CELESTIAL CONFORMAL PRIMARIES IN EFTs

Prahar Mitra

April 9, 2024

University of Amsterdam

Simons Collaboration on Celestial Holography Satellite Meeting

0. Introduction

0.1) Scattering Amplitudes in $d + 2$ Dimensions

- A massless n-point scattering amplitude in $d + 2$ dimensions is given by

$$A_n(p_1, \cdots, p_n) = \langle\, \mathcal{O}_1(p_1) \cdots \mathcal{O}_n(p_n) \,\rangle,$$

where $\langle \cdots \rangle$ denotes the time-ordered vacuum correlation function and the operators $\mathcal{O}_k$ are defined via the LSZ reduction formula

$$\mathcal{O}_k(\pm p) = -i \int \mathrm{d}^{d+2} X \, \Phi_k^{\pm}(p|X) \partial^2 \varphi_k(X),$$

with

$$\Phi_k^{\pm}(p|X) = \varepsilon_k^{\pm}(p) e^{\mp i p \cdot X}, \qquad p^2 = 0.$$

0.1) Scattering Amplitudes in $d+2$ Dimensions

- A massless n-point scattering amplitude in $d+2$ dimensions is given by

$$A_n(p_1, \cdots, p_n) = \langle\, \mathcal{O}_1(p_1) \cdots \mathcal{O}_n(p_n) \,\rangle,$$

where $\langle \cdots \rangle$ denotes the time-ordered vacuum correlation function and the operators $\mathcal{O}_k$ are defined via the LSZ reduction formula

$$\mathcal{O}_k(\pm p) = -i \int \mathrm{d}^{d+2}X \, \Phi_k^{\pm}(p|X) \partial^2 \varphi_k(X),$$

with

$$\Phi_k^{\pm}(p|X) = \varepsilon_k^{\pm}(p) e^{\mp i p \cdot X}, \qquad p^2 = 0.$$

- The plane wave operators $\mathcal{O}_k(p)$ diagonalize the translation generators,

$$P_\mu \cdot \mathcal{O}_k(p) = -p_\mu \mathcal{O}_k(p).$$

0.2) Celestial Amplitudes in $d + 2$ Dimensions

- A massless n-point celestial amplitude in $d + 2$ dimensions is given by

$$\mathcal{A}_n(\Delta_1, \vec{x}_1, \epsilon_1; \cdots ; \Delta_n, \vec{x}_n, \epsilon_n) = \left\langle \mathcal{O}_1^{\epsilon_1}(\Delta_1, \vec{x}_1) \cdots \mathcal{O}_n^{\epsilon_n}(\Delta_n, \vec{x}_n) \right\rangle$$

where

$$\mathcal{O}_k^{\pm}(\Delta, \vec{x}) = -i \int \mathrm{d}^{d+2}X \, \Psi_k^{\pm}(\Delta, \vec{x}|X) \partial^2 \varphi_k(X),$$

with (for scalars) Pasterski, Shao [1705.01027]

$$\Psi^{\pm}(\Delta, \vec{x}|X) = \frac{\Gamma(\Delta)}{(\mp i \hat{q}(\vec{x}) \cdot X)^{\Delta}}, \qquad \hat{q}^{\mu}(\vec{x}) \equiv \left(\frac{1 + |\vec{x}|^2}{2}, \vec{x}, \frac{1 - |\vec{x}|^2}{2} \right).$$

0.2) Celestial Amplitudes in $d+2$ Dimensions

- A massless n-point celestial amplitude in $d+2$ dimensions is given by

$$\mathcal{A}_n(\Delta_1,\vec{x}_1,\epsilon_1;\cdots;\Delta_n,\vec{x}_n,\epsilon_n) = \left\langle \mathcal{O}_1^{\epsilon_1}(\Delta_1,\vec{x}_1)\cdots\mathcal{O}_n^{\epsilon_n}(\Delta_n,\vec{x}_n) \right\rangle$$

where

$$\mathcal{O}_k^{\pm}(\Delta,\vec{x}) = -i\int \mathrm{d}^{d+2}X\,\Psi_k^{\pm}(\Delta,\vec{x}|X)\partial^2\varphi_k(X),$$

with (for scalars) Pasterski, Shao [1705.01027]

$$\Psi^{\pm}(\Delta,\vec{x}|X) = \frac{\Gamma(\Delta)}{(\mp i\hat{q}(\vec{x})\cdot X)^{\Delta}}, \qquad \hat{q}^{\mu}(\vec{x}) \equiv \left(\frac{1+|\vec{x}|^2}{2},\vec{x},\frac{1-|\vec{x}|^2}{2}\right).$$

- The conformal primary operators $\mathcal{O}_k^{\pm}(\Delta,\vec{x})$ transform as conformal primaries, e.g.

$$D\cdot\mathcal{O}_k^{\pm}(\Delta,\vec{0}) = -i\Delta\mathcal{O}_k^{\pm}(\Delta,\vec{0}), \qquad D = M_{d+1,0},$$

$$K_a\cdot\mathcal{O}_k^{\pm}(\Delta,\vec{0}) = 0, \qquad K_a = M_{0,a}+M_{d+1,a}.$$

$(a,b,\ldots\in\{1,\cdots,d\})$

0.3) Scattering Amplitudes $\longleftrightarrow$ Celestial Amplitudes

- The direct relationship between the plane wave operators and conformal primary operators is given by the Mellin transform

$$\mathbb{O}^{\pm}(\Delta, \vec{x}) = \int_0^{\infty} \mathrm{d}\omega \omega^{\Delta-1} \mathcal{O}(\pm\omega \hat{q}(\vec{x})).$$

0.3) Scattering Amplitudes $\longleftrightarrow$ Celestial Amplitudes

- The direct relationship between the plane wave operators and conformal primary operators is given by the Mellin transform

$$\mathbb{O}^{\pm}(\Delta, \vec{x}) = \int_0^\infty \mathrm{d}\omega \omega^{\Delta-1} \mathcal{O}(\pm\omega \hat{q}(\vec{x})).$$

- Completeness and normalizability (w.r.t. Klein-Gordon norm) of conformal primary wavefunctions implies Pasterski, Shao [1705.01027]

$$\Delta \in \mathcal{C}_P \cong \frac{d}{2} + i\mathbb{R}.$$

0.3) Scattering Amplitudes $\longleftrightarrow$ Celestial Amplitudes

- The direct relationship between the plane wave operators and conformal primary operators is given by the Mellin transform

$$\mathbb{O}^{\pm}(\Delta, \vec{x}) = \int_0^\infty \mathrm{d}\omega \omega^{\Delta-1} \mathcal{O}(\pm\omega \hat{q}(\vec{x})).$$

- Completeness and normalizability (w.r.t. Klein-Gordon norm) of conformal primary wavefunctions implies Pasterski, Shao [1705.01027]

$$\Delta \in \mathcal{C}_P \cong \frac{d}{2} + i\mathbb{R}.$$

- The inverse transform is given by

$$\mathcal{O}\left(\pm\omega \hat{q}(\vec{x})\right) = \int_{\mathcal{C}_P} \frac{\mathrm{d}\Delta}{2\pi i} \omega^{-\Delta} \mathbb{O}^{\pm}(\Delta, \vec{x}).$$

0.4) Some Observations

- The Mellin integral requires us to integrate the energy ω over $(0, \infty)$
  $\implies$ Celestial amplitudes cannot be defined for effective field theories.

0.4) Some Observations

- The Mellin integral requires us to integrate the energy ω over $(0, \infty)$ $\implies$ Celestial amplitudes cannot be defined for effective field theories.
- Completeness and normalizability restricts $\Delta \in \frac{d}{2} + i\mathbb{R}$. Many operators of interest do *not* live here, e.g. $w_{1+\infty}$ generators have $\Delta \in 2 - \mathbb{N}$.

0.4) Some Observations

- The Mellin integral requires us to integrate the energy ω over $(0, \infty)$ $\implies$ Celestial amplitudes cannot be defined for effective field theories.
- Completeness and normalizability restricts $\Delta \in \frac{d}{2} + i\mathbb{R}$. Many operators of interest do *not* live here, e.g. $w_{1+\infty}$ generators have $\Delta \in 2 - \mathbb{N}$.
  They are constructed by analytically continuing $\Delta \in \mathbb{C}$ and then extracting the residue of the poles at $\Delta \in 2 - \mathbb{N}$.

0.4) Some Observations

- The Mellin integral requires us to integrate the energy ω over $(0, \infty)$ $\implies$ Celestial amplitudes cannot be defined for effective field theories.
- Completeness and normalizability restricts $\Delta \in \frac{d}{2} + i\mathbb{R}$. Many operators of interest do *not* live here, e.g. $w_{1+\infty}$ generators have $\Delta \in 2 - \mathbb{N}$.
  They are constructed by analytically continuing $\Delta \in \mathbb{C}$ and then extracting the residue of the poles at $\Delta \in 2 - \mathbb{N}$.
- The bulk translation generators P_μ map $\Delta \to \Delta + 1$ which takes us off the principal series axis, $\mathcal{C}_P$.

0.4) Some Observations

- The Mellin integral requires us to integrate the energy ω over $(0, \infty)$ $\implies$ Celestial amplitudes cannot be defined for effective field theories.

- Completeness and normalizability restricts $\Delta \in \frac{d}{2} + i\mathbb{R}$. Many operators of interest do *not* live here, e.g. $w_{1+\infty}$ generators have $\Delta \in 2 - \mathbb{N}$.
  They are constructed by analytically continuing $\Delta \in \mathbb{C}$ and then extracting the residue of the poles at $\Delta \in 2 - \mathbb{N}$.

- The bulk translation generators P_μ map $\Delta \to \Delta + 1$ which takes us off the principal series axis, $\mathcal{C}_P$.
  P_μ is not bounded, but $\exp(-ia^\mu P_\mu)$ is, so that the latter has a good action

$$\exp(-ia^\mu P_\mu) \cdot \mathbb{O}_k^\pm(\Delta, \vec{x}) = \int_{\mathcal{C}_P} \frac{d\Delta'}{2\pi i} \frac{\Gamma(\Delta - \Delta')}{[\mp ia \cdot \hat{q}(\vec{x}) + \epsilon]^{\Delta - \Delta'}} \mathbb{O}_k^\pm(\Delta', \vec{x}).$$

0.4) Some Observations

- The Mellin integral requires us to integrate the energy ω over $(0, \infty)$
  $\implies$ Celestial amplitudes cannot be defined for effective field theories.

- Completeness and normalizability restricts $\Delta \in \frac{d}{2} + i\mathbb{R}$. Many operators of interest do *not* live here, e.g. $w_{1+\infty}$ generators have $\Delta \in 2 - \mathbb{N}$.
  They are constructed by analytically continuing $\Delta \in \mathbb{C}$ and then extracting the residue of the poles at $\Delta \in 2 - \mathbb{N}$.

- The bulk translation generators P_μ map $\Delta \to \Delta + 1$ which takes us off the principal series axis, $\mathcal{C}_P$.
  P_μ is not bounded, but $\exp(-ia^\mu P_\mu)$ is, so that the latter has a good action

$$\exp(-ia^\mu P_\mu) \cdot \mathcal{O}_k^\pm(\Delta, \vec{x}) = \int_{\mathcal{C}_P} \frac{\mathrm{d}\Delta'}{2\pi i} \frac{\Gamma(\Delta - \Delta')}{[\mp ia \cdot \hat{q}(\vec{x}) + \epsilon]^{\Delta - \Delta'}} \mathcal{O}_k^\pm(\Delta', \vec{x}).$$

- *Is there a different definition of the conformal primary operators that removes these obstacles?*

0.4) Some Observations

- The Mellin integral requires us to integrate the energy ω over $(0,\infty)$ $\implies$ Celestial amplitudes cannot be defined for effective field theories.

- Completeness and normalizability restricts $\Delta \in \frac{d}{2} + i\mathbb{R}$. Many operators of interest do *not* live here, e.g. $w_{1+\infty}$ generators have $\Delta \in 2 - \mathbb{N}$.
  They are constructed by analytically continuing $\Delta \in \mathbb{C}$ and then extracting the residue of the poles at $\Delta \in 2 - \mathbb{N}$.

- The bulk translation generators P_μ map $\Delta \to \Delta + 1$ which takes us off the principal series axis, $\mathcal{C}_P$.
  P_μ is not bounded, but $\exp(-ia^\mu P_\mu)$ is, so that the latter has a good action

$$\exp(-ia^\mu P_\mu) \cdot \mathcal{O}_k^\pm(\Delta, \vec{x}) = \int_{\mathcal{C}_P} \frac{d\Delta'}{2\pi i} \frac{\Gamma(\Delta - \Delta')}{[\mp ia \cdot \hat{q}(\vec{x}) + \epsilon]^{\Delta - \Delta'}} \mathcal{O}_k^\pm(\Delta', \vec{x}).$$

- *Is there a different definition of the conformal primary operators that removes these obstacles?* Yes! (this talk) PM [2402.09256]

Approach

I. Generalize the integration contours used to define conformal primaries.

II. Impose constraints: 1) Completeness, 2) Consistency with symmetries, and 3) Normalizability $\implies$ Only two contour choices are allowed!

- $\mathcal{C}_\omega^\pm \cong \mathbb{R}_\pm$, $\mathcal{C}_\Delta \cong \frac{d}{2} + i\mathbb{R}$.
- $\mathcal{C}_\omega \cong \Lambda e^{i[0,2\pi)}$, $\mathcal{C}_\Delta \cong \mathbb{Z}$ $\leftarrow$ resolves aforementioned obstacles!

III. Summary

IV. Comments

I. Generalized Conformal Primaries

I. Generalized Definition for Conformal Primaries

- Consider a generalized definition for the conformal primaries

$$\mathcal{O}(\Delta, \vec{x}) = c \int_{\mathcal{C}_\omega} \mathrm{d}\omega \, \omega^{\Delta-1} \mathcal{O}(\omega \hat{q}(\vec{x})),$$
$$\mathcal{O}(\omega \hat{q}(\vec{x})) = c' \int_{\mathcal{C}_\Delta} \mathrm{d}\Delta \, \omega^{-\Delta} \mathcal{O}(\Delta, \vec{x}).$$

The kernels $\omega^{\Delta-1}$ and $\omega^{-\Delta}$ are fixed by Lorentz symmetry and cannot be altered. The only thing we get to choose are the contours of integration $\mathcal{C}_\omega$ and $\mathcal{C}_\Delta$.

I. Generalized Definition for Conformal Primaries

- Consider a generalized definition for the conformal primaries

$$\mathbb{O}(\Delta, \vec{x}) = c \int_{\mathcal{C}_\omega} \mathrm{d}\omega \omega^{\Delta-1} \mathcal{O}(\omega \hat{q}(\vec{x})),$$
$$\mathcal{O}(\omega \hat{q}(\vec{x})) = c' \int_{\mathcal{C}_\Delta} \mathrm{d}\Delta \omega^{-\Delta} \mathbb{O}(\Delta, \vec{x}).$$

  The kernels $\omega^{\Delta-1}$ and $\omega^{-\Delta}$ are fixed by Lorentz symmetry and cannot be altered. The only thing we get to choose are the contours of integration $\mathcal{C}_\omega$ and $\mathcal{C}_\Delta$.

- The analytic continuation to $\omega \in \mathbb{C}$ is defined via the LSZ reduction formula.

I. Generalized Definition for Conformal Primaries

- Consider a generalized definition for the conformal primaries

$$\mathbb{O}(\Delta, \vec{x}) = c \int_{\mathcal{C}_\omega} \mathrm{d}\omega \omega^{\Delta-1} \mathcal{O}(\omega \hat{q}(\vec{x})),$$
$$\mathcal{O}(\omega \hat{q}(\vec{x})) = c' \int_{\mathcal{C}_\Delta} \mathrm{d}\Delta \omega^{-\Delta} \mathbb{O}(\Delta, \vec{x}).$$

  The kernels $\omega^{\Delta-1}$ and $\omega^{-\Delta}$ are fixed by Lorentz symmetry and cannot be altered. The only thing we get to choose are the contours of integration $\mathcal{C}_\omega$ and $\mathcal{C}_\Delta$.

- The analytic continuation to $\omega \in \mathbb{C}$ is defined via the LSZ reduction formula.
- For now, we restrict ourselves to tree-level amplitudes so $\mathcal{O}(\omega \hat{q}(\vec{x}))$ are meromorphic functions of ω. We will discuss with $\ln \omega$-type terms later.

II.1) Constraints: Completeness

II.1) Completeness

- We recall

$$\widetilde{\mathcal{O}}(\Delta, \vec{x}) = c \int_{\mathcal{C}_\omega} \mathrm{d}\omega \omega^{\Delta-1} \mathcal{O}(\omega \hat{q}(\vec{x})),$$
$$\mathcal{O}(\omega \hat{q}(\vec{x})) = c' \int_{\mathcal{C}_\Delta} \mathrm{d}\Delta \omega^{-\Delta} \widetilde{\mathcal{O}}(\Delta, \vec{x}).$$

- The plane wave operators $\mathcal{O}(\omega \hat{q}(\vec{x}))$ are complete. Consequently, $\widetilde{\mathcal{O}}(\Delta, \vec{x})$ will also be complete if the basis transformations above are invertible. Substituting the second equation into the first and vice versa, we find the constraints

$$cc' \int_{\mathcal{C}_\Delta} \mathrm{d}\Delta \omega^{-\Delta} \omega'^{\Delta-1} = \delta_{\mathcal{C}_\omega}(\omega', \omega),$$
$$cc' \int_{\mathcal{C}_\omega} \mathrm{d}\omega \omega^{\Delta-\Delta'-1} = \delta_{\mathcal{C}_\Delta}(\Delta', \Delta).$$

II.1) Completeness

- To process the constraints, we set $\omega = e^t$. In terms of t, they read

$$cc' \int_{\mathcal{C}_t} \mathrm{d}t e^{(t-t')(\Delta-\Delta')} = \delta_{\mathcal{C}_\Delta}(\Delta', \Delta),$$

$$cc' \int_{\mathcal{C}_\Delta} \mathrm{d}\Delta e^{-(t-t')(\Delta-\Delta')} = \delta_{\mathcal{C}_t}(t', t).$$

II.1) Completeness

- To process the constraints, we set $\omega = e^t$. In terms of t, they read

$$cc' \int_{\mathcal{C}_t} \mathrm{d}t e^{(t-t')(\Delta-\Delta')} = \delta_{\mathcal{C}_\Delta}(\Delta', \Delta),$$

$$cc' \int_{\mathcal{C}_\Delta} \mathrm{d}\Delta e^{-(t-t')(\Delta-\Delta')} = \delta_{\mathcal{C}_t}(t', t).$$

- Compare these to the standard integral representation of the Dirac delta function,

$$\frac{1}{2\pi i} \int_{i\mathbb{R}} \mathrm{d}x e^{(x-x')(k-k')} = \delta_{\mathbb{R}}(k, k') = \delta(k - k').$$

II.1) Completeness

- To process the constraints, we set $\omega = e^t$. In terms of t, they read

$$cc' \int_{\mathcal{C}_t} \mathrm{d}t e^{(t-t')(\Delta-\Delta')} = \delta_{\mathcal{C}_\Delta}(\Delta', \Delta),$$
$$cc' \int_{\mathcal{C}_\Delta} \mathrm{d}\Delta e^{-(t-t')(\Delta-\Delta')} = \delta_{\mathcal{C}_t}(t', t).$$

- Compare these to the standard integral representation of the Dirac delta function,

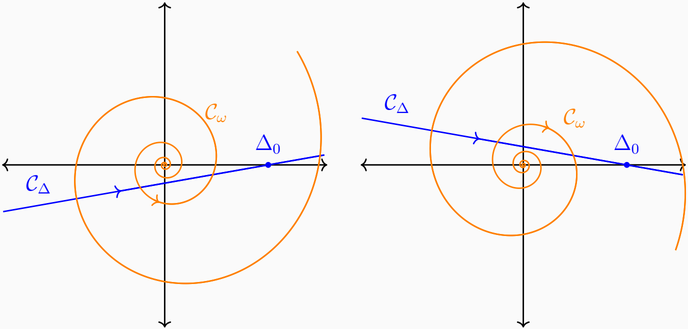
$$\frac{1}{2\pi i} \int_{i\mathbb{R}} \mathrm{d}x e^{(x-x')(k-k')} = \delta_{\mathbb{R}}(k, k') = \delta(k - k').$$

- Constraints are satisfied if

$$\mathcal{C}_\Delta \cong \Delta'_0 + e^{i\alpha}\mathbb{R}, \qquad \mathcal{C}_t \cong t'_0 + ie^{-i\alpha}\mathbb{R}, \qquad cc' = \frac{1}{2\pi i}.$$

for some fixed $\Delta'_0, t'_0 \in \mathbb{C}$ and $\alpha \in [0, \pi)$.

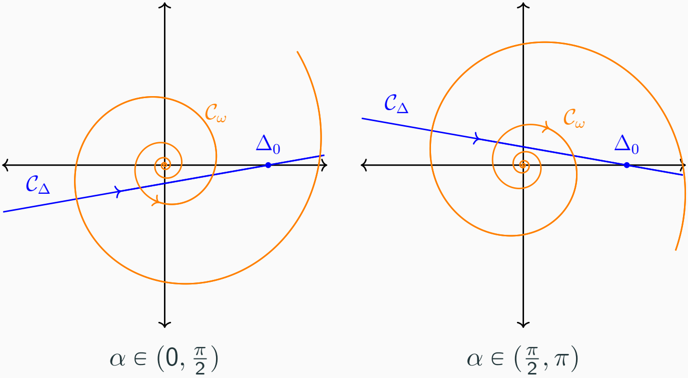
II.1) Completeness: $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$



$\alpha \in (0, \frac{\pi}{2})$ $\alpha \in (\frac{\pi}{2}, \pi)$

Contours in ω (orange) and Δ (blue) for $\alpha \neq 0$.

II.1) Completeness: $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$



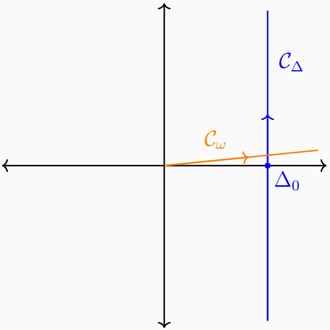
Contours in ω (orange) and Δ (blue) for $\alpha \neq 0$.

For fixed $\Delta_0, t_0 \in \mathbb{R}$, the contour is

$$\Delta = (\Delta_0 + \nu \cos \alpha) + i(\nu \sin \alpha), \qquad \nu \in \mathbb{R},$$

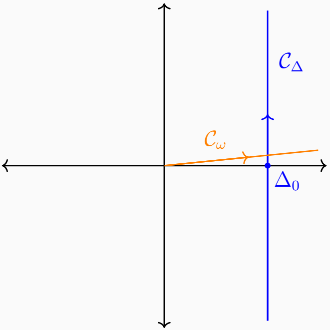
$$\omega = e^{\eta \sin \alpha} \cos(t_0 + \eta \cos \alpha) + i e^{\eta \sin \alpha} \sin(t_0 + \eta \cos \alpha), \qquad \eta \in \mathbb{R}.$$

II.1) Completeness: $\alpha = \frac{\pi}{2}$



Contours in ω (orange) and Δ (blue) for $\alpha = \frac{\pi}{2}$.

II.1) Completeness: $\alpha = \frac{\pi}{2}$

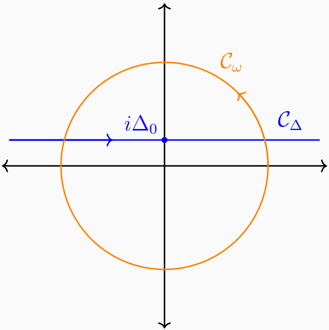


Contours in ω (orange) and Δ (blue) for $\alpha = \frac{\pi}{2}$.

For fixed $\Delta_0, t_0 \in \mathbb{R}$, the contour is

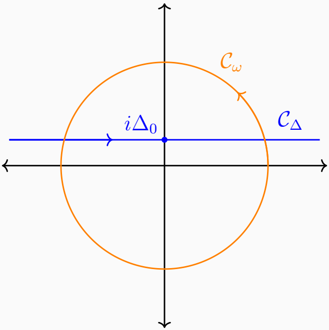
$$\Delta = \Delta_0 + i\nu, \qquad \nu \in \mathbb{R},$$
$$\omega = e^{\eta} \cos t_0 + i e^{\eta} \sin t_0, \qquad \eta \in \mathbb{R},$$

II.1) Completeness: $\alpha = 0$



Contours in ω (orange) and Δ (blue) for $\alpha = 0$.

II.1) Completeness: $\alpha = 0$



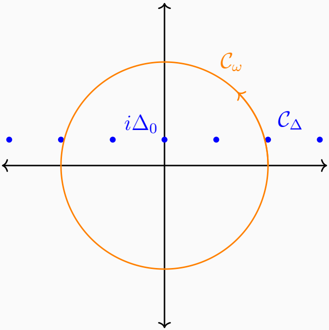
Contours in ω (orange) and Δ (blue) for $\alpha = 0$.

For fixed $\Delta_0, t_0 \in \mathbb{R}$, the contour is

$$\Delta = \nu + i\Delta_0, \qquad \nu \in \mathbb{R},$$
$$\omega = e^{t_0} e^{i\eta}, \qquad \eta \in \mathbb{R}.$$

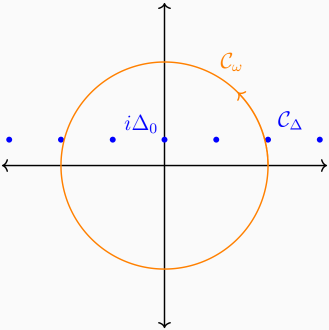
II.1) Completeness: $\alpha = 0$ + Analyticity

If the amplitude is analytic inside $\mathcal{C}_\omega$, then we can further restrict $\mathcal{C}_\Delta \in \mathbb{Z} + i\Delta_0$



II.1) Completeness: $\alpha = 0$ + Analyticity

If the amplitude is analytic inside $\mathcal{C}_\omega$, then we can further restrict $\mathcal{C}_\Delta \in \mathbb{Z} + i\Delta_0$



For fixed $\Delta_0, t_0 \in \mathbb{R}$, the contour is

$$\Delta = n + i\Delta_0, \qquad n \in \mathbb{Z},$$
$$\omega = e^{t_0} e^{i\eta}, \qquad \eta \in [0, 2\pi).$$

II.2) Constraints: Consistency with Symmetry

II.2) Consistency with Conformal Symmetry

We recall the definitions

$$\mathcal{O}(\Delta, \vec{x}) = c \int_{\mathcal{C}_\omega} \mathrm{d}\omega \omega^{\Delta-1} \mathcal{O}(\omega \hat{q}(\vec{x})),$$

$$\mathcal{O}(\omega \hat{q}(\vec{x})) = c' \int_{\mathcal{C}_\Delta} \mathrm{d}\Delta \omega^{-\Delta} \mathcal{O}(\Delta, \vec{x}).$$

Under boosts along the X^{d+1} direction,

$$D \cdot \mathcal{O}(\omega \hat{q}(\vec{0})) = i\omega \partial_\omega \mathcal{O}(\omega \hat{q}(\vec{0})).$$

It follows that

$$\begin{aligned} D \cdot \mathcal{O}(\Delta, \vec{0}) &= ic \int_{\mathcal{C}_\omega} \mathrm{d}\omega \omega^{\Delta} \partial_\omega \mathcal{O}(\omega \hat{q}(\vec{0})) \\ &= -i\Delta \mathcal{O}(\Delta, \vec{0}) + ic \left(\omega^{\Delta} \mathcal{O}(\omega \hat{q}(\vec{0})) \right) \Big|_{\partial \mathcal{C}_\omega} \end{aligned}$$

For $\mathcal{O}$ to transform as a conformal primary, the second term above must vanish.

II.2) Consistency with Conformal Symmetry ($\alpha \neq 0$)

- For the $\alpha \neq 0$ contours, $\partial \mathcal{C}_\omega = \{0, \infty\}$ so in this case, we must have

$$\lim_{\omega \to 0,\infty} [\omega^\Delta \mathcal{O}(\omega \hat{q}(\vec{x}))] = 0.$$

  We can't really say much about the UV limit without knowing the precise UV behavior of the theory. The IR limit is constrained by soft theorems which states $\mathcal{O}(\omega \hat{q}(\vec{x})) = \mathcal{O}(\omega^{-1})$.

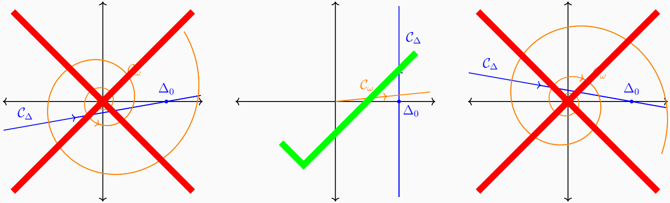
II.2) Consistency with Conformal Symmetry ($\alpha \neq 0$)

- For the $\alpha \neq 0$ contours, $\partial \mathcal{C}_\omega = \{0, \infty\}$ so in this case, we must have

$$\lim_{\omega \to 0, \infty} [\omega^\Delta \mathcal{O}(\omega \hat{q}(\vec{x}))] = 0.$$

  We can't really say much about the UV limit without knowing the precise UV behavior of the theory. The IR limit is constrained by soft theorems which states $\mathcal{O}(\omega \hat{q}(\vec{x})) = \mathcal{O}(\omega^{-1})$.

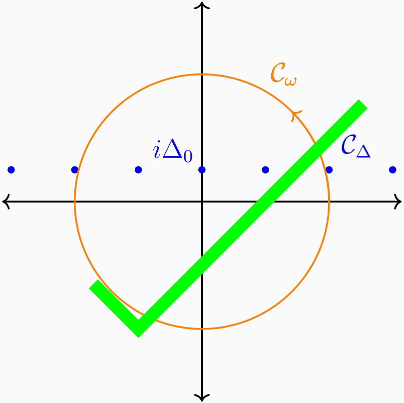
- The limit above holds iff. $\mathrm{Re}\,\Delta \geqslant 1$. It follows that only the $\alpha = \frac{\pi}{2}$ contour can be used with $\Delta_0 \geqslant 1$.



$\alpha \in (0, \frac{\pi}{2})$ $\qquad$ $\alpha = \frac{\pi}{2}$ $\qquad$ $\alpha \in (\frac{\pi}{2}, \pi)$

II.2) Consistency with Conformal Symmetry ($\alpha = 0$)

- For the $\alpha = 0$ contours, $\partial\mathcal{C}_\omega = \varnothing$ so in this case, there is no new constraint in this case.



II.2) Consistency with CPT

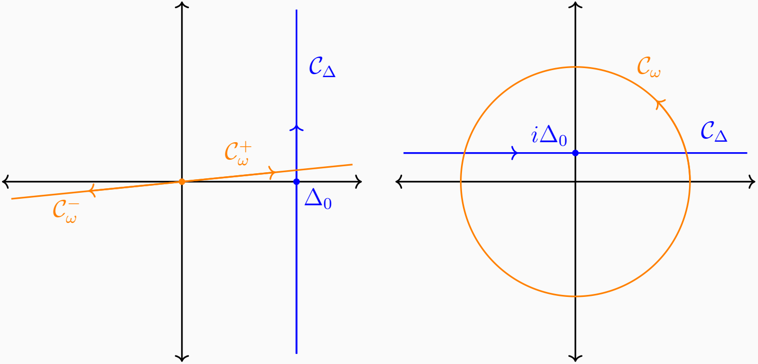
- CPT maps $\omega \to -\omega$.

II.2) Consistency with CPT

- CPT maps $\omega \to -\omega$.
- The $\alpha = \frac{\pi}{2}$ contour is *not* invariant under CPT, so we need to introduce a second set of conformal primary operators

II.2) Consistency with CPT

- CPT maps $\omega \to -\omega$.
- The $\alpha = \frac{\pi}{2}$ contour is *not* invariant under CPT, so we need to introduce a second set of conformal primary operators
- The $\alpha = 0$ contour is invariant under CPT, so in this case, we do not need to introduce a second set of conformal primary operators.



$\alpha = \frac{\pi}{2}$ $\alpha = 0$

II.3) Normalizability

II.3) Normalizability ($\alpha = \frac{\pi}{2}$)

- There are four different conformal primaries with $\alpha = \frac{\pi}{2}$, namely $\mathcal{O}^{\pm}(\Delta, \vec{x})$ and $\bar{\mathcal{O}}^{\pm}(\Delta, \vec{x})$

  (Roughly, these are the *in* creation and *out* annihilation operators for a particle and its CPT conjugate) .

II.3) Normalizability ($\alpha = \frac{\pi}{2}$)

- There are four different conformal primaries with $\alpha = \frac{\pi}{2}$, namely $\mathbb{O}^{\pm}(\Delta, \vec{x})$ and $\bar{\mathbb{O}}^{\pm}(\Delta, \vec{x})$ (Roughly, these are the *in* creation and *out* annihilation operators for a particle and its CPT conjugate) .
- The non-vanishing two-point function of the conformal primaries is given by

$$\langle\, \mathbb{O}^{+}(\Delta, \vec{x})\bar{\mathbb{O}}^{-}(\Delta', \vec{x}\,')\,\rangle = \int_{\mathcal{C}_\omega} \mathrm{d}\omega\omega^{\Delta-1} \int_{\mathcal{C}'_\omega} \mathrm{d}\omega'\omega'^{\Delta'-1} \langle\, \mathcal{O}(\omega, \vec{x})\bar{\mathcal{O}}(\omega', \vec{x}\,')\,\rangle.$$

II.3) Normalizability ($\alpha = \frac{\pi}{2}$)

- There are four different conformal primaries with $\alpha = \frac{\pi}{2}$, namely $\mathbb{O}^{\pm}(\Delta, \vec{x})$ and $\bar{\mathbb{O}}^{\pm}(\Delta, \vec{x})$
  (Roughly, these are the *in* creation and *out* annihilation operators for a particle and its CPT conjugate) .

- The non-vanishing two-point function of the conformal primaries is given by

$$\langle\, \mathbb{O}^{+}(\Delta, \vec{x})\bar{\mathbb{O}}^{-}(\Delta', \vec{x}\,')\,\rangle = \int_{\mathcal{C}_\omega} \mathrm{d}\omega\omega^{\Delta-1} \int_{\mathcal{C}'_\omega} \mathrm{d}\omega'\omega'^{\Delta'-1} \langle\, \mathcal{O}(\omega, \vec{x})\bar{\mathcal{O}}(\omega', \vec{x}\,')\,\rangle.$$

- The integrand is a $1 \to 1$ scattering amplitudes and is given by

$$\langle\, \mathcal{O}(\omega, \vec{x})\bar{\mathcal{O}}(\omega', \vec{x}\,')\,\rangle = 2(2\pi)^{d+1}(-\omega\omega')^{\frac{1}{2}(1-d)}\delta(\omega+\omega')\delta^{(d)}(\vec{x}-\vec{x}\,').$$

II.3) Normalizability ($\alpha = \frac{\pi}{2}$)

- There are four different conformal primaries with $\alpha = \frac{\pi}{2}$, namely $\mathbb{O}^{\pm}(\Delta, \vec{x})$ and $\bar{\mathbb{O}}^{\pm}(\Delta, \vec{x})$
  (Roughly, these are the *in* creation and *out* annihilation operators for a particle and its CPT conjugate) .

- The non-vanishing two-point function of the conformal primaries is given by

$$\langle\, \mathbb{O}^{+}(\Delta, \vec{x})\bar{\mathbb{O}}^{-}(\Delta', \vec{x}\,')\,\rangle = \int_{\mathcal{C}_\omega} \mathrm{d}\omega\omega^{\Delta-1} \int_{\mathcal{C}'_\omega} \mathrm{d}\omega'\omega'^{\Delta'-1} \langle\, \mathcal{O}(\omega, \vec{x})\bar{\mathcal{O}}(\omega', \vec{x}\,')\,\rangle.$$

- The integrand is a $1 \to 1$ scattering amplitudes and is given by

$$\langle\, \mathcal{O}(\omega, \vec{x})\bar{\mathcal{O}}(\omega', \vec{x}\,')\,\rangle = 2(2\pi)^{d+1}(-\omega\omega')^{\frac{1}{2}(1-d)}\delta(\omega + \omega')\delta^{(d)}(\vec{x} - \vec{x}\,').$$

- Analytically continue $\delta(\omega + \omega') \to \delta_{\mathcal{C}_\omega}(\omega, -\omega')$.

II.3) Normalizability ($\alpha = \frac{\pi}{2}$)

- It follows that

$$\langle\, \mathcal{O}^{+}(\Delta, \vec{x})\bar{\mathcal{O}}^{-}(\Delta', \vec{x}\,')\,\rangle = 2e^{\pi i(\Delta - d)}(2\pi)^{d+1}\delta^{(d)}(\vec{x} - \vec{x}\,') \times \int_{\mathcal{C}'_{\omega}} \mathrm{d}\omega' \omega'^{\Delta+\Delta-d-1}.$$

II.3) Normalizability ($\alpha = \frac{\pi}{2}$)

- It follows that

$$\langle\, \mathcal{O}^{+}(\Delta, \vec{x})\bar{\mathcal{O}}^{-}(\Delta', \vec{x}\,')\,\rangle = 2e^{\pi i(\Delta - d)}(2\pi)^{d+1}\delta^{(d)}(\vec{x} - \vec{x}\,') \times \int_{\mathcal{C}'_{\omega}} \mathrm{d}\omega' \omega'^{\Delta+\Delta-d-1}.$$

- The integral over ω' along the contour is given by

$$\int_{\mathcal{C}'_{\omega}} \mathrm{d}\omega' \omega'^{\Delta+\Delta'-d-1} = \int_{-\infty}^{\infty} \mathrm{d}\eta e^{(\eta + it_{\mathbf{0}})(2\Delta_{\mathbf{0}} - d + i(\nu+\nu'))}.$$

where we used the fact that in this case, $\Delta \in \Delta_0 + i\mathbb{R}$.

II.3) Normalizability ($\alpha = \frac{\pi}{2}$)

- It follows that

$$\langle \mathcal{O}^{+}(\Delta, \vec{x})\bar{\mathcal{O}}^{-}(\Delta', \vec{x}\,') \rangle = 2e^{\pi i(\Delta - d)}(2\pi)^{d+1}\delta^{(d)}(\vec{x} - \vec{x}\,') \\ \times \int_{\mathcal{C}'_\omega} \mathrm{d}\omega' \omega'^{\Delta+\Delta-d-1}.$$

- The integral over ω' along the contour is given by

$$\int_{\mathcal{C}'_\omega} \mathrm{d}\omega' \omega'^{\Delta+\Delta'-d-1} = \int_{-\infty}^{\infty} \mathrm{d}\eta e^{(\eta + it_{\mathbf{0}})(2\Delta_{\mathbf{0}} - d + i(\nu+\nu'))}.$$

where we used the fact that in this case, $\Delta \in \Delta_0 + i\mathbb{R}$.

- δ-function normalizability requires $\Delta_0 = \frac{d}{2} \implies \Delta \in \mathcal{C}_P$.

II.3) Normalizability ($\alpha = 0$)

- The argument is the same as before. The integral over ω' along the contour, in this case, is given by

$$\int_{\mathcal{C}_\omega} d\omega \omega^{\Delta+\Delta'-d-1} = i \int_{-\infty}^{\infty} d\eta e^{(t_0+i\eta)(2i\Delta_0+\nu+\nu'-d)}$$

where we used the fact that in this case, $\Delta \in \mathbb{R} + i\Delta_0$.

II.3) Normalizability ($\alpha = 0$)

- The argument is the same as before. The integral over ω' along the contour, in this case, is given by

$$\int_{\mathcal{C}_\omega} \mathrm{d}\omega \omega^{\Delta+\Delta'-d-1} = i \int_{-\infty}^{\infty} \mathrm{d}\eta e^{(t_0+i\eta)(2i\Delta_0+\nu+\nu'-d)}$$

  where we used the fact that in this case, $\Delta \in \mathbb{R} + i\Delta_0$.

- δ-function normalizability now implies $\Delta_0 = 0 \implies \Delta \in \mathbb{R}$.

III. Summary

III. Summary

Completeness, normalizability, and consistency with symmetries imply two possible definitions for conformal primary operators.

III. Summary

Completeness, normalizability, and consistency with symmetries imply two possible definitions for conformal primary operators.

- **$\alpha = \frac{\pi}{2}$:** In this case, we define *two* sets of conformal primary operators

$$\mathbb{O}^{\pm}(\Delta, \vec{x}) = \int_{e^{i(t_0 \pm \pi/2)}\mathbb{R}_+} \mathrm{d}\omega\, \omega^{\Delta-1} \mathcal{O}(\omega, \vec{x}), \qquad \Delta \in \frac{d}{2} + i\mathbb{R}.$$

for any $t_0 \in [-\pi/2, \pi/2)$. The Pasterski-Shao conformal primaries have $t_0 = -\pi/2$.

III. Summary

Completeness, normalizability, and consistency with symmetries imply two possible definitions for conformal primary operators.

- $\alpha = \frac{\pi}{2}$**:** In this case, we define *two* sets of conformal primary operators

$$\mathbb{O}^{\pm}(\Delta, \vec{x}) = \int_{e^{i(t_0 \pm \pi/2)}\mathbb{R}_+} d\omega \omega^{\Delta-1} \mathcal{O}(\omega, \vec{x}), \qquad \Delta \in \frac{d}{2} + i\mathbb{R}.$$

  for any $t_0 \in [-\pi/2, \pi/2)$. The Pasterski-Shao conformal primaries have $t_0 = -\pi/2$.

- $\alpha = 0$**:** In this case, we define just one set of conformal primary operator

$$\mathbb{O}(\Delta, \vec{x}) = \int_{\Lambda e^{i\mathbb{R}}} \frac{d\omega}{2\pi i} \omega^{\Delta-1} \mathcal{O}(\omega, \vec{x}), \qquad \Delta \in \mathbb{R}.$$

  for any $\Lambda \in \mathbb{R}_+$. If we additionally assume that $\mathcal{O}(\omega, \vec{x})$ is analytic in ω for $|\omega| < \Lambda$, then we can restrict the contour integral to $\Lambda e^{i[0,2\pi)}$ with $\Delta \in \mathbb{Z}$.

IV. Comments

IV. Comments

- The conformal primary operators with $\Delta \in \mathbb{Z}$ resolves all the obstacles we discussed previously.
  - The contour of integration in ω is $\Lambda e^{i[0,2\pi)}$. As long as $\Lambda < \Lambda_{\text{UV}}$, we can evaluate these integrals in EFTs.
  - All operators of interest (*so far*) are obtained from this definition. No analytic continuation in Δ is required.
  - P_μ has a perfectly well-defined action on the conformal primaries ($\Delta \to \Delta + 1$ makes sense!).

IV. Comments

- The conformal primary operators with $\Delta \in \mathbb{Z}$ resolves all the obstacles we discussed previously.
  - The contour of integration in ω is $\Lambda e^{i[0,2\pi)}$. As long as $\Lambda < \Lambda_{\text{UV}}$, we can evaluate these integrals in EFTs.
  - All operators of interest (*so far*) are obtained from this definition. No analytic continuation in Δ is required.
  - P_μ has a perfectly well-defined action on the conformal primaries ($\Delta \to \Delta + 1$ makes sense!).

- Fermionic plane wave operators are analytic in $\sqrt{\omega}$ so in that case, we have $\Delta \in \frac{1}{2}\mathbb{Z}$.

IV. Comments

- The conformal primary operators with $\Delta \in \mathbb{Z}$ resolves all the obstacles we discussed previously.
  - The contour of integration in ω is $\Lambda e^{i[0,2\pi)}$. As long as $\Lambda < \Lambda_{\rm UV}$, we can evaluate these integrals in EFTs.
  - All operators of interest (*so far*) are obtained from this definition. No analytic continuation in Δ is required.
  - P_μ has a perfectly well-defined action on the conformal primaries ($\Delta \to \Delta + 1$ makes sense!).

- Fermionic plane wave operators are analytic in $\sqrt{\omega}$ so in that case, we have $\Delta \in \frac{1}{2}\mathbb{Z}$.

- The operators have a topological dependence on Λ (radius of contour). For multiple insertions, the ordering of Λ defines the order of softness $\implies$ no ambiguity in $J_z J_{\bar{w}}$ OPE!

IV. Comments

- If $\mathcal{O}(\omega, \vec{x})$ has $\ln \omega$ terms in its soft expansion, then the contour integrals that define $\mathbb{O}(\Delta, \vec{x})$ are not defined due to the branch cut. This can be fixed by extending the definition to

$$\mathbb{O}^{(m)}(\Delta, \vec{x}) = \oint \frac{d\omega}{2\pi i} \omega^{\Delta - 1} (\omega \partial_\omega)^m \mathcal{O}(\omega, \vec{x})$$

The highest power m above equals the highest power of $\ln \omega$ that appears at $\mathcal{O}(\omega^{-\Delta})$ in the soft expansion.

IV. Comments

- If $\mathcal{O}(\omega, \vec{x})$ has $\ln \omega$ terms in its soft expansion, then the contour integrals that define $\mathbb{O}(\Delta, \vec{x})$ are not defined due to the branch cut. This can be fixed by extending the definition to

$$\mathbb{O}^{(m)}(\Delta, \vec{x}) = \oint \frac{d\omega}{2\pi i} \omega^{\Delta - 1} (\omega \partial_\omega)^m \mathcal{O}(\omega, \vec{x})$$

  The highest power m above equals the highest power of $\ln \omega$ that appears at $\mathcal{O}(\omega^{-\Delta})$ in the soft expansion.

- Conformal primaries with $\Delta \in \mathbb{Z}$ have appeared previously in Freidel, Pranzetti, Raclariu [2212.12469] and Cotler, Miller, Strominger [2302.04905]. These authors constructed conformal primary wavefunctions with $\Delta \in \mathbb{Z}$ that are normalizable w.r.t. modified norms (L^2-norm on Schwarz space and RSW norm). The operators constructed here are most certainly related to these.

Thank You!