Classical Worldlines from Scattering Amplitudes

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First Quantization in Strong Fields Conference, Edinburgh University, 27 Feb 2025

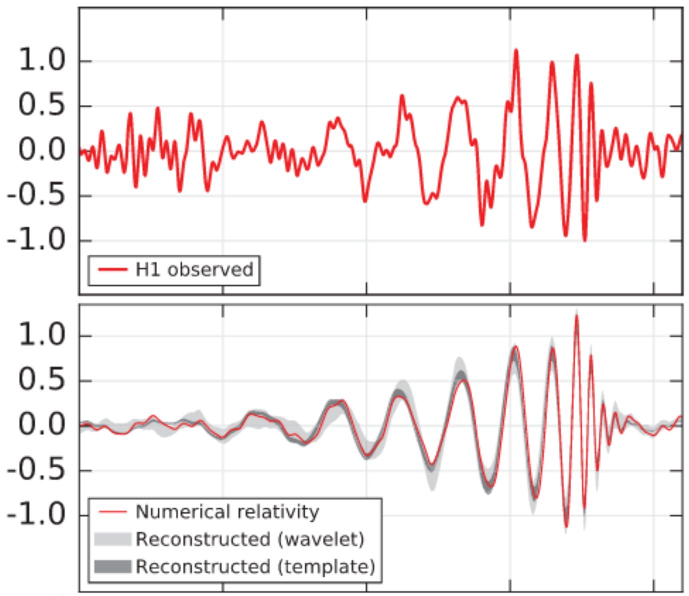
Talk based on arXiv:2412.10864, Zeno Capatti, MZ

Outline

- Background
- QFT approaches to classical dynamics in GR
  - 1st quantized – worldlines (WL)
  - 2nd quantized – scattering amplitudes
- Equivalence: re-deriving WL picture from Amplitudes
  - Emergence of the arrow of time
  - Bonus: manifest cancellation of $\hbar \to 0$ singularities

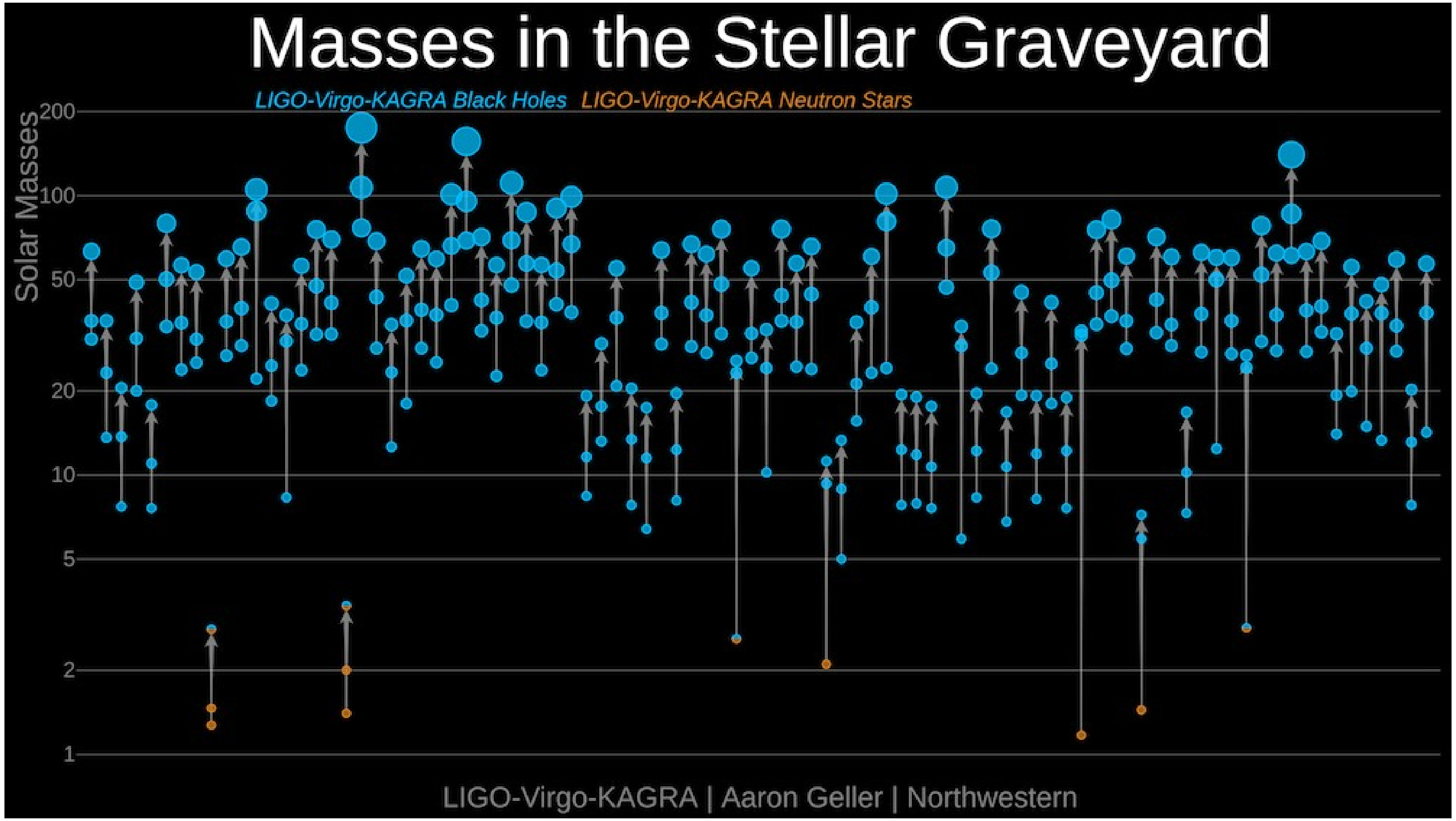
Background

Beginning of new era: GW detection, 2015-



- General relativity in strong-field regime
- Dense nuclear matter in neutron stars
- BSM compact objects, dark matter
- Black hole population & formation mechanisms
- ...

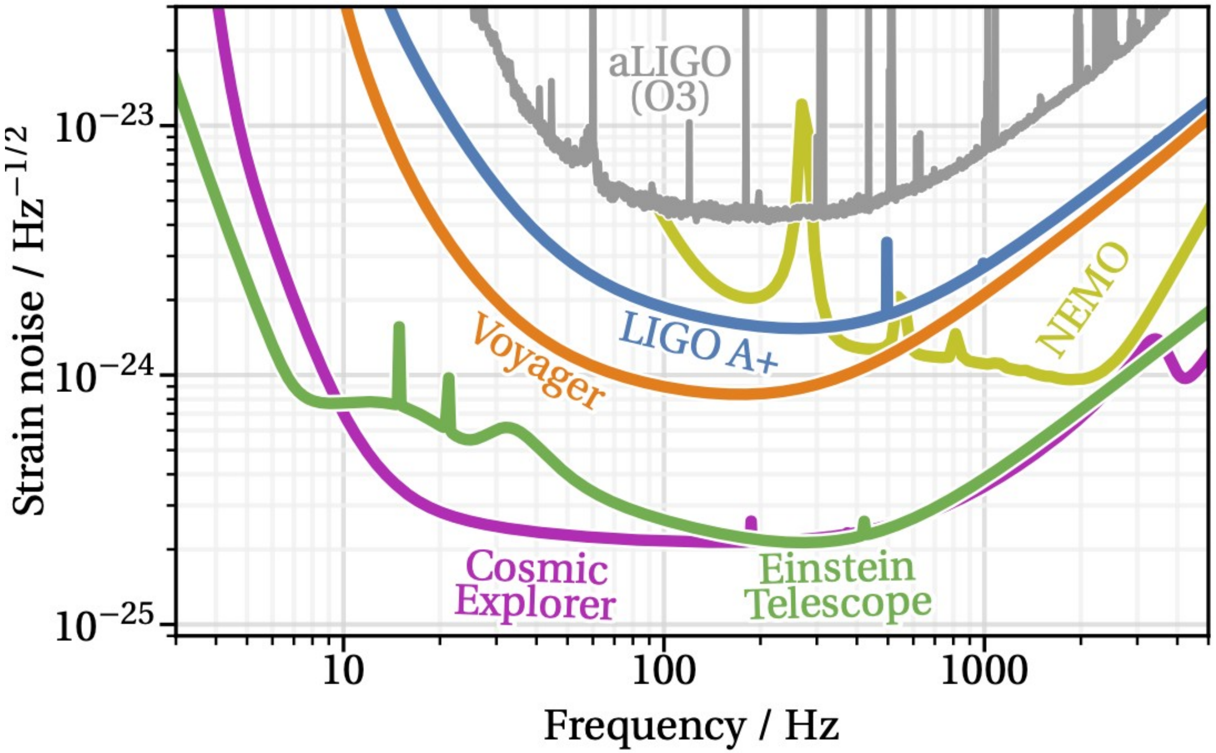
LIGO-VIRGO-KAGRA O3B Catalog (late 2021)

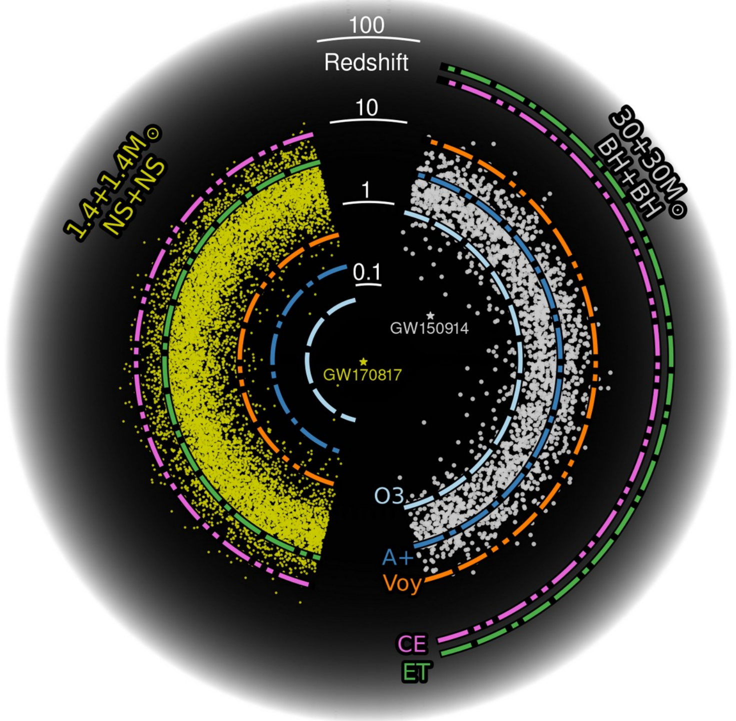


- **90 events** in above plot; expect **more than 200** at the end of O4 run this year!

Future gravitational wave detectors

- Ground based: Ongoing LIGO A+ upgrade, Einstein Telescope, Cosmic Explorer...
- ~100 times increase in strain sensitivity depending on frequency





Source: https://cosmicexplorer.org/sensitivity.html

Future gravitational wave detectors

- Space-based: LISA (2035+, approved in 2024), TianQin (2035+).

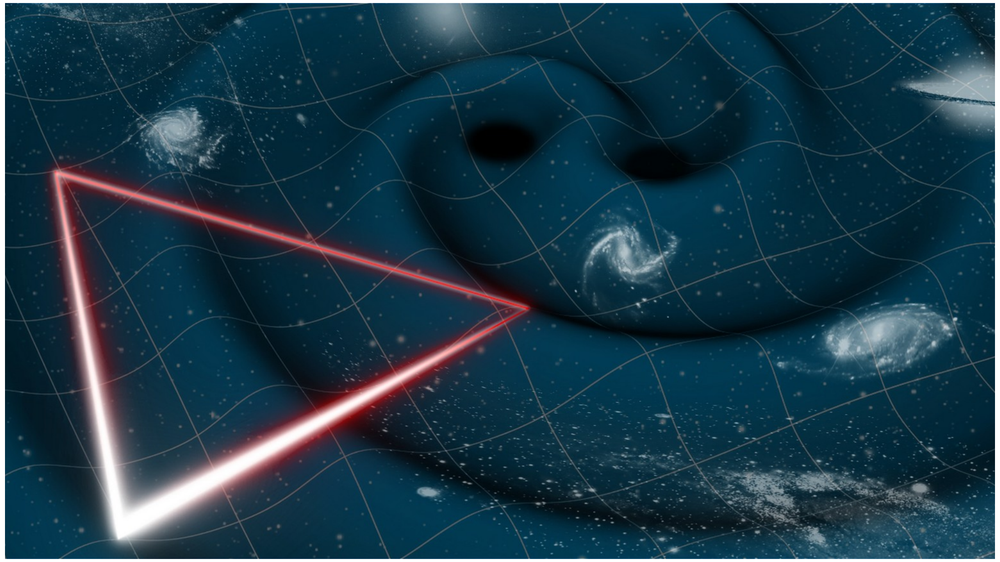
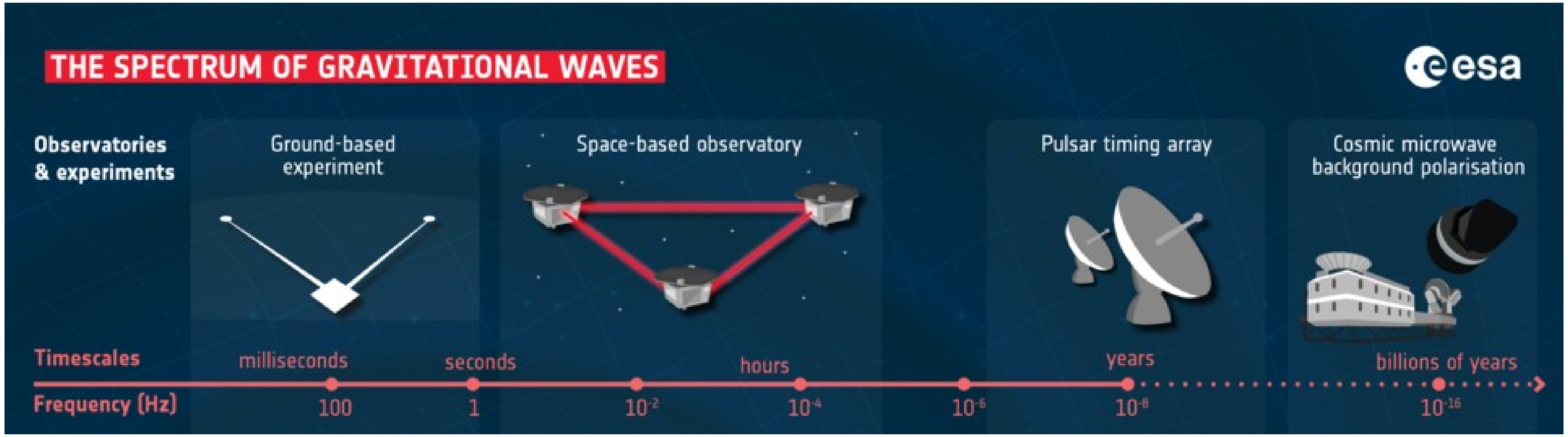
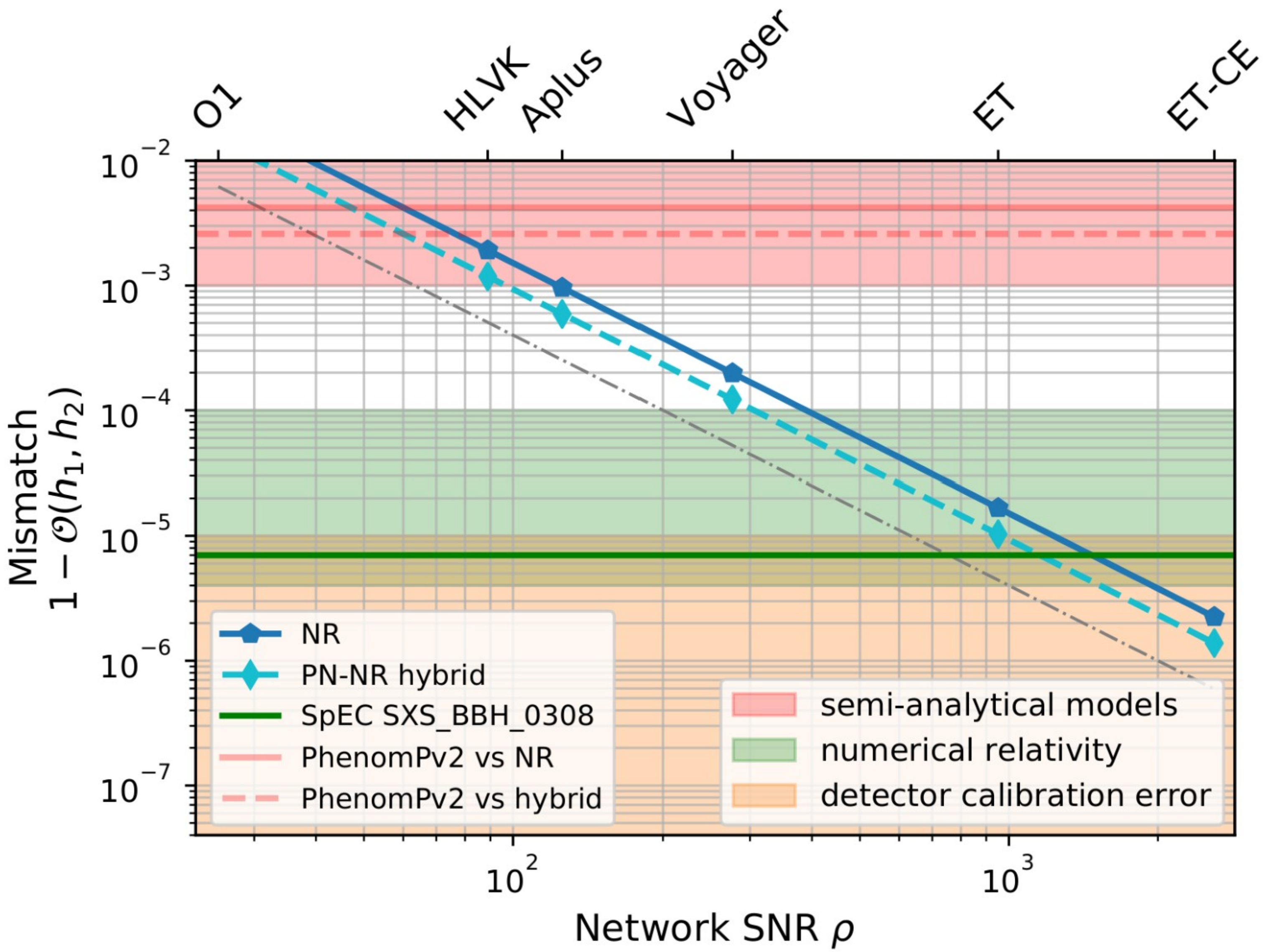


Image credit: ESA



Precision requirements for theory predictions

[Pürrer, Haster, '19]



Need orders of magnitude improvement

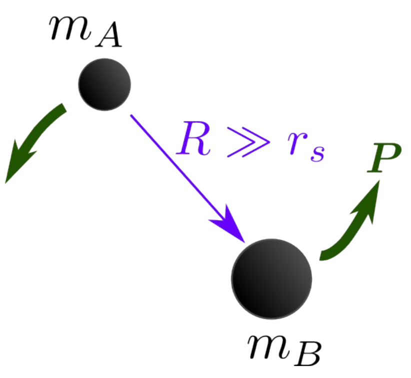
QFT approches to binary dynamics

- Perturbative GR is complicated; QFT insights have helped in multiple ways.
- Early success: non-relativistic general relativity (NRGR) [Goldberger, Rothstein '04]. Computes *post-Newtonian* potential between off-shell worldline sources.
- Recent breakthroughs: relativistic QFT for *post-Minkowskian* expansion. Some results far beyond classical GR. E.g., 2nd post-Minkowskian order **(2PM),** [Westpfahl, '85]
  - (On-shell) scattering amplitudes. BHs mapped to massive particles. E.g., **3PM,** '19. **4PM** (conservative) '21. **5PM in toy models** (EM, SUGRA), '24. **All-order in spin** at 1PM and 2PM (up to some Wilson coefficients). **$O(S1^4\ S2^0)$** at 3PM
  - Worldline methods for scattering observables (PMEFT, WQFT). E.g., **4PM** '21, '22. **5PM 1SF** '24. **Quartic-in-spin at** 2PM. **Quadratic-in-spin** at 3PM

Talk by Gustav Mogull

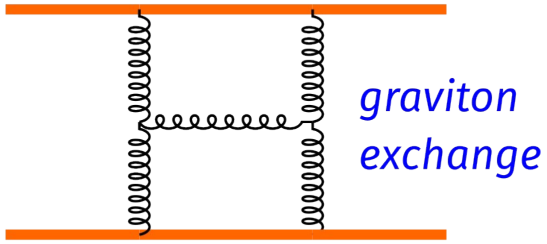
- Existing comparison between amplitude & WL approaches [Damgaard, Hansen, Planté, Vanhove, '23] does not establish diagrammatic equivalence before loop integration.

Point-particle effective field theory



point particle EFT

[Goldberger, Rothstein, '04]



- 1st quantized worldline approach (Deser / Polyakov action):

$$S = S_{\text{Einstein-Hilbert}} - \frac{m}{2}\int_{-\infty}^{\infty} d\tau \left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + 1\right)$$

arXiv:2412.10864,
Zeno Capatti, MZ

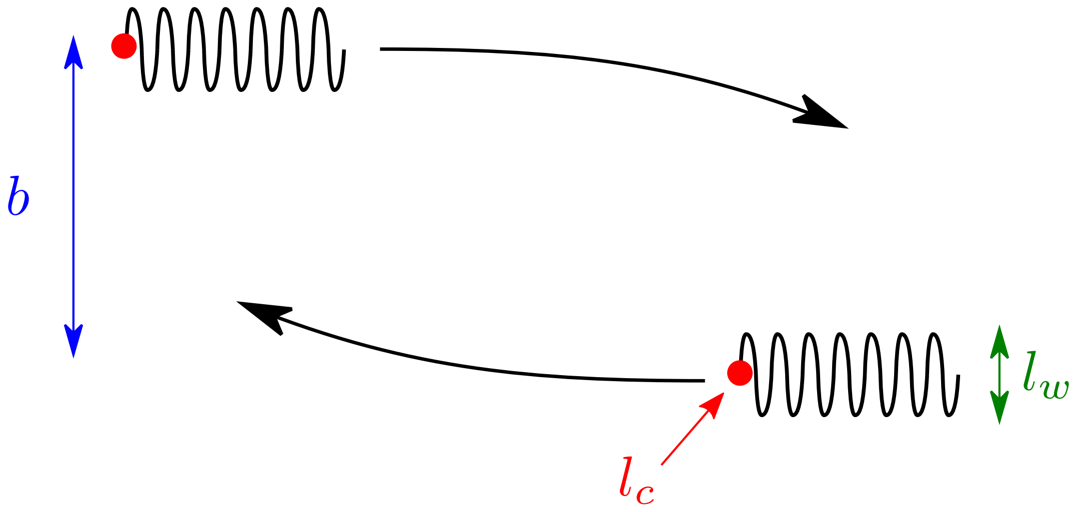
- 2nd quantized amplitude approach:

$$S = S_{\text{Einstein-Hilbert}} + \int d^4x\sqrt{-g}\left[-\frac{1}{2}\sum_i \left(\nabla^\mu \phi_i \nabla_\mu \phi_i + m^2\phi_i^2\right)\right]$$

Classical observables from S-matrix

[Kosower, Maybee, O'Connel, '18]

Wavepacket states with semi-classical localization in both position and momentum. "Goldilock condition": compton length $l_c \ll$ wavepacket spread $l_w \ll$ impact parameter



$$|\mathrm{in}\rangle \approx \int_{\boldsymbol{q}_T} e^{i\boldsymbol{b}\cdot\boldsymbol{q}_T} |\boldsymbol{p}+\boldsymbol{q}_T\rangle$$

Then compute expectation values of $\langle \mathrm{out}|\mathcal{O}|\mathrm{out}\rangle = \langle \mathrm{in}|\mathcal{S}^\dagger \mathcal{O}\mathcal{S}|\mathrm{in}\rangle$.

Practically: order-by-order evaluation of amplitudes, then integrate against plane wave profile.

Divergences in classical limit *q*→*0*

- Classical momentum p fixed. Momentum exchange *q* scales as $\mathcal{O}(\hbar/\mathrm{R})$.

- Compared with LO, NLO has one extra graviton-scalar vertex $\sim |p|^2$, and one matter propagator $\sim 1/[(p+q_1)^2 - m^2] \approx 1/(2p\cdot q_1) \sim 1/(|p||q_1|)$

▪ Overall enhancement $\sim |p|/|q|$ $\sim 1/\hbar$. Divergent correction!

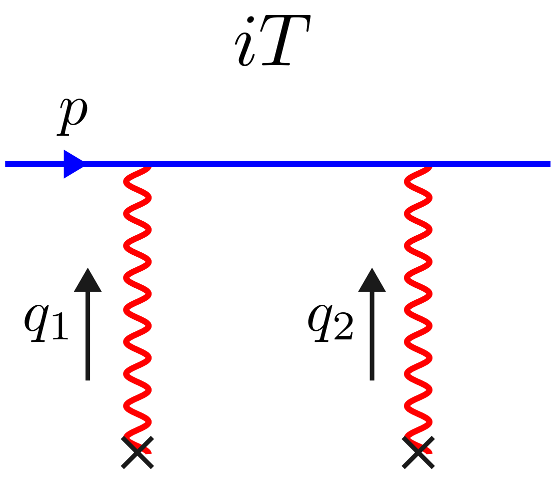
Classical observables from S-matrix

[Kosower, Maybee, O'Connel, '18]

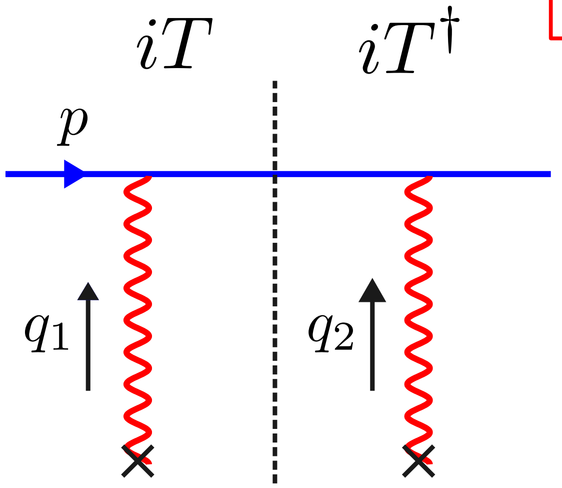
Compute *change* of observable (e.g. momentum of massive particle) during scattering

Simplified example: consider Δp^μ from scattering off a massive ***background source:***
(Our paper looks at two dynamic massive bodies)

Inspired by e.g. strong field study
[Adamo, Cristofoli, Ilderton, '22]



$\cdot\ (q_1^\mu + q_2^\mu)\ +$



$\cdot\ q_1^\mu$

linear-in-amplitude part

quadratic-in-amplitude part

Classical observables from S-matrix

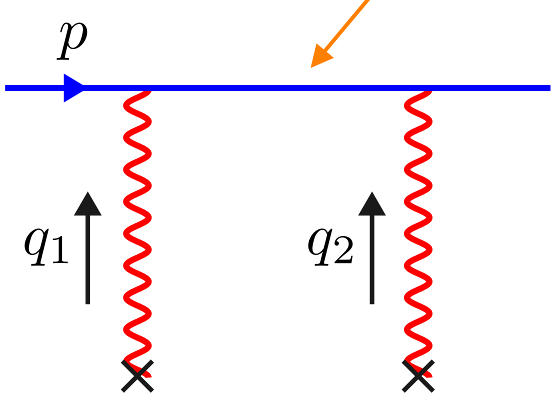
[Kosower, Maybee, O'Connel, '18]

Compute *change* of observable (e.g. momentum of massive particle) during scattering

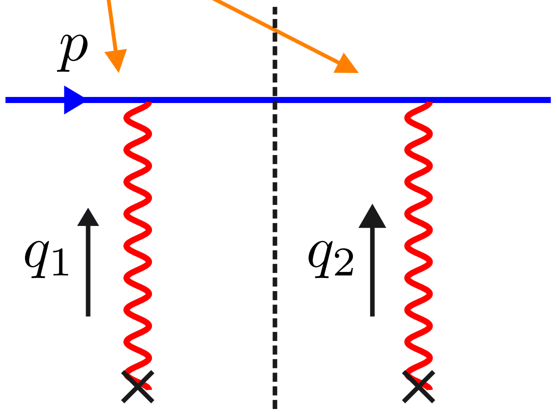
$$\Delta O = \langle \text{out}|\mathcal{O}|\text{out}\rangle - \langle \text{in}|\mathcal{O}|\text{in}\rangle = \langle \text{in}|(1 - iT^\dagger)\mathcal{O}(1 + iT^\dagger)|\text{in}\rangle - \langle \text{in}|\mathcal{O}|\text{in}\rangle$$

Using unitarity relation $iT^\dagger = -iT + T^\dagger T$,

$$\Delta O = \langle \text{in}|[\mathcal{O}, iT]|\text{in}\rangle + \langle \text{in}|(iT)^\dagger[\mathcal{O}, iT]|\text{in}\rangle$$



$\cdot\ (q_1^\mu + q_2^\mu)\ +$



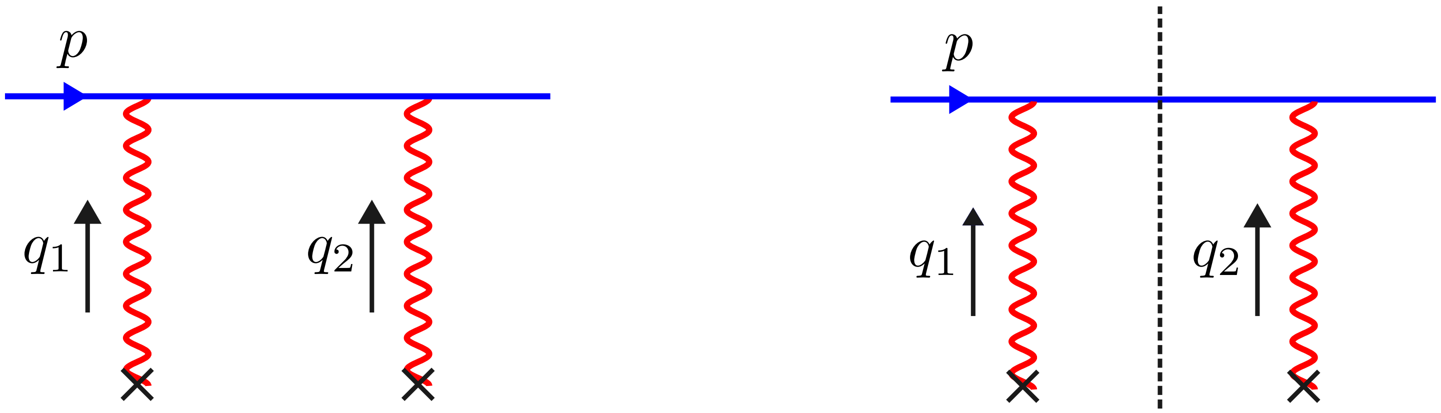
$\cdot\ q_1^\mu$

left of cut: amplitude
right of cut: conjugate amplitude

linear-in-amplitude part

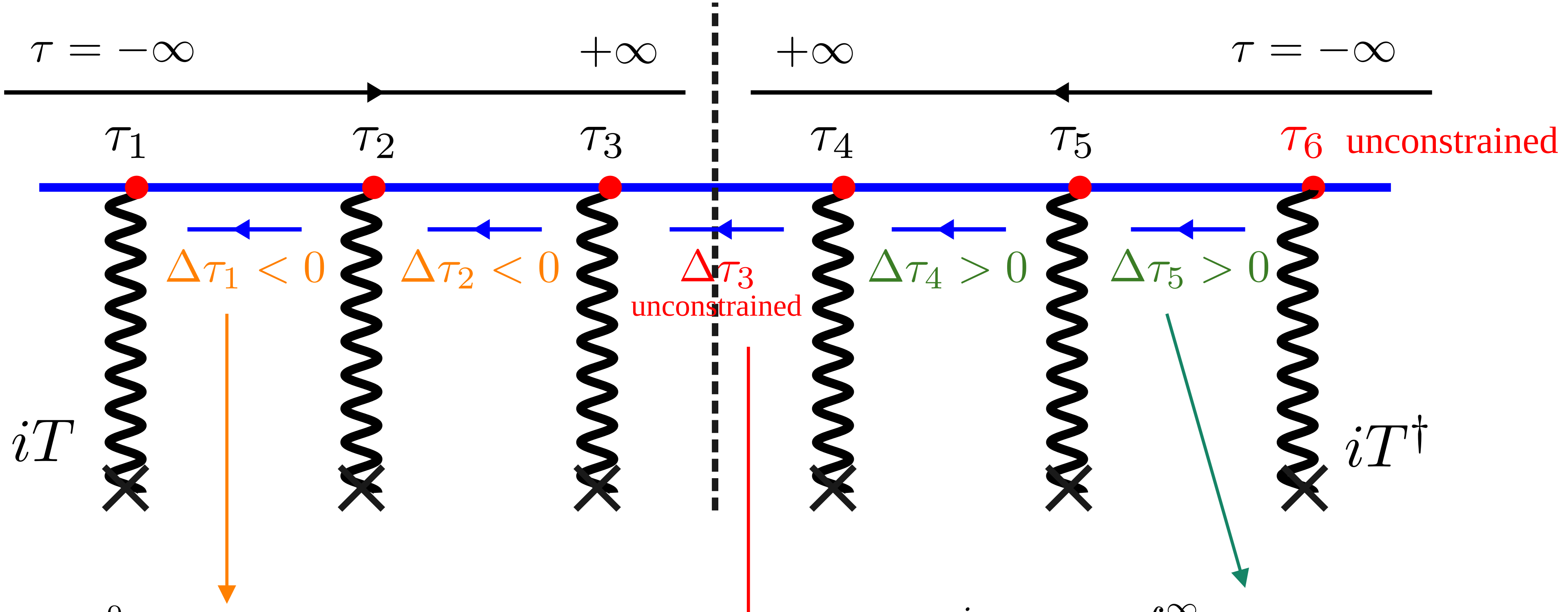
quadratic-in-amplitude part

Summing diagrams cancels divergence



- Leading order case already worked out in [Kosower, Maybee, O'Connel, '18]
- How to systematically generalize to higher orders?
- Our strategy: *quantum worldline representation* of the dressing of the massive matter propagator by interaction vertices with suitable time orderings. [Capatti, MZ, arXiv:2412.10864]
  - Inspired by parallel efforts for manifest **cancellation of IR singularities for collider observables:** loop-tree duality, local unitarity, cross-free family representation...

Worldline form of matter propagators

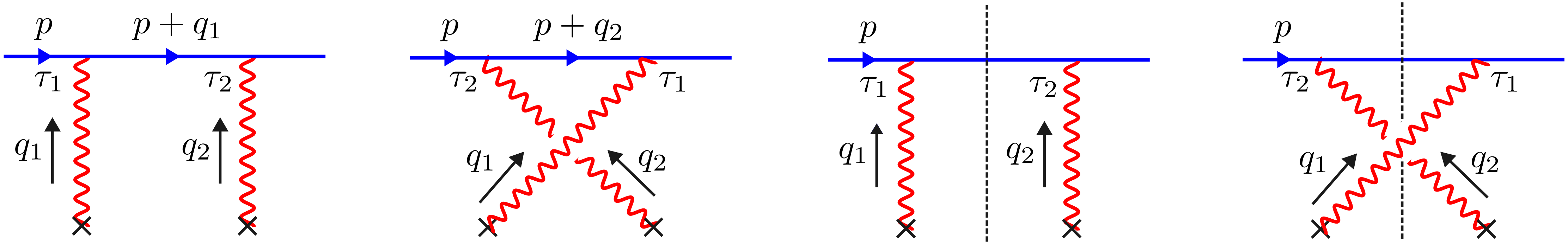


$$\frac{i}{k^2 - m^2 + i0} = \int_{-\infty}^{0} d\Delta\tau \exp\left[-i\Delta\tau(k^2 - m^2)\right]$$

$$\frac{-i}{k^2 - m^2 - i0} = \int_{0}^{\infty} d\Delta\tau \exp\left[-i\Delta\tau(k^2 - m^2)\right]$$

$$2\pi\delta(k^2 - m^2) = \int_{-\infty}^{\infty} d\Delta\tau \exp\left[-i\Delta\tau(k^2 - m^2)\right]$$

Cancellation of superclassical *q→0* divergence



(1): $\tau_1 < \tau_2$ (1'): $\tau_2 < \tau_1$ (2) & (2'): τ_1, τ_2 unordered

(1): $$(q_1^\mu + q_2^\mu) \int \mathrm{d}\tau_1 e^{-i\tau_1(2q_1\cdot p)} \int \mathrm{d}\tau_2 e^{-i\tau_2(2q_1\cdot p)}\, \theta(\tau_2 - \tau_1)$$

(1'): $$(q_1^\mu + q_2^\mu) \int \mathrm{d}\tau_1 e^{-i\tau_1(2q_1\cdot p)} \int \mathrm{d}\tau_2 e^{-i\tau_2(2q_1\cdot p)}\, \theta(\tau_1 - \tau_2)$$

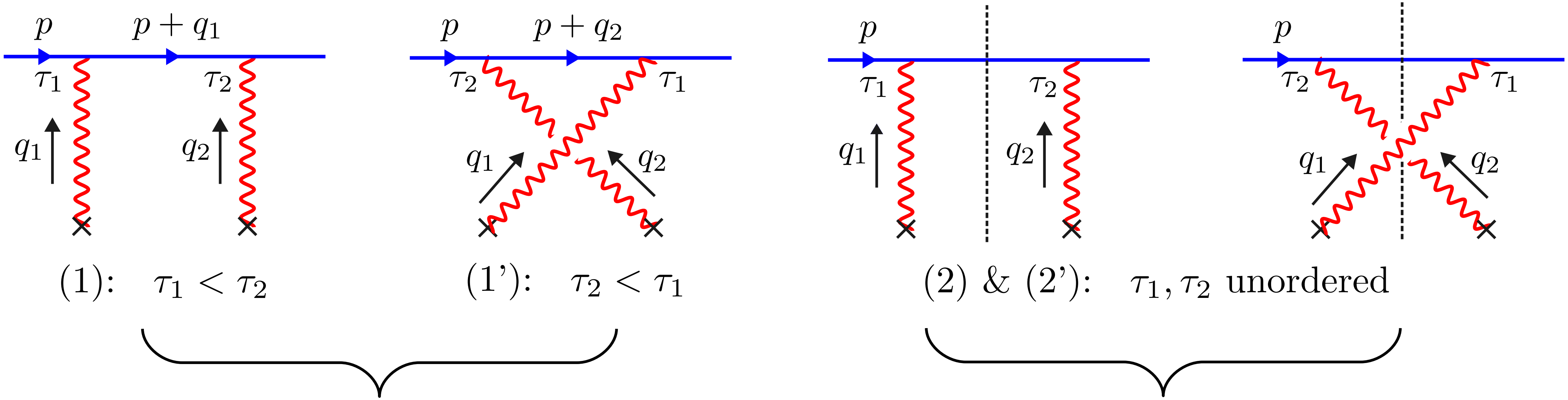
(2): $$-q_1^\mu \int \mathrm{d}\tau_1 e^{-i\tau_1(2q_1\cdot p)} \int \mathrm{d}\tau_2 e^{-i\tau_2(2q_1\cdot p)} \cdot 1$$

(2'): $$-q_2^\mu \int \mathrm{d}\tau_1 e^{-i\tau_1(2q_1\cdot p)} \int \mathrm{d}\tau_2 e^{-i\tau_2(2q_1\cdot p)} \cdot 1$$

Coupling to straightline WL with velocity p .

Sum of two time orderings exactly cancels cut contributions without ordering.

Quantum worldlines

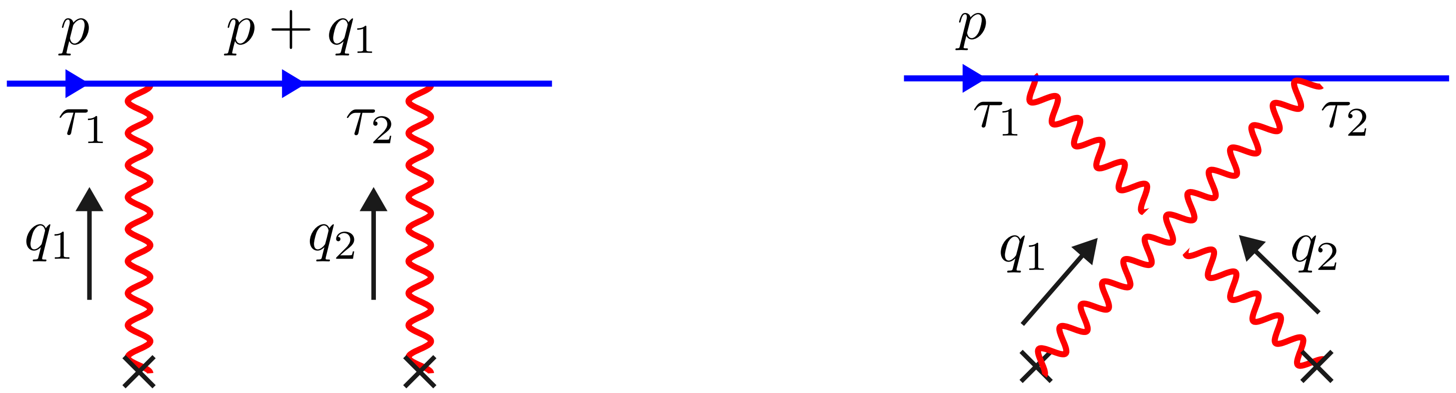


"(uncut) quantum worldline"

"cut quantum worldline"

- Defined as *dressed propagators with symmetrized attachments* on either side of cut.
- **Building blocks** for converting classical observables from amplitudes into WL form.

Subleading small-*q* expansion: classical order



$(1): \quad \tau_1 < \tau_2$ $\qquad$ $(1'): \quad \tau_2 < \tau_1$

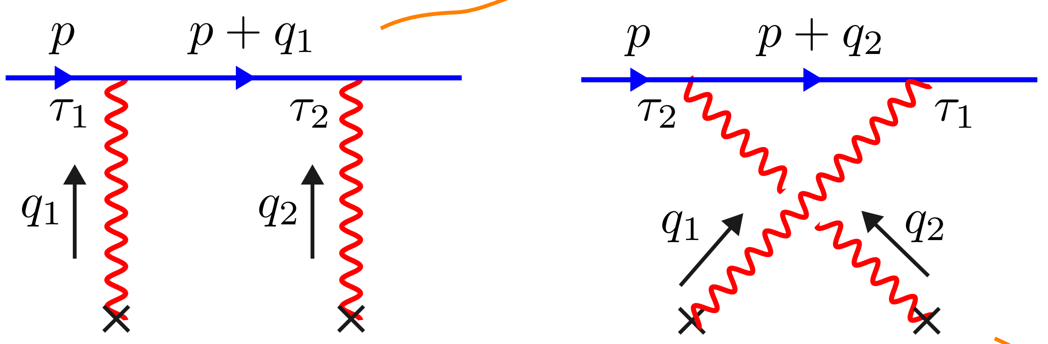
$$
\begin{aligned}
(1) + (1') &= \int \mathrm{d}\Delta\tau_{12} \int \mathrm{d}\tau_2 \, e^{-i\Delta\tau_{12}[(q_1+p)^2-m^2]-i\tau_2[(q_{12}+p)^2-m^2]} \\
&\quad \times \Big\{\theta(-\Delta\tau_{12}) + \theta(\Delta\tau_{12})[1 - 2i(q_1 \cdot q_2)\Delta\tau_{12}]\Big\}(q_1^\mu + q_2^\mu) \\
&= \int \mathrm{d}\Pi(p, q_i)\Big\{1 - 2i(q_1 \cdot q_2)\Delta\tau_{12}\theta(\Delta\tau_{12})\Big\}(q_1^\mu + q_2^\mu)
\end{aligned}
$$

untwisted diagram as "base measure"

leading small-*q* limit; superclassical, cancels with cut

Subleading piece

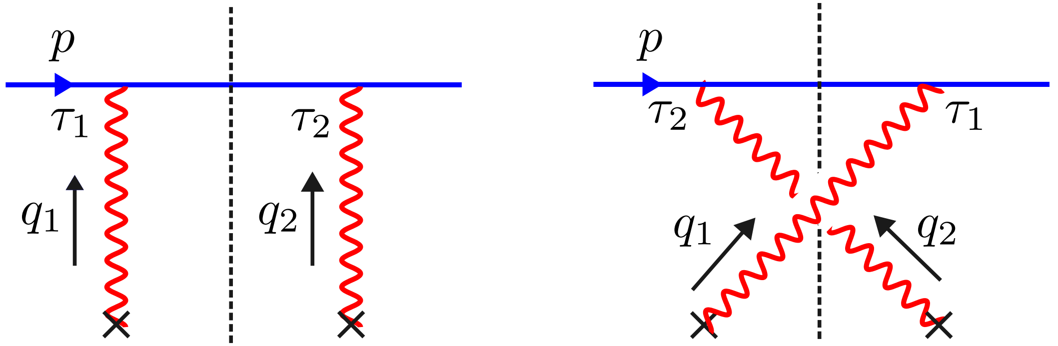
Subleading small-q expansion: cut + uncut



(1): $\tau_1 < \tau_2$ (1'): $\tau_2 < \tau_1$

$$= \text{leading} + (q_1^\mu + q_2^\mu) \int d\Pi(p, q_i)(-2i\, q_1 \cdot q_2)\Delta\tau_{12}\theta(\Delta\tau_{12})$$

Subleading order no longer annihilates the Θ function perfectly



(2) & (2'): τ_1, τ_2 unordered

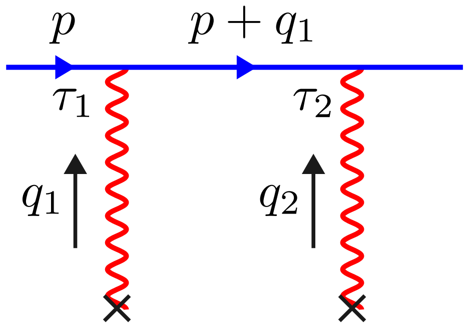
$$= -\text{leading} - q_2^\mu \int d\Pi(p, q_i)(-2i\, q_1 \cdot q_2)\Delta\tau_{12}$$

$1 - \theta(\Delta\tau_{12}) = \theta(\Delta\tau_{21})$

$$\Delta p^\mu \sim \int \mathrm{d}\tau_1 e^{-i\tau_1(2q_1\cdot p)} \int \mathrm{d}\tau_2 e^{-i\tau_2(2q_1\cdot p)} \left[q_1^\mu \Delta\tau_{12}\theta(\Delta\tau_{12}) + q_2^\mu \Delta\tau_{21}\theta(\Delta\tau_{21}) \right]$$

Emergence of causality: I. matter propagators

Previous slide:



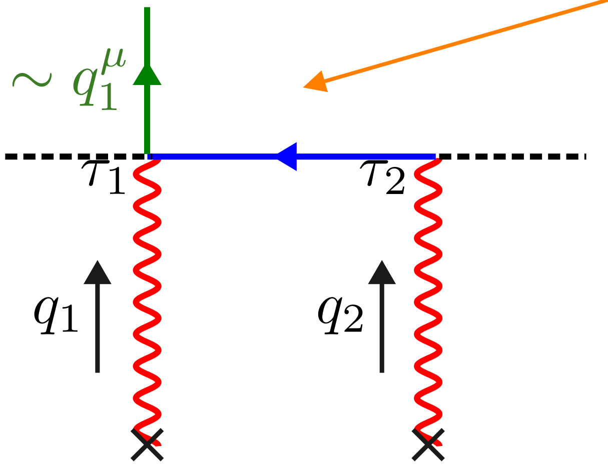
$+ \dots$

from e.g. WL theory Feynman rules

$$\Delta p^\mu \sim \int \mathrm{d}\tau_1 \int \mathrm{d}\tau_2 e^{-i\tau_1(2q_1\cdot p)-i\tau_2(2q_1\cdot p)} \left[q_1^\mu \Delta\tau_{12}\theta(\Delta\tau_{12}) + q_2^\mu \Delta\tau_{21}\theta(\Delta\tau_{21})\right](-2i\, q_1\cdot q_2)$$

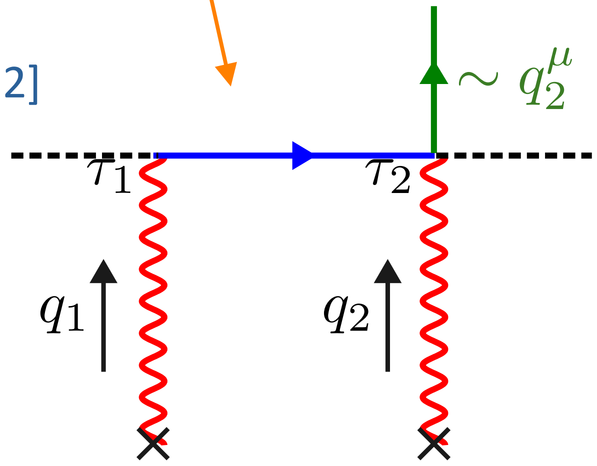
Retarded worldline propagator

[Mogull, Plefka, Steinhoff, '20.
Jakobsen, Mogull, Plefka, Sauer, '22]

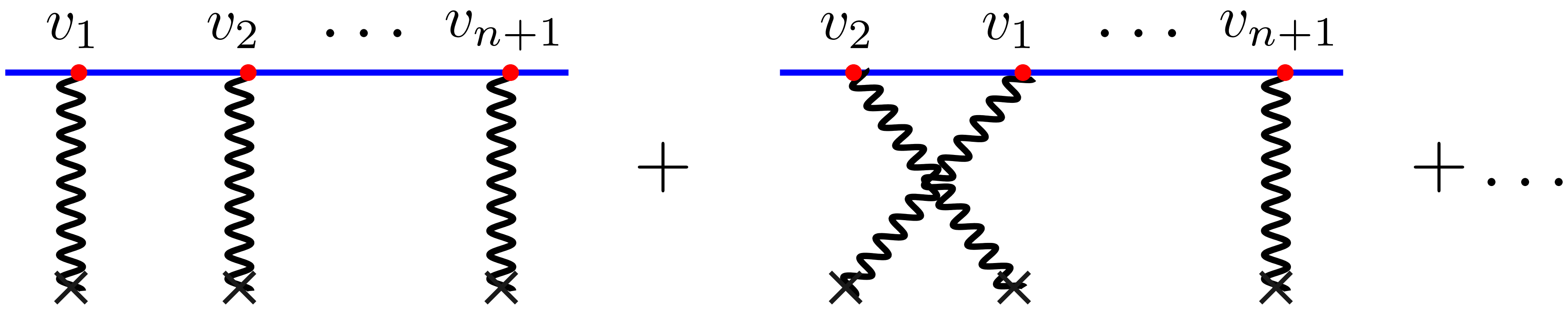


Only causal orderings remain after sum!

All time arrows flow towards observable



Compact all-multiplicity quantum WL expansion



- Symmetrized over vertices v_i with incoming momentum p_i and WL parameter τ_i. **Result: sum over forests**, each inducing a partial ordering of vertices resembling a causality flow.

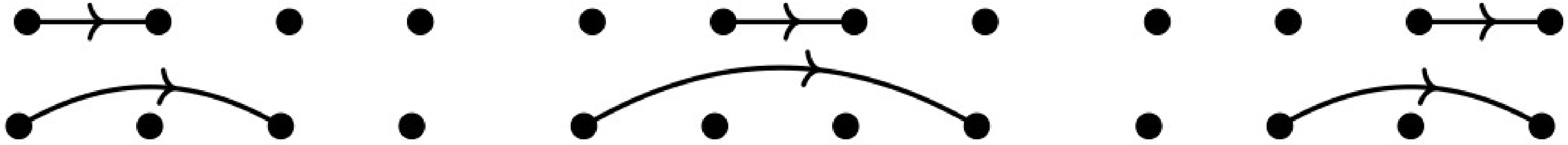
Compact all-multiplicity quantum WL expansion

- **Result:**

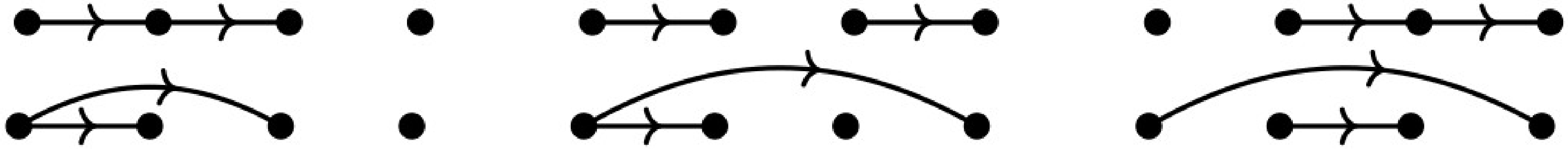
Reverse ordering for edges on the right of the cut; drop ordering if the edge crosses the cut

$$\sum_{\mathcal{F}_k \in \text{forests}} \prod_{e=(v,v')\in\mathcal{F}_k} \left(e^{-2i\tau_e(q_v\cdot q_{v'})} - 1\right)\Theta(\Delta\tau_e) + \text{quantum corrections}$$

(a) The forest contributing to $\mathscr{F}_0(V)$

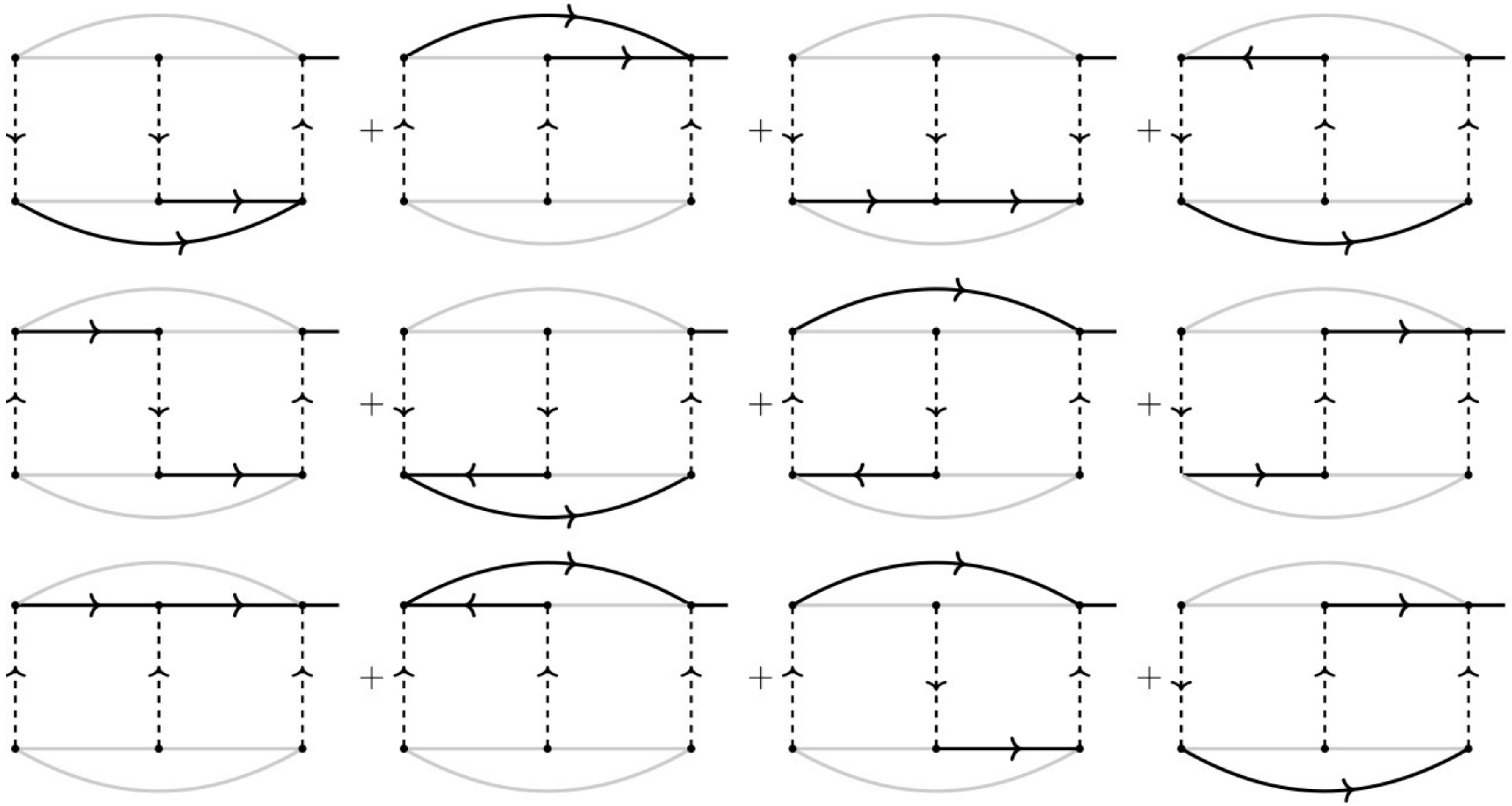


(b) The forests contributing to $\mathscr{F}_1(V)$



(c) A selection of the forests contributing to $\mathscr{F}_2(V)$

Sewing 2 WLs: NNLO conservative impulse



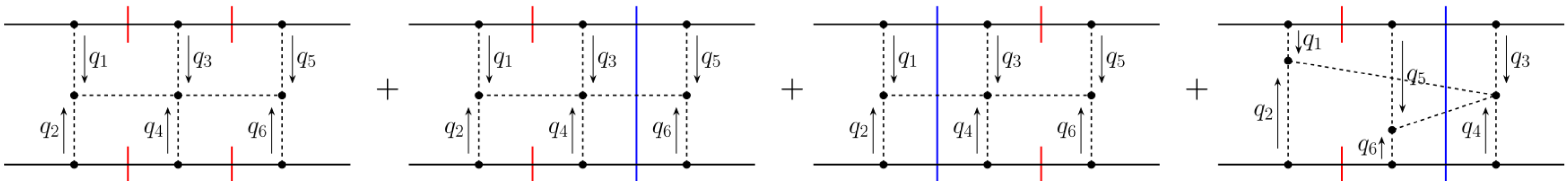
Manifestly finite. ***Identical*** to scalar theory WL integrand before any integration.

Emergence of causality: II. massless propagators

- Amplitude (KMOC) formalism contains only Feynman and cut propagators (and their complex conjugates).
- How to get retarded propagators in WL formalism? Use identity

$$G_F(q) + i\delta^+(q) = G_R(-q)$$

- Double ladders work to all orders! E.g., product of 2 retarded propagators in WL integrand (horizontal dashed lines below) become 4 terms in KMOC integrand:



Explicit checks

- **Manifest finiteness & causality** of classical limit after rewriting into WL form:
  - Scalar QED at 2 loops
  - Scalar theory at 2 loops
  - Scalar therory, 3-loop & 4-loop ladder (iteration) diagrams
  - Scalar theory, non-iteration double ladders to all orders
- **Identical loop integrand with WL** literature (PMEFT, WQFT)
  - Scalar QED at 2 loops (conservative part)
  - Scalar theory at 2 loops (full)
- **Gravity next;** though finiteness of classical limit, i.e. cancellation of iteration divergences, should have no essential difference with toy models

Conclusion

- **Scattering amplitudes** (KMOC) and **worldlines** have both become new powerful tools for post-Minkowskian expansion of gravitational binary dynamics.

- Formulated systematic method to establish their equivalence using Schwinger parametrization to rewrite scattering amplitude *integrands* into WL form.

- Bonus: manifest finiteness of KMOC formalism for extracting classical observables, previously only well-understood at 1 loop.

- Demonstrates emergence of classical causality starting from only the S matrix.

- Various follow-up directions, e.g. connections with generalized Wilson lines
  [White, Laenen, Stavenga, Bonocore, Kulesza, Pirsch ...]