Dynamical Edge Modes and Entanglement in Maxwell Theory

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Based on 2403.14542 with Albert Law and Gabriel Wong

Outline

- Context: What are edge modes?

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- Review entanglement entropy (EE)
  - Standard approach and its problems

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- A shrinkable boundary condition (BC) for Maxwell
- Bulk-edge split
- Resolving discrepancy

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- Hilbert space factorization? $\mathscr{H}_\Sigma = \mathscr{H}_A \otimes \mathscr{H}_{\bar{A}}$?

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  - Extended objects (e.g. Wilson lines) get cut open

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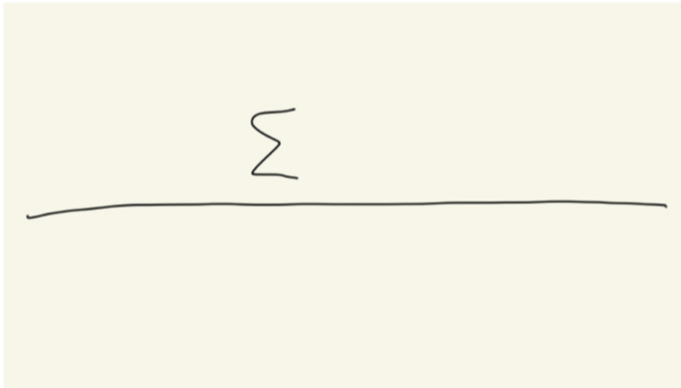
- Need to review path integral approach to EE first

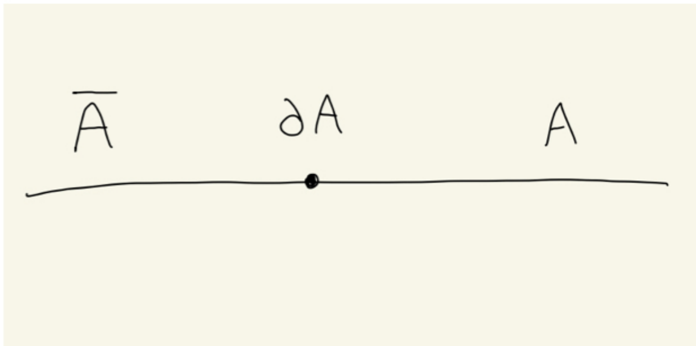
EE from path integral

For concreteness, let Σ be a slice of Minkowski



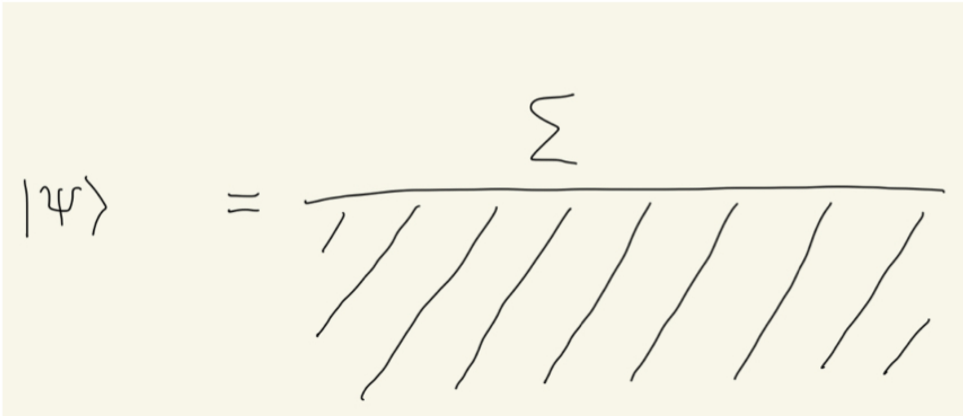
EE from path integral

Subregions and entangling surface



EE from path integral

Prepare vacuum with Euclidean half-plane



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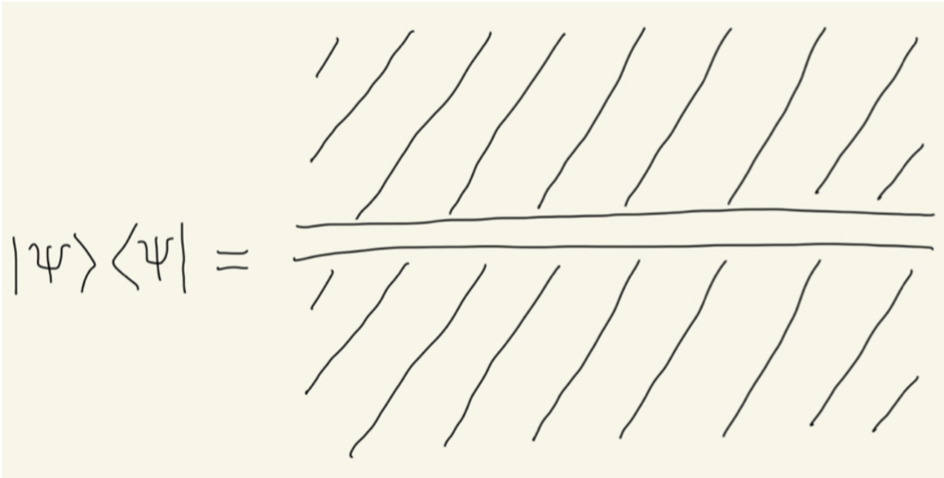
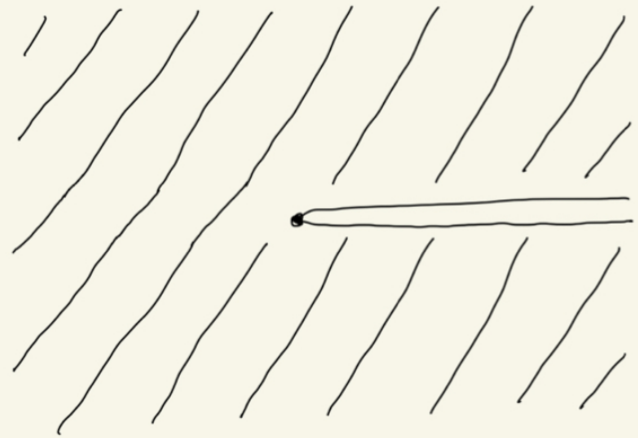


EE from path integral

$$\mathrm{Tr}_{\bar{A}} |\psi\rangle\langle\psi| =$$ 

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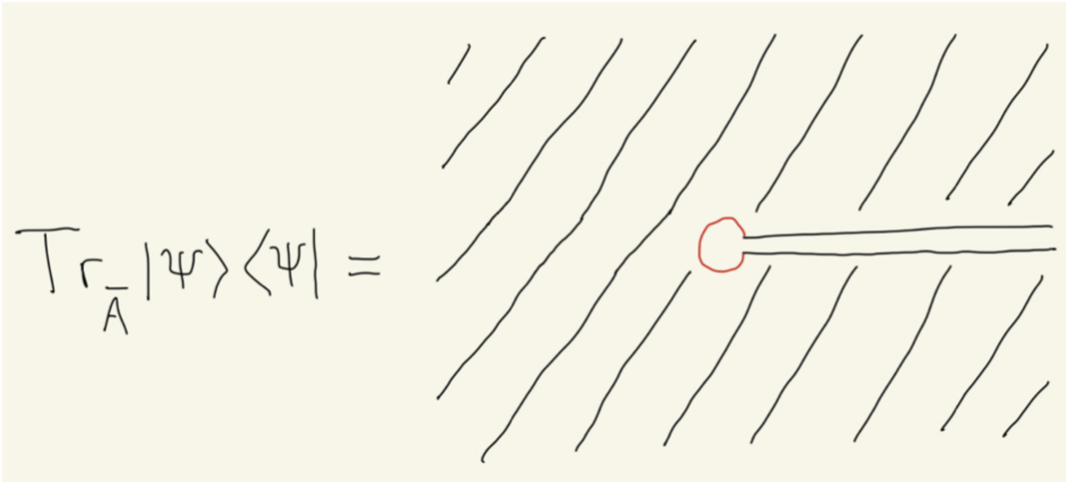
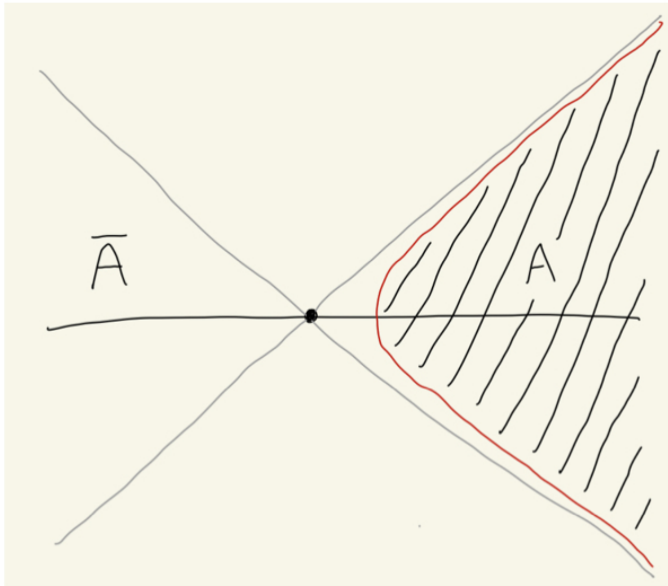


Cut out disk and choose a local BC.
Allows Hilbert space on radial slices

Brick wall



Lorentzian picture

EE from path integral

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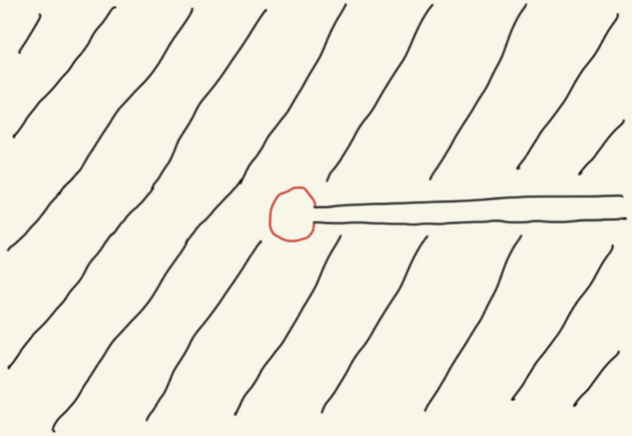
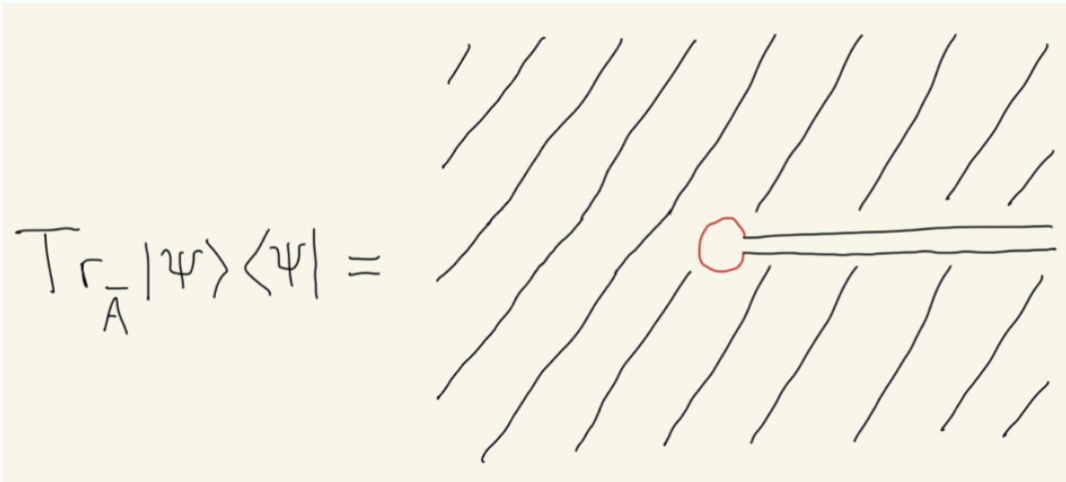


$$= \exp(-2\pi H_R)$$

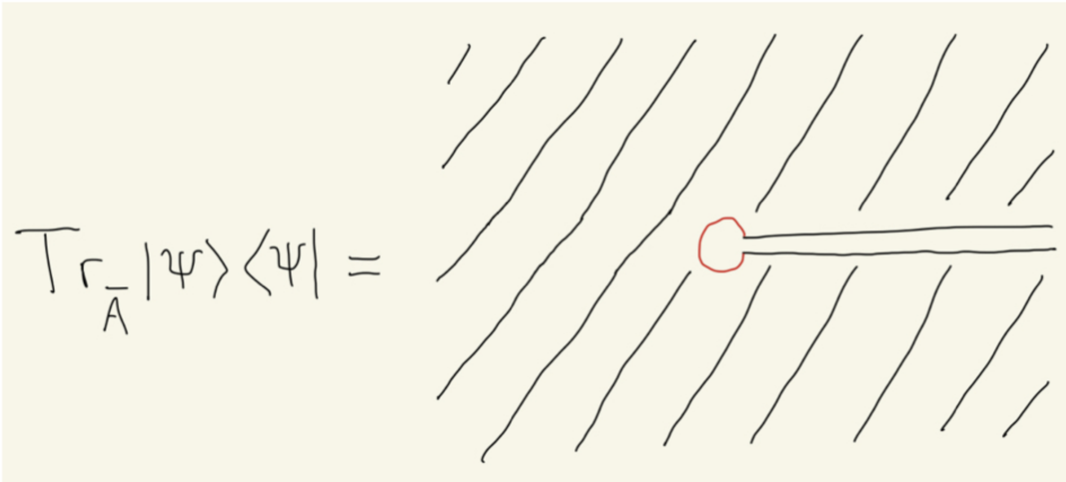
EE from path integral



$$= \exp(-2\pi H_R)$$

$$\rho_A \equiv \frac{\exp(-2\pi H_R)}{\mathrm{Tr}_A[\exp(-2\pi H_R)]} \qquad S_{\mathrm{vN}} = -\,\mathrm{Tr}_A[\rho_A \log \rho_A]$$

EE from path integral

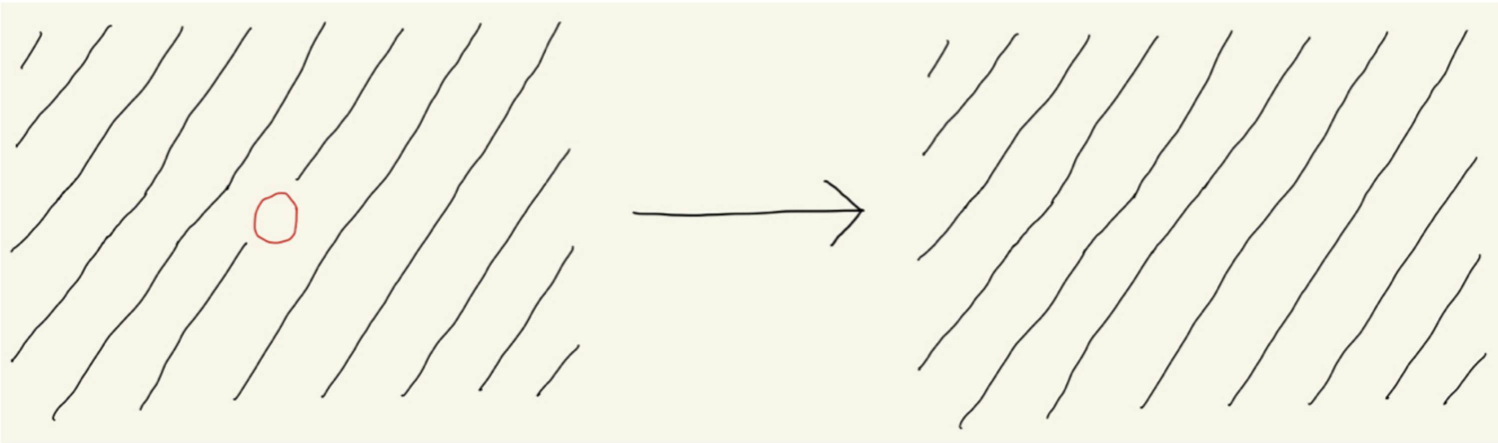


$= \exp(-2\pi H_R)$

$$S_{\mathrm{vN}} = S_{\mathrm{EE}}?$$

Shrinkability

A BC is shrinkable if it recovers no hole [Donnelly, Wong '18]



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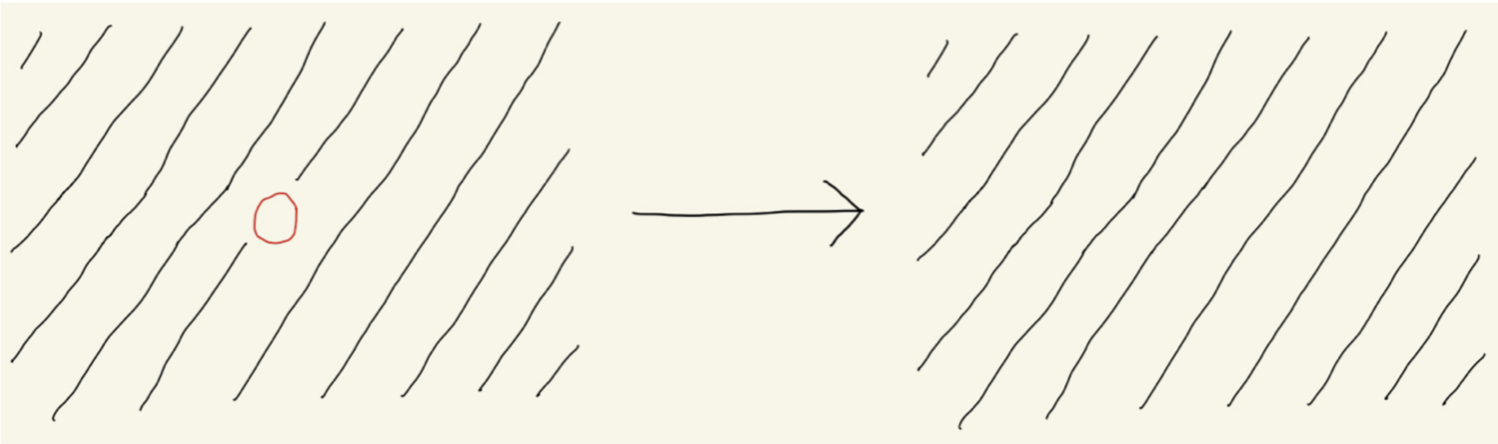

Shrinkable BC's should give correct EE

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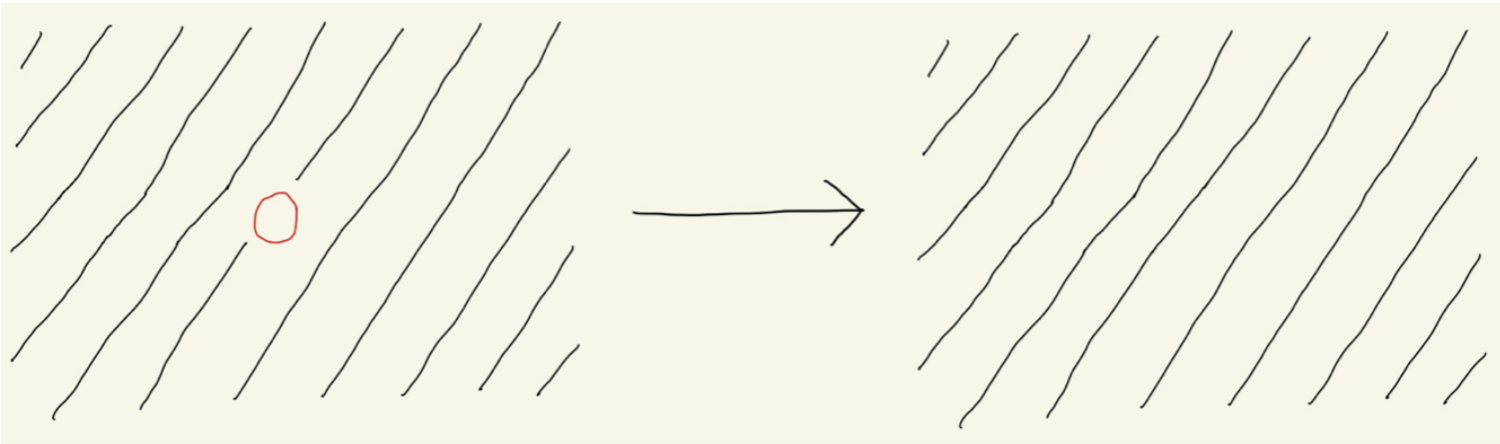

Spoiler: including edge modes yields shrinkability

EE in 4D Maxwell

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- 2D: $S_{\text{EE,scalar}} \sim \dfrac{1}{3} \log \dfrac{r}{\delta}$

General $Z(S^D)$

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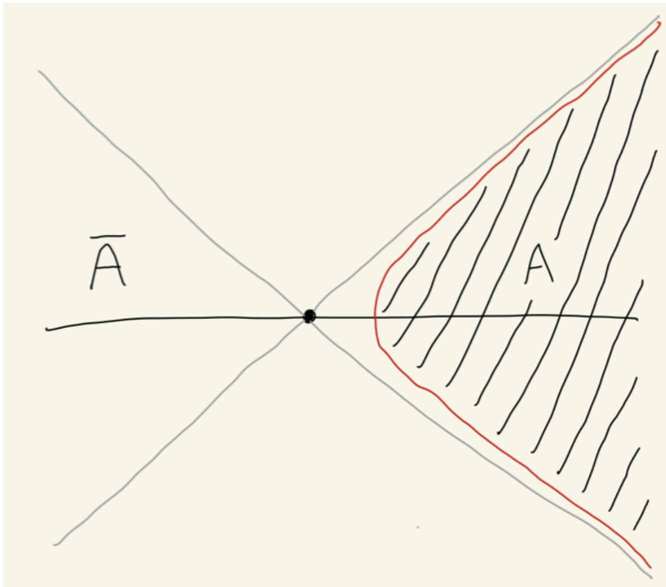
To resolve, need $Z_{\text{edge}}^{(D)} = 1/Z_{\text{scalar}}(S^{D-2})$

BC's for Maxwell

- Want a BC that allows edge modes. Recall they're associated with large gauge transformations



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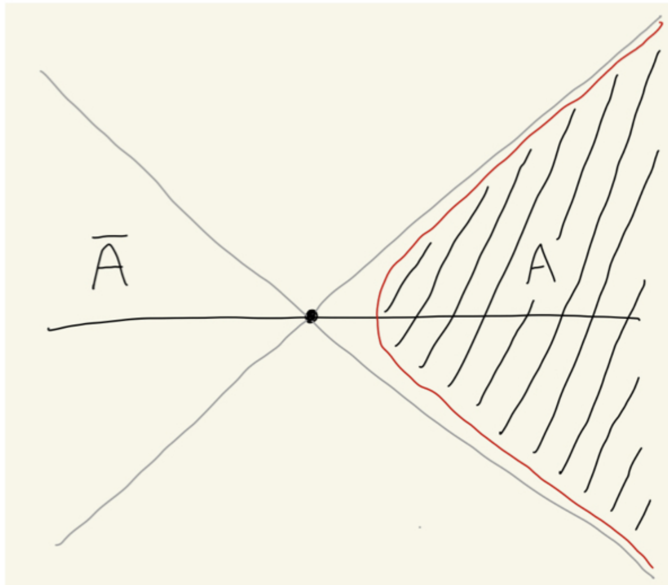

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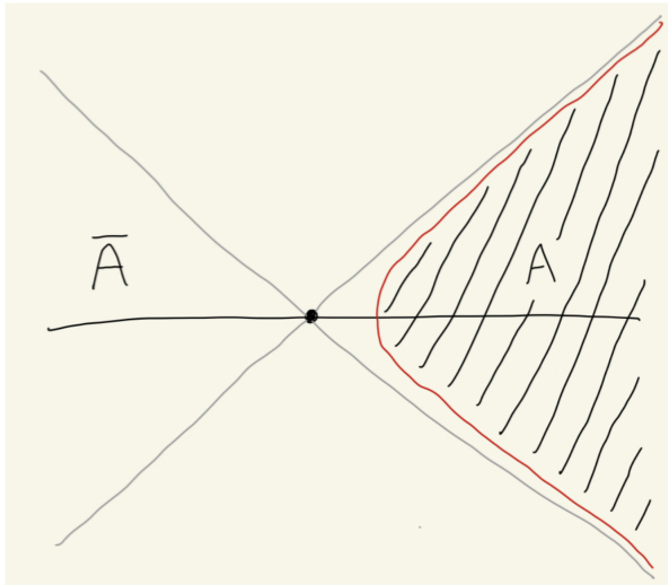

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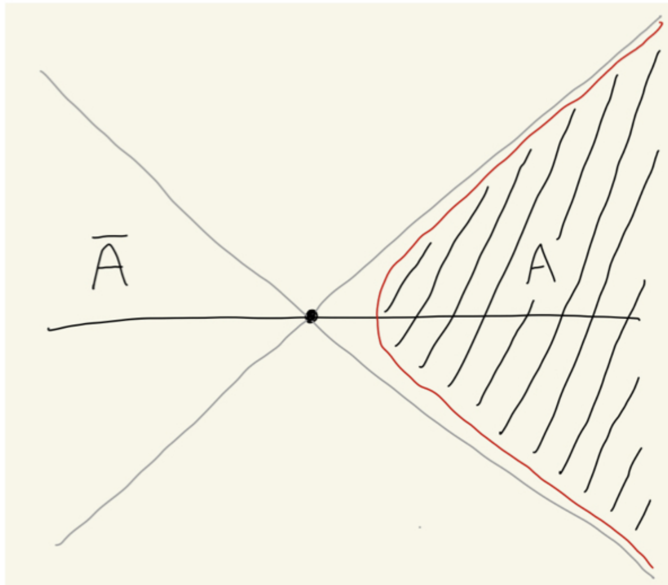
- Want a BC allowing both $\lambda|_{\partial A}$ and $E_n|_{\partial A}$



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  - Forces $E_n|_{\partial M} = 0$
- Dynamical edge mode (DEM) boundary condition: $F_{in}|_{\partial M} = 0 = A_t|_{\partial M}$
  - Allows $\lambda|_{\partial M}$ *and* $E_n|_{\partial M}$
  - Keeps all edge modes!

Dynamical Edge Modes

- With DEM BC, can parametrize data on A as
  - $A_i = \tilde{A}_i + \nabla_i \alpha \quad \text{where} \quad \nabla^i \tilde{A}_i = 0 = \tilde{A}_n|_{\partial A}$
  - $E_i = \tilde{E}_i + \nabla_i \beta \quad \text{where} \quad \nabla^i \tilde{E}_i = 0 = \tilde{E}_n|_{\partial A}$

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- Phase space factorizes,
  - $\Gamma_{\text{DEM}} = \Gamma_{\text{bulk}} \times \Gamma_{\text{edge}}$ also note $\Gamma_{\text{bulk}} = \Gamma_{\text{MC}}$

Dynamical Edge Modes

- Also have $H = \int_A (\frac{1}{2}\tilde{E}^i\tilde{E}_i + \frac{1}{4}\tilde{F}^{ij}\tilde{F}_{ij}) + \int_{\partial A} E_n \frac{1}{K} E_n$

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- K is the Dirichlet-to-Neumann operator. Maps harmonic Dirichlet data on ∂A to Neumann data on ∂A
  - Can write as integral kernel in terms of harmonic Green's function
  - In static horizon limit, simplifies to $K \leftrightarrow \log(\varepsilon^{-1})\,\Delta_{\partial A}$
    - Here ε is spatial distance from horizon to brick wall

Edge partition function

- $$\begin{aligned} Z_{\mathrm{DEM}}(\beta) &= \mathrm{Tr}\, e^{-\beta H} \\ &= \mathrm{Tr}_{\mathrm{bulk}}\, e^{-\beta H_{\mathrm{bulk}}}\ \mathrm{Tr}_{\mathrm{edge}}\, e^{-\beta H_{\mathrm{edge}}} \\ &= Z_{\mathrm{bulk}}(\beta)\, Z_{\mathrm{edge}}(\beta) \end{aligned}$$

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- Z_{bulk} is magnetically conducting case. Has been computed in examples
- Z_{edge} is our object of interest
  - $Z_{\mathrm{edge}}(2\pi) = \mathrm{Tr}_{\mathrm{edge}}\, e^{-2\pi H_{\mathrm{edge}}} \sim \det(K)^{+1/2} \sim \det(\Delta_{\partial A})^{+1/2} \sim 1/Z_{\mathrm{scalar}}(\partial A)$

Edge partition function

$$Z_{\text{edge}} = \text{Tr}_{\text{edge}}\, e^{-2\pi H_{\text{edge}}} \sim 1/Z_{\text{scalar}}(\partial A)$$

Found the codimension-two scalar
and wrote as trace!
Resolves discrepancies in literature.

4D: $-\frac{31}{45} = -\frac{16}{45} - \frac{1}{3}$

Any D: $Z(S^D) = Z_{\text{bulk}} + Z_{\text{edge}}$

Thank you!