The Statistical Interpretation of Semi-Classical Gravity

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-Alex Belin, JdB, arXiv:2006.05499
-Alex Belin, JdB, Diego Liska, arXiv:2110.14649
-Alex Belin, JdB, Pranyal Nayak,
 Julian Sonner, arXiv:2111.06373
-Tarek Anous, Alex Belin, JdB, Diego Liska, arXiv:2112.09143
-Alex Belin, JdB, Daniel Jafferis, Pranyal Nayak, Julian Sonner,
arXiv:2308.03829
-JdB, Jildou Hollander, Andrew Rolph, arXiv:2311.07655
-JdB, Diego Liska, Boris Post, Martin Sasieta, arXiv:2311.08132

New York, Simons Celestial Satellite, 9 April 2024

Semi-classical gravity (=sum over saddle points plus perturbative computations) provides approximate answers only and therefore contains incomplete information.

Lesson from physics: if you have incomplete information apply the principles of statistical physics.

A concrete approach (which is the basis for statistical physics and Bayesian inference) is to:

Build a description which maximizes ignorance/entropy while being compatible with observations/computations

Such a description may not be microscopically correct but it is the best we can do..

Example: Suppose that we have a quantum mechanical system whose Hamiltonian H we know, and suppose that we measure the energy of the system. What is the best possible description in the absence of any other information?

Answer: maximize

$$-\mathrm{Tr}(\rho\log\rho) + \lambda(\mathrm{Tr}(\rho H) - E)$$

Entropy = "ignorance" of the state of the system

Lagrange multiplier which enforces energy constraint.

Vary ρ to get

$$-\mathrm{Tr}(\delta\rho\log\rho) + \lambda\mathrm{Tr}(\delta\rho H) = 0 \Rightarrow \rho \sim e^{\lambda H} \Rightarrow \rho = Z^{-1}e^{-\beta(E)H}$$

One can play a similar game for much more general choices of data. Suppose for example that we know some correlators of an operator A and we want to extract a probability distribution $\mu[A]$ on the space of operators.

The general picture is one where if one e.g. inputs connected ≤k-point correlators, one gets a “matrix model” with up to k-th order interactions in the exponent.

Lagrange multipliers

$$\int dA d\lambda_i \left(-\mu[A] \log \mu[A] + \sum_i \lambda_i \, \mu[A] \left(f_i[A] - c_i \right) \right)$$

Shannon entropy

Input observations

$$\Rightarrow \mu[A] \sim e^{-\sum_i \lambda_i f_i[A]}$$

If we input only one- and two-point functions we get a quadratic matrix model. Famously, the spectrum has level repulsion. This is one way to interpret the success of random matrix theory to model the spectra of nuclei of heavy atoms by Wigner (1955). This interpretation goes back to Balian (1968).

If random matrix descriptions are related to ignorance, one might expect they play an important role in chaotic systems, as chaos makes systems unpredictable and renders the energy spectrum unsolvable.

Indeed, the Bohigas–Giannoni–Schmit (BGS) conjecture(1984) asserts that the spectral statistics of quantum systems whose classical counterparts exhibit chaotic behavior are described by random matrix theory.

Yet another example: if we apply this reasoning to a system where the information we have available are approximate finite temperature two-point functions of simple operators, the result is the so-called Eigenstate Thermalization Hypothesis:

$$\langle E_i | O_a | E_j \rangle = \delta_{ij} f_a(\bar{E}) + e^{-S(\bar{E})/2} g_a(\bar{E}, \Delta E) R^a_{ij}$$

Deutsch ’91
Srednicki ’94
Foini, Kurchan ‘19

$f_a(\bar{E})$: one point functions of simple operators
$g_a(\bar{E}, \Delta E)$: two point functions of simple operators
R^a_{ij} : Gaussian random variables

$$\langle R^a_{ij} \rangle = 0, \qquad \langle R^a_{ij} R^b_{kl} \rangle = \delta^{ab} \delta_{il} \delta_{jk}$$

JdB, Liska, Post, Sasieta, ‘23

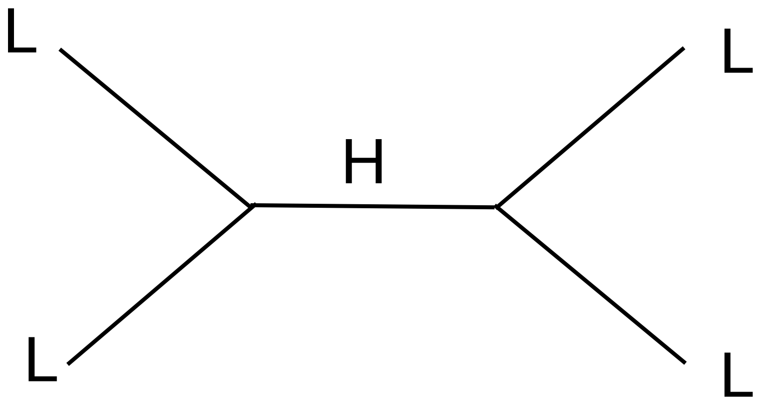
We now apply this logic to semi-classical gravity in AdS.

Semi-classical gravity is an incomplete theory so we will inevitably get a statistical description which contains all (not necessarily consistent) microscopic theories which are semi-classically indistinguishable.

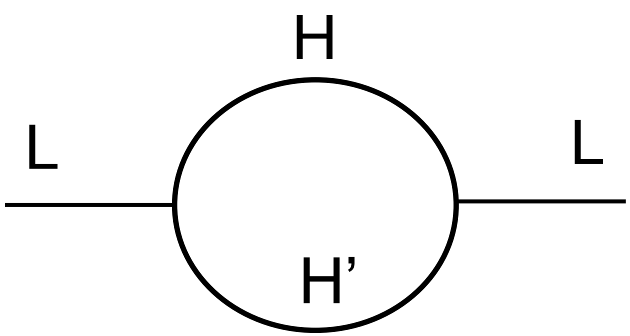
Semi-classical gravity involves a coarse-graining over high-energy microstates (e.g. semi-classical black hole entropy). Therefore the ignorance and statistical description will mostly involved the high-energy spectrum.

For 2d theories with a holographic AdS3 dual:

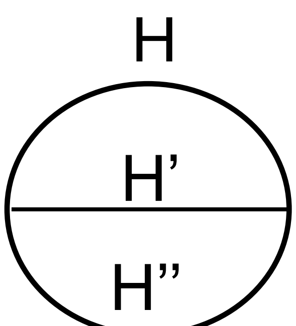
- we only have explicit access to low-lying operators (denoted L) and not to very high dimension operators corresponding to black holes (denoted H)
- we can compute correlation functions where the number of operators is <<c
- we can compute partition functions on surfaces with genus <<c
- we can compute correlation functions in Lorentzian signature as long as the center of mass is sub Planckian
- all computations are at best done up to non-perturbative errors of order e^{-c}



$$\sum_H C^2_{LLH}$$



$$\sum_{H,H'} C^2_{LHH'}$$



$$\sum_{H,H',H''} C^2_{HH'H''}$$

Input gives rise to quadratic matrix model for the C's

This is what gave rise to the OPE randomness hypothesis (Belin, JdB ’20):

$$C_{LLH} \sim f(E_H) R_H$$

$$C_{LHH'} \sim f(\bar{E_H}, \Delta E_H) R_{HH'}$$

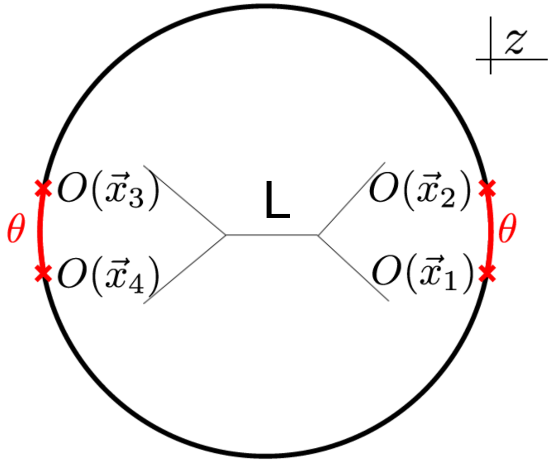
$$C_{HH'H''} \sim f(\bar{E_H}, \Delta E_H) R_{HH'H''}$$

Example of ETH

Slowly varying function of arguments

- Pseudorandom
- Mean=0
- Variance=1
- Can have higher moments which are exponentially suppressed.

Example:



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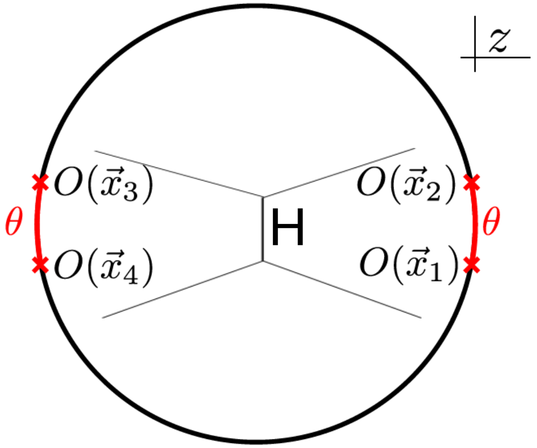


$$\frac{1}{\theta^{4\Delta_L}} \simeq \sum_H C_{LLH}^2 \left(\cos\frac{\theta}{2}\right)^{2\Delta_H}$$

$$\overline{|C_{LLH}|^2} \sim \frac{\Delta_H^{2\Delta_L-1}}{\Gamma(2\Delta_L)\rho(\Delta_H)}$$

Pappadopulo, Rychkov, Espin, Ratazzi ‘12

To model the spectrum of a 2d CFT (Cardy density of states) we can use a matrix representing the Hamiltonian. (cf JT)

In 2d, the result of all of this is a mixed matrix/tensor model which encodes statistics in the spectrum and statistics of OPE coefficients.

cf Jafferis, Kolchmeyer, Mukhametzhanov, Sonner ’22

In d>2, we do not know what the minimal set of data is to fully describe a CFT, but whatever those are, we get a corresponding statistical model. (Casimir energy on T^{d-1} is not obviously expressible in terms of Δ_i and C_{ijk})

Belin, JdB, Kruthoff, Michel, Shaghoulian, Shyani ’16
Belin, JdB, Kruthoff’’18

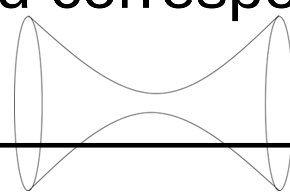
In this way we build a statistical model using gravitational computations with a single boundary.

This yields a “single trace” matrix/tensor model.

Statistical models predict correlations between multiple copies of the theory.

$$\int dH\,\mu[H]\mathrm{Tr}(e^{-\beta_1 H})\mathrm{Tr}(e^{-\beta_2 H}) - \int dH\,\mu[H]\mathrm{Tr}(e^{-\beta_1 H}) \int dH\,\mu[H]\mathrm{Tr}(e^{-\beta_2 H}) \neq 0$$

In gravity, these should correspond to connected wormhole geometries.



Conjecture: wormholes compute the correlations of the one-sided statistical model. They contain no new information.

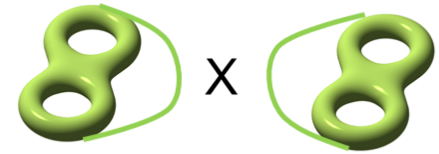
Intuition for the conjecture: one-sided computations allow one to reconstruct the bulk Lagrangian. Crossing symmetry is closely related to bulk locality. So all information which is needed to compute wormholes semi-classically is *in principle* available

This conjecture has been tested fairly extensively (Alex Belin, JdB '20; Chandra, Collier, Hartman, Maloney '22) but in remainder will instead describe:

- A wormhole prediction
- Connection to off-shell gravitational configurations
- A matrix/tensor model for 3d gravity?
- Statistical interpretation versus the page curve
- An alternative: state averaging

Prediction: a new wormhole?

$$Z_{g=2\times g=2} = \left\langle \left(\sum_{i,j,k} C_{iij}C^*_{jkk}e^{-3\beta\Delta}\right)\left(\sum_{lmn} C_{llm}C^*_{mnn}e^{-3\beta\Delta}\right)\right\rangle$$



$$Z_{g=2\times g=2} = \left\langle \left(\sum_{i,j,k} C_{iij}C^*_{jkk}e^{-3\beta\Delta}\right)\left(\sum_{l,m,n} C_{llm}C^*_{mnn}e^{-3\beta\Delta}\right)\right\rangle$$

$$Z_{g=2\times g=2} = \left\langle \left(\sum_{i,j,k} C_{iij}C^*_{jkk}e^{-3\beta\Delta}\right)\left(\sum_{l,m,n} C_{llm}C^*_{mnn}e^{-3\beta\Delta}\right)\right\rangle$$

????

Vertex computed independently from the genus 3 partition function

The quartic vertex dominates over the second wormhole contribution.

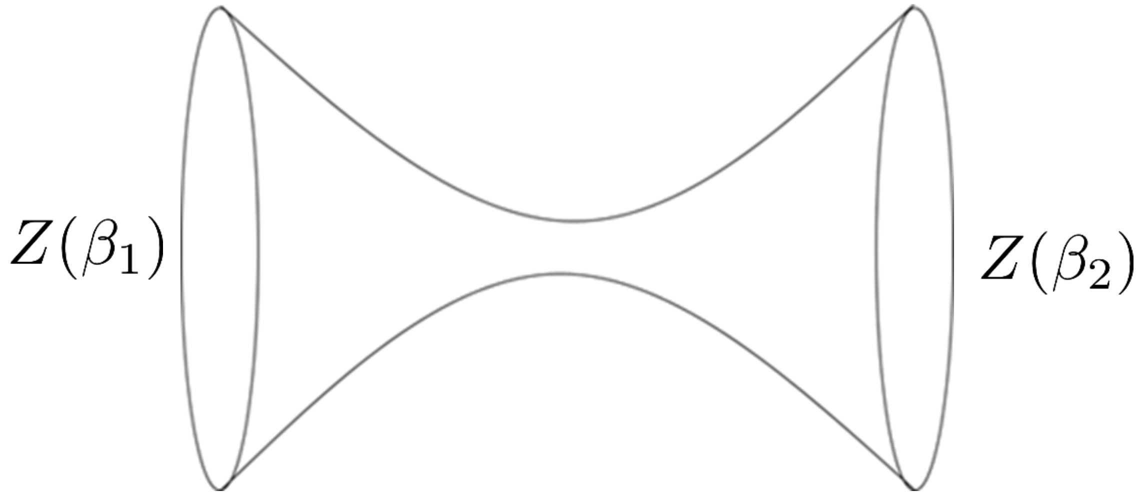
Suggests that there exists a new wormhole connecting two genus two Riemann surfaces with action

$$Z = e^{\frac{25c-360\Delta_\chi}{288}\frac{\pi^2}{\beta}}$$

lightest scalar

Result suggest that this is a wormhole supported by matter fields. Would be interesting to construct it explicitly. Can be constructed in Virasoro TQFT of (Collier, Eberhardt, Zhang ’23 ’24)

Applications to off-shell gravitational solutions?



Cotler, Jensen ’21 – see also Di Ubaldo, Perlmutter ’23 and Haehl, Reeves, Rozali ‘23

The off-shell gravity computation agrees to leading order with the universal random matrix theory result

$$\langle Z(\beta_1) Z(\beta_2)\rangle = Z(\beta_1) Z(\beta_2) + \frac{1}{2\pi} \frac{\sqrt{\beta_1 \beta_2}}{\beta_1 + \beta_2} + \dots$$

Ambjørn, Jurkiewicz, Makeenko ’90
Saad, Shenker, Sanford ’19

Individual off-shell geometries do not appear to have a simple interpretation in the dual CFTs.

JdB, Chandra, WIP

One can try to generalize the above by including a matter field to compute the Loschmidt Spectral Form Factor

$$\left\langle \mathrm{Tr} e^{-\beta_1 (H+\epsilon_1 \mathcal{O})} \mathrm{Tr} e^{-\beta_2 (H+\epsilon_2 \mathcal{O})} \right\rangle_c$$

Winer, Swingle ‘22

At late times this does not have a linear ramp but decays as $Te^{-\lambda T}$. Can be reproduced from wormhole with matter fields and also from

Cotler, Jensen, ‘21

$$\left\langle \mathrm{Tr} e^{-\beta_1 (H+\epsilon_1 \mathcal{O})} \mathrm{Tr} e^{-\beta_2 (H+\epsilon_2 \mathcal{O})} \right\rangle_c$$

JdB, Chandra, WIP

A Matrix/Tensor model for 3d gravity?

Recall that for 2d theories with a holographic AdS3 dual:

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- we can compute correlation functions where the number of operators is <<c
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In arXiv:2308.03829 we called a set of conformal dimensions and OPE coefficients for which these computations approximately obey the CFT axioms (crossing and modular invariance of 1-pt functions) an *approximate CFT*

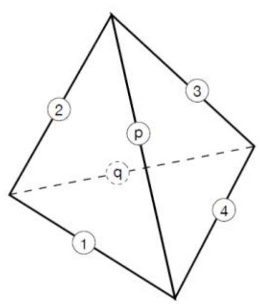
Note: by changing multiple conformal dimensions of heavy operators in a coordinated way, can prove every 2d CFT sits in an island of approximate CFTs

Note: the opposite is not obviously true. An approximate CFT may not be close to an actual CFT. Possible example: approximate 2d CFTs defined by pure 3d AdS gravity.

The idea is now to average over all CFT2 data with a spectrum which is very close to that of 3d gravity, and with a weight schematically of the form

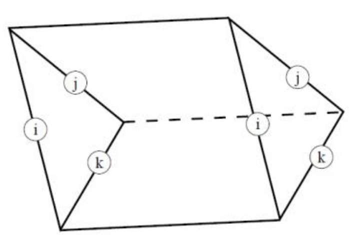
$$P(\Delta_i, C_{ijk}) \sim \exp\left(-a \sum (\text{axioms})^2\right)$$

Result is a quartic tensor model with Feynman rules



$$= \quad \begin{Bmatrix} \mathcal{O}_q & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O}_p & \mathcal{O}_4 & \mathcal{O}_3 \end{Bmatrix}$$

Virasoro 6j symbol



$$= \quad C_0\left(P_i, P_j, P_k\right) C_0\left(\bar{P}_i, \bar{P}_j, \bar{P}_k\right)$$

Crossing:

$$\sum_H C_{LLH} C_{LLH} \simeq C_{LL\mathbb{I}} C_{LL\mathbb{I}}$$

Square of crossing

$$\left(\sum_H C_{LLH} C_{LLH} - C_{LL\mathbb{I}} C_{LL\mathbb{I}} \right) \left(\sum_H C_{LLH} C_{LLH} - C_{LL\mathbb{I}} C_{LL\mathbb{I}} \right) \simeq 0$$

This is reminiscent of various other discrete descriptions of 3d gravity.

Regge ’61; Boulatov ‘92; Turaev Viro ‘92

It is also connected to the so-called Teichmüller TQFT (Andersen, Kishaev ‘11 ’13) which was recently connected to 3d gravity (Collier, Eberhardt, Zhang ’23 ’24) (Jafferis, Rosenberg, Wong to appear)

To be continued….

Statistical interpretation versus the Page curve

JdB, Hollander, Rolph, '23

If semi-classical gravity is described by a coarse-grained statistical ensemble, how can semi-classical gravity know about unitarity of black hole creation/evaporation?

Key: *time evolution maps an initial state to a classical statistical mixture of pure states*

Can be illustrated with a simple model which is a cartoon of the microcanonical Hilbert space of a black hole coupled to a bath

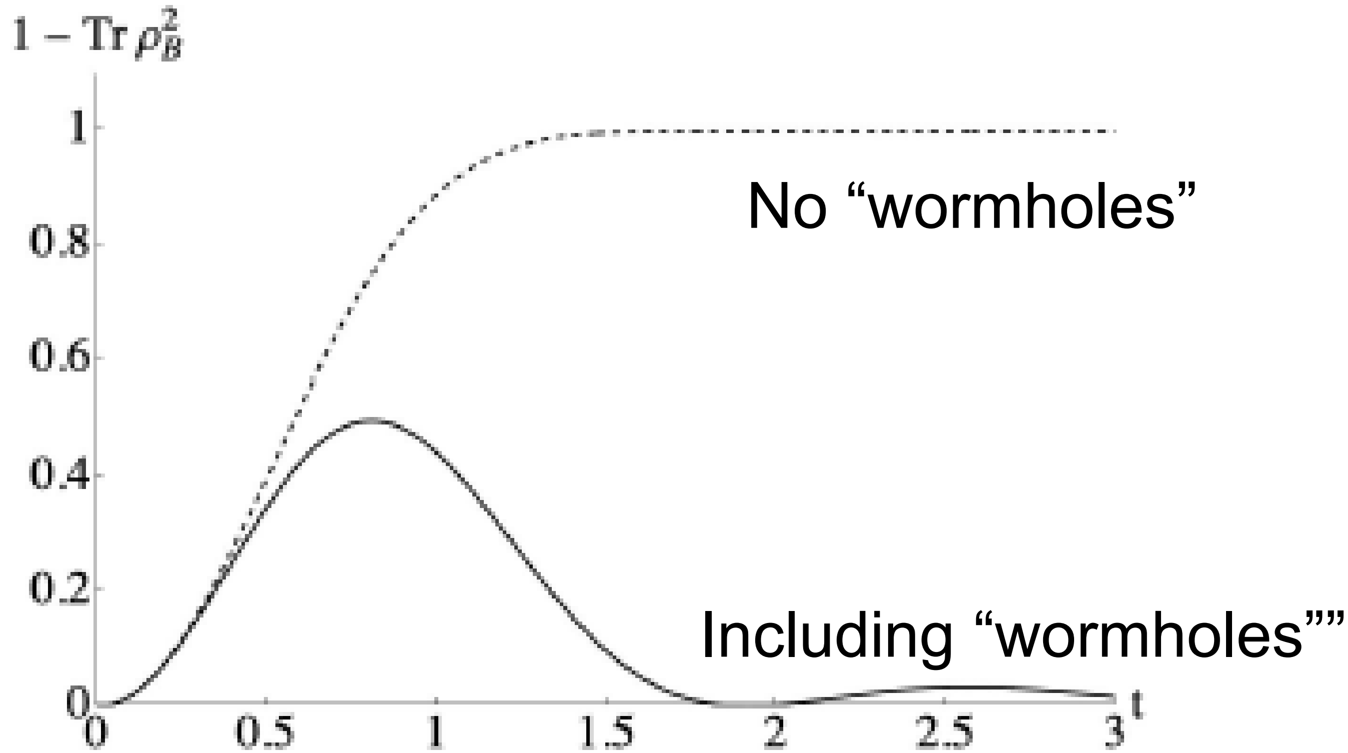
$$\mathcal{H} = (|1\rangle \otimes \mathcal{H}_1) \oplus (|0\rangle \otimes \mathcal{H}_0)$$

Black hole

Vacuum

with $\dim \mathcal{H}_1 \ll \dim \mathcal{H}_0$

Full random unitary dynamics:



Features:

$$g(t) = \mathrm{Tr}(e^{iHt})\mathrm{Tr}(e^{-iHt})$$

- Timescale is set by spectral form factor of matrix model

- “Wormholes” are related to types of contractions of random unitaries.

- Model has dynamics as opposed to many discrete qubit setups people have considered

- Mechanism is very simple and robust

- The final result is a classical statistical ensemble of pure states, not a mixed state. A replica computation like $\overline{\mathrm{Tr}(\rho^2)}$ can distinguish the two. A non-replica computation can not.

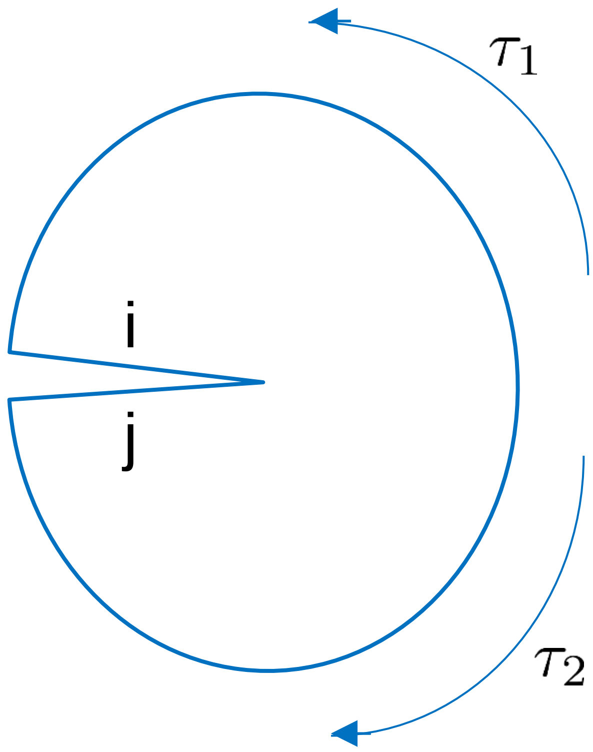
- No prediction for final state, just for unitarity.

State Averaging

So far we looked at semi-classical gravitational computations with a closed boundary.

However, we can also use gravitational path integrals with boundaries to semi-classically produce states.

Freivogel, Nikolapoulou, Rotundo ‘21
Chadra, Hartman ‘22
Penington, Shenker, Stanford, Yang ’19
Bah, Chen, Maldacena ’22
Goel, Lam, Turiaci, Verlinde ‘18
Balasubramanian, Lawrence, Magan, Sasieta ’22
JdB, Liska, Post, Sasieta ‘23



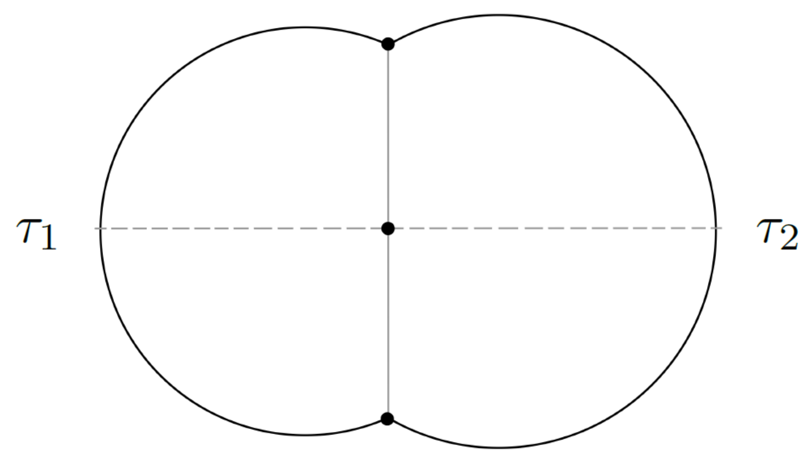
One can derive a suitable state-averaging ansatz for an open path integral.

More precisely, we consider purification of density matrices

$$|\Psi_{\mathrm{sc}}\rangle = \sum_{i,\alpha} A_{i\alpha} |E_i\rangle |E_\alpha\rangle \longrightarrow \rho = \sum_{i,j,\alpha} A_{i\alpha} A^\dagger_\alpha |E_i\rangle\langle E_j|$$

and assume such states can be prepared semiclassically (e.g. TFD state or PETS states). We then compute semiclassical overlaps of the form

$$\mathbf{Z}(\tau_1, \tau_2) := \left\langle \Psi_{\mathrm{sc}} \right| \mathrm{e}^{-\tau_1 H_L} \mathrm{e}^{-\tau_2 H_R} \left| \Psi_{\mathrm{sc}} \right\rangle =$$



and apply the maximal ignorance philosophy to obtain a quadratic matrix model for A.

Result:

$$\langle E_i|\rho|E_j\rangle = \delta_{ij}\,\bar{\rho}(E_i) + \frac{e^{-\beta\bar{E}_{ij}}}{Z(\beta)} e^{-S(\bar{E}_{ij})/2} j(\bar{E}_{ij},\omega_{ij})^{1/2} R_{ij}$$

Provides an alternative picture to OPE/spectral statistics. It more directly describes a coarse graining at the level of states. It is in particular useful for cutting/gluing constructions of correlation functions.

It reproduces many results of the OPE/spectral statistics picture.

Interesting feature: $\overline{S(\rho|\rho_\beta)} \sim \mathcal{O}(1)$

JdB, Liska, Post, Sasieta, ‘23

Interestingly, the matrix model for A also gives rise to non-gaussianities for ρ

$$\overline{\delta\rho_{ij}\delta\rho_{kl}\delta\rho_{mn}}^{\,\text{conn.}} = \sum_{\alpha,\beta,\gamma} A_{i\alpha}A^*_{j\alpha}A_{k\beta}A^*_{l\beta}A_{m\gamma}A^*_{n\gamma}$$

This makes the model subtly different from operator/spectral averages models, for example

$$\overline{\mathrm{Tr}(\rho\mathcal{O})\,\mathrm{Tr}(\rho\mathcal{O})}^{\,\text{conn.}} = \sum_{ij}\overline{|\delta\rho_{ij}|^2}\,\mathcal{O}_{ij}\mathcal{O}_{ji} \qquad \text{(state averaging)}$$

$$\langle\!\langle \mathrm{Tr}(\rho_\beta\mathcal{O})\,\mathrm{Tr}(\rho_\beta\mathcal{O})\rangle\!\rangle^{\text{conn.}} = \sum_i(\rho_\beta)^2_{ii}\langle\!\langle\mathcal{O}_{ii}\mathcal{O}_{ii}\rangle\!\rangle^{\text{conn.}} \qquad \text{(operator averaging).}$$

Upshot:

The statistical interpretation of semi-classical gravity as constructed from a single boundary may provide an interesting new (discrete? combinatorial?) description of gravity.

Beyond this, it is instrumental in interpreting and understanding the results obtained in any semi-classical computation involving copies of the theory, or replicas of the theory.

COMMENT #1

It is an interesting question what type of coarse graining underlies semi-classical gravity.

Both operator and state averaging are able to capture many semi-classical results.

Perhaps both are equally valid points of view and related by a suitable complicated change of variables?

COMMENT #2

None of the above implies that AdS/CFT fundamentally requires averaging. Averaging is purely a consequence of the semi-classical approximation. As one improves the description the averaging should become over increasingly smaller sets of data and ultimately disappear.

If the set of data would not become smaller as one would increase accuracy then the dual description would indeed be a proper average. This is what e.g. happens in topological theories like JT gravity.

But there currently is no evidence that anything like this is happening in standard examples of AdS/CFT.

COMMENT #3

It is an interesting question what the minimum number of ingredients are that we need to add to semiclassical gravity in order to uncover more detailed features of the UV and restore factorization.

Several suggestions exist in the literature, like half-wormholes, various branes, non-local interactions, ….

A simple universal explanation could be that wormholes are unstable due to brane creation by an analogue of Schwinger pair production. The Swampland cobordism conjecture suggests that such branes always exist.
cf Marolf Santos '21

Or overcounting? Eberhardt '20'21

See e.g.
Gao, Jafferis, Kolckmeyer '21
Saad, Shenker, Stanford, Yao '21
Blommaert, Kruthoff '21
Mukhametzhanov '21
Blommaert, Iliesiu, Kruthoff '21

Alternative: gauging higher-form symmetries.
Benini, Copetti, Di Pietro '22

COMMENT #4

The precise holographic definition of 3d pure AdS gravity remains confusing. The partition function of the putative dual theory naively has (i) a continuous spectrum and (ii) a non-positive spectral density (Maloney, Witten ’07).

Attempts to deal with this include adding heavy point particles, including off-shell configurations, finding the “minimal” modular invariant partition function with non-negative spectral density, etc.

Alternatively, is it an average (over approximate CFTs?)

COMMENT #5

Connection to α-vacua (Marolf, Maxfield ‘’20):

We average over a set of objects which are semi-classicaly indistinguishable, and the correlations in these averages give rise to wormholes.

The individual elements of the ensemble can loosely be thought of as corresponding to α-vacua, although they are not really different vacua in this description and we do not perform a third quantization.

COMMENT #6

What about celestial holograpy?

Inputs: S-matrix, black holes, what else? Black holes not dual to coarse graining an energy window.

What to model? Celestial OPE’s? What is Moore-Seiberg of CCFT? Or better to model a Carroll theory?

Are there any interesting replica computations for flat space holography?

COMMENT #7

What about de Sitter space?

Inputs: cosmological collider, black holes, cosmological horizon, what else?

What to model? Euclidean CFT? Non-isometric embedding of semi-classical theory in finite dimensional Hilbert space?

What are the interesting replica computations for de Sitter? Bra-Ket wormholes?

CONCLUSIONS

- Standard principles of statistical physics lead to a statistical maximal ignorance description of semi-classical gravity. In 3d one finds an interesting matrix/tensor model description.

- Wormholes correspond to correlations in this statistical description. Conjecturally, they contain no new information.

- It is not yet clear whether there is a simple universal mechanism which restores factorization in the UV.

- It would be interesting to apply this logic to (observer-centric approaches to) quantum gravity in flat space and de Sitter.

- Semi-classical gravity is averaging agnostic.

- Should we stop pretending we are meta-observers who can solve everything? Especially when we are part of a chaotic system ourselves?