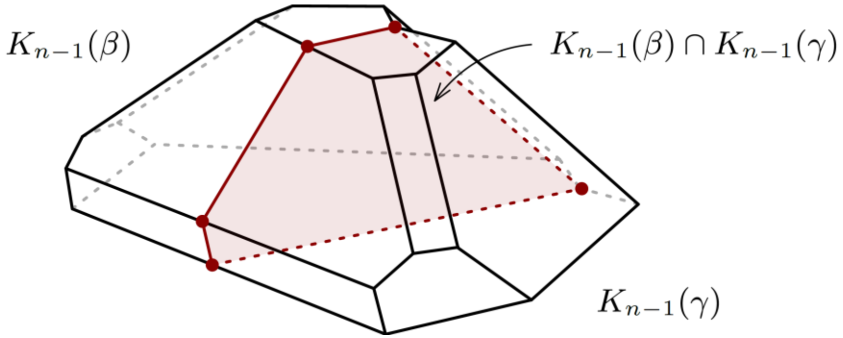
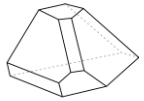


- Twisted cycles given by: $\mathsf{C}(\beta) =$  $_\beta \otimes \ldots$, where β is a colour ordering.
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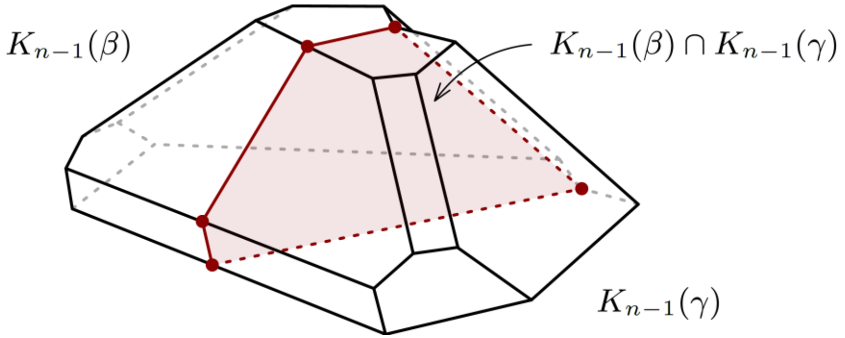
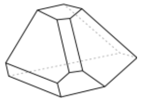


- Always given as tree diagrams with "propagators" $1/\sin(\pi\alpha' p^2/2)$ and $1/\tan(\pi\alpha' p^2/2)$. But we don't need to actually draw associahedra. Use diagrams instead, e.g., [SM 16, 17]

- Twisted cycles given by: $\mathsf{C}(\beta) =$  $_\beta \otimes \ldots$, where β is a colour ordering.
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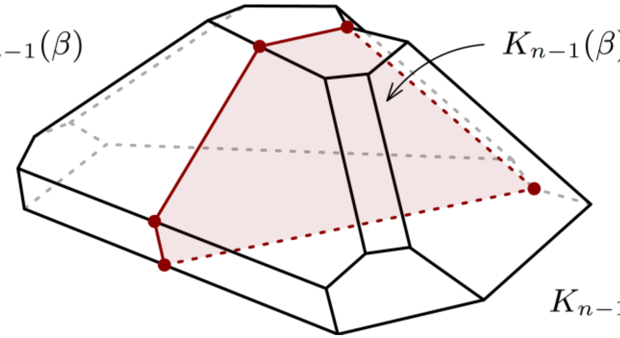

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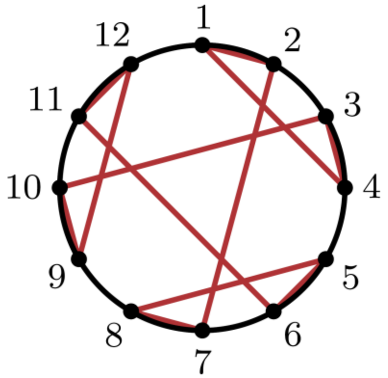

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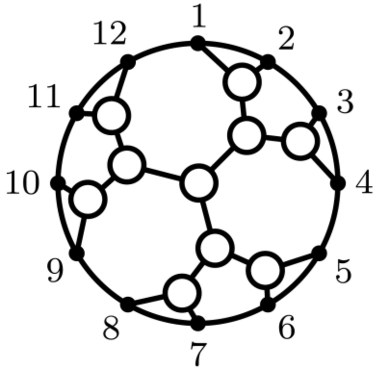

$$= -\frac{1}{\sin \pi s_{12}} \frac{1}{\sin \pi s_{34}} \frac{1}{\sin \pi s_{56}} \frac{1}{\sin \pi s_{78}} \cdots$$

- There are three types of pairings we can define:

$$\begin{array}{ccc} \varphi(z) & \longleftrightarrow & \varphi^{\vee}(z) \\ \updownarrow & & \updownarrow \\ \mathsf{C}(\beta) & \longleftrightarrow & \mathsf{C}(\gamma) \end{array}$$

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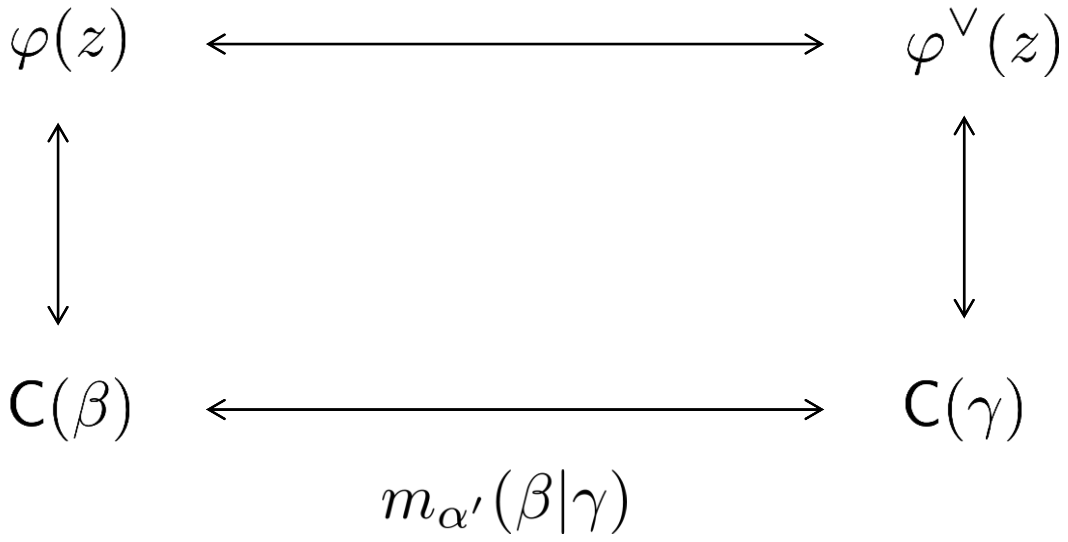

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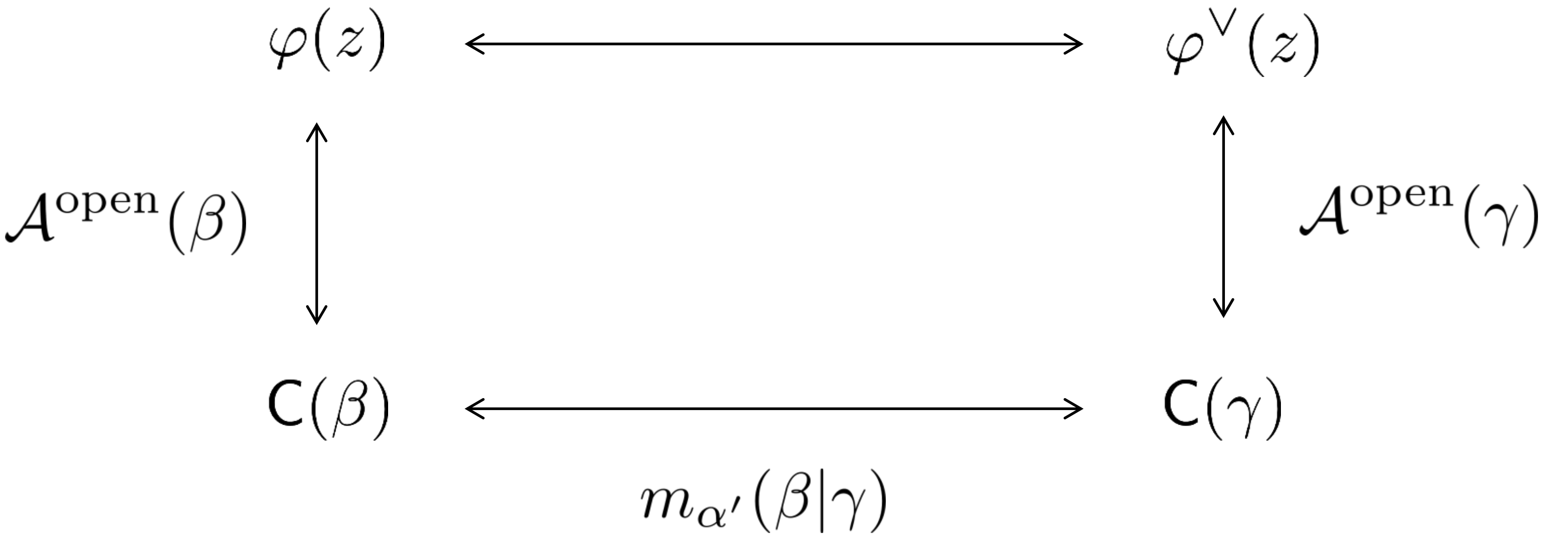

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- They are related by the *twisted period relations:* [Cho & Matsumoto 94, SM 16, 17]

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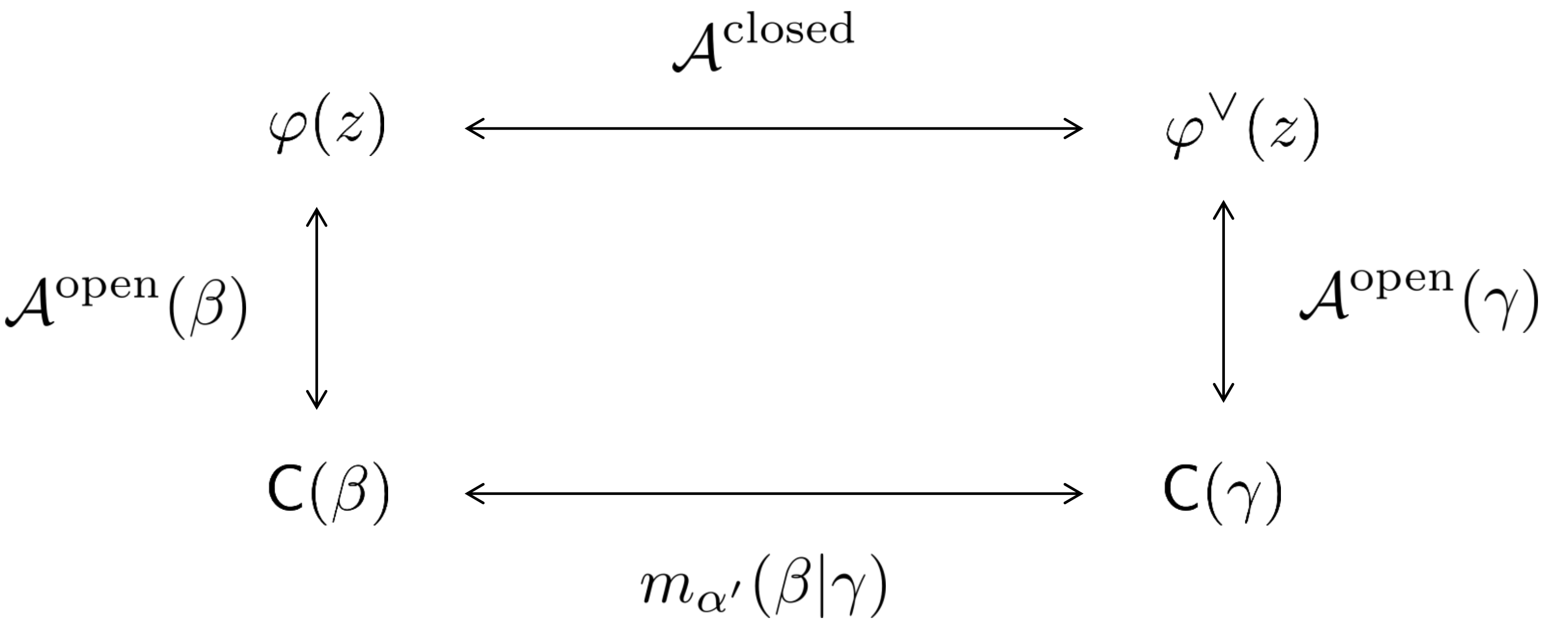

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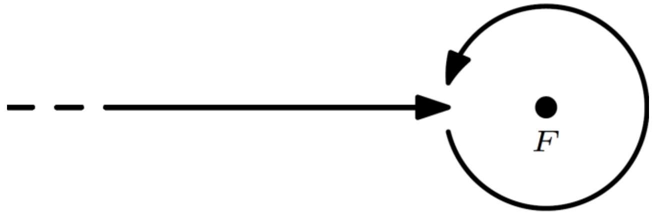
- These are the Kawai—Lewellen—Tye relations with KLT kernel given by $m_{\alpha'}^{-1}$!

- In the field theory limit, $\alpha' \to 0$, each facet contributes



$$= \frac{1}{e^{2\pi i \alpha' s_F} - 1} \longrightarrow \frac{1}{2\pi i\, \alpha' s_F} + \dots$$

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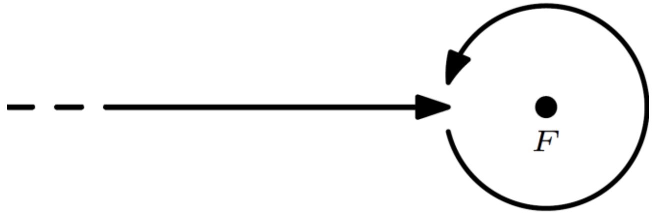

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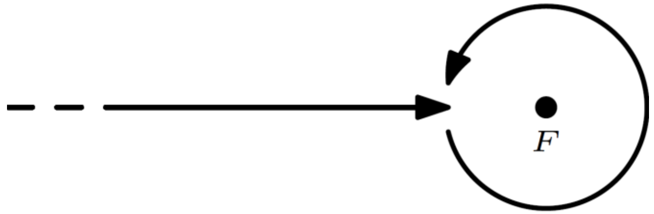

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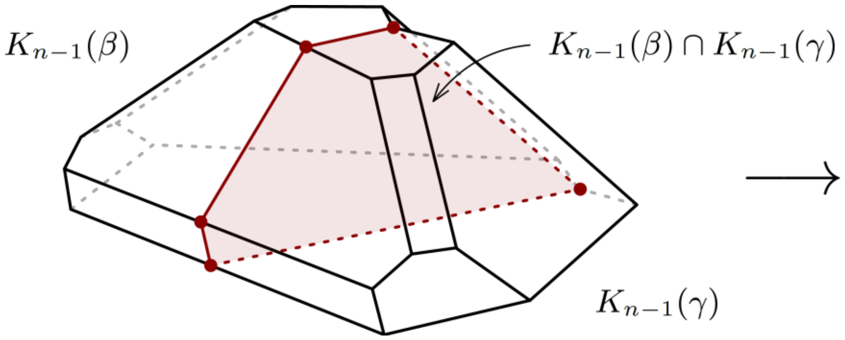

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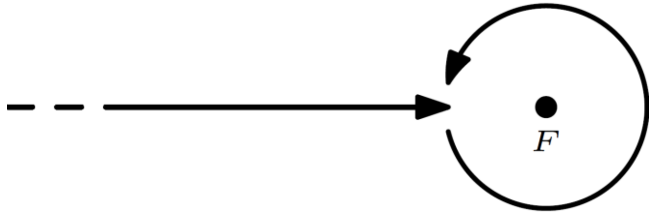

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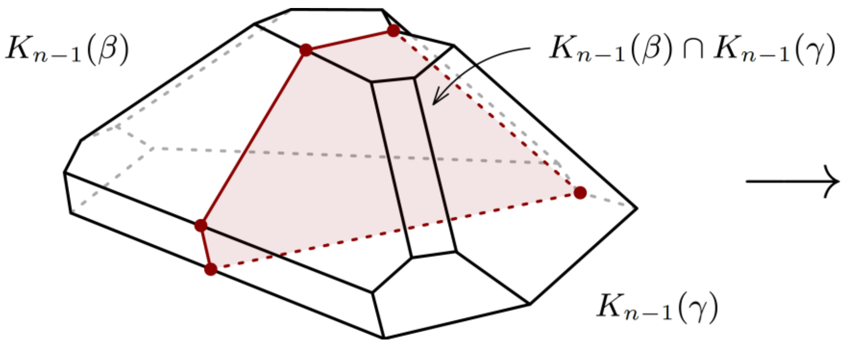

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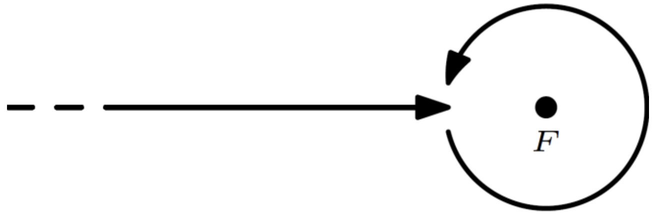

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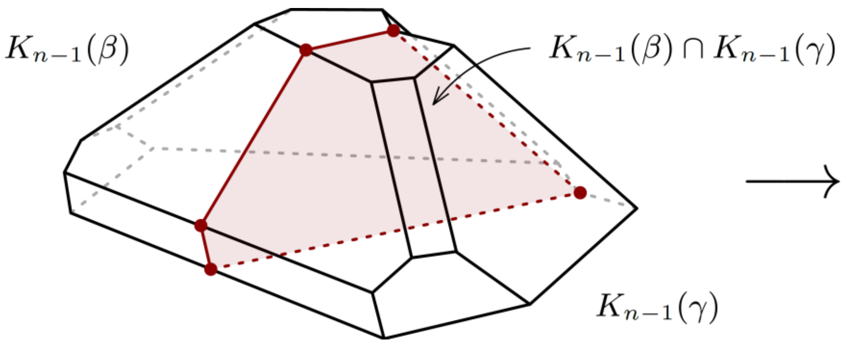

$\longrightarrow$ (sum over trivalent Feynman diagrams planar w.r.t. β, γ) $= m(\beta|\gamma)$

- Since $\mathcal{A}^{\text{closed}} \to \mathcal{A}^{\text{GR}}$ and $\mathcal{A}^{\text{open}} \to \mathcal{A}^{\text{YM}}$ in this limit, KLT relations become:

$$\mathcal{A}^{\text{GR}} = \sum_{\beta,\gamma} \mathcal{A}^{\text{YM}}(\beta)\, m^{-1}(\beta|\gamma)\, \mathcal{A}^{\text{YM}}(\gamma)$$

bi-adjoint scalar amplitudes

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  to colour-kinematics duality [Carrasco later this week]

Thank you

References

S. Mizera, *Inverse of the String Theory KLT Kernel*, [1610.04230]

S. Mizera, *Combinatorics and Topology of Kawai—Lewellen—Tye Relations*, [1706.08527]