Actions and S-matrices

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Based on:
- 2307.12368 (Seolhwa Kim, PK, Ruben Monten, Richard Myers)
- WIP with Richard Myers

Related:
- 2311.03443 (Jain, Kundu, Minwalla, Parrikar, Prabhu, Shrivastava)
- Talk by Minwalla

Outline

- We revisit and refine an old approach to the S-matrix based on computing the path integral subject to asymptotic boundary conditions
- Definition in terms of *asymptotic* data makes this approach well suited to studying implications of asymptotic symmetries
- Discuss relation to "Carrollian" picture of the S-matrix, and utility in understanding the Minkowski S-matrix from the flat space limit of AdS

Motivation from AdS/CFT

- AdS/CFT analogy: two ways to compute boundary correlators

GKP/W: $\phi_{\rm bulk}\,(r,\vec{x}) \sim r^{\Delta-d}\bar{\phi}(\vec{x}) + \ldots \quad (\text{ non-normalizable mode})$

$$Z[\bar{\phi}] = \int_{\phi\sim r^{\Delta-d}\bar{\phi}} \mathcal{D}\phi_{\rm bulk}\, e^{iI_{bulk}[\phi_{bulk}]} \quad \longrightarrow \quad \langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle \sim \frac{\delta}{\delta\bar{\phi}(x_1)}\cdots\frac{\delta}{\delta\bar{\phi}(x_n)} Z[\bar{\phi}]\Big|_{\bar{\phi}=0}$$

BDHM: $\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle = \lim_{r\to\infty} r_1^{\Delta}\ldots r_n^{\Delta}\langle\phi(r_1,x_1)\ldots\phi(r_n,x_n)\rangle_{bulk}$

Prescriptions are equivalent, possibly up to contact terms

Proof: consider bulk path integral with bulk source J and bndy condition $\bar{\phi}$

To establish equivalence one shows that $Z[\bar{\phi},J]$ depends on $(\bar{\phi},J)$ only through:

$$\bar{\phi}_J(x) = \underbrace{\bar{\phi}(x)} + \int d^{d+1}x'\sqrt{g}G(x;x')\,J(x')$$

$$\bar{\phi}(x) = \int_{\partial\mathrm{AdS}} d^d y'\sqrt{h}K(x;\vec{y}\,')\,\bar{\phi}(\vec{y}\,')$$

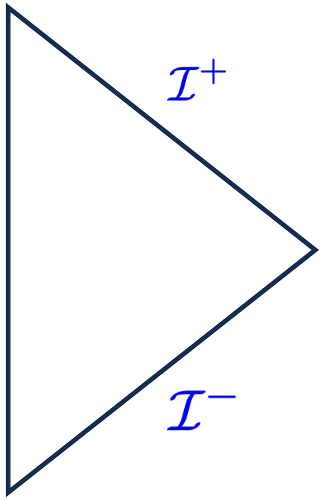
G = bulk-bulk propagator

K = bulk-boundary propagator

Use substitution $\int GJ \leftrightarrow \int K\bar{\phi}$ to toggle between $Z[0,J] \leftrightarrow Z[\bar{\phi};0]$

S-matrix

- BDHM clearly analogous to LSZ. Makes manifest that correlators are computed by Feynman/Witten diagrams
- What is the analog of GKP/W for the flat space S-matrix?
- Consider a massless scalar field on Minkowski space

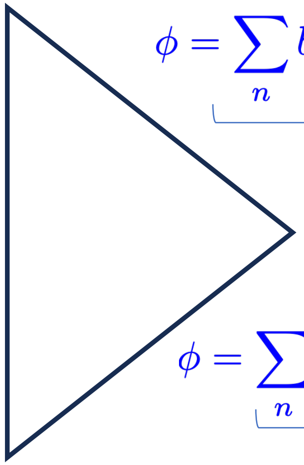


- Impose asymptotic boundary conditions at null infinity and evaluate action as functional of the boundary data

S-matrix

- Boundary conditions should fix unique classical solution, and for S-matrix should involve both past and future boundaries

- Fix positive(negative) frequency field content at past(future) null infinity



$$\phi = \underbrace{\sum_n b_n^\dagger e^{i\omega_n t}}_{\bar{\phi}_- \,=\, \text{fixed}} + \text{pos. freq. part}$$

$$\phi = \underbrace{\sum_n b_n e^{-i\omega_n t}}_{\bar{\phi}_+ \,=\, \text{fixed}} + \text{neg. freq. part}$$

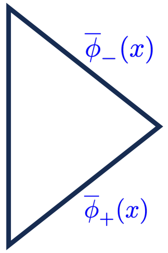
$b_n \; , \; b_n^\dagger$ independent

- Action: $I[\phi, \bar{\phi}] = \int d^4x \left(\frac{1}{2}\phi \nabla^2 \phi - V(\phi) \right) + I_{\text{bndy}}[\phi, \bar{\phi}]$

Boundary terms

- Boundary terms can be deduced by demanding good variational principle

$$I = \int d^4x \left(\frac{1}{2} \phi \nabla^2 \phi - V(\phi) \right) + I_{bndy}$$



Demand $\delta I = (\mathrm{EoM})$ when $\delta\overline{\phi}_\pm = 0$

$\longrightarrow$ $I_{bndy} = (\overline{\phi}_-, \phi)_{\mathcal{I}^+} - (\overline{\phi}_+, \phi)_{\mathcal{I}^-}$

where $(\phi_1, \phi_2)_\Sigma = \frac{1}{2} \int d^3x \sqrt{h} n^\mu (\phi_1 \partial_\mu \phi_2 - \partial_\mu \phi_1 \phi_2)$

- Same basic story for more complicated theories (gauge theory, gravity)

S-matrix

- Path integral with prescribed boundary data

$$Z[\overline{\phi}_+, \overline{\phi}_-] = \int_{\overline{\phi}_+, \overline{\phi}_-} \mathcal{D}\phi e^{iI[\phi]}$$

- The claim is that this gives the S-matrix operator according to

$$\boxed{\hat{S}[\hat{\overline{\phi}}] =: e^{-iI_{\rm bndy}[\hat{\overline{\phi}}, \hat{\overline{\phi}}]} Z[\hat{\overline{\phi}}] :}$$

$$\hat{\overline{\phi}} = \hat{\overline{\phi}}_+ + \hat{\overline{\phi}}_- = \sum_n (\hat{b}_n e^{-i\omega_n t} + \hat{b}_n^\dagger e^{i\omega_n t})$$

I.e. $\langle p'_1, \ldots p'_2; \text{out}|p_1, p_2 \ldots; \text{in}\rangle = \langle p'_1, p'_2, \ldots |\hat{S}[\hat{\overline{\phi}}]|p_1, p_2, \ldots \rangle$

- Equivalently, $Z[\overline{\phi}]$ serves as generating function for S-matrix

$$\langle p'_1, \ldots p'_2; \text{out}|p_1, p_2 \ldots; \text{in}\rangle = \left[\frac{\delta}{\delta b(p'_1)} \frac{\delta}{\delta b(p'_2)} \cdots \frac{\delta}{\delta b^\dagger(p_1)} \frac{\delta}{\delta b^\dagger(p_2)} \cdots Z[\overline{\phi}] \right]_{\overline{\phi}=0}$$

- Agreement with LSZ (when both are defined) can be established

  by same argument as for GKP/W = BDHM

Comments

- Via these arguments we rediscover an old proposal Arefeva, Faddeev, Slavnov (1974) (AFS) who worked with the coherent state matrix elements of the S-operator
- This formulation of the S-matrix pops up occasionally but is rarely used, because in most cases evaluation of the action involves the standard Feynman diagram expansion.
- We take the perspective that the AFS formulation is more holographic in spirit, so may provide insight into holography, asymptotic symmetries, etc.

Free theory: Particle creation in curved space

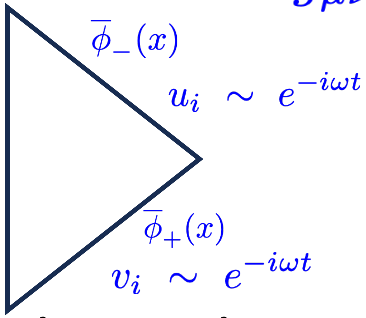
- For a free scalar field the action reduces to boundary terms:

$$I = (\overline{\phi}_-, \phi_+)_{\mathcal{I}^+} + (\overline{\phi}_+, \phi_-)_{\mathcal{I}^-}$$

- Obviously boring in Minkowski space, but considered curved metric:

$$g_{\mu\nu}(x) \longrightarrow \eta_{\mu\nu} \text{ in any direction}$$

- Mode solutions:



$$u_i = \sum_j (\alpha_{ij} v_j + \beta_{ij} v_j^*)$$

Just need to find the right combination of {u,u*} or {v,v*} that obeys boundary conditions $\{\overline{\phi}_+, \overline{\phi}_-\}$. Plugging $\bar{\phi}_- = \sum b_i^\dagger u_i^*$, $\bar{\phi}_+ = \sum b_i v_i$ into bndy action gives S-matrix:

$$\hat{S} = C : e^{i(I - I_0)} : \quad = \frac{1}{\sqrt{|\det \alpha|}} : \exp\left\{ b^\dagger \left(\alpha^{-1} - I\right) b + \frac{1}{2} b \alpha^{-1} \beta b - \frac{1}{2} b^\dagger \beta^* \alpha^{-1} b \right\}$$

- Yields known result obtained more laboriously from operator methods (DeWitt)

Bulk and boundary actions

- Path integral gives "partition function" defined on $\mathcal{I} = \mathcal{I}^- \cup \mathcal{I}^+$
- Gives partition function of hypothetical "Carrollian theory" (Barnich, Troessaert; Bagchi, Banerjee, Basu, Dutta; Donnay, Fiorucci, Herfray, Ruzziconi; Mason, Ruzziconi, Srikant …)
- Scalar action: $I_\phi = \sum_{m,n=1}^{\infty} I_\phi^{(m,n)}[\bar{\phi}]$ $\qquad \phi(x) \sim \frac{1}{r}\bar{\phi}(u,\Omega)$

$$I_\phi^{(m,n)} = \left(\prod_{i=1}^{m}\int_{I^+} d\Omega_i du_i \bar{\phi}_{i-}(u_i,\Omega_i)\frac{1}{2}\overleftrightarrow{\partial}_{u_i}\right)\left(\prod_{j=1}\int_{I^-} d\Omega_j dv_j \bar{\phi}_{i+}(v_j,\Omega_j)\frac{1}{2}\overleftrightarrow{\partial}_{v_j}\right)\hat{G}_\phi^{(m,n)}(u_i,\Omega_i;v_j,\Omega_j)$$

$$\sim \int_{\mathcal{I}} \partial_{u_i}\bar{\phi}_i(u_i,\Omega_i)\hat{G}_\phi^{(m,n)}(u,\Omega)$$

- Boundary correlators encode on-shell Feynman amplitudes $\tilde{G}_\phi^{(m,n)}(p_i)$

$$\hat{G}_\phi^{(m,n)}(u,\Omega) = \left(\int_{-\infty}^{\infty}\prod_i d\omega_i e^{-i\omega_i u_i}\right)\hat{\tilde{G}}_\phi^{(m,n)}(\omega_i,\omega\Omega_i)$$

- Explicit examples in recent literature

Lorentz invariance

- In usual SL(2,C) description $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1$ that acts at large r as

$$r \to r_\Lambda = \frac{r}{f_\Lambda(z,\bar{z})}$$

$$u \to u_\Lambda = f_\Lambda(z,\bar{z}) u$$

$$z \to z_\Lambda = \frac{az+b}{cz+d}$$

$$d\Omega^2 = \frac{2}{(1+z\bar{z})^2} dz d\bar{z} \to \left(f_\Lambda(z,\bar{z})\right)^2 d\Omega^2$$

$$f_\Lambda(z,\bar{z}) = \frac{1+z\bar{z}}{|az+b|^2 + |cz+d|^2}$$

- Boundary data transforms as

$$\bar{\phi}(u,z,\bar{z}) \;\to\; f_\Lambda(z,\bar{z}) \bar{\phi}\left(u_\Lambda, z_\Lambda, \bar{z}_\Lambda\right)$$

- Lorentz invariance of S-matrix encoded in following relation obeyed by boundary amplitudes: $\hat{G}^N_\phi\left(u_i, z_i, \bar{z}_i\right) = \left(\prod_{i=1}^N f_\Lambda\left(z_i, \bar{z}_i\right)\right) \hat{G}^N_\phi\left(u_{\Lambda i}, z_{\Lambda i}, \bar{z}_{\Lambda i}\right)$

Scalar QED

$$I=\int d^4x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+|D_\mu\phi|^2\right)+I_{bndy}$$ with $$D_\mu=(\partial_\mu-ieA_\mu)\phi \qquad \nabla^\mu A_\mu=0$$

- Boundary data $\overline{A}_A^+(v,\Omega)$, $\overline{A}_A^-(u,\Omega)$, $\overline{\phi}^+(v,\Omega)$, $\overline{\phi}^-(u,\Omega)$

  sphere index

$\overline{A}_A$ includes (antipodally matched) large gauge (Goldstone) mode: $\overline{A}_A=\nabla_A\Phi(\Omega)+\text{radiative}$

- Boundary action now takes form: $I_{\phi\gamma}=\int_{\mathcal{I}}\left(\prod_i\partial_{u_i}\overline{\phi}_i(u_i,\Omega_i)\right)\overline{A}_A(u,\Omega)\partial_u G^A(u,\Omega;u_i,\Omega_i)$

- Relation to standard amplitudes given by

$$G^A(u,\Omega;u_i,\Omega_i)=\left(\int_{-\infty}^{\infty}\prod_i d\omega_i e^{-i\omega_i u_i}\right)\sum_\alpha\left((\varepsilon^{*\alpha})_\mu\tilde{G}^\mu(q,p)\right)\hat{\varepsilon}_\alpha^A$$

- Lorentz transformation: $\bar{A}_z(u,z,\bar{z})\ \rightarrow\ \frac{1}{(cz+d)^2}\bar{A}_z(u_\Lambda,z_\Lambda,\bar{z}_\Lambda)$

- Lorentz invariance implies:

$$(f_\Lambda(z,\bar{z}))^2\left(\prod_{i=1}^N f_\Lambda(z_i,\bar{z}_i)\right]\Big)G^z(u_\Lambda,z_\Lambda,\bar{z}_\Lambda,u_{i\Lambda},z_{i\Lambda},\bar{z}_{i\Lambda})=\frac{1}{(cz+d)^2}G^z(u,z,\bar{z},u_i,z_i,\bar{z}_i)$$

QED Ward identity and soft theorem

- Bulk action is invariant under large gauge transformations that act on boundary data as:

$$\bar{A}_A \;\to\; \bar{A}_A + \nabla_A \lambda(\Omega)$$
$$\bar{\phi}_i \;\to\; e^{iq_i\lambda(\Omega)}\bar{\phi}_i$$

- Boundary action:

$$I = \int_{\mathcal{I}} \left(\prod_i \partial_{u_i}\bar{\phi}_i\,(u_i,\Omega_i)\right)\hat{G}(u,\Omega) + \int_{\mathcal{I}} \left(\prod_i \partial_{u_i}\bar{\phi}_i\,(u_i,\Omega_i)\right)\bar{A}_A(u,\Omega)\partial_u\hat{G}^A\,(u,\Omega;u_i,\Omega_i) + \ldots$$

Invariance under LGT implies Ward identity

$$\hat{\nabla}_A^{(y)}\int du\partial_u G^A_{\phi\gamma}\,(u,\hat{y};u_i,\hat{x}_i) = ie\Big[\frac{1}{\sqrt{\gamma}}\sum_{i=1}^{N} q_i\delta^2\,(\hat{y}-\hat{x}_i)\Big]G_\phi\,(u_i,\hat{x}_i)$$

Solving gives leading soft photon theorem, as in (He, Mitra, Porfyriadis, Strominger)

Subleading soft theorem in scalar QED

- Known subleading soft theorem implies that classical on-shell action must be invariant under (as in Lysov, Pasterski, Strominger)

$$\delta \bar{A}_z(u,\hat{y}) = u\hat{\nabla}_z\hat{\nabla}_A\lambda^A(\hat{y})\ , \quad \lambda^{\bar{z}} = 0$$

$$\delta \bar{A}_{\bar{z}}(u,\hat{y}) = 0$$

$$\delta\partial_u\bar{\phi}_i(u,\hat{x}) = ieq_i\left[\partial_u\left(u\hat{\nabla}_A\lambda^A(\hat{x})\bar{\phi}(u,\hat{x})\right) + \lambda^A(\hat{x})\hat{\nabla}_A\bar{\phi}(u,\hat{x})\right]$$

- Invariance of boundary action implies Ward identity:

$$\partial_{u_j}\hat{\nabla}^{(y)}_z\hat{\nabla}^{(y)}_z\int du G^z\left(u,\hat{y};u_i,\hat{x}_i\right)$$

$$= -ieq_j\left[\partial_{u_j}\left(u_j G_\phi\left(u_i,\hat{x}_i\right)\right)\hat{\nabla}^{(y)}_z\frac{\delta^2\left(\hat{y}-\hat{x}_j\right)}{\sqrt{\gamma(y)}} - \hat{\nabla}^{(x_j)}_z G_\phi\left(u_i,\hat{x}_i\right)\frac{\delta^2\left(\hat{y}-\hat{x}_j\right)}{\sqrt{\gamma(y)}}\right]$$

Solution gives Low subleading soft theorem. Invariance of the action is not obvious to the eye.

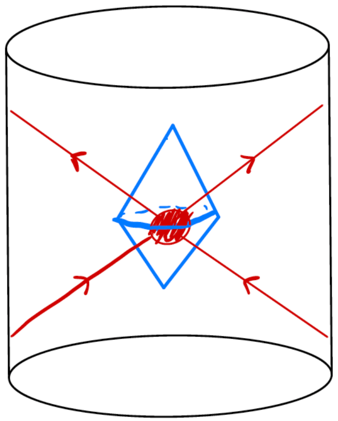
(Campiglia, Laddha
Himwich, Strominger
Choi, Laddha, Puhm)

Flat limit of AdS

- An old idea is to extract the Minkowski S-matrix from the flat limit of AdS correlators

(Polchinski
Giddings
…
Hijano, Neuenfeld)

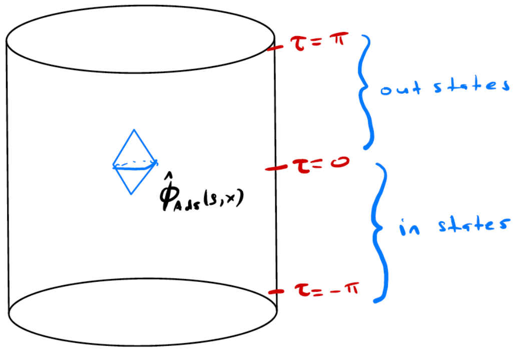
- Use boundary sources to create/destroy particles arranged to collide in region << R



- Since Minkowski S-matrix is equal to Minkowski path integral with bndy conds on $\mathcal{I}$, while AdS boundary correlators are equal to path integral with AdS boundary conditions, we essentially just need to relate the two sets of boundary conditions.

Flat limit of AdS (Hijano, Neuenfeld)

- HKLL relation: $\hat{\phi}_{\text{AdS}}\left(\rho, x\right) = \int_{\mathcal{T}} d\tau' \int d^2\Omega \left[K_+\left(\rho, x; x'\right)\mathcal{O}^+\left(x'\right) + K_-\left(\rho, x; x'\right)\mathcal{O}^-\left(x'\right)\right]$



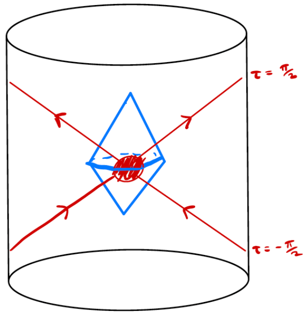
Boundary integral taken over $\tau \in [-\pi, 0]$ for in states and $\tau \in [0, \pi]$ for out states

- Take large R limit while keeping bulk point in region near AdS origin. Write as expression for in/out flat space creation/annihilation operators:

$$a_{\text{in},\vec{p}} = \int_{-\pi}^{0} d\tau e^{i\omega_p R\left(\tau + \frac{\pi}{2}\right)} \mathcal{O}^-(\tau, -\hat{p})$$

$$a_{\text{out},\vec{p}} = \int_{0}^{\pi} d\tau e^{i\omega_p R\left(\tau - \frac{\pi}{2}\right)} \mathcal{O}^-(\tau, \hat{p})$$

- For wavepacket states integrals localize to $\tau = \pm\dfrac{\pi}{2}$



Flat limit of AdS

- Flat space boundary action can be interpreted as S-matrix in coherent space basis. Flat <-> AdS data map equivalent to map between flat and AdS coherent states. Flat space: $a_{\vec{p}}|\alpha\rangle = \alpha(\vec{p})|\alpha\rangle$

AdS state created by boundary source: $S = \int d^3x \phi_0 O$

- AdS and flat partition functions related as $Z_{\text{flat}}[\alpha(\phi_0)] = Z_{\text{AdS}}[\phi_0]$
- Operator mapping implies

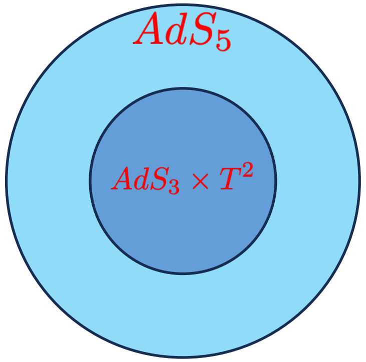
$$\alpha(\vec{p}) \sim \omega \int_{-\infty}^{\infty} \mathrm{d}u e^{-i\omega u} \phi_0 \left(-\frac{\pi}{2} + \frac{u}{R}, -\hat{p}\right).$$

- Within "scattering" subspace of Hilbert space, this provides a map between flat S-matrix and AdS correlators

IR symmetry in AdS

- BMS must arise from flat limit of AdS, though latter has only finite dimensional asymptotic symmetry algebra above D=3.
- Toy example: Virasoro as IR symmetry of AdS_5

  Magnetic brane solution (D'Hoker, PK)
- Asymptotically AdS_5 solution of D=5 Einstein-Maxwell with boundary B-field (dual to N=4 SYM with external B-field coupled to R-current).
- Describes RG flow from AdS_5 -> AdS_3 x T^2



Brown-Henneaux Virasoro should be visible from AdS_5 boundary. Microscopically described by effective D=1+1 CFT of fermions in Landau levels

IR symmetry in AdS

- Direct approach: stress tensor correlators (D'Hoker, PK,Shah)

$$\langle T_{zz}(z)T_{zz}(0)\rangle \underset{|z|\to\infty}{=} \frac{c_{BH}/2}{z^4}$$

- Alternatively, extend AdS_3 boundary gravitons to AdS_5 (Kim, PK,Myers)

Background solution: $ds^2 = \frac{dr^2}{L(r)^2} + 2L(r)dx^+dx^- + e^{2V}dx^i dx^i \,, \quad F = b\,dx^1 \wedge dx^2$

Perturbation: $ds^2 \;\to\; ds^2 + M(x^\mu)(dx^+)^2$

↳ $\approx -\frac{1}{2b^2}\partial_+^3\epsilon(x^+)$ as $r\to 0$ (BH diff)

Extend perturbation to full spacetime: $M = \frac{1}{b}L^c(r)\partial_+^3\epsilon(x^+)$ $\quad L^c(r) = L(r)\int_\infty^r \frac{dr'}{L(r')^2 e^{2V(r')}}$

↳ not pure diff mode

- Full spacetime symplectic form found to reduce to Brown-Henneaux:

$$\Omega_{D=5} = c_{BH}\int d\phi\,\partial_+\delta\epsilon \wedge \partial_+^2\delta\epsilon = \Omega_{BH}$$

The End