Eikonal probe scattering on the celestial sphere

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Based on 2404.xxxxx with Tim Adamo, Wei Bu, and Piotr Tourkine

More on the Celestial and Carrollian world

- Tomasz Taylor, BZ, *w(1+infinity) Algebra with a Cosmological Constant and the Celestial Sphere*, 2312.00876
- Stephan Stieberger, Tomasz Taylor, BZ, *Carrollian Amplitudes from Strings,* 2402.14062

Outline

- Motivation
- Eikonal amplitudes from curved backgrounds
- Celestial probe amplitudes in backgrounds to all orders in G
- Dispersion relations of celestial probe amplitudes in backgrounds and celestial eikonal amplitudes
- Summary and a road map for *Adamo, Bu, Tourkine, BZ '24*

Motivation

There are two closely related aspects as motivation of this work:

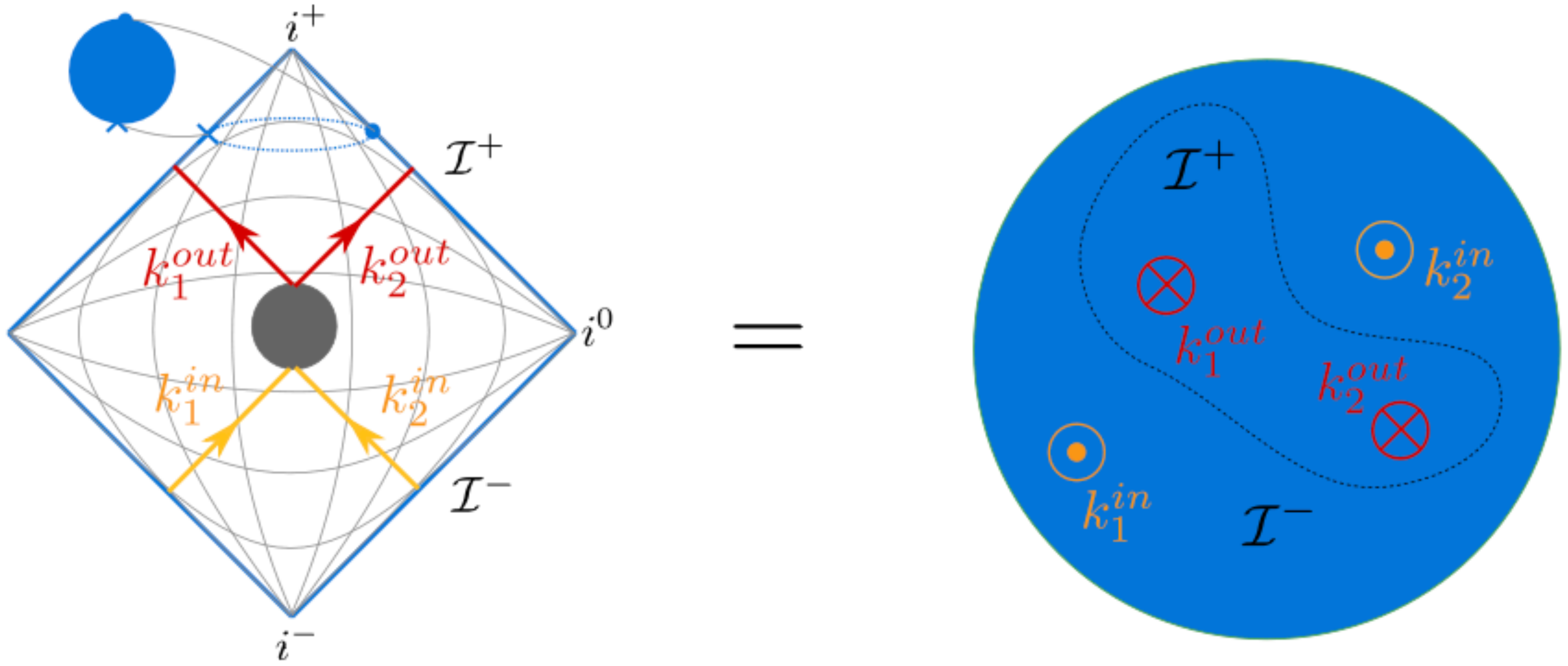
- Eikonal amplitudes or Eikonal approaches in general
- Celestial holography on backgrounds

See the massive review by Di Vecchia, Heissenberg, Russo, Veneziano '23

I will mainly focus on the second one

Celestial Holography

- Celestial Holography: The S-matrix in 4D asymptotically flat spacetime is described by a putative celestial CFT (CCFT) living on the celestial sphere
- Bottom-up approach: Understand universal properties of celestial amplitudes (CCFT correlators) obtained by a change of basis: momentum eigenstate basis to boost eigenstate basis



- Celestial amplitudes of massless particles

$$\left\langle \prod_{n=1}^{N} O^{a_n}_{\Delta_n, J_n}(z_n, \bar{z}_n) \right\rangle = \prod_{n=1}^{N} \int_0^\infty d\omega_n \omega_n^{\Delta_n - 1} \delta^{(4)}\Big(\sum_{i=1}^{N} \epsilon_i \omega_i q_i\Big) \mathcal{M}(\omega_n, z_n, \bar{z}_n, J_n, a_n),$$

Pasterski, Shao, Strominger '17

where $\mathcal{M}(\omega_n, z_n, \bar{z}_n, J_n, a_n)$ are scattering amplitudes in written in momentum basis

Motivation

- We have learned a lot about the symmetries of CCFT, e.g., BMS, super BMS, w(1+infinity), s-algebra

  However, CCFT has the following exotic property: low-point celestial amplitudes are distributional

- Consider (1,3) signature: celestial sphere $z \in \mathbb{C}$

$$\left.\begin{aligned} &\langle O_{\Delta_1,-} O_{\Delta_2,-} O_{\Delta_3,+} \rangle = 0 \\ &\langle O_{\Delta_1,-} O_{\Delta_2,-} O_{\Delta_3,+} O_{\Delta_4,+} \rangle \sim \delta(x - \bar{x}) \end{aligned}\right\}$$

Due to momentum conservation in 4D

Conformal invariant cross ratio: $x = \dfrac{z_{12} z_{34}}{z_{13} z_{24}}$

- In (2,2) signature, celestial three-point gluon amplitudes

$$A_{--+}(\Delta_k, z_k, \bar{z}_k) \sim \frac{\delta(\bar{z}_{13})\delta(\bar{z}_{12})\delta(h_1 + h_2 + h_3 - 1)}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{31}^{h_3+h_1-h_2}}$$

Pasterski, Shao, Strominger '17

- In standard CFTs:

$$\langle O_1 O_2 O_3 \rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}$$

determined by conformal symmetry $SL(2,\mathbb{C})$, up to a constant

$\langle O_1 O_2 O_3 O_4 \rangle \sim$ One single-valued function $G(x, \bar{x})$

Celestial holography on backgrounds

- Recent attempts: Break translation invariance in controllable ways

Couple the Yang-Mills system to dilaton backgrounds

Fan, Fotopolous, Stieberger, Taylor, BZ '22

Casali, Melton, Strominger '22

- Resultant celestial amplitudes on backgrounds take the standard forms of low-point correlators in CFTs

$$\langle O_1 O_2 O_3 \rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}$$

determined by conformal symmetry $SL(2,\mathbb{C})$, up to a constant

✦ No δ function singularity in the coordinates of the celestial sphere

Basic questions and motivation

- Is it possible to relate the celestial amplitudes on non-trivial backgrounds to ordinary CFTs?

Celestial n-point MHV gluon amplitudes in a specific dilaton background are related to Liouville CFT

Stieberger, Taylor, BZ, '22

Tree-level: Light operators in the semiclassical (large central charge) limit of Liouville CFT play a role

Taylor, BZ, '23

Stieberger, Taylor, BZ '23

The semiclassical (large central charge) limit of Liouville CFT also appears in celestial leaf amplitudes

Melton, Sharma, Strominger, Wang '24

- What happens to the celestial chiral algebras in backgrounds?

w(1+infinity) algebra and the s-algebra are undeformed on self-dual radiative backgrounds

Adamo, Bu, BZ '23

- Scattering amplitudes or observables in general on strong field backgrounds are important and interesting. See Tim's talk at the celestial kickoff meeting.

Main message: Celestial holography and strong field QFT can teach each other

- For more examples and details of celestial holography on backgrounds, talk to

  Fan, Fotopolous, Stieberger, Taylor, BZ, Casali, Melton, Strominger, Narayanan, de Gioia, Raclariu, Gonzo, McLoughlin, Puhm, Pasterski, Costello, Paquette, Sharma, Garner, Banerjee, Mandal, Manu, Paul, Adamo, Bu, Ball, De, Srikant, Volovich, Bittleston, Heuveline, Skinner, Bogna, Kmec, Mason, Crawley, Guevara, Himwich, Cristofoli, Tourkine….

- Celestial scatterings in non-trivial backgrounds

  Celestial probe amplitudes (two-point) in shockwaves, Schwarzchild, and Kerr backgrounds were computed. *Gonzo, McLoughlin, Puhm '22*

  Black holes were treated as point-like objects. Namely, it was to the leading order in G (Born amplitudes)

  A proposed matching between the celestial probe amplitude in shockwaves and the four-point celestial eikonal amplitude at leading order in G. *de Gioia, Raclariu '22*

- What happens if we include higher-order terms in G? How to include them?

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Eikonal amplitudes from curved backgrounds I

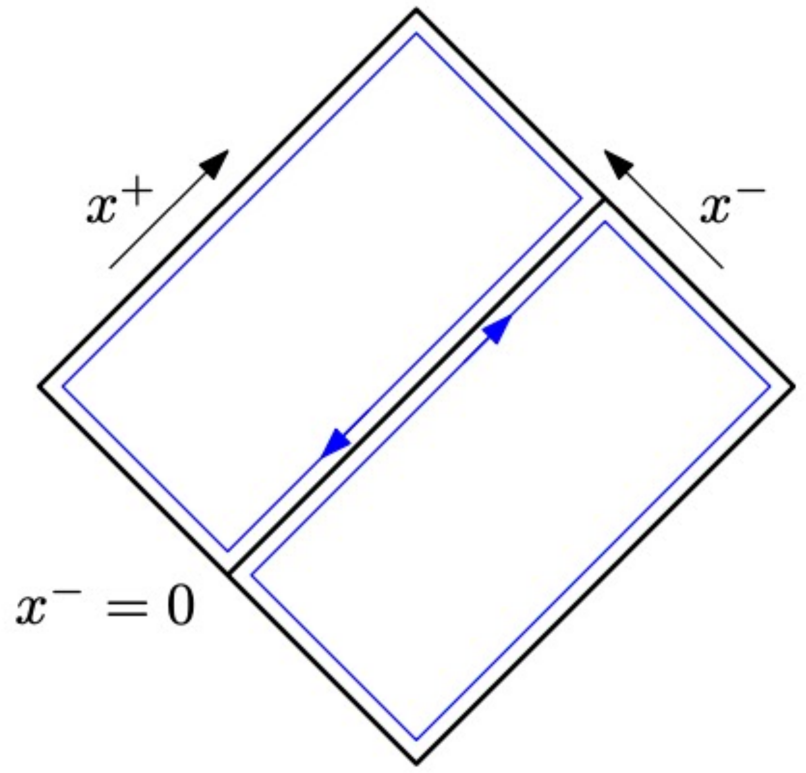
- Eikonal amplitudes of 2 → 2 scatterings of massless scalars
- When $s \gg -t$, the ladder diagrams with graviton exchanges can be resummed

The eikonal amplitudes are controlled by the eikonal phase $e^{\mathrm{i}\delta(x^\perp)}$. Perform Fourier transforms with respect to the $x^\perp$, one finds

$$\mathrm{i}\,\mathcal{M}_{\mathrm{eik}} = 4\,(p_1 \cdot p_2) \int d^2x^\perp \exp\left(\mathrm{i}\,\delta_0(x^\perp) - \mathrm{i}\,q_\perp x^\perp\right) = \mathrm{i}\,\underbrace{\frac{8\pi\,G\,s^2}{t}}_{\text{Born}}\,\underbrace{\frac{\Gamma(-\mathrm{i}\,G\,s)}{\Gamma(\mathrm{i}\,G\,s)}\left(\frac{4\,\mu^2}{-t}\right)^{-\mathrm{i}\,G\,s}}_{\text{phase}}$$

- Classical 1 → 1 probe amplitudes in a fixed, classical background sourced by another particle, evaluated at a large impact parameter

t' Hooft (1987)



In the case of shockwaves

$$M_2 = \frac{\pi\,\delta(p_{1,+} - p_{2,+})}{4\,P_-}\mathcal{M}_{\mathrm{eik}}$$

Eikonal amplitudes from curved backgrounds II

- For the cases of Schwarzchild and Kerr, using the perturbiner method, it was shown that

$$M_2 = \frac{\pi\delta(p_1^0 - p_2^0)}{2M}\mathcal{M}_{\text{eik}}$$

Adamo, Cristofoli, Tourkine '21

When the probe is a massless particle

$$\mathcal{M}_{\text{eik,Sch}}(q_\perp) = -\underbrace{\frac{32\pi G(p\cdot P)^2}{q_\perp^2}}_{\text{Born}}\underbrace{\frac{\Gamma(-\mathrm{i}\alpha(s))}{\Gamma(\mathrm{i}\alpha(s))}\left(\frac{4\mu^2}{q_\perp^2}\right)^{-\mathrm{i}\alpha(s)}}_{\text{phase}}$$

where $\alpha(s) = G(s - M^2)$

$$\mathcal{M}_{\text{eik,Kerr}}(q_\perp) = e^{-\mathrm{i}q_\perp\cdot a_\perp}\mathcal{M}_{\text{eik,Sch}}(q_\perp)$$

Celestial probe scattering amplitudes in Shockwaves

- To compute the celestial probe scattering amplitudes in shockwave backgrounds

  We use the kinematic variables written in terms of the light cone energies and coordinates on the celestial sphere for the problem

$$s = \omega_1 \, P_-$$

$$t = -\, \omega_1 \omega_2 \, z_{12} \, \bar{z}_{12}$$

Celestial probe amplitude is given by

$$\widetilde{M}_2 = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \, \omega_1^{\Delta_1 - 1} \omega_2^{\Delta_2 - 1} \, \frac{\pi \, \delta(p_{1,+} - p_{2,+})}{4 \, P_-} \, \mathscr{M}_{\text{eik}}$$

$$= \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \, \omega_1^{\Delta_1 - 1} \omega_2^{\Delta_2 - 1} \, \frac{\pi \, \delta(p_{1,+} - p_{2,+})}{4 \, P_-} \, \frac{8\pi \, G \, s^2}{t} \, \frac{\Gamma(-\mathrm{i} \, G \, s)}{\Gamma(\mathrm{i} \, G \, s)} \left(\frac{4 \, \mu^2}{-t} \right)^{-\mathrm{i} \, G \, s}$$

Celestial probe scattering amplitudes in shockwaves

Use $$-\frac{\Gamma(-\mathrm{i}\,G\,s)}{\Gamma(\mathrm{i}\,G\,s)} = = \exp(\mathrm{i}\,2\,\gamma_E\,G\,s)\,\exp\left[\sum_{k\geq 1}\frac{2\zeta(2k+1)}{2k+1}(\mathrm{i}\,G\,s)^{2k+1}\right]$$

We obtain

$$\widetilde{M}_2 = \frac{2\pi^2\,GP_-}{z_{12}\bar{z}_{12}} \times \exp\left[2\mathrm{i}\,G\,P_-\,\exp^{\partial_{\Delta_1+\Delta_2}}\,\partial_{\Delta_1+\Delta_2}\right]\exp\left[\sum_{k\geq 1}\frac{2\,\zeta(2k+1)}{2k+1}(\mathrm{i}\,G\,\,P_-\,\exp^{\partial_{\Delta_1+\Delta_2}})^{2k+1}\right]\mathscr{F}(\Delta_1+\Delta_2)$$

where $$\mathscr{F}(\Delta_1+\Delta_2) = \int_0^{+\infty} d\omega_1\omega_1^{\Delta_1+\Delta_2-2}\exp\left[\mathrm{i}\,2\gamma_E\,G\,\omega_1 P_- - \mathrm{i}G\,\omega_1\,P_-\log\left(\frac{4\mu^2}{z_{12}\bar{z}_{12}}\right)\right]$$

$$= \Gamma(\Delta_1+\Delta_2-1)\frac{\mathrm{i}^{\Delta_1+\Delta_2-1}}{\left(G\,P_-\,\log\left(\frac{C_E\,z_{12}\bar{z}_{12}}{4\mu^2}\right)\right)^{\Delta_1+\Delta_2-1}}$$

- Similar calculations can be done in Schwarzchild and Kerr black holes

Celestial eikonal amplitudes (4-point)

- We also compute the celestial eikonal amplitudes of massless scalars in a similar way

- Due to conformal symmetries on the celestial sphere, the dynamical contribution to the four-point is controlled by a function $\mathscr{A}_{\rm Eik}(\beta, z)$

where $\beta = \sum_{i=1}^{4} (\Delta_i - 1)\,, z$ is the four-point cross-ratio, following the notations in

Arkani-Hamed, Pate, Raclariu, Strominger '20

Dispersion relations

- Dispersion relation of celestial probe amplitudes in shockwave backgrounds

$$\widetilde{M}_2(\Delta_1+\Delta_2-2, z_{12}\bar{z}_{12}) = e^{i\pi(\Delta_1+\Delta_2-2)}\widetilde{M}_2^*(\Delta_1^*+\Delta_2^*-2, z_{12}\bar{z}_{12})$$

- Dispersion relation of celestial 4-point eikonal amplitudes

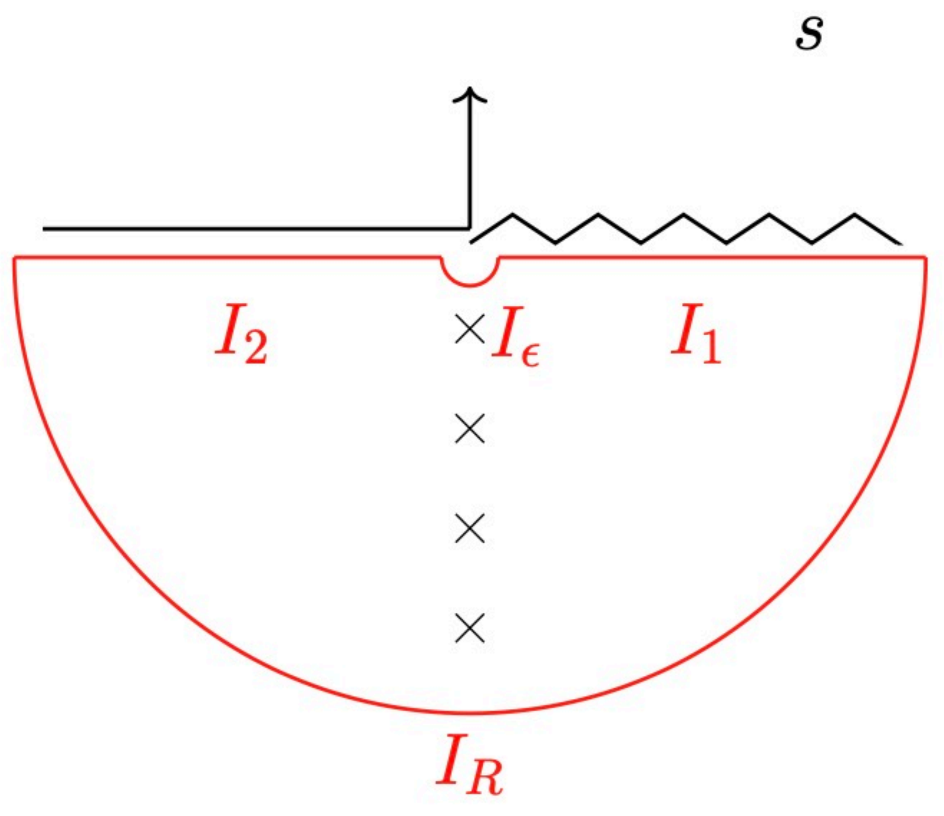


Figure 1. Contour of celestial four-point eikonal amplitudes in the complex s plane.

$$-e^{i\pi\beta}\mathscr{A}_{\text{Eik}}(\beta, z) + e^{\frac{i\pi\beta}{2}}\mathscr{A}^*_{\text{Eik}}(\beta^*, -z) = \frac{\pi}{z}\sum_{n\in\mathbb{Z}_+}\frac{i^n}{n!(n-1)!}\left(\frac{-i\,n}{G}\right)^{\frac{\beta}{2}}\left(\frac{n\,z}{4\mu^2 G}\right)^n$$

Summary and a road map for

Adamo, Bu, Tourkine, BZ '24

- We computed celestial probe amplitudes in shockwaves, Schwarzchild, and Kerr backgrounds to all orders in G. Dispersion relations were obtained.

We also

  - Find a replacement map between the celestial eikonal amplitude and the celestial probe amplitude on shockwave backgrounds
  - Compute Carrollian probe amplitudes
  - Discuss the E&M cases

Questions

- Are there any generic properties of celestial amplitudes on backgrounds? E.g., dispersion relations?

Chang, Huang, Huang, Li '21
García-Sepúlveda, Guevara, Kulp, Wu '22
Ghosh, Raman, Sinha '22

- Is there a celestial dual of the Schwarzschild black hole, and how do we describe it?

Thank you!

Celestial probe scattering amplitudes in Schwarzschild

$$\widetilde{M}_{2,\mathrm{Sch}} = \frac{8\pi^2 GM}{z_{12}\bar{z}_{12}}(1+|z_1|^2)\left(\frac{1+|z_1|^2}{1+|z_2|^2}\right)^{\Delta_2} \exp\left[2\mathrm{i}GM(1+|z_1|^2)\partial_{\Delta_1+\Delta_2}\mathrm{e}^{\partial_{\Delta_1+\Delta_2}}\right]$$
$$\times \exp\left[\sum_{k\geq 1}\frac{2\zeta(2k+1)}{2k+1}(\mathrm{i}GM(1+|z_1|^2)\mathrm{e}^{\partial_{\Delta_1+\Delta_2}})^{2k+1}\right]\mathcal{F}(\Delta_1+\Delta_2)_{\mathrm{Sch}}\,, \tag{3.27}$$

where the 'primary' function becomes a conformal two point function with slightly more complicated kinematics dependence:

$$\mathcal{F}(\Delta_1+\Delta_2)_{\mathrm{Sch}} =$$
$$= \int_0^\infty d\omega_1 \omega_1^{\Delta_1+\Delta_2-2}\exp\left[2\mathrm{i}\gamma_E GM(1+|z_1|^2)\omega_1 - \mathrm{i}GM(1+|z_1|^2)\omega_1\log\left(\frac{4\mu^2}{z_{12}\bar{z}_{12}}\frac{1+|z_2|^2}{1+|z_1|^2}\right)\right]$$
$$= \Gamma(\Delta_1+\Delta_2-1)\frac{\mathrm{i}^{\Delta_1+\Delta_2-1}}{\left[GM(1+|z_1|^2)\log\left(\frac{c_E z_{12}\bar{z}_{12}(1+|z_1|^2)}{4\mu^2(1+|z_2|^2)}\right)\right]^{\Delta_1+\Delta_2-1}}\,. \tag{3.28}$$

Replacement map

$$\mathscr{A}(\beta, z) \quad \longrightarrow \quad \widetilde{M}_2(\Delta_1 + \Delta_2 - 2, z_{12}\bar{z}_{12})$$

$$\begin{aligned} G &\to GP_- \,, \\ z &\to z_{12}\bar{z}_{12} \,, \\ \beta &\to 2(\Delta_1 + \Delta_2 - 2) \,, \end{aligned} \tag{4.11}$$

Modulo a caveat of one term in the exponent