SUPERTRANSLATIONS, ANGULAR MOMENTUM, AND COVARIANCE IN 4D ASYMPTOTICALLY FLAT SPACE

BASED ON

Reza Javadinezhad and MP
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earlier work with R. Javadinezhad and U. Kol

BUT MOSTLY ON:

Reza Javadinezhad and MP arXiv:2312.02458, arXiv:24xx.xxxxx

The metric of an asymptotically flat spacetime can be written in $(u = t - r, r, \theta^A)$ coordinates in the Bondi gauge as

$$ds^2 = -du^2 - 2dudr + r^2\left(h_{AB} + \frac{C_{AB}}{r}\right)d\theta^A d\theta^B + \frac{2m}{r}du^2 + \\ + \frac{1}{r}\left(\frac{4}{3}(N_A + u\partial_A m) - \frac{1}{8}\partial_A(C_{BD}C^{BD})\right)dud\theta^A + \dots.$$

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This metric is invariant under:

1) Lorentz transformations, that act as conformal isometries of the celestial sphere $\theta^A \to \theta^A + V^A(\theta)$

2) Supertranslations. At leading order in r they are: $u \to u + f(\theta)$ They transform the shear as $C_{AB} \to C_{AB} + (-2D_A D_B + h_{AB}D^2)f$

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The supertranslation charges are

$$Q[f] = \frac{1}{4\pi G}\int d^2\theta\sqrt{h}f(\theta)m(\theta, u)$$

and $Q(\theta) = P^0 + P_m Y_{1\,m}(\theta)$ is the energy-momentum 4-vector

The Bondi Lorentz charges at retarded time u are

$$J_Y(u) = \frac{1}{8\pi G}\int d^2\theta\sqrt{h}\,Y^A N_A(\theta,u)$$

Y^A are the 6 conformal Killing vectors of the celestial sphere obeying $D_A Y_B + D_B Y_A = h_{AB} D_C Y^C$

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Under supertranslations $u \to u + f(\theta)$ the angular momentum aspect and Bondi charges at $u = \pm\infty$ transform as

$$N_A(\theta, \pm\infty) \to N_A(\theta, \pm\infty) + 3\partial_A f(\theta) m(\theta, \pm\infty)$$

$$J_Y(\pm\infty) \to J_Y(\pm\infty) + \frac{1}{4\pi G}\int d^2\theta\sqrt{h}Y^A m(\theta, \pm\infty)\partial_A f(\theta)$$

The flux can be changed by supertranslations, i.e. by adding an infinite-wavelength gravitational wave!

$$\Delta J_Y \to \Delta J_Y + \frac{1}{4\pi G}\int d^2\theta \sqrt{h} Y^A \Delta m(\theta) \partial_A f(\theta)$$

Not crazy ($p \times x$ can be large even when the momentum is small) yet not useful. We want to be able to tell wheat from chaff separating the radiation due to a scattering process (e.g. black hole mergers) from unobservable backgrounds

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Soft dynamics is universal and there exists an automorphism of the algebra of u-local operators that makes them independent of supertranslations and boundary graviton degrees of freedom

R. Bousso, M. P. CQG 34 (2017) 20, 204001 arXiv:1706.00436
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Shouldn't it be possible to do the same for angular momentum &c.?

Several choices of angular momentum exist in the literature.
We choose one proposed by Chen, Wang, Wang and Yau

$$J_Y^{CWWY}(\pm\infty) = J_Y^{\pm} - j_Y(m^{\pm}, C^{\pm})$$

$$j_Y(A,B) = \frac{1}{4\pi G}\int d^2\theta\sqrt{h}A(\delta_Y^{-1/2}B) = -\frac{1}{4\pi G}\int d^2\theta\sqrt{h}(\delta_Y^{3/2}A)B$$

$$D^AD^BC_{AB}(\pm\infty,\theta) \equiv D^2(D^2+2)C^{\pm}$$

$$\delta_Y^w X \equiv wD_CY^CX + Y^CD_CX$$

invariant because $\delta_f C^{\pm} = f, \quad \delta_f m^{\pm} = 0$

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Other definitions dress the angular momentum by using only the boundary graviton C^- as additional degree of freedom. Some do not capture $O(G^2)$ contributions to the flux needed to explain radiative back-reaction effects in gravitational scattering.

THE CWWY FLUX IS A GOOD CANDIDATE

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The flux ΔJ_Y^{CWWY} computes:

final angular momentum in the supertranslation frame where the final metric is $h_{AB} + O(1/r^2)$

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THESE ARE THE FRAMES WHERE THE BONDI CHARGES J_Y COINCIDE WITH CANONICAL ADM CHARGES
(Venziano and Vilkovisky following Ashtekar et al.)
This may explain why they coincide with scattering amplitudes computations and other perturbative computations that implicitly or explicitly work in the "round-metric" canonical frame.

THE CWWY FLUX AND COVARIANCE

The flux ΔJ_Y^{CWWY} is covariant if m^+, m^-, C^+, C^- transform covariantly i.e.

$$\delta_Y^{3/2} m^\pm = \frac{3}{2} D_A Y^A m^\pm + Y^A D_A m^\pm$$

$$\delta_Y^{-1/2} C^\pm = -\frac{1}{2} D_A Y^A C^\pm + Y^A D_A C^\pm$$

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These transformations follow from the asymptotic form of Lorentz transformations on coordinates

$$u = K(\bar{\mathbf{x}})\bar{u}, \quad \mathbf{x} = g(\bar{\mathbf{x}}), \quad r = K(\bar{\mathbf{x}})$$

$\mathbf{x}$ and $\bar{\mathbf{x}}$ are unit vectors parametrizing the celestial sphere

Generated by J_Y^- not by $J_Y^- - j_Y(m^-, C^-)$

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supertranslation invariance + covariance = translation invariance

invariant flux is ALSO TRANSLATION INVARIANT

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JACOBI:

$$[[F^{\mu\nu}, J^{\rho\sigma}], S^a] + [[S^a, F^{\mu\nu}], J^{\rho\sigma}] + [[J^{\rho\sigma}, S^a], F^{\mu\nu}] = 0$$

COVARIANCE

$$[F^{\mu\nu}, J^{\rho\sigma}] = -\eta^{\mu\rho}F^{\nu\sigma} - \eta^{\nu\sigma}F^{\mu\rho} + \eta^{\nu\rho}F^{\mu\sigma} + \eta^{\mu\sigma}F^{\nu\rho}$$

SUPERTRANSLATION INVARIANCE

$$[S^a, F^{\mu\nu}] = 0$$

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SUPERTRANSLATION INVARIANCE

$$[S^a, F^{\mu\nu}] = 0$$

On the other hand:

$$[J^{\rho\sigma}S^a] = supertranslations + TRANSLATIONS$$

so Jacobi implies $[F^{\mu\nu}, P^\rho] = 0$

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We define it first in the initial center of mass rest frame

$$m^{-}_{1,m} \equiv \int d^2\Theta \sqrt{h}\, \mathrm{Y}_{1m} m(-\infty, \Theta) = 0$$

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We fix the origin of the the initial center of mass rest frame by requiring

$$J^{-}_{\bar{Y}} - j_{\bar{Y}}[m^{-}, C^{-}] = 0$$

for all boosts, which can be written as

$$\bar{Y}^A = \text{boost} = D^A\psi, \quad D^2\psi = -2\psi$$

$$J_{\bar{Y}}^{-} - j_{\bar{Y}}[m^{-}, C^{-}] = 0$$

defines the l=1 components of $C^{-} = C(-\infty)$. After some straightforward calculations:

$$\frac{3m_0^{-}}{4\pi G} C_{1m}^{-} = J_{\bar{Y}^A}^{-} - j_{\bar{Y}^A}[m^{-}, C^{-}|_{l>1}], \quad \bar{Y}^A = D^A \mathrm{Y}_{1-m}$$

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In tensor notation we have just set $J_{i0} = 0$

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Equivalently, define the vector

$$*(J_{Y}^{-} - j_{Y}[m^{-}, C^{-}]) \equiv *J \in \{Conformal\ Killing\ Vectors\}$$

and impose

$$\delta_{*J} m^{-}|_{l} = 0 \quad l = 0,1$$

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To completely define the a covariant and (super)translation invariant flux we have to define also $C^+|_{l\leq 1}$

We can use either:

A: $C^+|_{l\leq 1} = C^-|_{l\leq 1}$

or

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It is easy to show that the two prescriptions agree to $O(G^2)$

After $C^{\pm}|_{l\leq 1}$ is fixed by either prescription A or B, the covariant flux is $\Delta J_Y^{CWWY} = J_Y^{+} - J_Y^{-}$ when both $J_Y^{\pm}$ are computed in the metric $h_{AB} + O(1/r^2)$

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It was recently argued that J_Y^- in the $h_{AB} + O(1/r^2)$ frame is the ADM angular momentum (using results by Ashtekar et al.)
So $\Delta J_Y^{CWWY} = J_Y^+ - J_Y^-$ is the change in the total ADM angular momentum. Riva Vernizzi and Wong verified in some cases that it coincides with the formulas given by Bini, Damour and Manohar et al.

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This identification is valid only for the ADM Lorentz generators defined with special asymptotic boundary conditions that forbid supertranslations (there is no single angular momentum otherwise)

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generate the Poincaré algebra. How is this consistent with all that we just found?

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Because $\mathfrak{J}_Y \equiv J^-_Y - j_Y(m^-, C^-|_{l>1})$ does not generate the Lorentz transformations induced on the asymptotic metric by the coordinate change $u = K(\bar{\mathbf{x}})\bar{u}, \quad \mathbf{x} = g(\bar{\mathbf{x}}), \quad r = K(\bar{\mathbf{x}})$

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This is obvious because supertranslation invariance implies

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In fact $\mathfrak{J}_Y$ belongs the the universal enveloping algebra of (generalized) BMS but it is not in the BMS algebra

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