Effective Actions in a Supermanifold

Viola Gattus

First Quantisation for Physics in Strong Fields, Edinburgh, UK

26 February 2025

Based on Phys. Rev. D 110, 105006 (2024) with Prof Apostolos Pilaftsis

MANCHESTER
1824
The University of Manchester

CENTRE FOR THEORETICAL PHYSICS
Higgs

In this talk

I. Heat Kernels

- Standard Heat Kernel
- Intuitive Heat Kernel Method
- Dim-reg in position space

II. QFT Applications

- Limitations of standard QFT
- Ordinary Effective Action
- QFT in Field Space
- Covariant Effective Action
- Covariant interactions
- QEA with curvature
- Fermionic one-loop QEA
- Outlook
- Next up

I. Heat Kernels

Standard Heat Kernel Method

Flat spacetime

One-loop scalar effective action

$$\Gamma^{(1)} = \frac{i}{2}\left[\ln \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)} - \ln \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=0}\right]$$

Standard Heat Kernel Method

Flat spacetime

One-loop scalar effective action

$$\Gamma^{(1)} = \frac{i}{2}\left[\ln \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)} - \ln \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=0}\right]$$

Simple example

$$S \;=\; \int d^d x \left[\frac{1}{2}(\partial_\mu\phi)^2 \;-\; \frac{1}{2}m^2\phi^2 \;-\; U(\phi(x))\right]$$

Standard Heat Kernel Method

Flat spacetime

One-loop scalar effective action

$$\Gamma^{(1)} = \frac{i}{2}\left[\ln\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)} - \ln\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=0}\right]$$

Simple example

$$S \;=\; \int d^d x \left[\frac{1}{2}(\partial_\mu\phi)^2 \;-\; \frac{1}{2}m^2\phi^2 \;-\; U(\phi(x))\right]$$

Q.M. Hamiltonian

$$K(x,y) \;=\; \frac{\delta^2 S}{\delta\phi(x)\,\delta\phi(y)} \;=\; \left(-\partial^2 \;-\; m^2 - V\right)\delta(x-y)$$

Standard Heat Kernel Method

Flat spacetime

One-loop scalar effective action

$$\Gamma^{(1)} = \frac{i}{2}\left[\ln \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)} - \ln \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=0}\right]$$

Simple example

$$S = \int d^d x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - U(\phi(x))\right]$$

Q.M. Hamiltonian

$$K(x,y) = \frac{\delta^2 S}{\delta\phi(x)\,\delta\phi(y)} = \left(-\partial^2 - m^2 - \boxed{V}\right)\delta(x-y)$$

$$\boxed{V \equiv U''}$$

Standard Heat Kernel Method

Flat spacetime

One-loop scalar effective action

$$\Gamma^{(1)} = -\frac{i}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{tr}\left(e^{-tK} - e^{-tK_0}\right)$$

$$= -\frac{i}{2}\int d^dx \int d^dy \int_0^\infty \frac{dt}{t}\left[\langle x|\, e^{-tK}\,|y\rangle - \langle x|\, e^{-tK_0}\,|y\rangle\right]$$

Standard Heat Kernel Method

Flat spacetime

One-loop scalar effective action

$$\Gamma^{(1)} = -\frac{i}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{tr}\left(e^{-tK} - e^{-tK_0}\right)$$

$$= -\frac{i}{2}\int d^dx \int d^dy \int_0^\infty \frac{dt}{t}\left[\langle x|\, e^{-tK}\,|y\rangle - \langle x|\, e^{-tK_0}\,|y\rangle\right]$$

Schrödinger equation

$$\left[-\partial_x^2 + V(x)\right]\langle x|\, e^{-tK}\,|y\rangle = -\frac{\partial}{\partial t}\langle x|\, e^{-tK}\,|y\rangle$$

Standard Heat Kernel Method

Flat spacetime

One-loop scalar effective action

$$\Gamma^{(1)} = -\frac{i}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{tr}\left(e^{-tK} - e^{-tK_0}\right)$$

$$= -\frac{i}{2}\int d^dx \int d^dy \int_0^\infty \frac{dt}{t}\left[\langle x|\, e^{-tK}\,|y\rangle - \langle x|\, e^{-tK_0}\,|y\rangle\right]$$

Schrödinger equation

$$\left[-\partial_x^2 + V(x)\right]\boxed{\langle x|\, e^{-tK}\,|y\rangle} = -\frac{\partial}{\partial t}\langle x|\, e^{-tK}\,|y\rangle$$

$$\boxed{\equiv e^{-\sigma(x,y;t)}}$$

Synge world function

Standard Heat Kernel Method

Flat spacetime

One-loop scalar effective action

$$\Gamma^{(1)} = -\frac{i}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{tr}\left(e^{-tK} - e^{-tK_0}\right)$$

$$= -\frac{i}{2}\int d^dx \int d^dy \int_0^\infty \frac{dt}{t}\left[\langle x|\, e^{-tK}\,|y\rangle - \langle x|\, e^{-tK_0}\,|y\rangle\right]$$

Schrödinger equation

$$\partial^2\sigma - (\partial\sigma)^2 + V = \frac{\partial\sigma}{\partial t}$$

Standard Heat Kernel Method

Flat spacetime

One-loop scalar effective action

$$\Gamma^{(1)} = -\frac{i}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{tr}\left(e^{-tK} - e^{-tK_0}\right)$$

$$= -\frac{i}{2}\int d^dx \int d^dy \int_0^\infty \frac{dt}{t}\left[\langle x|\, e^{-tK}\,|y\rangle - \langle x|\, e^{-tK_0}\,|y\rangle\right]$$

Schrödinger equation

$$\partial^2\sigma - (\partial\sigma)^2 + V = \frac{\partial\sigma}{\partial t}$$

Small t expansion

$$\sigma = \frac{1}{4t}(x-y)^2 + \frac{d}{2}\ln 4\pi t + At + Bt^2 + Ct^3 + \mathcal{O}(t^4)$$

Standard Heat Kernel Method

Flat spacetime

HaMiDeW coefficients

$$\Gamma^{(1)} = -\frac{i}{2}\int d^dx \int_0^\infty \frac{dt}{t}\frac{e^{-m^2t}}{(4\pi t)^{d/2}}\left[Vt + \frac{1}{2}V^2t^2 + \frac{1}{6}\left(V^3 - (\partial_\mu V)^2\right)t^3 + \mathcal{O}(t^4)\right]$$

Standard Heat Kernel Method

Flat spacetime

HaMiDeW coefficients

$$\Gamma^{(1)} = -\frac{i}{2}\int d^dx \int_0^\infty \frac{dt}{t}\frac{e^{-m^2 t}}{(4\pi t)^{d/2}}\left[Vt + \frac{1}{2}V^2t^2 + \frac{1}{6}\left(V^3 - (\partial_\mu V)^2\right)t^3 + \mathcal{O}(t^4)\right]$$

Evaluate t integral in dim reg in $d = 4 - 2\epsilon$

$$\begin{aligned}\Gamma^{(1)} \;&=\; -\frac{i}{2}\int \frac{d^dx}{(4\pi)^{d/2}}\left[(-m^2)^{(-1+d/2)}V\,\Gamma\!\left(1-\frac{d}{2}\right) \;+\; \frac{1}{2}(-m^2)^{(-4+d)/2}V^2\,\Gamma\!\left(2-\frac{d}{2}\right)\right.\\ &\left.\quad + \frac{1}{6}(-m^2)^{(-6+d)/2}\left(V^3-(\partial_\mu V)^2\right)\Gamma\!\left(3-\frac{d}{2}\right)\right].\end{aligned}$$

Standard Heat Kernel Method

Flat spacetime

HaMiDeW coefficients

$$\Gamma^{(1)} = -\frac{i}{2}\int d^dx \int_0^\infty \frac{dt}{t}\frac{e^{-m^2 t}}{(4\pi t)^{d/2}}\left[Vt + \frac{1}{2}V^2t^2 + \frac{1}{6}\left(V^3 - (\partial_\mu V)^2\right)t^3 + \mathcal{O}(t^4)\right]$$

Evaluate t integral in dim reg in $d = 4 - 2\epsilon$

UV divergent

$$\Gamma^{(1)} = -\frac{i}{2}\int \frac{d^dx}{(4\pi)^{d/2}}\left[\boxed{(-m^2)^{(-1+d/2)}V\,\Gamma\left(1-\frac{d}{2}\right)} + \boxed{\frac{1}{2}(-m^2)^{(-4+d)/2}V^2\,\Gamma\left(2-\frac{d}{2}\right)}\right.$$
$$\left. + \frac{1}{6}(-m^2)^{(-6+d)/2}\left(V^3 - (\partial_\mu V)^2\right)\Gamma\left(3-\frac{d}{2}\right)\right].$$

Standard Heat Kernel Method

Flat spacetime

HaMiDeW coefficients

$$\Gamma^{(1)} = -\frac{i}{2}\int d^dx \int_0^\infty \frac{dt}{t}\frac{e^{-m^2 t}}{(4\pi t)^{d/2}}\left[Vt + \frac{1}{2}V^2t^2 + \frac{1}{6}\left(V^3 - (\partial_\mu V)^2\right)t^3 + \mathcal{O}(t^4)\right]$$

Evaluate t integral in dim reg in $d = 4 - 2\epsilon$

UV divergent

$$\Gamma^{(1)} = -\frac{i}{2}\int \frac{d^dx}{(4\pi)^{d/2}}\left[\boxed{(-m^2)^{(-1+d/2)}V\,\Gamma\left(1-\frac{d}{2}\right)} + \boxed{\frac{1}{2}(-m^2)^{(-4+d)/2}V^2\,\Gamma\left(2-\frac{d}{2}\right)}\right.$$
$$\left. + \boxed{\frac{1}{6}(-m^2)^{(-6+d)/2}\left(V^3 - (\partial_\mu V)^2\right)\Gamma\left(3-\frac{d}{2}\right)}\right].$$

UV finite

Intuitive Heat Kernel Method

Extracting UV behaviour

Use HK to compute one-loop effective action in x-space

1. Represent $\ln \Delta$ as integral over t
2. Obtain UV divergences as $1/\varepsilon$-poles in $t \to 0^+$ limit
3. Solve iteratively a diffusion-type equation in powers of t

Intuitive Heat Kernel Method

Extracting UV behaviour

Use HK to compute one-loop effective action in x-space

1. Represent $\ln \Delta$ as integral over t

2. Obtain UV divergences as $1/\varepsilon$-poles in $t \to 0^+$ limit

3. ~~Solve iteratively a diffusion-type equation in powers of t~~

Intuitive Heat Kernel Method

Extracting UV behaviour

Use HK to compute one-loop effective action in x-space

1. Represent $\ln \Delta$ as integral over t
2. Obtain UV divergences as $1/\varepsilon$-poles in $t \to 0^+$ limit
3. Use Zassenhaus formula to derive HK coefficients

$$\exp\left[t(X+Y)\right] = \exp(tX)\exp(tY)\exp\left(-\frac{t^2}{2}[X,Y]\right)\exp\left[\frac{t^3}{6}\left(2[Y,[X,Y]]+[X,[X,Y]]\right)\right]\ldots$$

Intuitive Heat Kernel Method

Extracting UV behaviour

Use HK to compute one-loop effective action in x-space

1. Represent $\ln\Delta$ as integral over t
2. Obtain UV divergences as $1/\varepsilon$-poles in $t \to 0^+$ limit
3. Use Zassenhaus formula to derive HK coefficients

$$\exp\left[t(X+Y)\right] = \exp(tX)\,\exp(tY)\,\exp\left(-\frac{t^2}{2}[X,Y]\right)\,\exp\left[\frac{t^3}{6}\Big(2[Y,[X,Y]]+[X,[X,Y]]\Big)\right]\ldots$$

E.g. for $\widehat{X} = -\partial^2 - m^2$ and $\widehat{Y} = -V$

$$\langle x|e^{-t(-\partial^2-m^2)}|y\rangle = \frac{e^{-\frac{1}{4t}(x-y)^2+m^2 t}}{(4\pi t)^{d/2}}$$

$$\langle x|e^{-t(-\partial^2-m^2-V)}|x'\rangle = \int d^d y\, \boxed{\langle x|e^{t(\partial^2+m^2)}|y\rangle}$$

$$\times\, \langle y|e^{tV} e^{-\frac{t^2}{2}[\partial^2,V]} e^{\frac{t^3}{6}\left(2[V,[\partial^2,V]]+[\partial^2,[\partial^2,V]]\right)} e^{\mathcal{O}(t^4)}|x'\rangle$$

Dim-Reg in Position Space

Massless Heat Kernel

Massless HK integral

$$\int_0^\infty dt\, \frac{e^{-(x_A-x_B)^2/4t}}{(4\pi t)^{d/2}}\, t^n \;=\; 2^{-2+2n}\,\pi^{-d/2}\,|x_A-x_B|^{-d+2+2n}\,\Gamma\left(-1+\frac{d}{2}-n\right)$$

Dim-Reg in Position Space

Massless Heat Kernel

Massless HK integral

$$\int_0^\infty dt\, \frac{e^{-(x_A-x_B)^2/4t}}{(4\pi t)^{d/2}}\, t^n \;=\; 2^{-2+2n}\, \pi^{-d/2}\, |x_A - x_B|^{-d+2+2n}\, \Gamma\left(-1+\frac{d}{2}-n\right)$$

Momentum space integral in d dimensions

$$\int \frac{d^dk}{(2\pi)^d}\, \frac{1}{(k^2-\Delta)^\alpha} \;=\; \frac{i\,(-1)^\alpha}{(4\pi)^{d/2}}\, \frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)} \left(\frac{1}{\Delta}\right)^{\alpha-d/2}$$

Dim-Reg in Position Space

Massless Heat Kernel

Massless HK integral

$$\int_0^\infty dt\, \frac{e^{-(x_A-x_B)^2/4t}}{(4\pi t)^{d/2}}\, t^n \;=\; 2^{-2+2n}\, \pi^{-d/2}\, |x_A - x_B|^{-d+2+2n}\, \Gamma\left(-1+\frac{d}{2}-n\right)$$

Momentum space integral in d dimensions

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-\Delta)^\alpha} \;=\; \frac{i\,(-1)^\alpha}{(4\pi)^{d/2}} \frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)} \left(\frac{1}{\Delta}\right)^{\alpha-d/2}$$

Equivalent position space integral

$$\int d^dx\, \frac{\delta^{(d)}(x)}{[x^2]^{d/2-\alpha}} \;=\; i\,(-1)^\alpha 4^{\alpha-d/2} \frac{\Gamma(\alpha-d/2)}{\Gamma(d/2-\alpha)} \lim_{\Delta\to 0} \left(\frac{1}{\Delta}\right)^{\alpha-d/2}$$

Dim-Reg in Position Space

Massless Heat Kernel

Massless HK integral

$$\int_0^\infty dt \, \frac{e^{-(x_A-x_B)^2/4t}}{(4\pi t)^{d/2}} \, t^n \;=\; 2^{-2+2n} \, \pi^{-d/2} \, |x_A - x_B|^{-d+2+2n} \, \Gamma\left(-1 + \frac{d}{2} - n\right)$$

Momentum space integral in d dimensions

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^\alpha} = \frac{i\,(-1)^\alpha}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - d/2)}{\Gamma(\alpha)} \left(\frac{1}{\Delta}\right)^{\alpha - d/2}$$

Equivalent position space integral

$$\int d^d x \, \frac{\delta^{(d)}(x)}{[x^2]^{d/2-\alpha}} \;=\; i\,(-1)^\alpha 4^{\alpha - d/2} \frac{\Gamma(\alpha - d/2)}{\Gamma(d/2 - \alpha)} \lim_{\Delta \to 0} \left(\frac{1}{\Delta}\right)^{\alpha - d/2}$$

Vanishing tadpole condition $\alpha = 1$

$0 \le \alpha < 2$	vanishes
$\alpha = 2$	$-i$
$\alpha > 2$	IR divergent

Dim-Reg in Position Space

Massive Heat Kernel

Massive HK integral

$$\int_0^\infty \frac{dt}{t}\,\frac{e^{-t\mu^2-(x-y)^2/4t}}{(4\pi t)^{d/2}}\,t^n \;=\; (\mu^2)^{\frac{1}{4}(d-2n)}\,2^{1-d/2-n}\,\pi^{-d/2}\,|x-y|^{n-d/2}\,K_\nu\left(\sqrt{\mu^2}|x-y|\right)$$

Dim-Reg in Position Space

Massive Heat Kernel

Massive HK integral

$$\int_0^\infty \frac{dt}{t}\,\frac{e^{-t\mu^2-(x-y)^2/4t}}{(4\pi t)^{d/2}}\,t^n \;=\; (\mu^2)^{\frac{1}{4}(d-2n)}\,2^{1-d/2-n}\,\pi^{-d/2}\,|x-y|^{n-d/2}\,K_\nu\left(\sqrt{\mu^2}|x-y|\right)$$

Define ‘reduced’ Bessel function

$$z^{-\nu}K_\nu(\mu z) \;=\; \frac{1}{2}\,\mu^\nu 2^{-\nu}\hat{K}_\nu(\mu z)$$

Dim-Reg in Position Space

Massive Heat Kernel

Massive HK integral

$$\int_0^\infty \frac{dt}{t} \frac{e^{-t\mu^2-(x-y)^2/4t}}{(4\pi t)^{d/2}}\, t^n \;=\; (\mu^2)^{\frac{1}{4}(d-2n)}\, 2^{1-d/2-n}\, \pi^{-d/2}\, |x-y|^{n-d/2}\, K_\nu\left(\sqrt{\mu^2}|x-y|\right)$$

Define 'reduced' Bessel function

$$z^{-\nu} K_\nu(\mu z) \;=\; \frac{1}{2}\, \mu^\nu 2^{-\nu} \hat{K}_\nu(\mu z)$$

If $\nu < 0$,

$$\lim_{z\to 0} \hat{K}_\nu(\mu z) \;=\; \Gamma(-\nu)$$

II. QFT application

Limitations of standard QFT

Motivation

Off-shell calculations sensitive to choice of parametrisation

Limitations of standard QFT

Motivation

Off-shell calculations sensitive to choice of parametrisation

Gauge-fixing in gauge theory and quantum gravity [Barvinsky, Vilkovisky (1985), Ellicott, Toms (1989), Burgess, Kunstatter (1987), Odintsov (1990)]

Quantum frame problem [Burns, Karamitsos, Pilaftsis (2016), Falls, Herrero-Valea (2019), Finn, Karamitsos, Pilaftsis (2020) ..]

Parametrisation dependent SMEFT expansion [Alonso, Jenkins, Manohar (2016), Cohen, Craig, Sutherland (2021), Talbert (2023), Assi, Helset, Manohar, Pagès, Shen (2023) ...]

Limitations of standard QFT

Motivation

Off-shell calculations sensitive to choice of parametrisation

Gauge-fixing in gauge theory and quantum gravity [Barvinsky, Vilkovisky (1985), Ellicott, Toms (1989), Burgess, Kunstatter (1987), Odintsov (1990)]

Quantum frame problem [Burns, Karamitsos, Pilaftsis (2016), Falls, Herrero-Valea (2019), Finn, Karamitsos, Pilaftsis (2020) ..]

Parametrisation dependent SMEFT expansion [Alonso, Jenkins, Manohar (2016), Cohen, Craig, Sutherland (2021), Talbert (2023), Assi, Helset, Manohar, Pagès, Shen (2023) ...]

Solution?

Limitations of standard QFT

Motivation

Off-shell calculations sensitive to choice of parametrisation

Gauge-fixing in gauge theory and quantum gravity [Barvinsky, Vilkovisky (1985), Ellicott, Toms (1989), Burgess, Kunstatter (1987), Odintsov (1990)]

Quantum frame problem [Burns, Karamitsos, Pilaftsis (2016), Falls, Herrero-Valea (2019), Finn, Karamitsos, Pilaftsis (2020) ..]

Parametrisation dependent SMEFT expansion [Alonso, Jenkins, Manohar (2016), Cohen, Craig, Sutherland (2021), Talbert (2023), Assi, Helset, Manohar, Pagès, Shen (2023) ...]

Solution?

Covariant differential geometric approach to QFT

Ordinary Effective Action

Non-covariance of standard formulation

Path integral formulation of QFT

$$Z[J] \equiv \exp\left(\frac{i}{\hbar}W[J]\right) = \int [\mathrm{d}\phi]\sqrt{\det(G)}\exp\left[\frac{i}{\hbar}S[\phi] + \frac{i}{\hbar}J_a\phi^a\right]$$

Effective Action (EA) definition

$$\exp\left(\frac{i}{\hbar}\Gamma[\varphi]\right) = \int [\mathrm{d}\phi]\ \exp\left\{\frac{i}{\hbar}\left[S[\phi] + \frac{\delta\Gamma[\varphi]}{\delta\varphi^A}(\varphi^A - \phi^A)\right]\right\}$$

Ordinary Effective Action

Non-covariance of standard formulation

Path integral formulation of QFT

$$Z[J] \equiv \exp\left(\frac{i}{\hbar}W[J]\right) = \int [\mathrm{d}\phi]\sqrt{\det(G)}\,\exp\left[\frac{i}{\hbar}S[\phi] + \frac{i}{\hbar}J_a\phi^a\right]$$

Effective Action (EA) definition

$$\exp\left(\frac{i}{\hbar}\Gamma[\varphi]\right) = \int [\mathrm{d}\phi]\ \exp\left\{\frac{i}{\hbar}\left[S[\phi] + \boxed{\frac{\delta\Gamma[\varphi]}{\delta\varphi^A}}\,\boxed{(\varphi^A - \phi^A)}\right]\right\}$$

not a vector

ordinary functional derivatives

Ordinary Effective Action

Non-covariance of standard formulation

Path integral formulation of QFT

$$Z[J] \equiv \exp\left(\frac{i}{\hbar}W[J]\right) = \int [\mathrm{d}\phi]\sqrt{\det(G)}\exp\left[\frac{i}{\hbar}S[\phi] + \frac{i}{\hbar}J_a\phi^a\right]$$

Effective Action (EA) definition

$$\exp\left(\frac{i}{\hbar}\Gamma[\varphi]\right) = \int [\mathrm{d}\phi]\ \exp\left\{\frac{i}{\hbar}\left[S[\phi] + \boxed{\frac{\delta\Gamma[\varphi]}{\delta\varphi^A}}\boxed{(\varphi^A - \phi^A)}\right]\right\}$$

ordinary functional derivatives

not a vector

→ Vilkovisky-DeWitt formalism in field space

QFT in Field Space

Supermanifolds

Field-space supermanifold of dimension (N|8M) in 4d spacetime

Fields are the chart [DeWitt (2012)]

$$\boldsymbol{\Phi} \equiv \{\Phi^a\} = \begin{pmatrix} \phi^A \\ \psi^X \\ \overline{\psi}^{Y\,\mathsf{T}} \end{pmatrix}$$

Field reparameterization = diffeomorphism

$$\Phi^a \to \widetilde{\Phi}^a = \widetilde{\Phi}^a(\boldsymbol{\Phi})$$

QFT in Field Space

Supermanifolds

Field-space supermanifold of dimension (N|8M) in 4d spacetime

Fields are the chart

[DeWitt (2012)]

$$\mathbf{\Phi} \equiv \{\Phi^a\} = \begin{pmatrix} \phi^A \\ \psi^X \\ \overline{\psi}^{Y\,\mathsf{T}} \end{pmatrix}$$

Field reparameterization = diffeomorphism

$$\Phi^a \to \widetilde{\Phi}^a = \widetilde{\Phi}^a(\mathbf{\Phi})$$

Diffeormophically - or frame invariant Lagrangian

[Finn, Karamitsos, Pilaftsis (2021), VG, Finn, Karamitsos, Pilaftsis (2022)]

$$\boxed{\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi^A \,{}_A k_B(\mathbf{\Phi})\, \partial_\nu \Phi^B + \frac{i}{2} \zeta^\mu_A(\mathbf{\Phi})\, \partial_\mu \Phi^A - U(\mathbf{\Phi})}$$

QFT in Field Space

Supermanifolds

Supermanifold metric

$$ {}_AG_B = ({}_AG_B)^{\mathsf{sT}} $$

symmetric rank-2 FS tensor
ultralocal
determined from action

QFT in Field Space

Supermanifolds

Supermanifold metric

$${}_AG_B = ({}_AG_B)^{\mathsf{sT}}$$

- symmetric rank-2 FS tensor
- ultralocal
- determined from action

Global metric found from vielbeins and local metric

[Finn, Karamitsos, Pilaftsis (2021), VG, Finn, Karamitsos, Pilaftsis (2022)]

$${}_AG_B = {}_Ae^{\widehat{M}} \boxed{{}_{\widehat{M}}H_{\widehat{N}}} {}^{\widehat{N}}e_B^{\mathsf{sT}}$$

$${}_{\widehat{A}}H_{\widehat{B}} \equiv \begin{pmatrix} \mathbf{1}_N & 0 & 0 \\ 0 & 0 & \mathbf{1}_{dM} \\ 0 & -\mathbf{1}_{dM} & 0 \end{pmatrix}$$

Covariant Effective Action

Implicit equation

Implicit equation for effective action using VDW

[Vilkovisky (1984), DeWitt (1985), Finn, Karamitsos, Pilaftsis (2022)]

$$\exp\left(\frac{i}{\hbar}\Gamma[\mathbf{\Phi}]\right) = \int \sqrt{|\,\mathrm{sdet}\, G|}\,[\mathcal{D}\mathbf{\Phi}_q]\exp\left(\frac{i}{\hbar}S\left[\mathbf{\Phi}_q\right] + \frac{i}{\hbar}\int \delta^4 x\sqrt{-g}\;\Gamma[\mathbf{\Phi}]_{,a}\,\Sigma^a\left[\mathbf{\Phi},\mathbf{\Phi}_q\right]\right)$$

Covariant Effective Action

Implicit equation

Implicit equation for effective action using VDW

[Vilkovisky (1984), DeWitt (1985), Finn, Karamitsos, Pilaftsis (2022)]

$$\exp\left(\frac{i}{\hbar}\Gamma[\mathbf{\Phi}]\right) = \int \sqrt{|\,\mathrm{sdet}\,G|}\,[\mathcal{D}\mathbf{\Phi}_q]\exp\left(\frac{i}{\hbar}S\left[\mathbf{\Phi}_q\right] + \frac{i}{\hbar}\int \delta^4 x\sqrt{-g}\,\Gamma[\mathbf{\Phi}]_{,a}\,\Sigma^a\left[\mathbf{\Phi},\mathbf{\Phi}_q\right]\right)$$

Master functional differential equation for 1PI QEA at all orders

[Kim (2006)]

$$2i\left(\Gamma^{(n)}\frac{\overleftarrow{\delta}}{\delta\Delta^{ab}}\right)\delta\Delta^{ba} = \mathrm{str}\left[\Delta^{-1}\sum_{k=1}^{n-1}(-1)^k\,\Delta\,\boxed{\Pi^{(l_1)}}\,\Delta\,\Pi^{(l_2)}\boxed{\Delta}\ldots\Pi^{(l_k)}\left(\delta\Delta\right)\right]$$

1PI self-energy

covariant propagator

Covariant Effective Action

Implicit equation

Implicit equation for effective action using VDW

[Vilkovisky (1984), DeWitt (1985), Finn, Karamitsos, Pilaftsis (2022)]

$$\exp\left(\frac{i}{\hbar}\Gamma[\mathbf{\Phi}]\right) = \int \sqrt{|\,\mathrm{sdet}\, G|}\,[\mathcal{D}\mathbf{\Phi}_q]\exp\left(\frac{i}{\hbar}S\left[\mathbf{\Phi}_q\right] + \frac{i}{\hbar}\int \delta^4 x\sqrt{-g}\,\Gamma[\mathbf{\Phi}]_{,a}\,\Sigma^a\left[\mathbf{\Phi},\mathbf{\Phi}_q\right]\right)$$

Master functional differential equation for 1PI QEA at all orders

[Kim (2006)]

$$2i\left(\Gamma^{(n)}\frac{\overleftarrow{\delta}}{\delta\Delta^{ab}}\right)\delta\Delta^{ba} = \mathrm{str}\left[\Delta^{-1}\sum_{k=1}^{n-1}(-1)^k\,\Delta\,\Pi^{(l_1)}\Delta\,\Pi^{(l_2)}\Delta\ldots\Pi^{(l_k)}\left(\delta\Delta\right)\right]$$

One and two loops expressions

$$\Gamma^{(1)}[\mathbf{\Phi}] = \frac{i}{2}\ln\mathrm{sdet}\left({}^aS_b\right) = \frac{i}{2}\mathrm{str}\left(\ln{}^aS_b\right)$$

$$\Gamma^{(2)}[\mathbf{\Phi}] = -\frac{1}{8}S_{\{abcd\}}\,\Delta^{dc}\Delta^{ba} + \frac{1}{12}(-1)^{bc+m(b+d)}\,S_{\{mca\}}\,\Delta^{ab}\Delta^{cd}\Delta^{mn}{}_{\{ndb\}}S$$

Covariant Effective Action

Implicit equation

Implicit equation for effective action using VDW

[Vilkovisky (1984), DeWitt (1985), Finn, Karamitsos, Pilaftsis (2022)]

$$\exp\left(\frac{i}{\hbar}\Gamma[\mathbf{\Phi}]\right) = \int \sqrt{|\,\mathrm{sdet}\, G|}\,[\mathcal{D}\mathbf{\Phi}_q]\exp\left(\frac{i}{\hbar}S\left[\mathbf{\Phi}_q\right] + \frac{i}{\hbar}\int \delta^4 x\sqrt{-g}\,\Gamma[\mathbf{\Phi}]_{,a}\,\Sigma^a\left[\mathbf{\Phi},\mathbf{\Phi}_q\right]\right)$$

Master functional differential equation for 1PI QEA at all orders

[Kim (2006)]

$$2i\left(\Gamma^{(n)}\frac{\overleftarrow{\delta}}{\delta\Delta^{ab}}\right)\delta\Delta^{ba} = \mathrm{str}\left[\Delta^{-1}\sum_{k=1}^{n-1}(-1)^k\,\Delta\,\Pi^{(l_1)}\Delta\,\Pi^{(l_2)}\Delta\ldots\Pi^{(l_k)}\left(\delta\Delta\right)\right]$$

One and two loops expressions

$${}_aS_b \equiv {}_a\overrightarrow{\nabla} S\overleftarrow{\nabla}_b \equiv {}_a\Delta_b^{-1}$$

$$\Gamma^{(1)}[\mathbf{\Phi}] = \frac{i}{2}\ln\mathrm{sdet}\left({}^aS_b\right) = \frac{i}{2}\mathrm{str}\left(\ln\boxed{{}^aS_b}\right)$$

$$\Gamma^{(2)}[\mathbf{\Phi}] = -\frac{1}{8}S_{\{abcd\}}\,\Delta^{dc}\boxed{\Delta^{ba}} + \frac{1}{12}(-1)^{bc+m(b+d)}\,S_{\{mca\}}\,\Delta^{ab}\Delta^{cd}\Delta^{mn}\,{}_{\{ndb\}}S$$

$$\Delta^{ac}\,{}_cS_b = {}^a\delta_b$$

Covariant Interactions

Scalar-Fermion Theories

D operator

[Ecker, Honerkamp, (1972)]

$$(\partial_\mu \Phi^a)_{;b} \;=\; \left(\delta^A{}_B\, \partial^{(A)}_\mu + \Gamma^A{}_{BM}\, \partial_\mu \Phi^M\right) \delta(x_A - x_B) \;\equiv\; (D_\mu)^a{}_b$$

Covariant Interactions

Scalar-Fermion Theories

D operator

[Ecker, Honerkamp, (1972)]

$$(\partial_\mu \Phi^a)_{;b} \;=\; \left(\delta^A{}_B\, \partial_\mu^{(A)} + \Gamma^A{}_{BM}\, \partial_\mu \Phi^M\right) \delta(x_A - x_B) \;\equiv\; (D_\mu)^a{}_b$$

Important supergeometric identity

$$\boxed{(D_\mu)_{ab;c} \;=\; R_{abcm}\, \partial_\mu \Phi^m}$$

Covariant Interactions

Scalar-Fermion Theories

D operator

[Ecker, Honerkamp, (1972)]

$$(\partial_\mu \Phi^a)_{;b} \;=\; \left(\delta^A{}_B\, \partial_\mu^{(A)} + \Gamma^A{}_{BM}\, \partial_\mu \Phi^M\right) \delta(x_A - x_B) \;\equiv\; (D_\mu)^a{}_b$$

Important supergeometric identity

$$(D_\mu)_{ab;c} \;=\; R_{abcm}\, \partial_\mu \Phi^m$$

Complete covariant inverse superpropagator

[VG, Pilaftsis (2024)]

$$\begin{aligned} S_{ab} = {} & (-1)^{am} (D_\mu)^m{}_a\, {}_m k_n\, (D^\mu)^n{}_b \;+\; \partial_\mu \Phi^m \Big({}_m k_n\, R^n{}_{abp}\, \partial^\mu \Phi^p \\ & + (-1)^{an}\, {}_m k_{n;[a}\, (D^\mu)^n{}_{b]} + \frac{1}{2} (-1)^{n(a+b)}\, {}_m k_{n;ab}\, \partial^\mu \Phi^n \Big) \\ & + i(-1)^a \left({}_a\lambda^\mu_m (D_\mu)^m{}_b + (-1)^{bm}\, {}_a\lambda^\mu_{m;b}\, \partial_\mu \Phi^m \right) - U_{ab} \end{aligned}$$

Covariant Interactions

Scalar-Fermion Theories

D operator

[Ecker, Honerkamp, (1972)]

$$(\partial_\mu \Phi^a)_{;b} \;=\; \left(\delta^A{}_B\, \partial_\mu^{(A)} + \Gamma^A{}_{BM}\, \partial_\mu \Phi^M\right) \delta(x_A - x_B) \;\equiv\; (D_\mu)^a{}_b$$

Important supergeometric identity

$$(D_\mu)_{ab;c} \;=\; R_{abcm}\, \partial_\mu \Phi^m$$

Complete covariant inverse superpropagator

[VG, Pilaftsis (2024)]

bosonic sector

$$\begin{aligned} S_{ab} = \;& \boxed{(-1)^{am} (D_\mu)^m{}_a\, {}_m k_n\, (D^\mu)^n{}_b \;+\; \partial_\mu \Phi^m \left({}_m k_n\, R^n{}_{abp}\, \partial^\mu \Phi^p + (-1)^{an}\, {}_m k_{n;[a}\, (D^\mu)^n{}_{b]} + \frac{1}{2}(-1)^{n(a+b)}\, {}_m k_{n;ab}\, \partial^\mu \Phi^n \right)} \\ & \boxed{+\, i(-1)^a \left({}_a\lambda^\mu_m (D_\mu)^m{}_b + (-1)^{bm}\, {}_a\lambda^\mu_{m;b}\, \partial_\mu \Phi^m \right)} \boxed{- U_{ab}} \end{aligned}$$

fermionic sector

mixed potential

QEA with Curvature

Pure scalar theory

Scalar frame invariant Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^A \; {}_A G_B(\phi)\, \partial_\nu \phi^B$$

Mixed-rank covariant inverse propagator

$${}^a S_b \;=\; -(D_\mu)^a{}_m \,(D^\mu)^m{}_b - R^a{}_{mbp} (\partial_\mu \phi^m)(\partial^\mu \phi^p) - U^a{}_b$$

QEA with Curvature

Pure scalar theory

Scalar frame invariant Lagrangian

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^A \, {}_A G_B(\phi) \, \partial_\nu \phi^B$$

Mixed-rank covariant inverse propagator

$${}^a S_b = -(D_\mu)^a{}_m \, (D^\mu)^m{}_b - R^a{}_{mbp} (\partial_\mu \phi^m)(\partial^\mu \phi^p) - U^a{}_b$$

Project D^2 operator with vielbeins

$$(D^2)^A{}_B \equiv (D_\mu)^A{}_M \, (D^\mu)^M{}_B = {}^A e_{\widehat{A}} \, (\widehat{D}^2)^{\widehat{A}}{}_{\widehat{B}} \, {}^{\widehat{B}} e^{\mathsf{sT}}_B$$

QEA with Curvature

Pure scalar theory

Scalar frame invariant Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^A \; {}_A G_B(\phi)\, \partial_\nu \phi^B$$

Mixed-rank covariant inverse propagator

$${}^a S_b \;=\; -(D_\mu)^a{}_m \,(D^\mu)^m{}_b - R^a{}_{mbp}(\partial_\mu \phi^m)(\partial^\mu \phi^p) - U^a{}_b$$

Project D^2 operator with vielbeins

$$(D^2)^A{}_B \equiv (D_\mu)^A{}_M \,(D^\mu)^M{}_B \;=\; {}^A e_{\widehat{A}} \,(\widehat{D}^2)^{\widehat{A}}{}_{\widehat{B}} \; {}^{\widehat{B}} e^{\mathsf{sT}}_B$$

Extract covariant heat kernel

$$\langle x_A | e^{tD^2} | x_B \rangle \;=\; K^A{}_B \, \frac{e^{-(x_A - x_B)^2/4t}}{(4\pi t)^{d/2}} \;+\; W^A{}_B$$

QEA with Curvature

Pure scalar theory

Scalar frame invariant Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^A \; {}_A G_B(\phi)\, \partial_\nu \phi^B$$

Mixed-rank covariant inverse propagator

$${}^a S_b \;=\; -(D_\mu)^a{}_m\, (D^\mu)^m{}_b - R^a{}_{mbp} (\partial_\mu \phi^m)(\partial^\mu \phi^p) - U^a{}_b$$

Project D^2 operator with vielbeins

$$(D^2)^A{}_B \equiv (D_\mu)^A{}_M\, (D^\mu)^M{}_B \;=\; {}^A e_{\widehat{A}}\, (\widehat{D}^2)^{\widehat{A}}{}_{\widehat{B}}\; {}^{\widehat{B}} e^{\mathsf{T}}_B$$

Extract covariant heat kernel

$$\langle x_A | e^{tD^2} | x_B \rangle \;=\; \boxed{K^A{}_B} \frac{e^{-(x_A - x_B)^2/4t}}{(4\pi t)^{d/2}} + \boxed{W^A{}_B}$$

$$\boxed{K^A{}_B \;\equiv\; {}^A e_{\widehat{A}}(x_A)\; {}^{\widehat{A}} e^{\mathsf{T}}_B(x_B) \;\to\; \delta^A_B} \qquad \boxed{W^A{}_B \to 0}$$

QEA with Curvature

Pure scalar theory

Covariant heat diffusion equation

$$\frac{\partial}{\partial t}\langle x_A|e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,D^2\,e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,e^{tD^2}\,D^2|x_B\rangle$$

QEA with Curvature

Pure scalar theory

Covariant heat diffusion equation

$$\frac{\partial}{\partial t}\langle x_A|e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,D^2\,e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,e^{tD^2}\,D^2|x_B\rangle$$

Explicit computation

$$\Gamma^{(1)}[\boldsymbol{\Phi}] \;=\; \frac{i}{2}\ln\operatorname{sdet}\left({}^aS_b\right) \;=\; \frac{i}{2}\operatorname{str}\left(\ln{}^aS_b\right)$$

QEA with Curvature

Pure scalar theory

Covariant heat diffusion equation

$$\frac{\partial}{\partial t}\langle x_A|e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,D^2\,e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,e^{tD^2}\,D^2|x_B\rangle$$

Series expansion in t

$$\begin{aligned}\Gamma^{(1)} \;=\; & -\frac{i}{2}\int_{x_A,x_B}\int_0^\infty \frac{dt}{t}\frac{e^{-(x_A-x_B)^2/4t}}{(4\pi t)^{d/2}}\Big[t\left(R_{MN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N+U^A{}_A\right)+\frac{t^2}{2}U^A{}_M U^M{}_A \\ & +\frac{t^2}{2}\left(R^A{}_{MPN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\,U^P{}_A+U^A{}_P R^P{}_{MAN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\right) \\ & +\frac{t^2}{2}R^A{}_{MPN}\,\partial_\mu\phi^M\,\partial^\mu\phi^N\,R^P{}_{SAT}\,\partial_\nu\phi^S\,\partial^\nu\phi^T\Big]\,\delta(x_B-x_A)\;+\;\Delta\Gamma^{(1)}\end{aligned}$$

QEA with Curvature

Pure scalar theory

Covariant heat diffusion equation

$$\frac{\partial}{\partial t}\langle x_A|e^{tD^2}|x_B\rangle \;=\; \langle x_A|\, D^2\, e^{tD^2}|x_B\rangle \;=\; \langle x_A|\, e^{tD^2}\, D^2|x_B\rangle$$

Integrate over t

$$\begin{aligned}\Gamma^{(1)}_{\mathrm{UV}} = &-\frac{i}{16\pi^2}\int_{x_A,x_B}\left[(x_A-x_B)^2\right]^{-1+\epsilon}\left(R_{MN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N + U^A{}_A\right)\delta(x_B-x_A)\\ &+\frac{i}{64\pi^2\epsilon}\int_{x_A,x_B}\left[(x_A-x_B)^2\right]^{\epsilon}\Big\{U^A{}_M U^M{}_A + R^A{}_{MPN}\,\partial_\mu\phi^M\,\partial^\mu\phi^N\,R^P{}_{SAT}\,\partial_\nu\phi^S\,\partial^\nu\phi^T\\ &\qquad + R^A{}_{MPN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\,U^P{}_A + U^A{}_P R^P{}_{MAN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\Big\}\,\delta(x_B-x_A)\,.\\ &+\;\Delta\Gamma^{(1)}\end{aligned}$$

QEA with Curvature

Pure scalar theory

Covariant heat diffusion equation

$$\frac{\partial}{\partial t}\langle x_A|e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,D^2\,e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,e^{tD^2}\,D^2|x_B\rangle$$

Integrate over t

$$\Gamma^{(1)}_{\mathrm{UV}} = -\frac{i}{16\pi^2}\int_{x_A,x_B} \overset{0}{\boxed{\left[(x_A-x_B)^2\right]^{-1+\epsilon}\left(R_{MN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N + U^A{}_A\right)\delta(x_B-x_A)}}$$
$$+\frac{i}{64\pi^2\epsilon}\int_{x_A,x_B}\left[(x_A-x_B)^2\right]^{\epsilon}\Big\{U^A{}_M U^M{}_A + R^A{}_{MPN}\,\partial_\mu\phi^M\,\partial^\mu\phi^N\,R^P{}_{SAT}\,\partial_\nu\phi^S\,\partial^\nu\phi^T$$
$$+R^A{}_{MPN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\,U^P{}_A + U^A{}_P R^P{}_{MAN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\Big\}\,\delta(x_B-x_A)\,.$$
$$+\boxed{\Delta\Gamma^{(1)}}$$

$$\boxed{\begin{aligned}\Delta\Gamma^{(1)} = -\frac{i}{2}\int_{x_A,x_B}\int_0^\infty \frac{dt}{t}\frac{e^{-(x_A-x_B)^2/4t}}{(4\pi t)^{d/2}}\,\frac{t^2}{2}\,&\langle x_B|\left(R^A{}_{MPN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N + U^A{}_P\right)(D_\mu)^P{}_C(D^\mu)^C{}_A\\ &-(D_\mu)^A{}_C(D^\mu)^C{}_P\left(R^P{}_{MAN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N + U^P{}_A\right)|x_A\rangle\end{aligned}}$$

QEA with Curvature

Pure scalar theory

Covariant heat diffusion equation

$$\frac{\partial}{\partial t}\langle x_A|e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,D^2\,e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,e^{tD^2}\,D^2|x_B\rangle$$

Integrate over t

$$\begin{aligned}\Gamma^{(1)}_{\mathrm{UV}} = &-\frac{i}{16\pi^2}\int_{x_A,x_B}\overbrace{\boxed{\left[(x_A-x_B)^2\right]^{-1+\epsilon}\left(R_{MN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N+U^A_{\ A}\right)\,\delta(x_B-x_A)}}^{0}\\ &+\frac{i}{64\pi^2\epsilon}\int_{x_A,x_B}\left[(x_A-x_B)^2\right]^{\epsilon}\Big\{U^A_{\ M}U^M_{\ A}+R^A_{\ MPN}\,\partial_\mu\phi^M\,\partial^\mu\phi^N\,R^P_{\ SAT}\,\partial_\nu\phi^S\,\partial^\nu\phi^T\\ &\qquad+R^A_{\ MPN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\,U^P_{\ A}+U^A_{\ P}R^P_{\ MAN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\Big\}\,\delta(x_B-x_A)\,.\\ &+\underbrace{\boxed{\Delta\Gamma^{(1)}}}_{0}\end{aligned}$$

QEA with Curvature

Pure scalar theory

Covariant heat diffusion equation

$$\frac{\partial}{\partial t}\langle x_A|e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,D^2\,e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,e^{tD^2}\,D^2|x_B\rangle$$

Integrate over t

$$\begin{aligned}\Gamma^{(1)}_{\mathrm{UV}} = &-\frac{i}{16\pi^2}\int_{x_A,x_B}\left[(x_A-x_B)^2\right]^{-1+\epsilon}\left(R_{MN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N + U^A{}_A\right)\delta(x_B-x_A)\\ &+\frac{i}{64\pi^2\epsilon}\int_{x_A,x_B}\left[(x_A-x_B)^2\right]^{\epsilon}\Big\{U^A{}_M U^M{}_A + R^A{}_{MPN}\,\partial_\mu\phi^M\,\partial^\mu\phi^N\,R^P{}_{SAT}\,\partial_\nu\phi^S\,\partial^\nu\phi^T\\ &\qquad + R^A{}_{MPN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\,U^P{}_A + U^A{}_P R^P{}_{MAN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\Big\}\,\delta(x_B-x_A)\,.\end{aligned}$$

terms with $1/\epsilon$- poles

QEA with Curvature

Pure scalar theory

Covariant heat diffusion equation

$$\frac{\partial}{\partial t}\langle x_A|e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,D^2\,e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,e^{tD^2}\,D^2|x_B\rangle$$

Integrate over t

$$\Gamma^{(1)}_{\rm UV} = -\frac{i}{16\pi^2}\int_{x_A,x_B}\left[(x_A-x_B)^2\right]^{-1+\epsilon}\left(R_{MN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N + U^A{}_A\right)\delta(x_B-x_A)$$

$$\boxed{\begin{aligned}&+\frac{i}{64\pi^2\epsilon}\int_{x_A,x_B}\left[(x_A-x_B)^2\right]^{\epsilon}\Big\{U^A{}_M U^M{}_A + R^A{}_{MPN}\,\partial_\mu\phi^M\,\partial^\mu\phi^N\,R^P{}_{SAT}\,\partial_\nu\phi^S\,\partial^\nu\phi^T\\&\qquad + R^A{}_{MPN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\,U^P{}_A + U^A{}_P R^P{}_{MAN}\,\partial_\nu\phi^M\,\partial^\nu\phi^N\Big\}\,\delta(x_B-x_A)\,.\end{aligned}}$$

terms with $1/\epsilon$- poles

[Alonso, Jenkins, Manohar (2016), Jenkins, Manohar, Naterop, Pagès (2024)...]

UV contributing terms

$$\boxed{\begin{aligned}\Gamma^{(1)}_{\rm UV} \;=\; -\frac{1}{64\pi^2\epsilon}\int_{x_A}\Big(&U^A{}_M U^M{}_A + R^A{}_{MBN}\,\partial_\mu\phi^M\,\partial^\mu\phi^N\,U^B{}_A + U^A{}_B R^B{}_{MAN}\,\partial_\mu\phi^M\,\partial^\mu\phi^N\\&+R^A{}_{MBN}\,\partial_\mu\phi^M\,\partial^\mu\phi^N\,R^B{}_{SAT}\,\partial_\nu\phi^S\,\partial^\nu\phi^T\Big)\,.\end{aligned}}$$

Fermionic One-loop QEA

New Clifford Algebra and Bosonization

Fermionic invariant Lagrangian

$$\mathcal{L} \;=\; \frac{i}{2}\,\zeta^{\mu}_{A}(\boldsymbol{\Phi})\,\partial_\mu \Phi^A \;-\; U(\boldsymbol{\Phi})$$

Mixed-rank covariant inverse propagator

$${}^aS_b \;=\; {}^a\lambda^{\mu}_{m}(iD_\mu)^m{}_b \;+\; (-1)^{bm}\,{}^a\lambda^{\mu}_{m;b}\,i\partial_\mu\Phi^m \;-\; {}^aU_b$$

Fermionic One-loop QEA

New Clifford Algebra and Bosonization

Fermionic invariant Lagrangian

$$\mathcal{L} = \frac{i}{2}\zeta^{\mu}_{A}(\mathbf{\Phi})\,\partial_\mu \Phi^A - U(\mathbf{\Phi})$$

Mixed-rank covariant inverse propagator

$${}^{a}S_{b} = \boxed{{}^{a}\lambda^{\mu}_{m}}(iD_\mu)^{m}_{\ b} + (-1)^{bm}\,{}^{a}\lambda^{\mu}_{m;b}\,i\partial_\mu\Phi^m - {}^{a}U_{b}$$

$${}_{a}\lambda^{\mu}_{b} \equiv \frac{1}{2}\left({}_{a,}\zeta^{\mu}_{b} - (-1)^{a+b+ab}\,{}_{b,}\zeta^{\mu}_{a}\right)$$

Fermionic One-loop QEA

New Clifford Algebra and Bosonization

Fermionic invariant Lagrangian

$$\mathcal{L} = \frac{i}{2} \zeta^{\mu}_{A}(\mathbf{\Phi})\, \partial_\mu \Phi^A - U(\mathbf{\Phi})$$

Mixed-rank covariant inverse propagator

$${}^{a}S_{b} = {}^{a}\lambda^{\mu}_{m} (iD_\mu)^{m}_{\ b} + (-1)^{bm}\, {}^{a}\lambda^{\mu}_{m;b}\, i\partial_\mu \Phi^m - {}^{a}U_{b}$$

New Clifford algebra in FS

$${}^{a}\lambda^{\mu}_{m}\, {}^{m}\bar{\lambda}^{\nu}_{b} + {}^{a}\lambda^{\nu}_{m}\, {}^{m}\bar{\lambda}^{\mu}_{b} = 2\eta^{\mu\nu}\, {}^{a}\delta_{b}$$

Fermionic One-loop QEA

New Clifford Algebra and Bosonization

Fermionic invariant Lagrangian

$$\mathcal{L} = \frac{i}{2}\,\zeta^{\mu}_{A}(\mathbf{\Phi})\,\partial_\mu \Phi^A - U(\mathbf{\Phi})$$

Mixed-rank covariant inverse propagator

$${}^{a}S_{b} = {}^{a}\lambda^{\mu}_{m}(iD_\mu)^{m}{}_{b} + (-1)^{bm}\,{}^{a}\lambda^{\mu}_{m;b}\,i\partial_\mu\Phi^m - {}^{a}U_{b}$$

New Clifford algebra in FS

$${}^{a}\lambda^{\mu}_{m}\,{}^{m}\bar{\lambda}^{\nu}_{b} + {}^{a}\lambda^{\nu}_{m}\,\boxed{{}^{m}\bar{\lambda}^{\mu}_{b}} = 2\eta^{\mu\nu}\,{}^{a}\delta_{b}$$

$$\boxed{\bar{\lambda}^{\mu} = \left\{(\lambda^0)^{-1}, -(\lambda^i)^{-1}\right\}}$$

Fermionic One-loop QEA

New Clifford Algebra and Bosonization

Fermionic invariant Lagrangian

$$\mathcal{L} = \frac{i}{2}\zeta^\mu_A(\mathbf{\Phi})\,\partial_\mu\Phi^A - U(\mathbf{\Phi})$$

Mixed-rank covariant inverse propagator

$${}^aS_b = {}^a\lambda^\mu_m(iD_\mu)^m{}_b + (-1)^{bm}\,{}^a\lambda^\mu_{m;b}\,i\partial_\mu\Phi^m - {}^aU_b$$

New Clifford algebra in FS

$${}^a\lambda^\mu_m\;{}^m\bar{\lambda}^\nu_b + {}^a\lambda^\nu_m\;{}^m\bar{\lambda}^\mu_b = 2\eta^{\mu\nu}\,{}^a\delta_b$$

Bosonise to extract HK

$$\ln\operatorname{sdet}(i\lambda^\mu D_\mu) = \frac{1}{2}\ln\operatorname{sdet}\left(-\lambda^\mu D_\mu\bar{\lambda}^\nu D_\nu\right) = \frac{1}{2}\ln\operatorname{sdet}\left(-D^2 - A\right)$$

Fermionic One-loop QEA

New Clifford Algebra and Bosonization

Fermionic invariant Lagrangian

$$\mathcal{L} \;=\; \frac{i}{2}\,\zeta^{\mu}_{A}(\mathbf{\Phi})\,\partial_\mu \Phi^A \;-\; U(\mathbf{\Phi})$$

Mixed-rank covariant inverse propagator

$${}^{a}S_{b} \;=\; {}^{a}\lambda^{\mu}_{m}(iD_\mu)^{m}{}_{b} \;+\; (-1)^{bm}\,{}^{a}\lambda^{\mu}_{m;b}\, i\partial_\mu \Phi^m \;-\; {}^{a}U_{b}$$

New Clifford algebra in FS

$${}^{a}\lambda^{\mu}_{m}\;{}^{m}\bar{\lambda}^{\nu}_{b} \;+\; {}^{a}\lambda^{\nu}_{m}\;{}^{m}\bar{\lambda}^{\mu}_{b} \;=\; 2\eta^{\mu\nu}\,{}^{a}\delta_{b}$$

Bosonise to extract HK

kernel

covariant operator

$$\ln \mathrm{sdet}\,(i\lambda^\mu D_\mu) \;=\; \frac{1}{2}\ln \mathrm{sdet}\left(-\lambda^\mu D_\mu \bar{\lambda}^\nu D_\nu\right) = \frac{1}{2}\ln \mathrm{sdet}\left(\boxed{-D^2}\,\boxed{-\,A}\right)$$

$$\boxed{A \;\equiv\; \lambda^\mu[D_\mu\,,\bar{\lambda}^\nu]\,D_\nu \;+\; M}$$

Fermionic One-loop QEA

Extracting divergent terms

Fermionic one-loop effective action

$$\Gamma^{(1)} = -\frac{i}{4}\,\mathrm{tr}\ln\left(-D^2 - A\right) - \frac{i}{2}\,\mathrm{tr}\ln\left[\mathbf{1} + i\bar{\lambda}^\mu D_\mu \left(-D^2 - A\right)^{-1} V\right]$$

Fermionic One-loop QEA

Extracting divergent terms

$$V \equiv (-1)^{bm}\, {}^{a}\lambda^{\mu}_{m;b}\, i\partial_\mu \Phi^m - {}^{a}U_b$$

Fermionic one-loop effective action

$$\Gamma^{(1)} = -\frac{i}{4}\,\mathrm{tr}\ln\left(-D^2 - A\right) - \frac{i}{2}\,\mathrm{tr}\ln\left[\mathbf{1} + i\bar{\lambda}^{\mu} D_\mu \left(-D^2 - A\right)^{-1} \boxed{V}\right]$$

Fermionic One-loop QEA

Extracting divergent terms

Fermionic one-loop effective action

$$\Gamma^{(1)} = -\frac{i}{4}\,\mathrm{tr}\ln\left(-D^2 - A\right) - \frac{i}{2}\,\mathrm{tr}\ln\left[\mathbf{1} + i\bar{\lambda}^{\mu} D_{\mu}\left(-D^2 - A\right)^{-1} V\right]$$

Series expansion in t

$$\Gamma^{(1)} = \frac{i}{4}\,\mathrm{tr}(I_1 - 2I_2 - I_3 + \ldots)$$

Fermionic One-loop QEA

Extracting divergent terms

Fermionic one-loop effective action

$$\Gamma^{(1)} = -\frac{i}{4}\,\mathrm{tr}\ln\left(-D^2 - A\right) - \frac{i}{2}\,\mathrm{tr}\ln\left[\mathbf{1} + i\bar{\lambda}^\mu D_\mu\left(-D^2 - A\right)^{-1} V\right]$$

Series expansion in t

$$\Gamma^{(1)} = \frac{i}{4}\,\mathrm{tr}(\boxed{I_1} - 2I_2 - I_3 + \ldots) \qquad B \equiv A^2 - [D^2, A]$$

$$\boxed{I_1 = \int_{x_A, x_B} \langle x_B| \left(\frac{1}{\pi^2}|x_A - x_B|^{-4+2\epsilon} + \frac{1}{4\pi^2}|x_A - x_B|^{-2+2\epsilon} A - \frac{1}{2}\frac{1}{16\pi^2\epsilon}|x_A - x_B|^{2\epsilon} B\right)|x_A\rangle}$$

$$\equiv I_1^0 + I_1^A + I_1^B$$

Fermionic One-loop QEA

Extracting divergent terms

Fermionic one-loop effective action

$$\Gamma^{(1)} = -\frac{i}{4}\,\mathrm{tr}\ln\left(-D^2 - A\right) - \frac{i}{2}\,\mathrm{tr}\ln\left[\mathbf{1} + i\bar{\lambda}^\mu D_\mu \left(-D^2 - A\right)^{-1} V\right]$$

Series expansion in t

$$\Gamma^{(1)} = \frac{i}{4}\,\mathrm{tr}(I_1 - 2\boxed{I_2} - I_3 + \ldots)$$

$$I_1 = \int_{x_A, x_B} \langle x_B| \left(\frac{1}{\pi^2}|x_A - x_B|^{-4+2\epsilon} + \frac{1}{4\pi^2}|x_A - x_B|^{-2+2\epsilon} A - \frac{1}{2}\frac{1}{16\pi^2\epsilon}|x_A - x_B|^{2\epsilon} B\right)|x_A\rangle$$

$$\boxed{I_2 = \int_{x_A, x_B, x_C} \langle x_A| i\bar{\lambda}^\mu D_\mu |x_B\rangle \langle x_C| \left(\frac{1}{4\pi^2}|x_B - x_C|^{-2+2\epsilon} - \frac{1}{16\pi^2\epsilon}|x_B - x_C|^{2\epsilon} A + \frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_B - x_C|^{2+2\epsilon} B\right) V\,|x_A\rangle}$$

$$\equiv I_2^0 + I_2^A + I_2^B$$

Fermionic One-loop QEA

Extracting divergent terms

Fermionic one-loop effective action

$$\Gamma^{(1)} = -\frac{i}{4}\,\mathrm{tr}\ln\left(-D^2 - A\right) - \frac{i}{2}\,\mathrm{tr}\ln\left[\mathbf{1} + i\bar{\lambda}^\mu D_\mu \left(-D^2 - A\right)^{-1} V\right]$$

Series expansion in t

$$\Gamma^{(1)} = \frac{i}{4}\mathrm{tr}(I_1 - 2I_2 - \boxed{I_3} + \ldots)$$

$$I_1 = \int_{x_A,x_B} \langle x_B| \left(\frac{1}{\pi^2}|x_A - x_B|^{-4+2\epsilon} + \frac{1}{4\pi^2}|x_A - x_B|^{-2+2\epsilon}A - \frac{1}{2}\frac{1}{16\pi^2\epsilon}|x_A - x_B|^{2\epsilon}B\right)|x_A\rangle$$

$$I_2 = \int_{x_A,x_B,x_C} \langle x_A|\, i\bar{\lambda}^\mu D_\mu\, |x_B\rangle \langle x_C| \left(\frac{1}{4\pi^2}|x_B - x_C|^{-2+2\epsilon} - \frac{1}{16\pi^2\epsilon}|x_B - x_C|^{2\epsilon}A + \frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_B - x_C|^{2+2\epsilon}B\right) V\, |x_A\rangle$$

$$\boxed{\begin{aligned} I_3 = \int_{\substack{x_A,x_B,x_C\\x_D,x_E,x_F}} &\langle x_A|\, i\bar{\lambda}^\mu D_\mu\, |x_B\rangle \langle x_C| \left(\frac{1}{4\pi^2}|x_B - x_C|^{-2+2\epsilon} - \frac{1}{16\pi^2\epsilon}|x_B - x_C|^{2\epsilon}A + \frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_B - x_C|^{2+2\epsilon}B\right) V\, |x_D\rangle \\ &\langle x_D|\, i\bar{\lambda}^\nu D_\nu\, |x_E\rangle \langle x_F| \left(\frac{1}{4\pi^2}|x_E - x_F|^{-2+2\epsilon} - \frac{1}{16\pi^2\epsilon}|x_E - x_F|^{2\epsilon}A + \frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_E - x_F|^{2+2\epsilon}B\right) V\, |x_A\rangle \end{aligned}}$$

Fermionic One-loop QEA

Extracting divergent terms

Only terms with scaling factor $|x_A - x_B|^{2\epsilon}$ are non zero

$$I_1 = \int_{x_A,x_B} \langle x_B| \left(\frac{1}{\pi^2}|x_A - x_B|^{-4+2\epsilon} + \frac{1}{4\pi^2}|x_A - x_B|^{-2+2\epsilon} A - \frac{1}{2}\frac{1}{16\pi^2\epsilon}|x_A - x_B|^{2\epsilon} B\right)|x_A\rangle$$

$$I_2 = \int_{x_A,x_B,x_C} \langle x_A| i\bar{\lambda}^\mu D_\mu |x_B\rangle \langle x_C| \left(\frac{1}{4\pi^2}|x_B - x_C|^{-2+2\epsilon} - \frac{1}{16\pi^2\epsilon}|x_B - x_C|^{2\epsilon} A + \frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_B - x_C|^{2+2\epsilon} B\right) V |x_A\rangle$$

Fermionic One-loop QEA

Extracting divergent terms

Only terms with scaling factor $|x_A - x_B|^{2\epsilon}$ are non zero

$$I_1 = \int_{x_A, x_B} \langle x_B | \left(\cancel{\frac{1}{\pi^2} |x_A - x_B|^{-4+2\epsilon}} + \cancel{\frac{1}{4\pi^2} |x_A - x_B|^{-2+2\epsilon}} A - \frac{1}{2} \frac{1}{16\pi^2 \epsilon} |x_A - x_B|^{2\epsilon} B \right) |x_A\rangle$$

$$I_2 = \int_{x_A, x_B, x_C} \langle x_A | i\bar{\lambda}^\mu D_\mu | x_B \rangle \langle x_C | \left(\cancel{\frac{1}{4\pi^2} |x_B - x_C|^{-2+2\epsilon}} - \frac{1}{16\pi^2 \epsilon} |x_B - x_C|^{2\epsilon} A + \frac{1}{2} \frac{1}{64\pi^2 \epsilon} |x_B - x_C|^{2+2\epsilon} B \right) V |x_A\rangle$$

Fermionic One-loop QEA

Extracting divergent terms

Only terms with scaling factor $|x_A - x_B|^{2\epsilon}$ are non zero

$$I_1 = \int_{x_A,x_B} \langle x_B| \left(\cancel{\frac{1}{\pi^2}|x_A - x_B|^{-4+2\epsilon}} + \cancel{\frac{1}{4\pi^2}|x_A - x_B|^{-2+2\epsilon}} A - \frac{1}{2}\frac{1}{16\pi^2\epsilon}|x_A - x_B|^{2\epsilon} B \right) |x_A\rangle$$

$$I_2 = \int_{x_A,x_B,x_C} \langle x_A| i\bar{\lambda}^\mu D_\mu |x_B\rangle \langle x_C| \left(\cancel{\frac{1}{4\pi^2}|x_B - x_C|^{-2+2\epsilon}} - \frac{1}{16\pi^2\epsilon}|x_B - x_C|^{2\epsilon} A + \frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_B - x_C|^{2+2\epsilon} B \right) V |x_A\rangle$$

No leftover δ- function integration $\longrightarrow$ keep $\mathcal{O}(D^2)$ terms

$$I_3 = \int_{\substack{x_A,x_B,x_C\\x_D,x_E,x_F}} \langle x_A| i\bar{\lambda}^\mu D_\mu |x_B\rangle \langle x_C| \left(\frac{1}{4\pi^2}|x_B - x_C|^{-2+2\epsilon} - \frac{1}{16\pi^2\epsilon}|x_B - x_C|^{2\epsilon} A + \frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_B - x_C|^{2+2\epsilon} B \right) V |x_D\rangle$$
$$\langle x_D| i\bar{\lambda}^\nu D_\nu |x_E\rangle \langle x_F| \left(\frac{1}{4\pi^2}|x_E - x_F|^{-2+2\epsilon} - \frac{1}{16\pi^2\epsilon}|x_E - x_F|^{2\epsilon} A + \frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_E - x_F|^{2+2\epsilon} B \right) V |x_A\rangle$$

Fermionic One-loop QEA

Extracting divergent terms

Only terms with scaling factor $|x_A - x_B|^{2\epsilon}$ are non zero

$$I_1 = \int_{x_A, x_B} \langle x_B| \left(\cancel{\frac{1}{\pi^2}|x_A - x_B|^{-4+2\epsilon}} + \cancel{\frac{1}{4\pi^2}|x_A - x_B|^{-2+2\epsilon}} A - \frac{1}{2}\frac{1}{16\pi^2\epsilon}|x_A - x_B|^{2\epsilon} B \right) |x_A\rangle$$

$$I_2 = \int_{x_A, x_B, x_C} \langle x_A| i\bar{\lambda}^\mu D_\mu |x_B\rangle \langle x_C| \left(\cancel{\frac{1}{4\pi^2}|x_B - x_C|^{-2+2\epsilon}} - \frac{1}{16\pi^2\epsilon}|x_B - x_C|^{2\epsilon} A + \frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_B - x_C|^{2+2\epsilon} B \right) V |x_A\rangle$$

No leftover δ- function integration $\longrightarrow$ keep $\mathcal{O}(D^2)$ terms

$$I_3 = \int_{\substack{x_A, x_B, x_C \\ x_D, x_E, x_F}} \langle x_A| i\bar{\lambda}^\mu D_\mu |x_B\rangle \langle x_C| \left(\frac{1}{4\pi^2}|x_B - x_C|^{-2+2\epsilon} - \cancel{\frac{1}{16\pi^2\epsilon}|x_B - x_C|^{2\epsilon} A} + \cancel{\frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_B - x_C|^{2+2\epsilon} B} \right) V |x_D\rangle$$

$$\langle x_D| i\bar{\lambda}^\nu D_\nu |x_E\rangle \langle x_F| \left(\frac{1}{4\pi^2}|x_E - x_F|^{-2+2} \quad \cancel{\frac{1}{16\pi^2\epsilon}|x_E - x_F|^{2\epsilon} A} + \cancel{\frac{1}{2}\frac{1}{64\pi^2\epsilon}|x_E - x_F|^{2+2\epsilon} B} \right) V |x_A\rangle$$

Fermionic One-loop QEA

Divergent terms

Many new fermionic operators in the UV

from I_1

$$(I_1^B)_{\mathrm{UV}} = \frac{1}{64\pi^2\epsilon}\int_{x_A}{}^{A}\Big\{\left[D^\mu\left(\lambda^\alpha(D_\alpha\bar{\lambda}^\beta)\right) + 2\Sigma^{\mu\nu}D_\nu\left(\lambda^\alpha(D_\alpha\bar{\lambda}^\beta)\right)\right]\mathcal{R}_{\mu\beta} - 2\lambda^\mu D_\mu\left(\bar{\lambda}^\nu(D_\nu M)\right) + \lambda^\mu D_\mu\left(\bar{\lambda}^\nu\lambda^\alpha(D_\alpha\bar{\lambda}^\beta)\right)\mathcal{R}_{\nu\beta}\Big\}_A$$

from I_2

$$(I_2^A)_{\mathrm{UV}} = \frac{i}{32\pi^2\epsilon}\int_{x_A}{}^{A}\left[(D^\mu V)\bar{\lambda}^\nu\mathcal{R}_{\mu\nu} + \lambda^\mu D_\mu(\bar{\lambda}^\nu V)\bar{\lambda}^\alpha\mathcal{R}_{\nu\alpha} + 2\Sigma^{\mu\nu}(D_\nu V)\bar{\lambda}^\alpha\mathcal{R}_{\mu\alpha}\right]_A$$

Fermionic One-loop QEA

Divergent terms

Many new fermionic operators in the UV

from I_2

$$(I_2^B)_{\rm UV} = \frac{i}{64\pi^2\epsilon}\int_{x_A}{}^{A}\Big\{\lambda^\mu D_\mu\left[\bar\lambda^\nu\lambda^\alpha D_\alpha(\bar\lambda^\beta D_\beta V)\right]\bar\lambda_\nu + \lambda^\mu D_\mu\left[\bar\lambda^\nu D_\nu\left(\lambda^\alpha D_\alpha(\bar\lambda^\beta V)\right)\right]\bar\lambda_\beta$$

$$+ D^\mu\left[\lambda^\nu D_\nu(\bar\lambda^\alpha D_\alpha V)\right]\bar\lambda_\mu + 2\Sigma^{\mu\nu}D_\nu\left[\lambda^\alpha D_\alpha(\bar\lambda^\beta D_\beta V)\right]\bar\lambda_\mu$$

$$+ \lambda^\mu D_\mu\left[\bar\lambda^\nu D_\nu\left(\lambda^\alpha\bar\lambda^\beta D_\beta V\right)\right]\bar\lambda_\alpha\Big\}_A$$

from I_3

$$(I_3)_{\rm UV} = -\frac{1}{32\pi^2\epsilon}\int_{x_A}{}^{A}\left[\bar\lambda^\mu V\bar\lambda_\mu(D^2V) + (D^2\bar\lambda^\mu)V\bar\lambda_\mu V + 2(D_\nu\bar\lambda^\mu)V\bar\lambda_\mu(D^\nu V)\right]_A$$

Outlook

QFT

Geometric QFT

Outlook

QFT → Geometric QFT

Field-space metric

Covariant effective action

Outlook

QFT → Geometric QFT

Field-space metric

Covariant effective action

New Tools

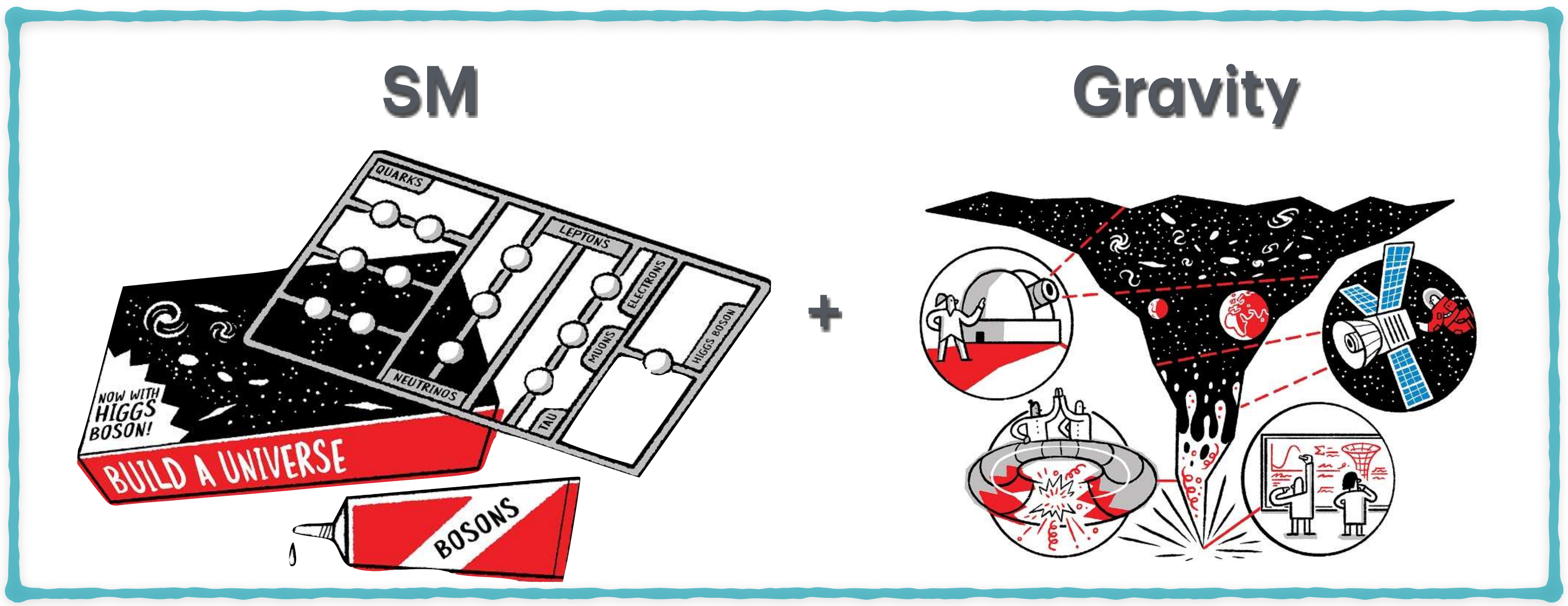
- Covariant D operator language
- Intuitive approach to Schwinger-DeWitt HK technique
- Field dependent Clifford algebra

Next up

Off-shell covariant vertices ✓

Geometric interactions ✓

UV covariant EA ✓



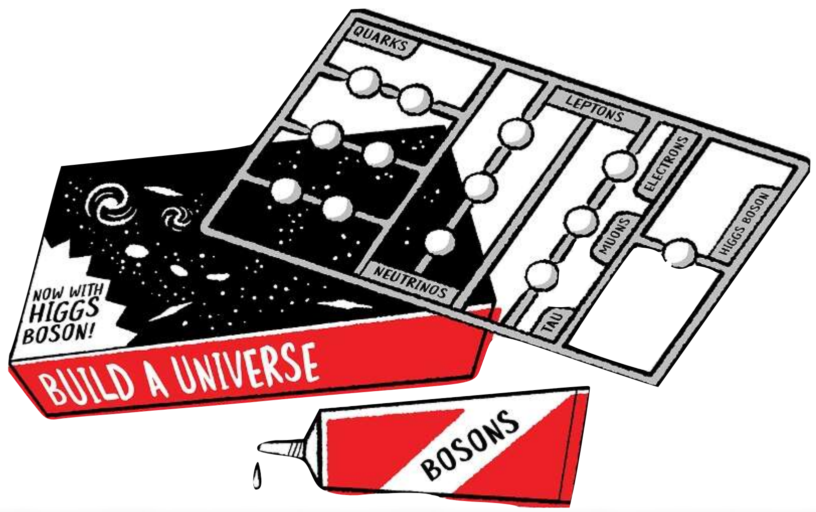
Next up

Off-shell covariant vertices ✓

Geometric interactions ✓

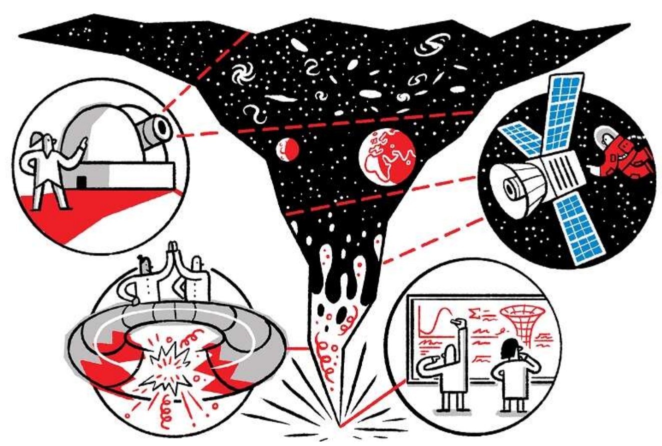
UV covariant EA ✓

SM



+

Gravity



Global/local symmetry ?

EFT applications ?

new portal to dark-sector dynamics ?

Thank you!

Backup slides

Supermanifolds

Field-space supermanifold of dimension (N|8M) in 4d spacetime

Fields are the chart

[DeWitt (2012)]

$$\mathbf{\Phi} \;\equiv\; \{\Phi^a\} \;=\; \begin{pmatrix} \phi^A \\ \psi^X \\ \overline{\psi}^{Y\,\mathsf{T}} \end{pmatrix}$$

Field reparameterization = diffeomorphism

$$\Phi^a \;\to\; \widetilde{\Phi}^a \;=\; \widetilde{\Phi}^a(\mathbf{\Phi})$$

Diffeormophically - or frame invariant Lagrangian

[Finn, Karamitsos, Pilaftsis (2021), VG, Finn, Karamitsos, Pilaftsis (2022)]

$$\boxed{\mathcal{L} \;=\; \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi^A \;{}_A k_B(\mathbf{\Phi})\, \partial_\nu \Phi^B \;+\; \frac{i}{2} \zeta^\mu_A(\mathbf{\Phi})\, \partial_\mu \Phi^A \;-\; U(\mathbf{\Phi})}$$

$$\zeta^{\mu}_{B}{}^{\;B} \left(\overleftarrow{\Sigma}_\mu\right)_A \;=\; \zeta_A \quad \text{where} \quad \overleftarrow{\Sigma}_\mu = \frac{1}{d} \begin{pmatrix} \overleftarrow{\delta}/\delta\gamma^\mu & 0 \\ 0 & \Gamma_\mu \end{pmatrix}$$

Covariant Effective Action

Master Equation for All Loop Orders

Goal: Recursive relation for QEA at all orders

Covariant generalisation of the identity

$$ {}_a\Gamma_m \, \Gamma^{mb} = - {}_a\delta^b $$

Use 1PI - 2PI EA equivalence at extrema

$$ \Gamma^{ab} \equiv \sum_{k=0}^{n-1} \Gamma^{(k)\,ab} = 2i \sum_{l=1}^{n} \Gamma^{(l)} \frac{\overleftarrow{\delta}}{\boxed{\delta\,{}_a\Delta_b^{-1}}} $$

covariant inverse propagator

Write $(n-1)$th loop order EA as

$$ \Gamma^{(n-1)\,ab} = {}^a\Big[\Delta \sum_{k=1}^{n-1} (-1)^k \boxed{\Pi^{(l_1)}} \Delta\, \Pi^{(l_2)} \ldots \Delta\, \Pi^{(l_k)} \Delta\Big]^b $$

1PI self-energy

Covariant Effective Action

Master Equation for All Loop Orders

After some algebra...

Master functional differential equation for 1PI QEA at all orders

$$2i\left(\Gamma^{(n)}\frac{\overleftarrow{\delta}}{\delta\Delta^{ab}}\right)\delta\Delta^{ba} = \mathrm{str}\left[\Delta^{-1}\sum_{k=1}^{n-1}(-1)^k\,\Delta\,\Pi^{(l_1)}\Delta\,\Pi^{(l_2)}\Delta\,\ldots\Pi^{(l_k)}\left(\delta\Delta\right)\right]$$

E.g. $n = 1$

$$\text{RHS} = \mathrm{str}[\Delta^{-1}\,\delta\Delta]$$

E.g. $n = 2$

$$\text{RHS} = -(-1)^c{}_c\left(\Pi^{(1)}\,\delta\Delta\right)^c$$

Covariant Interactions

Scalar-Fermion Theories

D operator

[Ecker, Honerkamp, (1972)]

$$(\partial_\mu \Phi^a)_{;b} \;=\; \left(\delta^A{}_B\,\partial_\mu^{(A)} + \Gamma^A{}_{BM}\,\partial_\mu \Phi^M\right)\delta(x_A - x_B) \;\equiv\; (D_\mu)^a{}_b$$

Important supergeometric identity

$$(D_\mu)_{ab;c} \;=\; R_{abcm}\,\partial_\mu \Phi^m$$

Complete covariant inverse superpropagator

[VG, Pilaftsis (2024)]

$$\begin{aligned} S_{ab} \;=& (-1)^{am}(D_\mu)^m{}_a\,{}_m k_n\,(D^\mu)^n{}_b \;+\; \partial_\mu \Phi^m \Bigg({}_m k_n\, R^n{}_{abp}\,\partial^\mu \Phi^p \\ &+ (-1)^{an}\,{}_m k_{n;[a}\,(D^\mu)^n{}_{b]} + \frac{1}{2}(-1)^{n(a+b)}\,{}_m k_{n;ab}\,\partial^\mu \Phi^n \Bigg) \\ &+ i(-1)^a \left(\boxed{{}_a\lambda^\mu_m} (D_\mu)^m{}_b + (-1)^{bm}\,{}_a\lambda^\mu_{m;b}\,\partial_\mu \Phi^m \right) - U_{ab} \end{aligned}$$

$${}_a\lambda^\mu_b \;\equiv\; \frac{1}{2}\left({}_{a,}\zeta^\mu_b \;-\; (-1)^{a+b+ab}\,{}_{b,}\zeta^\mu_a\right)$$

Covariant Heat Kernel Method

Integral identities

$(D_\mu)^A{}_B$ field-dependent differential operator in single-state Hilbert space

$$\langle x_A|x_B\rangle \;=\; \delta(x_A - x_B)\,, \quad \langle p|x\rangle \;=\; e^{ip_\mu x^\mu}\,, \quad \hat{x}^\mu|x_A\rangle = x^\mu_A|x_A\rangle\,, \quad \hat{p}_\mu \;=\; i\frac{\overrightarrow{\partial}}{\partial x^\mu}$$

Covariant heat diffusion equation

$$\frac{\partial}{\partial t}\langle x_A|e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,D^2\,e^{tD^2}|x_B\rangle \;=\; \langle x_A|\,e^{tD^2}\,D^2|x_B\rangle$$

Useful identities

$$\int_0^\infty \frac{dt}{t}\,\frac{t^2}{2}\,\langle x_A|\,D^2\,e^{tD^2}\,|x_B\rangle\,\delta(x_A - x_B) \;\sim\; \frac{1}{\varepsilon_{\rm IR}}$$

$$\int_0^\infty \frac{dt}{t}\,\frac{t^2}{2}\,\langle x_A|\,(D^2)^2\,e^{tD^2}\,|x_B\rangle\,\delta(x_A - x_B) \;=\; 0$$

QEA with Curvature

Minimal fermionic model with FS curvature

Previously: fermionic model with FS curvature

$$\mathcal{L} \;=\; \frac{1}{2}k(\phi)(\partial_\mu\phi)\,(\partial^\mu\phi) \;+\; \frac{i}{2}\left(g_0 + g_1\bar\psi\psi\right)\left[\bar\psi\gamma^\mu(\partial_\mu\psi) \;-\; (\partial_\mu\bar\psi)\gamma^\mu\psi\right] \;-\; U$$

The supermetric

$$\boldsymbol{G} \;=\; \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & f^{\mathsf{T}} \\ 0 & -\boxed{f} & 0 \end{pmatrix} \qquad f \;=\; \left(g_0 + g_1\bar\psi\psi\right)\mathbf{1}_d + g_1\psi\bar\psi$$

Mixed Lorentz and FS-tensor

$${}_a\lambda^\mu_b \;=\; \boxed{g_0\begin{pmatrix} 0 & \gamma^\mu_{ba} \\ \gamma^\mu_{ab} & 0 \end{pmatrix}}$$

flat part

curvature part

$$\boxed{-\,\frac{g_1}{2}\begin{pmatrix} \bar\psi_a(\bar\psi\gamma^\mu)_b + \bar\psi_b(\bar\psi\gamma^\mu)_a & \bar\psi_a(\gamma^\mu\psi)_b + (\bar\psi\gamma^\mu)_a\psi_b - 2(\bar\psi\psi)\gamma^\mu_{ba} \\ \bar\psi_b(\gamma^\mu\psi)_a + (\bar\psi\gamma^\mu)_b\psi_a - 2(\bar\psi\psi)\gamma^\mu_{ab} & -\bar\psi_a(\bar\psi\gamma^\mu)_b - \bar\psi_b(\bar\psi\gamma^\mu)_a \end{pmatrix}}$$

Fermionic One-loop QEA

More on Bosonization

Equivalence of the EA

$$\operatorname{sdet}\left({}^{a}\lambda^{\mu}_{m}\,(iD_{\mu})^{m}{}_{b}\right) \;=\; \operatorname{sdet}\left({}^{a}\bar{\lambda}^{\nu}_{m}\,(iD_{\nu})^{m}{}_{b}\right) \;=\; (p^{2})^{d} \;+\; f(\psi,\bar{\psi})$$

One-loop EA

$$\Gamma^{(1)} \;=\; \frac{i}{2}\,\ln \operatorname{sdet}\left(i\lambda^{\mu}D_{\mu}\right) \;+\; \frac{i}{2}\,\ln \operatorname{sdet}\left[\mathbf{1} + \left(i\lambda^{\mu}D_{\mu}\right)^{-1} V\right]$$

Bosonisation for inverse differential operator

$$(i\lambda^{\mu}D_{\mu})^{-1} = \; i\bar{\lambda}^{\mu}D_{\mu}\left(-D^{2} - A\right)^{-1}$$

Fermionic One-loop QEA

Extracting divergent terms

A new magnetic-moment-type operator

$$ {}^{A}M_{B} \;\equiv\; {}^{A}(\Sigma^{\mu\nu}\,\mathcal{R}_{\mu\nu})_{B} $$

New mixed differential two-form rank-2 field-space tensor

$$ {}^{A}(\mathcal{R}_{\mu\nu})_{B} \;\equiv\; {}^{A}[D_{\mu}\,,\,D_{\nu}]_{B} \;=\; (-1)^{MN}\,R^{A}{}_{BMN}\,\partial_{\mu}\Phi^{M}\,\partial_{\nu}\Phi^{N} $$

$\rightarrow$ novel feature of SG-QFTs with non-zero fermionic curvature

Higher-Point Covariant Interactions

Covariant Three-Vertex

Notation

$$[abc] = abc + (-1)^{ab}bac + (-1)^{c(a+b)}cab \qquad \text{(symmetrisation)}$$

$$(abc) = abc + (-1)^{a(b+c)}bca + (-1)^{c(a+b)}cab \qquad \text{(cyclic permutation)}$$

Three-vertex

$$\begin{aligned}
S_{abc} \;=\; & \partial_\mu\Phi^m \left((-1)^{cp}\, {}_m k_n\, R^n{}_{abp;c} + (-1)^{an}\, {}_m k_{n;[a}\, R^n{}_{bc]p} + \frac{1}{2}(-1)^{p(a+b+c)}\, {}_m k_{p;abc} \right) \partial^\mu\Phi^p \\
& + (-1)^{am}(D_\mu)^m{}_{[a} \left({}_m k_n\, R^n{}_{bc]p} + (-1)^{p(b+c)}\, {}_m k_{p;bc]} \right) \partial^\mu\Phi^p + \partial_\mu\Phi^m\, {}_m k_n\, R^n{}_{abp}(D^\mu)^p{}_c \\
& + (-1)^{am+bn}(D_\mu)^m{}_{(a}\, {}_m k_{n;b}\, (D^\mu)^n{}_{c)} + i(-1)^a \left({}_a\lambda^\mu_m\, R^m{}_{bcp}\partial_\mu\Phi^p + (-1)^{bm}\, {}_a\lambda^\mu_{m;[b}(D_\mu)^m{}_{c]} \right. \\
& \left. + (-1)^{m(b+c)}\, {}_a\lambda^\mu_{m;bc}\partial_\mu\Phi^m \right) \;-\; U_{abc}\,.
\end{aligned}$$

Spin connection in SG-QFTs

Inertial effects

Spin connection encodes the inertial effects

$$\omega_B{}^{\widehat{A}}{}_{\widehat{B}} = (-1)^{B(\widehat{A}+\widehat{B})}\,\Gamma^{\widehat{A}}{}_{\widehat{B}\widehat{C}}\,{}^{\widehat{C}}e^{\mathsf{sT}}{}_B \qquad \omega_C{}^{\widehat{A}\widehat{B}} = -(-1)^{\widehat{A}\widehat{B}}\omega_C{}^{\widehat{B}\widehat{A}}$$

Tetrad postulate in field space

$$ {}^{\widehat{A}}e^{\mathsf{sT}}{}_{A\,;B} \;=\; {}^{\widehat{A}}e^{\mathsf{sT}}{}_{A\,,B} - {}^{\widehat{A}}e^{\mathsf{sT}}{}_M\,\Gamma^M{}_{AB} + (-1)^{B(\widehat{A}+A)}\,\omega_B{}^{\widehat{A}}{}_{\widehat{B}}\,{}^{\widehat{B}}e^{\mathsf{sT}}{}_A \;=\; 0$$

Projected D operator

$$(D_\mu)^a{}_b \;=\; {}^Ae_{\widehat{A}}(x_A)\left[{}^{\widehat{A}}\delta_{\widehat{B}}\,\partial_\mu^{(A)}\delta(x_A - x_B) + \omega_\mu{}^{\widehat{A}}{}_{\widehat{B}}(x_A)\,\delta(x_A - x_B)\right]{}^{\widehat{B}}e^{\mathsf{sT}}{}_B(x_B)$$

Spin connection in SG-QFTs

Inertial effects

Covariant HK

$$\langle x_A | e^{tD^2} | x_B \rangle \;=\; K^A{}_B \, \frac{e^{-(x_A - x_B)^2/4t}}{(4\pi t)^{d/2}} \;+\; \boxed{W^A{}_B}$$

$$W^A_B \;=\; \int_{x_M} \frac{e^{-(x_A - x_M)^2/4t}}{(4\pi t)^{d/2}} \; {}^A e_{\widehat{A}}(x_A) \left\langle x_M \middle| \, t \, {}^{\widehat{A}}\Omega_{\widehat{B}} + \frac{t^2}{2} \left({}^{\widehat{A}}\Omega_{\widehat{M}} \, {}^{\widehat{M}}\Omega_{\widehat{B}} - \left[\partial^2 , {}^{\widehat{A}}\Omega_{\widehat{B}} \right] \right) \middle| x_B \right\rangle \, {}^{\widehat{B}} e^{\mathsf{sT}}_{B}(x_B)$$

where

$$\Omega \;\equiv\; 2\, \omega_\mu \, \partial^\mu \;+\; (\partial^\mu \omega_\mu) \;+\; \omega_\mu \, \omega^\mu$$

$$\omega_\mu{}^{\widehat{A}}{}_{\widehat{B}} \;\equiv\; (-1)^C \, \partial_\mu \Phi^C \;\, \omega_C{}^{\widehat{A}}{}_{\widehat{B}}$$