From Correlators to massive amplitudes in $\mathcal{N} = 4$ SYM

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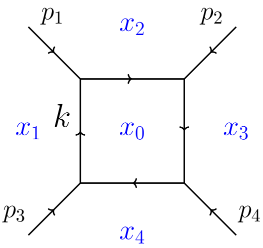
Solving $\mathcal{N} = 4$ SYM in the planar limit

- Simplest gauge theory in four dimensions: maximal susy, conformal and more hidden symmetries.
- Holographic dual to type IIB strings in $AdS_5 \times S^5$.
- Laboratory for testing perturbative (high-loop amplitudes, EEC) and non-perturbative techniques (conformal bootstrap).
- In the planar 't Hooft limit, the $SU(N_c)$ theory with $N_c \to \infty$, is regarded as solvable due to enhanced symmetries: integrability.
- Many observables addressed by integrability techniques: correlators, Wilson loops, massless amplitudes, form factors, etc.
- The Coulomb Branch of N=4 SYM is still largely unexplored by integrability.
- This talk: massive amplitudes in CB = correlators of operators with large scaling dimension.

Dual Conformal symmetry of SYM amplitudes

- Hidden symmetry revealed in dual coordinate space: $p_i \equiv x_i - x_{i+1}$.



$$\int \frac{d^4k\,(p_1+p_2)^2(p_2+p_3)^2}{k^2(k+p_1)^2(k+p_1+p_2)^2(k-p_4)^2}$$

$$\updownarrow \quad \begin{array}{c} p_i \equiv x_i - x_{i+1} \\ k \equiv x_1 - x_0 \end{array}$$

$$\int \frac{d^4x_0\; x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$$

- The loop-integrand is covariant under conformal inversion:

$$x^\mu \to \frac{x^\mu}{x^2}: \qquad \frac{1}{x_{ij}^2} \to \frac{x_i^2 x_j^2}{x_{ij}^2} \qquad d^4x_0 \to \frac{d^4x_0}{(x_0^2)^4}$$

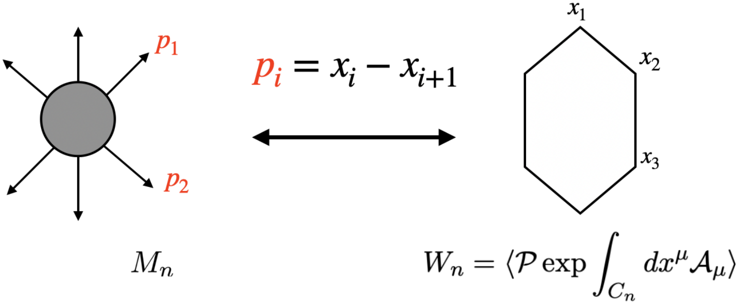
- Hidden dual conformal symmetry present at all loops.
- At integrated level, the symmetry is broken by IR singularities of amplitude.

Duality with Light-like Wilson Loops

- Exponentiation of IR singularities controled by Γ_{cusp} (also in QCD)

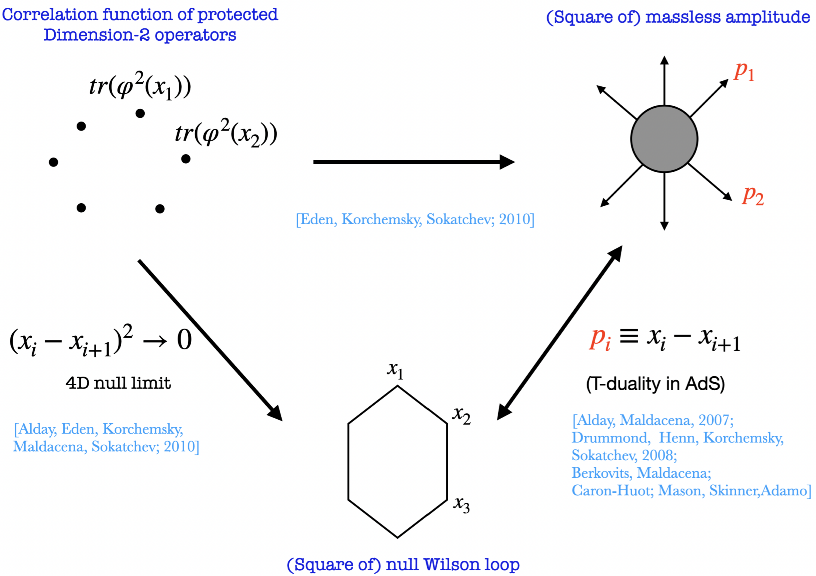
$$\log M_n = \sum_{l=1}^{\infty} (g^2)^l \left[\frac{\Gamma_{cusp}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma_{col}^{(l)}}{l\epsilon} \right] \sum_{i=1}^{n} \left(\frac{\mu_{IR}^2}{-s_{i,i+1}} \right)^{l\epsilon} + \text{Finite}(p_i, g^2)$$

- In planar SYM, also the finite part of the amplitude and Wilson loop are identical after the change to dual x-space:



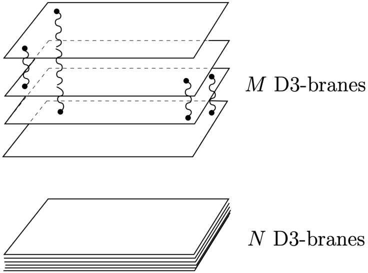
- Non-perturbative techniques based on integrability (GKP flux-tube).

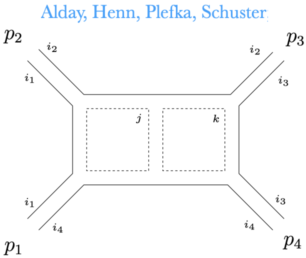
Triality in planar $\mathcal{N} = 4$ SYM



Massive amplitudes in the Coulomb branch

- Massive particles can be introduced via Higgs mechanism. We consider the symmetry breaking $U(N_c+4) \to U(N_c) \times U(1)^4$
$$\langle \vec{\Phi} \rangle = \text{diag}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \underbrace{\vec{0}, \cdots, \vec{0}}_{N_c})$$
- Each $\vec{v}_i$ is a 6-dim vector of the $SO(6)$ R-charge.
- New massive fields: $m^{int} = \vec{v}_i.\vec{v}_i$ and $m^{ext} = (\vec{v}_i - \vec{v}_j)^2$.





Extended dual conformal symmetry

- At one-loop massive integrand (with $p_i \equiv x_i - x_{i+1}$)

$$M_4^{(1)}(x_i, m_i) = \int \frac{d^4x_0\,(x_{13}^2 + (m_1 - m_3)^2)(x_{24}^2 + (m_2 - m_4)^2)}{(x_{10}^2 + m_1^2)(x_{20}^2 + m_2^2)(x_{30}^2 + m_3^2)(x_{40}^2 + m_4^2)}$$

- Same as massless integrand under replacement:

$$x^\mu \to X^{\hat\mu} = (x^\mu, m)$$

- Higher-dimensional dual conformal symmetry:

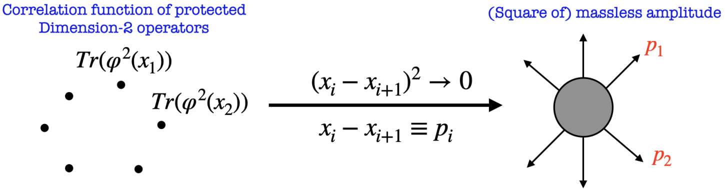
$$X^{\hat\mu} \to \frac{X^{\hat\mu}}{X^2}$$

[Alday, Henn, Plefka, Schuster; Caron-Huot,O'Connell; Bern, Carrasco, Dennen, Huang, Ita]

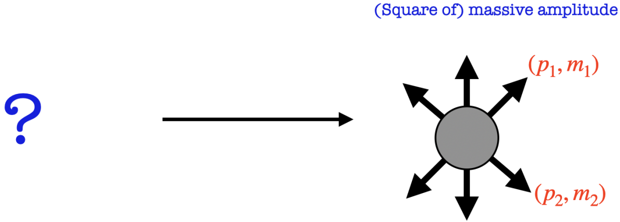
- The amplitude is IR finite and the symmetry holds at integrated level

Duality for massive case?

- Remember massless case:



- What is the correlator dual to Coulomb branch amplitudes?



- Should be a new correlator with higher dimensional structure.

Going beyond the stress tensor multiplet

- All single-trace protected operators:

$$\text{Tr}(y.\Phi(x))^k$$

Vector of six scalars

fields = scaling dimension

6D null polarization vector $y.y = 0$

4D spacetime position

Tr $\equiv$ trace over $N_c \times N_c$ gauge group.

- $k = 2$: stress tensor multiplet in SYM

$$\mathcal{T}(x, y, \theta, \bar{\theta}) = \text{Tr}(y.\Phi)^2 + \cdots + (\theta.y.\theta)^2 \mathcal{L}(x) + \cdots$$

- 4D null limit of $k = 2$ correlator dual to massless gluon amplitude.
- $k = 2$ is holographic dual of the graviton multiplet in $AdS_5 \times S^5$
- Higher "k" are dual to Kaluza-Klein modes of the graviton in S^5.
- Can we restore the 10D structure by resumming the KK tower?

Generating function in free theory

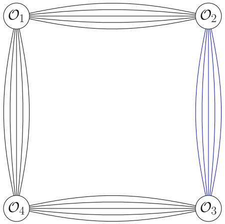
We define the determinant operator the resumms all half-BPS ops.

$$\mathsf{O}(x,y) = \mathsf{Tr}(y.\Phi(x)) + \frac{1}{2}\mathsf{Tr}(y.\Phi(x))^2 + \frac{1}{3}\mathsf{Tr}(y.\Phi(x))^3 + \cdots \quad \text{all KK tower}$$

In the free theory: Wick contractions using scalar propagator

$$\langle \mathsf{O}_1\, \mathsf{O}_2\, \mathsf{O}_3\, \mathsf{O}_4 \rangle^{(0)} = \prod_{i=1}^{4} \frac{-y_{i,i+1}^2}{\left(x_{i,i+1}^2 + y_{i,i+1}^2\right)} + (\text{other planar topologies})$$

Emergence of 10D poles from geometric series resummation.



4D scalar propagator: $\langle y_1.\phi(x_1)\; y_2.\phi(x_2)\rangle_{\text{free}} = \frac{-y_{12}^2}{x_{12}^2}$

For each bundle of propagators: 10D pole

$$\longrightarrow \left(\frac{-y_{ij}^2}{x_{ij}^2}\right) + \left(\frac{-y_{ij}^2}{x_{ij}^2}\right)^2 + \left(\frac{-y_{ij}^2}{x_{ij}^2}\right)^3 + \cdots = \frac{-y_{ij}^2}{x_{ij}^2 + y_{ij}^2}$$

Ten-dimensional poles of loop integrands

The loop-integrands through Lagrangian insertion method:

$$g^2 \frac{\partial}{\partial g^2} \langle \mathsf{O}_1 \cdots \mathsf{O}_n \rangle_{\mathsf{SYM}} = \int d^4 x_{n+1} \langle \mathsf{O}_1 \cdots \mathsf{O}_n \, \mathcal{L}(x_{n+1}) \rangle_{\text{self-dual SYM}}$$

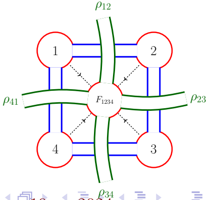
We define the supersymmetric determinant: $\mathbb{D} = \det(1 - y.\Phi) +$ susy.
Notice $\mathsf{O} = -\log\det(1 - y.\Phi)$. In self-dual SYM we use matrix duality:

$$\left\langle \prod_{i=1}^{n} \mathbb{D}(x_i, y_i, \theta_i) \right\rangle_{\mathsf{SDYM}} = \int D\rho \, e^{-N_c \sum_{i<j}^{n} \frac{\rho_{ij}\,\rho_{ji}}{2} \frac{x_{ij}^2}{-y_{ij}^2}} \det(1 - \Delta\rho)^{N_c}$$

$$\overset{N_c \to \infty}{=} \int D\rho \, e^{-N_c \sum_{i<j}^{n} \frac{\rho_{ij}\,\rho_{ji}}{2} \frac{x_{ij}^2 + y_{ij}^2}{-y_{ij}^2} + N_c \left(\frac{1}{3} Tr(\Delta\rho)^3 + \frac{1}{4} Tr(\Delta\rho)^4 + \cdots \right)}$$

The loop-integrands as projections on θ and y:

$$\left\langle \prod_{i=1}^{n+l} \mathbb{D}_i \right\rangle \longrightarrow G_n^{(l)} = \langle \mathsf{O}_1 \cdots \mathsf{O}_n \underbrace{\mathcal{L}_{n+1} \cdots \mathcal{L}_{n+l}}_{l \text{ lagrangians}} \rangle$$

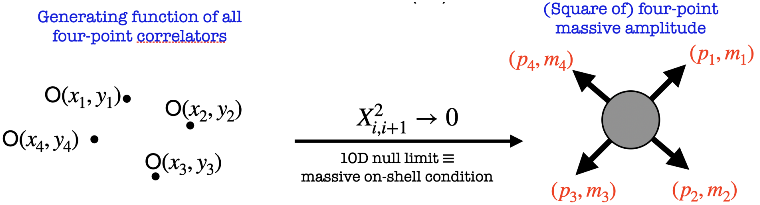


Conjecture: 10D null limit → amplitude

- The correlator of $\mathrm{O}(x,y) = -\log\det(1 - y.\Phi(x))$ has an emergent 10D structure that combines spacetime and R-charge distances:

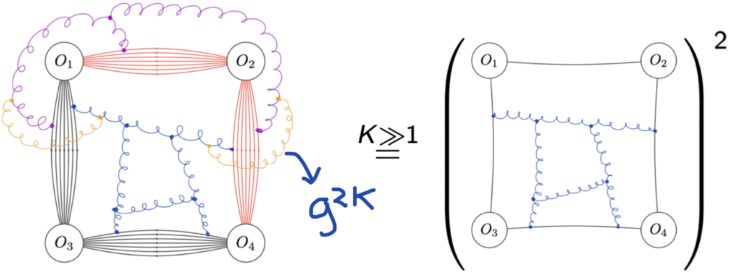
$$X^2_{i,i+1} \equiv (x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 \overset{duality}{=} p_i^2 + m_i^2$$

- The R-charge vector y_i is identified with a complex VEV $\vec{v}_i$.
- The 10D null limit of the correlator is equal to a (squared) scattering amplitude of massive W-bosons in the Coulomb branch.



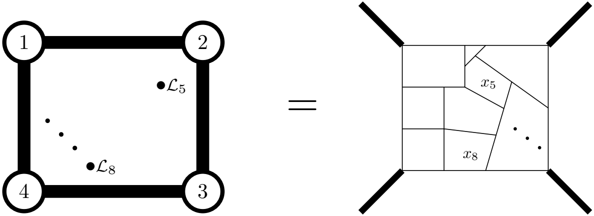
10D null limit: Large R-charge correlators

- The limit $y_{i,i+1}^2 \to 0$ gives small R-charge.
- The 10D null limit: $x_{i,i+1}^2 + y_{i,i+1}^2 \to 0$ is dominated by the tail of BPS operators with large R-charge.
- In the planar limit, these large R-charge correlators factorize in two disks: "interior" and "exterior" of the graph perimeter.



Duality of "Square" with 4-point amplitude

- Duality at loop-integrand and integrated level:



- Bootstrap to all loops: single-valuedness, Steinmann relations, exponentiation of IR divergences in the massless limit.

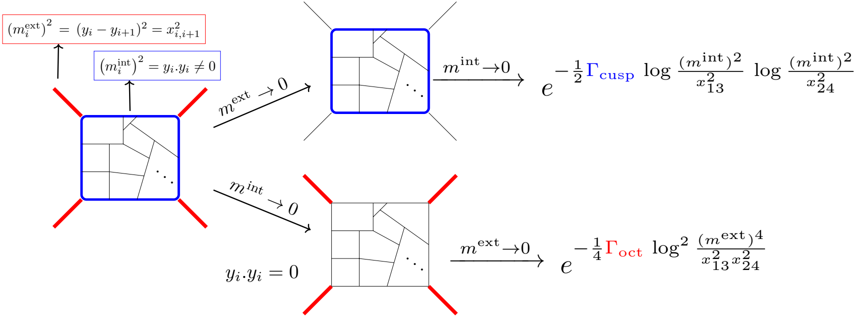
$$\lim_{(-y^2_{i,i+1}=x^2_{i,i+1})\to 0} \text{Square} = e^{-\frac{\Gamma_{oct}(g)}{16}\log^2\left[\frac{x^2_{12}x^2_{23}x^2_{34}x^2_{41}}{(x^2_{13}x^2_{24})^2}\right]}$$

[FC; Kostov, Petkova, Serban; Belitsky, Korchemsky]

- But the coefficient Γ_{oct} controlling the IR divergences is not Γ_{cusp}!

Comparisson in massless limit

- Half-BPS condition = restricted VEVs. Difference in order of limits:



- Γ_{cusp} and Γ_{oct} satisfy the same equation: deformed BES equation

[Beisert, Eden, Staudacher]

$$\Gamma_{cusp} = 4\,g^2 + (-8\zeta_2)g^4 + (88\zeta_4)g^6 + \cdots = \Gamma(\alpha = \pi/4)$$

$$\Gamma_{oct} = 4\,g^2 + (-16\zeta_2)g^4 + (256\zeta_4)g^6 + \cdots = \Gamma(\alpha = 0) = \frac{2\log\cosh(2\pi g)}{\pi^2}$$

[Basso, Dixon, Papathanasious]

Summary and outlook

- 10D null limit of generating function is equal to massive amplitude:

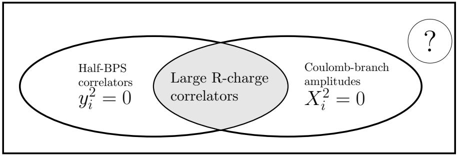
Correlator — Amplitude

$$10D: \quad \langle \mathbb{D} \cdots \mathbb{D} \rangle \xrightarrow{(x^2_{i,i+1} + y^2_{i,i+1}) \to 0} [M(x,y)]^2 \qquad p_i = x_i - x_{i+1}, \quad m_i = y_i - y_{i+1}$$

$$\downarrow y^2_{ij} \to 0 \qquad\qquad\qquad \downarrow y^2_{ij} \to 0$$

$$4D: \quad \langle \mathcal{T} \cdots \mathcal{T} \rangle \xrightarrow[x^2_{i,i+1} \to 0]{} [M(x,0)]^2 \qquad p_i = x_i - x_{i+1}$$

- Dictionary for polarizations in massive amplitudes.
- Understand masless limit: interpolation between Γ_{cusp} an Γ_{oct}.
- Relaxing BPS condition $y_i^2 = 0$?



- Bootstrapping higher polygons and integrability for Coulomb branch.