Worldline Formalism and Gauge/Gravity Correspondence

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Introduction

The Worldline formalism is a powerfull tool for studying the strong coupling regime of QCD.

Together with the AdS/CFT correspondence, it can lead to novel non-trivial results.

In this talk I will review some of the powerful insights that the wordline formalism offers us, including:

- Better understanding of colour screening and confinement.
- The lower boundary of the conformal window in QCD
- Meson scattering and holographic corrections to the Veneziano amplitude

Worldline and Wilson loops

Consider an observable $\mathcal{O}$ in QCD

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}}\int DA_\mu\,\mathcal{O}\,\exp\left(-S_{\rm YM}\right)\det{}^{N_f}\left(i\not{D}-m\right)\,.$$

Let us use the worldline formalism in order to express the fermionic determinant in terms of Wilson loops

$$\det\left(i\not{D}-m\right) = \exp\Gamma[A]\,,$$

where

$$\begin{aligned}\Gamma[A] &= -\frac{1}{2}\int_0^\infty\frac{dT}{T}\\ &\times\int\mathcal{D}x\mathcal{D}\psi\,\exp\left\{-\int_\epsilon^T d\tau\,\left(\frac{1}{2}\dot{x}^\mu\dot{x}^\mu+\frac{1}{2}\psi^\mu\dot{\psi}^\mu-\frac{1}{2}m^2\right)\right\}\\ &\times\ {\rm Tr}\,\mathcal{P}\exp\left\{i\int_0^T d\tau\,\left(A_\mu\dot{x}^\mu-\frac{1}{2}\psi^\mu F_{\mu\nu}\psi^\nu\right)\right\}\,,\end{aligned}$$

Worldline and Wilson loops

Thus $\Gamma[A]$ is a sum over (super)-Wilson loops. The sum is over contours of all sizes and shapes. Large contours are, however, suppressed by the quark mass, which serves as an IR cut-off. The sum can be written schematically as $\Gamma[A] = \sum_{\mathcal{C}} W$. In this notation the fermionic determinant is

$$\det\left(i\not{D} - m\right) = \exp \sum_{\mathcal{C}} W = \sum_{n} \frac{1}{n!} \left(\sum_{\mathcal{C}} W \right)^{n} .$$

The quenched theory is obtained by approximating the above exponent by 1. Let's improve the approximation by adding the first term

$$\langle \mathcal{O} \rangle \approx \langle \mathcal{O} \rangle_{\mathrm{YM}} + \frac{N_f}{N_c} \sum_{\mathcal{C}} \langle \mathcal{O} W \rangle^{\mathrm{conn.}}_{\mathrm{YM}} .$$

Namely, we suggest to improve the calculation of an observable $\mathcal{O}$ by adding to the quenched value a sum of its correlator with all possible Wilson loops.

Wilson loops in the AdS/CFT correspondence

Let us choose a specific observable: a certain large Wilson loop

$$\mathcal{O} = w$$

How can we calculate $\langle w \rangle$ and its correlators with Wilson loops at strong coupling? One way is to use the AdS/CFT correspondence.

We need to understand how Wilson loops are computed within the AdS/CFT correspondence.

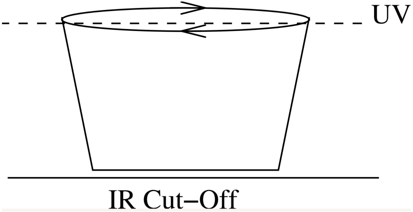
Wilson loops are string worldsheets that terminate on the boundary of AdS space (UV). In order to be concrete let us choose the following 'hardwall' model metric:

$$ds^2 = \frac{du^2}{u^2} + u^2 dx_\mu^2$$

Let us choose $\infty > u > u_0$, where the UV boundary is at $u = \infty$ and the IR (confining scale) is at $u = u_0$.

Wilson loops in the AdS/CFT correspondence

A typical circular Wilson loop takes the following form



The expectation value of the wilson loop is computed by the Nambu-Goto action

$$\langle w \rangle = \exp - I_{\mathrm{N.G.}} \sim N_c \exp - \sigma \mathcal{A}$$

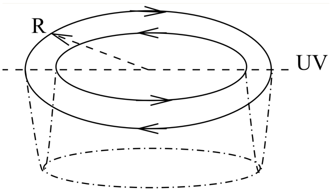
In a confining theory the string worldsheet is dominated by the contribution from the string that rests on the IR cut-off. That's how we obtain an area law.

Wilson loops in the AdS/CFT correspondence

Let us also calculate the correlator

$$\langle \mathcal{O} W \rangle_{\mathrm{YM}} = \langle w W \rangle_{\mathrm{YM}}$$

It is given by the Nambu-Goto action of the following string worldsheet



AdS Space

The relevant string worldsheet in this case is a tiny strip along the perimeter of w.

Screening

The result, hence, is a perimeter law

$$\langle wW \rangle = N_f \exp(-\mu\, 2\pi R)$$

Altogether we obtain

$$\langle w \rangle = N_c \exp(-\sigma \mathcal{A}) + N_f \exp(-\mu P)$$

Very similar to the lattice strong coupling expansion.

The result, obtained using the worldline and string theory, is that the planar theory, $N_c \to \infty$, is confining, where the finite N_c theory is screening, as anticipated.

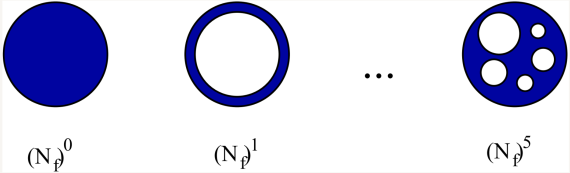
Quark pair production, which is a $1/N_c$ effect, screens the heavy quark source, leading to a perimeter law.

A comment about convergence

The expansion

$$\det\left(i \not{D} - m\right) = \sum_n \frac{1}{n!}\left(\sum_{\mathcal{C}} W\right)^n .$$

Has finite radius of converge.



It converges as long as $N_f/N_c \lesssim 4$, namely when QCD is 'below' the conformal window. In fact, using the worldline formalism it is possible to estimate the value above which QCD flows in the IR to a non-trivial intereacting fixed point.

Meson Scattering in QCD

Consider the scattering amplitude of four scalar mesons

$$\mathcal{A}(x_1, x_2, x_3, x_4) = \langle \bar{q}q(x_1)\bar{q}q(x_2)\bar{q}q(x_3)\bar{q}q(x_4)\rangle \,.$$

In order to calculate the amplitude we add to the QCD action a source of the form $\int d^4x\, J(x)\bar{q}q$

$$\mathcal{Z}_J = \int DA_\mu \exp(-S_{\mathrm{YM}})\left(\det(i\not{D} + J(x))\right)^{N_f}$$

and differentiate the partition function with respect to $J(x_1), J(x_2), J(x_3), J(x_4)$

$$\langle \bar{q}q(x_1)\bar{q}q(x_2)\bar{q}q(x_3)\bar{q}q(x_4)\rangle\rangle = \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}\log \mathcal{Z}_J|_{J=0}$$

Meson Scattering using the worldline formalism

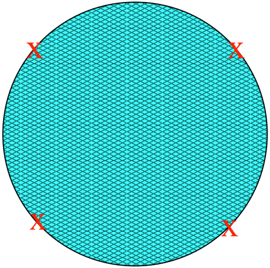
In the worldline formalism, such a variation *excludes* Wilson loops whose contours do not pass through $\{x_1, x_2, x_3, x_4\}$.

In the 't Hooft limit the leading large-N_c expression for the scattering amplitude can schematically be written as

$$\mathcal{A}(x_1, x_2, x_3, x_4) = \frac{1}{2}\int_0^\infty \frac{dT}{T}\int \mathcal{D}x \, \exp\left(-\int d\tau \, \tfrac{1}{2}\dot{x}_\mu^2\right)\langle W(x_1, x_2, x_3, x_4)$$

The Wilson loops are calculated in the large-N pure Yang-Mills theory. The sum is over all sizes and shapes of Wilson loops that pass through the points $\{x_1, x_2, x_3, x_4\}$. The QCD amplitude admits the topology of a disk and it resembles the string disk amplitude.

Meson Scattering using the worldline formalism

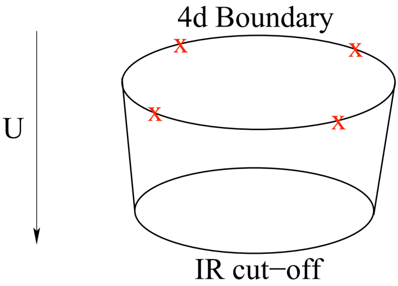


How can we calculate such a horrendous sum? Let's use holography, where Wilson loops are string worldsheets.

Meson Scattering using the worldline formalism

In holography we need to sum over all possible string worldsheets that terminate on the UV boundary and pass through the points $\{x_1, x_2, x_3, x_4\}$.

A typical (but not necessarily circular) string worldsheet looks like that



Meson Scattering and holography

It is difficult to carry out a sum over all string worldsheets in AdS space, but a certain crude approximation leads to an interesting result.

Let us assume that the IR cut-off is close to the 4d boundary. It means that all Wilson loops, however small, yield an area law.

Essentially we push Λ_{QCD} to the UV cut-off scale. This is not QCD, but rather similar to the lattice strong coupling expansion.

We obtain

$$\mathcal{A}(x_1, x_2, x_3, x_4) = \int \mathcal{D}x^\mu \, \exp\left(-\Sigma \int d^2\sigma \, \partial_\alpha x^\mu \partial^\alpha x_\mu \right)$$

Meson Scattering and holography

The calculation in Fourier space

$$\mathcal{A}(k_1, k_2, k_3, k_4) =$$
$$\int \mathcal{D}x^\mu \prod_{i=1,..,4} \int dy_i \exp\left(ik_i x(y_i)\right) \exp\left(-\Sigma \int d^2\sigma\, \partial_\alpha x^\mu \partial^\alpha x_\mu\right) ,$$

Yields the celebrated Veneziano amplitude!

$$\mathcal{A}(k_1, k_2, k_3, k_4) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}$$

It is actually possible to do better than that to include holographic corrections that take into account the curvature of space, but I will not review it here.

Conclusions

The worldline formalism, due to its expansion via Wilson loop, is a natural bridge between ordinary field theory and the AdS/CFT correspondence.

The QCD partition function is mapped into a sum over string worldsheets in AdS space.

In particular it leads to an understanding of the physics of the Veneziano amplitude: scattering of mesons in a confining theory at all scales.

Thank you!