The AdS Virasoro-Shapiro Amplitude

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Based on work with Tobias Hansen.

What will this talk be about?

A set of tools to compute String Theory amplitudes on AdS

More specifically:

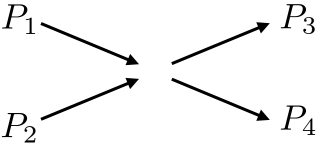
Scattering of four massless closed strings (gravitons) on $AdS_5 \times S^5$.

- Flat space review.
- AdS curvature corrections (efficiently).

Scattering amplitudes

Scattering Amplitudes

Probability that two particles/strings colliding (with momenta p_1, p_2) result into two other particles (with momenta p_3, p_4).



$$= \quad A(g, s, t, u)$$

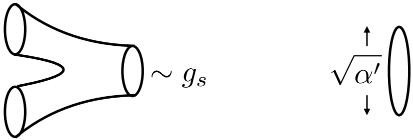
- $A(g, s, t, u)$ depends on:
  - The particles you are scattering (their masses, polarisations, etc)
  - The parameters of your theory g.
  - The momenta of the particles being scattered:

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2$$

Four-graviton amplitude - Flat space

4pt graviton amplitude in string theory in flat space

- The parameters of the theory are g_s and α'.

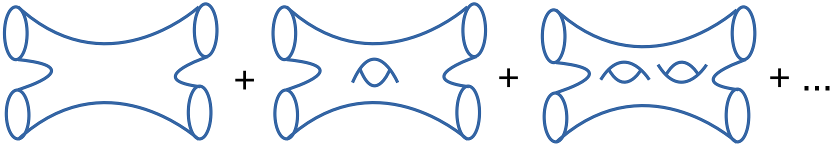


- The amplitude depends on the momenta p_i and polarisations ϵ_i of the (external) gravitons.
- SUSY fixes the dependence on the polarisations:

$$A(g_s, \alpha', p_i, \epsilon_i) = \underbrace{pref(\epsilon_i, p_i)}_{\text{simple prefactor}} \times \underbrace{A(g_s, \alpha', s, t, u)}_{\text{we focus on this}}$$

String theory scattering amplitudes

- The computation organises in a genus expansion



$$A^{(genus\ 0)}(\alpha', s, t, u) + g_s^2 A^{(genus\ 1)}(\alpha', s, t, u) + g_s^4 A^{(genus\ 2)}(\alpha', s, t, u) + \cdots$$

- In flat space we can use the world-sheet theory to compute these amplitudes:

$$A^{(genus\ 0)}(\alpha', s, t, u) \sim \int_{CP^1} |z|^{2\alpha' s-2} |1-z|^{2\alpha' t-2} d^2 z$$

- Note: already at genus-one the expressions are tremendously complicated!

Four-graviton amplitude - Flat space

Leading order in g_s: Virasoro-Shapiro amplitude

$$A_{VS}(\alpha', s, t, u) = \alpha'^3 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)\Gamma(-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}$$

- Crossing symmetric $(s+t+u=0)$
- Poles due to the exchange of particles (of mass $m = 2\sqrt{n/\alpha'}$ and spin ℓ)

$$A_{VS}(\alpha', s, t, u) \sim \frac{P_\ell(t,u)}{\alpha' s - n}$$

- Regge behaviour

$$A_{VS}(\alpha', s, t, u) \sim s^{-2+\alpha' \frac{t}{2}}, \quad \text{for large } |s|$$

- Low energy expansion (powers of α')

$$A_{VS}(\alpha', s, t, u) \sim \underbrace{\frac{1}{s\,t\,u}}_{\text{sugra}} + \underbrace{2\zeta(3)\alpha'^3 + 2\zeta(5)\alpha'^5(s^2+t^2+u^2) + \cdots}_{\text{stringy corrections}}$$

VS and single-valued periods

- Less appreciated: only odd ζ-values appear in the α' expansion.

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

- Zeta values (MZV) can be defined in terms of polylogarithms evaluated at $z = 1$

$$Li_n(z) = \sum_{k=1} \frac{z^k}{k^n} \to Li_n(1) = \zeta(n)$$

- While these series converge for $|z| < 1$, polylogarithms can be analytically continued to the whole complex plane:

$$Li_1(z) = -\log(1-z), \quad Li_n(z) = \int_0^z Li_{n-1}(t) \frac{dt}{t}$$

- However these functions are not single-valued.

VS and single-valued periods

- Unique map from multi-valued to single-valued polylogarithms

$$Li_n(z) \xrightarrow{SV} \mathcal{L}_n(z,\bar{z})$$

- Such that the 'weight' and differential relations are preserved.
- If we evaluate single-valued polylogarithms at $z=\bar{z}=1$, we only get a subset of the usual zeta values.

$$\mathcal{L}_2(1)=0, \quad \mathcal{L}_3(1)=2\zeta(3)$$

- More generally
  - $\zeta(2n+1)$ **are** single-valued ✓
  - But $\zeta(2n)$ **are not** ✗

Important message

The α' expansion of the VS amplitude contains only single-valued zetas.

Virasoro Shapiro on AdS

Can we compute string amplitudes on curved space-time?

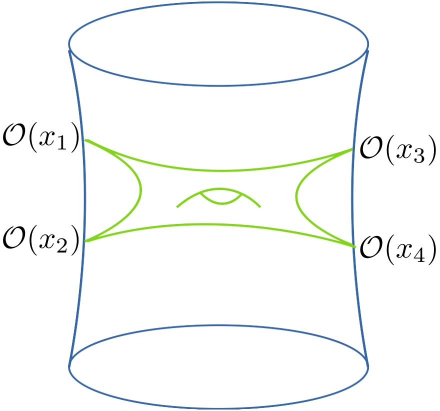
- One of the biggest challenges of string theory is to understand it on curved backgrounds.
- No world-sheet theory to compute amplitudes in general.
- The highly symmetric case $AdS_5 \times S^5$ is particularly interesting, as here we have powerful tools!

Can we compute the AdS Virasoro-Shapiro amplitude?

i.e. tree-level amplitude for four gravitons in string theory on $AdS_5 \times S^5$.

AdS/CFT

String amplitudes on *AdS* $\leftrightarrow$ Correlators of local operators in the CFT at the boundary.



$$\mathcal{A}(g_s, \alpha', s, t, u) \leftrightarrow \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$$

AdS/CFT

String theory on $AdS_5 \times S^5$ $\leftrightarrow$ 4d $\mathcal{N}=4$ SYM

(g_s, R) $\qquad$ (g_{YM}, N)

$$g_s \approx \frac{1}{N}, \qquad \frac{R^2}{\alpha'} = \sqrt{g_{YM}^2 N} \equiv \sqrt{\lambda}$$

String amplitudes on $AdS_5 \times S^5$	Correlators in $\mathcal{N}=4$ SYM
Genus expansion	$1/N$ expansion
Stringy corrections to sugra	$1/\lambda$ corrections
Graviton on AdS	$\mathcal{O}_2$: Scalar operator of dim. 2 in the stress-tensor multiplet

Consider $\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$ in a $1/N$ expansion.

4d $\mathcal{N} = 4$ SYM - 4p correlator

- Conformal symmetry + susy:

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle = pref \times \mathcal{G}(U,V)$$

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- Tree-level amplitude $\rightarrow$ leading non-trivial term in the $1/N$ expansion:

$$\mathcal{G}(U,V) = \underbrace{\mathcal{G}_{disc}(U,V)}_{disconnected} + \frac{1}{N^2}\underbrace{\mathcal{G}_{tree}(U,V)}_{tree-level} + \cdots$$

- Complicated function of λ. We keep the full dependence.

$$\mathcal{G}_{tree}(U,V) = \underbrace{\underbrace{\mathcal{G}^{(sugra)}(U,V)}_{\text{computed 22 years ago}} + \frac{\mathcal{G}^{(1)}(U,V)}{\lambda^{3/2}} + \frac{\mathcal{G}^{(2)}(U,V)}{\lambda^{5/2}} + \cdots}_{\text{we want the whole tower}}$$

The right language: Mellin space

$\mathcal{G}_{tree}(U,V) \to \mathcal{M}_{tree}(s,t,u)$, with $s+t+u=4$.

$$\mathcal{G}_{tree}(U,V) = \int_{-i\infty}^{i\infty} dsdt U^s V^t \underbrace{\Gamma(s,t,u)}_{\text{prefactor}} \quad \underbrace{\mathcal{M}_{tree}(s,t,u)}_{\text{VS amplitude in } AdS_5\times S^5}$$

1. Crossing symmetric.
2. Exchanged operators lead to simple poles:

$$\mathcal{M}_{tree}(s,t) = C^2_{\Delta,\ell} \sum_{m=0}^{\infty} \frac{Q_{\ell,m}(u,t)}{s-(\Delta-\ell)-2m} + \text{regular}$$

3. Regge limit

$$\mathcal{M}_{tree}(s,t) \sim s^{-2}, \quad \text{for large } |s| \text{ and } Re(t) < 2$$

4. Low energy expansion

$$\mathcal{M}_{tree}(s,t) = \frac{1}{(s-2)(t-2)(u-2)} + \text{stringy corrections on } AdS$$

AdS Virasoro-Shapiro around flat space

5. Flat space limit (large s, t, λ with $s/\sqrt{\lambda}, t/\sqrt{\lambda}$ fixed) $\rightarrow$ usual VS.

Consider $\mathcal{M}_{tree}(s,t)$ around flat space

$$\mathcal{M}_{tree}(s,t) = \underbrace{A^{(0)}(s,t)}_{\text{VS in flat space}} + \underbrace{\frac{\alpha'}{R^2}A^{(1)}(s,t) + \frac{\alpha'^2}{R^4}A^{(2)}(s,t) + \cdots}_{\text{curvature corrections}}$$

- Where each bit admits a low energy expansion

$$A^{(0)}(s,t) = \frac{1}{s\,t\,u} + 2\zeta(3)\alpha'^3 + 2\zeta(5)\alpha'^5(s^2+t^2+u^2) + \cdots$$

$$A^{(1)}(s,t) = \underbrace{\frac{s^2+t^2+u^2}{(s\,t\,u)^2}}_{\text{gravity on } AdS} + \underbrace{\alpha_1\alpha'^4 + \alpha_2\alpha'^6(s^2+t^2+u^2) + \cdots}_{\text{unknown coefficients}}$$

Key assumption: unknown coefficients are also single-valued zetas!

AdS Virasoro-Shapiro around flat space

Very powerful when supplemented with the correct structure of poles!

- While $A^{(0)}(s,t)$ has single poles, corrections are more complicated:

$$A^{(1)}(s,t) \sim \frac{r_n^{(0)}(t)}{(\alpha' s - n)^4} + \frac{r_n^{(1)}(t)}{(\alpha' s - n)^3} + \cdots$$

- This follows from the AdS-propagator around flat-space (and also the dispersive sum rules). In general

$$\mathcal{M}_{tree}(s,t) = \underbrace{A^{(0)}(s,t)}_{\text{simple poles}} + \frac{\alpha'}{R^2} \underbrace{A^{(1)}(s,t)}_{\text{quartic poles}} + \frac{\alpha'^2}{R^4} \underbrace{A^{(2)}(s,t)}_{\text{seventh order poles}} + \cdots$$

AdS Virasoro-Shapiro amplitude

Poles + Single-valuedness + World-sheet intuition

$\Downarrow$

Proposal order by order

$$A^{(0)}(s,t) = \int_{CP^1} d^2z |z|^{2\alpha' s-2} |1-z|^{2\alpha' t-2}$$

$$A^{(1)}(s,t) = \int_{CP^1} d^2z |z|^{2\alpha' s-2} |1-z|^{2\alpha' t-2} \underbrace{W_3(z,\bar{z})}_{\text{SV polylogs of weight 3}}$$

$$A^{(2)}(s,t) = \int_{CP^1} d^2z |z|^{2\alpha' s-2} |1-z|^{2\alpha' t-2} \underbrace{W_6(z,\bar{z})}_{\text{SV polylogs of weight 6}}$$

$$\vdots$$

Also consistent with soft graviton theorems.

AdS Virasoro-Shapiro amplitude

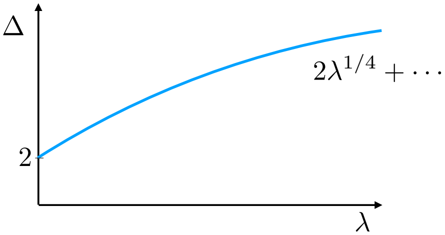
- At a given order only a finite number undetermined coefficients and the structure of poles is already very constraining.

We fixed $A^{(0)}(s,t), A^{(1)}(s,t), A^{(2)}(s,t)$ fully by our procedure!

- From the answer we can read of a wealth of CFT-data, *e.g.* there is an intermediate operator with

$$\Delta_{\mathcal{K}} = 2\lambda^{1/4} - 2 + \frac{2}{\lambda^{1/4}} + \frac{1/2 - 3\zeta(3)}{\lambda^{3/4}} + \cdots$$

But this is the Konishi operator!



Conclusions

Computing the full AdS VS amplitude seems now within reach!

- Single valuedness plays an important role in understanding and constructing scattering amplitudes in flat space. Now also in *AdS*!
- New connections between standard bootstrap techniques, localisation, integrability and number theory.
- The high energy limit is super interesting **[See Maria Nocchi's poster]**.
- This has been extended to open strings. **[2403.13877]**.

Using this answer as a guiding principle, can we reproduce this with a direct World-sheet computation?