Low-energy master formulae on the loop and line in a homogeneous electromagnetic field

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First Quantisation for Physics in Strong Fields
Higgs Centre, University of Edinburgh
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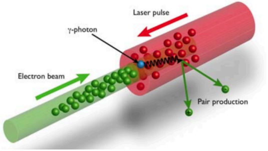
Outline

QED

- Treat perturbatively in α, with high precision.

SFQED

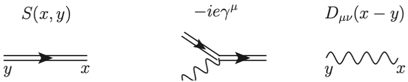
- QED + background field, e.g. ultra-intensity lasers



[M. Marklund, J. Lundin, Eur. Phys. J.D. 55, 319326 (2009).]

- May scale larger that α: $\chi^{2/3}\alpha$? [V. I. Ritus, Sov. Phys. JETP 30, 1181 (1970); N. B. Narozhnyi, Phys. Rev. D 21, 1176 (1980).]
- Must treat non-perturbatively!

- Particles dressed with background field + perturbative photons [W. H. Furry, Phys. Rev. 81 (1951)]:



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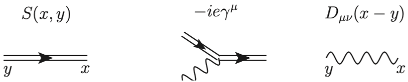

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  1. Unrealistic modeling of e.g. high intensity lasers
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  3. Resummation required

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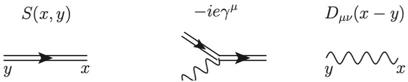

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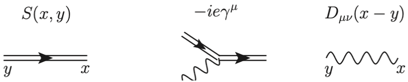

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Make use of simplifying **low-energy** expansion for *all* multiplicity!

- To illustrate the *external photon* **low-energy expansion** look at effective action and propagator

$$\mathcal{D}_N^{x'x}[A] = (-ie)^N \int_0^\infty dT \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau)\, \mathrm{e}^{iS} \prod_{i=1}^{N} V[f_i]$$

$$\Gamma_N[A] = (-ie)^N \int_0^\infty \frac{dT}{T} \oint \mathcal{D}x(\tau)\, \mathrm{e}^{iS} \prod_{i=1}^{N} V[f_i]$$

- Worldline action: $S = -\int_0^T d\tau\, [m^2 + \frac{1}{4}\dot{x}^2 + eA\cdot\dot{x}]$
- Vertex operator (about a standard photon expansion gauge):

$$V[f_i] = \int_0^T d\tau\, a_i\cdot\dot{x} = \int_0^T d\tau\, \varepsilon_i\cdot\dot{x}\, \mathrm{e}^{ik\cdot x}$$

- For the effective action we have *periodic boundary conditions*: $x(0) = x(T)$.
- In the low energy approximation, small k_i:

$$V[f_i] = \int_0^T d\tau \, \frac{1}{2} x(\tau) \cdot f_i \cdot \dot{x}(\tau) + \mathcal{O}((k_i^\mu)^2)$$

where $f_i^{\mu\nu} := i[k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu]$

- Constant field! [L. C. Martin, C. Schubert, V. M. Villanueva San- doval, Nucl. Phys. B (2003); J. P. Edwards, A. Huet, and C. Schubert, Nucl. Phys. B (2018); N. Ahmadiniaz, M. A. Lopez-Lopez, C. Schubert, Phys. Lett. B 852, 138610 (2024); M. A. Lopez-Lopez, Phys. Lett. B (2025).]

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- But what about the **propagator**
- Boundary contribution already at *zeroth* order in photon energy with standard gauge: $\varepsilon \cdot (x' - x)$.

- Instead of standard photon gauge $\rightarrow$ **Fock-Schwinger gauge**

$$a_\mu(x) = -\sum_{n=0}^{\infty} \frac{1}{n!(n+2)} (x-\hat{x})^\nu (x-\hat{x})^{\nu_1} \dots (x-\hat{x})^{\nu_n} \times ik_{\nu_1} \dots ik_{\nu_n} f_{\mu\nu} ,$$

- In the low-energy limit

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- In the low-energy limit

$$a_\mu(x) = -\frac{1}{2}(x-\hat{x})^\nu f_{\mu\nu}$$

- *Any* reference point, $\hat{x}$, vertex operator is linear in f

$$V[f_i] = \int_0^T d\tau \, \frac{1}{2}(x(\tau)-\hat{x}) \cdot f_i \cdot \dot{x}(\tau) + \mathcal{O}((k_i^\mu)^2)$$

- Suggestively re-exponentiate the vertices via a linear operator

$$\prod_{i=1}^{N} (-ie)^N V[f_i] = \exp\left[-ie \int_0^T d\tau \, a \cdot \dot{x}\right]\Big|_{\mathrm{lin}N}$$

with **effective homogeneous field**

$$a^\mu := \sum_{i=1}^{N} a_i^\mu = -\frac{1}{2} \sum_{i=1}^{N} f^{\mu\nu}(x - \hat{x})_\nu$$

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- Evaluate low-energy $N-$photon coupling **just by coupling to a homogeneous field**

$$\mathcal{D}_N^{x'x}[A] = \mathcal{D}^{x'x}[A+a]\Big|_{\mathrm{lin}N}, \; \Gamma_N[A] = \Gamma[A+a]\Big|_{\mathrm{lin}N}$$

- Holds for *any* background, A.
(Though, we treat the case of a homogeneous background)

Outline

- Well known solution:

$$\Gamma[A+a] = -i\mathcal{V}\int_0^\infty \frac{dT}{(4\pi)^2 T^3}\, \mathrm{e}^{-im^2 T}\, K[F+f]$$

$$K[F+f] = \frac{\mathrm{Det}^{-\frac{1}{2}}[\hat{\partial}_\tau^2 - 2e(F+f)\hat{\partial}_\tau]}{\mathrm{Det}^{-\frac{1}{2}}[\hat{\partial}_\tau^2]}$$

$$= \det{}^{-\frac{1}{2}}\left[\frac{\sinh(Z+z)}{Z+z}\right]$$

with $Z := eFT$ and $z := efT$

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- The trick is how to expand about f. Re-express the determinant as

$$K[F+f] = K[F]\frac{\mathrm{Det}^{-\frac{1}{2}}[\hat{\partial}_\tau^2 - 2e(F+f)\hat{\partial}_\tau]}{\mathrm{Det}^{-\frac{1}{2}}[\hat{\partial}_\tau^2 - 2eF\hat{\partial}_\tau]}$$
$$= K[F]\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\ln\left[1-2e(\hat{\partial}_\tau^2-2eF\hat{\partial}_\tau)^{-1}f\hat{\partial}_\tau\right]}$$

then expand the log. *Will need the Green function:*

- Periodic Green function in functional form:

$$\mathcal{G}(\tau, \tau') = \langle \tau | \frac{2}{\hat{\partial}_\tau^2 - 2eF\hat{\partial}_\tau} | \tau' \rangle$$

and solution is [M. G. Schmidt and C. Schubert, PLB 1993.]

$$\mathcal{G}_{ij} = \frac{T}{2Z^2}\left(1 - \frac{Z}{\sinh(Z)} e^{-Z\dot{G}_{ij}} - Z\dot{G}_{ij}\right)$$

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- Then we can write for the log expansion

$$K[F+f] = K[F]\mathrm{e}^{\sum_{n=1}^{\infty} \frac{(-ie)^n}{2n}\Delta_n}$$

where

$$\Delta_n = i^n \prod_{i=1}^{n} \int_0^T d\tau_i \,\mathrm{tr}(\dot{\mathcal{G}}_{12} f \dot{\mathcal{G}}_{23} f \ldots \dot{\mathcal{G}}_{n1} f)$$

- Finally $K_N[F] = K[F+f]|_{\mathrm{lin}N}$.

- Let us confirm this at the path integral level.
  Expand about the usual $x(\tau) = x_0 + q(\tau)$.
  Path integral with expectation value about q is

$$\Gamma_N[A] = -i\mathcal{V}_4 \int_0^\infty dT \, \frac{\mathrm{e}^{-im^2 T}}{(4\pi)^2 T^3} K[F] \, (-ie)^N \langle \prod_{i=1}^N V[f_i] \rangle_{\text{PBC}}$$

  vertices become: $V[f_i] = \int_0^T d\tau \, \frac{1}{2} q(\tau) \cdot f_i \cdot \dot{q}(\tau)$.

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- Perform the Wick contractions to find for $N = 1, 2$

$$K_1[F] = \frac{e}{2} K[F] \int_0^T d\tau_1 \mathrm{tr}[\dot{\mathcal{G}}_{11} f_1]$$

$$K_2[F] = \frac{e^2}{2} K[F] \int_0^T d\tau_1 d\tau_2 \left[\mathrm{tr}(\dot{\mathcal{G}}_{11} f_1) \mathrm{tr}(\dot{\mathcal{G}}_{22} f_2) + \mathrm{tr}(\dot{\mathcal{G}}_{12} f_1 \dot{\mathcal{G}}_{21} f_2) \right]$$

in agreement with previous log expansion

How to perform parameter integrals

-or- how to expand in matrix form without the integrals!

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How to perform parameter integrals

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  For arbitrary F and f

$$Ff - \widetilde{f}\widetilde{F} = -I_{fF}\,, \quad \widetilde{F}f + \widetilde{f}F = 2I_{f\widetilde{F}}\,,$$

  and the following built from the above

$$[F^2, f] = [\widetilde{F}^2, f] = F\widetilde{f}\widetilde{F} - \widetilde{F}\widetilde{f}F$$
$$\{F^2 + \widetilde{F}^2, f\} = 2[2I_{f\widetilde{F}}\widetilde{F} - I_{fF}F]$$

$I_{Ff} = \frac{1}{2}f_{\mu\nu}F^{\mu\nu}$, and $I_{\widetilde{F}f} = -\frac{1}{4}\widetilde{F}_{\mu\nu}f^{\mu\nu} = -\frac{1}{4}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}f^{\mu\nu}$

- **Commutes with F and $\widetilde{F}$**

We first show how to do the **parameter integrals**.

- Recall we have $i^n \prod_{i=1}^{n} \int_0^T d\tau_i \, \mathrm{tr}(\dot{\mathcal{G}}_{12} f \dot{\mathcal{G}}_{23} f \ldots \dot{\mathcal{G}}_{n1} f)$

$$f = \left(\{f, F^2 + \widetilde{F}^2\} + [f, F^2 + \widetilde{F}^2]\right) \frac{1}{F^2 + \widetilde{F}^2}$$

- Split the parameter integral f insertions into **commutable w/ F** and non-commutable parts.
- Commutable part:

$$\int_0^T d\tau' \dot{\mathcal{G}}_{\tau\tau'} \{f, F^2 + \widetilde{F}^2\} \dot{\mathcal{G}}_{\tau'\tau''} = \int_0^T d\tau' \dot{\mathcal{G}}_{\tau\tau'} \dot{\mathcal{G}}_{\tau'\tau''} 2[2 I_{f\widetilde{F}} \widetilde{F} - I_{fF} F]$$

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- Non-commutable part, employ identity:

$$\int_0^T d\tau' \dot{\mathcal{G}}_{\tau\tau'}[f, F^2 + \widetilde{F}^2]\dot{\mathcal{G}}_{\tau'\tau''} = -2[\dot{\mathcal{G}}_{\tau\tau''}, \{eF, f\}]$$

Let us now show the same structure through a **matrix expansion**.

- One may equally write

$$K[F+f] = e^{f\cdot\partial_F} K[F]\,,\ K_N[F] = \frac{1}{N!}(f\cdot\partial_F)^N K[F]\Big|_{\mathrm{lin}\,N}$$

- Can build *recursively* higher order K_N by performing N ∂_F. *However derivative of F and F in general don't commute!*

Scalar effective action

Matrix expansion and parameter integrals

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- Start w/ $K_1[F]$. If **only 1 f under the trace**, straightforward expansion. First use Jacobi's identity:

$$\begin{aligned} K_1[F] &= \tfrac{1}{2} K[F]\mathrm{tr}\left[\frac{\sinh(Z)}{Z} f_1\cdot\partial_F \frac{Z}{\sinh(Z)}\right] \\ &= \tfrac{e}{2} K[F]\, T\, \mathrm{tr}[(Z^{-1} - \coth(Z)) f_1] \end{aligned}$$

Even though $[f_1, F] \neq 0$, by cyclicity of trace, above is possible.

- Onto $K_2[F]$; insert $\frac{F^2+\widetilde{F}^2}{F^2+\widetilde{F}^2}$ under trace.

$$
\begin{aligned}
K_1[F] &= \tfrac{e}{2} K[F]\, T\, \mathrm{tr}[\frac{F^2+\widetilde{F}^2}{F^2+\widetilde{F}^2}(Z^{-1}-\coth(Z)) f_1] \\
&= \tfrac{e}{4} K[F]\, T\, \mathrm{tr}[\frac{1}{F^2+\widetilde{F}^2}(Z^{-1}-\coth(Z))\{f_1, F^2+\widetilde{F}^2\}] \\
&= \tfrac{e}{4} K[F]\, T\, \mathrm{tr}[\frac{1}{F^2+\widetilde{F}^2}(Z^{-1}-\coth(Z)) 2[2 I_{f_1 \widetilde{F}} \widetilde{F} - I_{f_1 F} F]]
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- Now everything under the trace **commutes**!
- Apply same procedure used to get $K_1[F]$ to find $K_2[F] = \frac{1}{2}f_2\cdot\partial_F K_2[F]$.

  Straightforward ∂_F derivative under trace!

- Continue to find higher order N.

Outline

We can do the same for the **scalar propagator**

- Solution well-known [J. Schwinger, Phys. Rev. (1951).] in homogeneous fields

$$D^{x'x}[A+a] = -i\int_0^\infty \frac{dT}{(4\pi T)^2} e^{-im^2 T} K^{x'x}[F+f]$$

$$K^{x'x}[F+f] = K[F+f]\mathrm{e}^{-i\frac{x_-^2}{4T}+ix_-\cdot\frac{(Z+z)^2}{T^4}\int_0^T d\tau d\tau' \underset{\smile}{\Delta}(\tau,\tau'|F+f)\cdot x_-}$$

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- For the determinant, $K[F+f]$, can use the same treatment as used for the effective action to expand about f.
- *Even though Dirchilet BCs, expanded determinant same w/ periodic BCs.*

- Expand the new **Green function obeying Dirichlet BCs,**

$$\underset{\smile}{\Delta}(\tau,\tau'|F+f) = \langle\tau|\frac{1}{\hat{\partial}_\tau^2 - 2e(F+f)\hat{\partial}_\tau}|\tau'\rangle$$

with solution $2\underset{\smile}{\Delta}_{ij} = \mathcal{G}_{ij} - \mathcal{G}_{i0} - \mathcal{G}_{0j} + \mathcal{G}_{00}$.

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- First rewrite as

$$\frac{1}{\hat{\partial}_\tau^2 - 2e(F+f)\hat{\partial}_\tau} = \frac{1}{1 - 2e\frac{1}{\hat{\partial}_\tau^2 - 2eF\hat{\partial}_\tau} f\hat{\partial}_\tau}\frac{1}{\hat{\partial}_\tau^2 - 2eF\hat{\partial}_\tau}$$

- Then perform the geometric series, $\underset{\smile}{\Delta}(\tau,\tau'|F) =: \underset{\smile}{\Delta}(\tau,\tau')$

$$\underset{\smile}{\Delta}(\tau,\tau'|F+f) = \underset{\smile}{\Delta}(\tau,\tau') + \sum_{n=1}^{\infty}(-ie)^n \underset{\smile}{\Delta}^{(n)}(\tau,\tau')$$

$$\underset{\smile}{\Delta}^{(n)}(\tau,\tau') = (2i)^n \prod_{i=1}^{n}\int_0^T d\tau_i\, \underset{\smile}{\Delta}_{\tau 1} f\,{}^\bullet\underset{\smile}{\Delta}_{12} f \ldots {}^\bullet\underset{\smile}{\Delta}_{n\tau'}$$

Scalar propagator

Functional expansion

- Gathering the coordinate dependent part and the determinant we can write the **entire expression**

$$K^{x'x}[F+f] = K^{x'x}[F] \times \mathrm{e}^{ix_-\cdot z\frac{2Z+z}{T^4}\circ\underset{\smile}{\Delta}^{\circ}\cdot x_- + \sum_{n=1}^{\infty}(-ie)^n\left[\frac{1}{2n}\Delta_n + ix_-\cdot\frac{(Z+z)^2}{T^4}\circ\underset{\smile}{\Delta}^{\circ(n)}\cdot x_-\right]}$$

- To compute $D_N^{x'x}[A]$, or $K_N^{x'x}[F] = K^{x'x}[F+f]|_{\text{lin}N}$, just truncate the sum in the exponent to N. Keep all f^n for all $n \leq N$ in the exponent.

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- First order expression:

$$K_1^{x'x}[F] = -ieK^{x'x}[F]\left(\frac{1}{2}\Delta_1 - x_- \cdot \frac{Z}{T^4}[2fT \circ \underset{\smile}{\Delta} \circ - iZ \circ \underset{\smile}{\Delta}^{\circ(1)}] \cdot x_-\right)\Big|_{\text{lin}1}$$

- Let's confirm using the path integral approach (expand about straight line, and integrate out fluctuations)

$$\mathcal{D}_N^{x'x}[A] = -\,i(-ie)^N \int_0^\infty \frac{dT}{(4\pi T)^2}\,\mathrm{e}^{-im^2 T}\,\mathrm{e}^{-i\frac{x_-^2}{4T}} \times K[F]\Big\langle \mathrm{e}^{\frac{ie}{T}\int_0^T d\tau\, x_-\cdot F\cdot q} \prod_{i=1}^N V[f_i]\Big\rangle_{\mathrm{DBC}}$$

with vertices $V[f_i] = \int_0^T d\tau_i\, \frac{1}{2}\Big(q_i\cdot f_i\cdot \dot{q}_i - \frac{2}{T}x_-\cdot f_i\cdot q_i\Big)$

Scalar propagator

Photon vertices

- Let's confirm using the path integral approach (expand about straight line, and integrate out fluctuations)

$$\mathcal{D}_N^{x'x}[A] = -\, i(-ie)^N \int_0^\infty \frac{dT}{(4\pi T)^2}\, \mathrm{e}^{-im^2T}\, \mathrm{e}^{-i\frac{x_-^2}{4T}} \times K[F] \Big\langle \mathrm{e}^{\frac{ie}{T}\int_0^T d\tau\, x_-\cdot F\cdot q} \prod_{i=1}^N V[f_i] \Big\rangle_{\mathrm{DBC}}$$

with vertices $V[f_i] = \int_0^T d\tau_i\, \frac{1}{2}\Big(q_i\cdot f_i\cdot \dot{q}_i - \frac{2}{T}x_-\cdot f_i\cdot q_i\Big)$

- First order term of expectation value is found as

$$\Big\{ \int_0^T d\tau_1 \Big[i\,\mathrm{tr}(f_1\cdot {}^{\bullet}\underset{\smile}{\Delta}_{11}) - \frac{2e}{T^2}\, x_-\cdot f_1 \underset{\smile}{\Delta}{}^{\circ}_1 F\cdot x_- - \frac{2e^2}{T^2}\, x_-\cdot F{}^{\circ}\underset{\smile}{\Delta}_1 f_1 {}^{\bullet}\underset{\smile}{\Delta}{}^{\circ}_1 F\cdot x_- \Big] \Big\} \mathrm{e}^{i(\frac{e}{T})^2 x_-\cdot F^{2\,\circ}\underset{\smile}{\Delta}{}^{\circ}\cdot x_-}$$

in agreement with previous expression

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- First order term is simply

$$K_1^{x'x}[F] = K^{x'x}[F] \Big(\frac{e}{2} T \mathrm{tr}[(Z^{-1} - \coth(Z)) f_1]$$
$$- \frac{i}{4T} f_1 \cdot \partial_F x_- \cdot Z \coth(Z) \cdot x_- \Big)$$

Outline

Spinor effective action

- *Intuitively, a similar construction exists for spinors.*
- Effective action is very simple due to the **replacement rule**!

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$$\Gamma_N^{\text{spin}}[A] = \frac{i}{2}\int_0^\infty \frac{dT}{T^3(4\pi)^2} e^{-im^2 T} \mathrm{det}^{1/2}[(Z+z)\coth(Z+z)]\Big|_{\text{lin} N}$$

- Breakup the determinant into the **spin factor** and **scalar** contributions so that

$$\frac{\mathrm{Det}^{\frac{1}{2}}[\hat{\partial}_\tau - 2e(F+f)]}{\mathrm{Det}^{\frac{1}{2}}[\hat{\partial}_\tau^2 - 2e(F+f)\hat{\partial}_\tau]}$$

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- Can immediately see the replacement rule

$$\det{}^{1/2}[(Z+z)\coth(Z+z)] = \det{}^{1/2}[(Z)\coth(Z)]\mathrm{e}^{\sum_{n=1}^\infty \frac{(-ie)^n}{2n}\Delta_n - \Delta_{Fn}}$$

where analogously for the antiperiodic Green function, $\mathcal{G}_F$,

$$\Delta_{Fn} = i^n \prod_{i=1}^n \int_0^T d\tau_i\, \mathrm{tr}(\mathcal{G}_{F12} f \mathcal{G}_{F23} f \ldots \mathcal{G}_{Fn1} f)$$

- Similar applications to the matrix expansion apply!

Spinor propagator in coordinate space

- Treat the spacetime kernel: $\mathcal{S}_N^{x'x} = (i\not{D}_F + m)\mathcal{K}_N^{x'x} - e\not{a}\mathcal{K}_{N-1}^{x'x}$

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$$\mathrm{Symb}^{-1}\left\{e^{-\eta\cdot\tanh(Z+z)\cdot\eta}\right\} = \mathrm{Symb}^{-1}\left\{e^{-\eta\cdot[Z+z-\frac{1}{T^2}(Z+z)^2\circ\mathcal{G}_F^{\circ}]\cdot\eta}\right\}$$

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- Therefore, need only additionally expand

$$\mathcal{G}_F(\tau,\tau'|F+f) = \mathcal{G}_F(\tau,\tau') + \sum_{n=1}^{\infty}(-ie)^n\mathcal{G}_F^{(n)}(\tau,\tau')$$

$$\mathcal{G}^{(n)}_{F\tau\tau'} = i^n\prod_{i=1}^{n}\int_0^T d\tau_i\,\mathcal{G}_{F\tau 1}f\mathcal{G}_{F12}f\dots\mathcal{G}_{Fn\tau'}$$

- Same rules apply to find $\mathcal{K}_N^{x'x}$ and its analogous matrix expansion.

Outline

- We can perform an analogous expansion for the (e.g. scalar here) propagator, but in **momentum space**.
- Define momentum space propagator by Fourier transform:

$$\mathcal{D}^{p'p}[F+f] = \int d^4x \int d^4x'\, \mathrm{e}^{i(p'\cdot x' - p\cdot x)} \int_0^\infty dT\, \mathrm{e}^{-m^2 T} \mathrm{e}^{-i\frac{x_-^2}{4T}}$$
$$\int_{\mathrm{DBC}} \mathcal{D}q\, \mathrm{e}^{-i\int_0^T d\tau\, (\frac{\dot{q}^2}{4} + \frac{e}{2} q\cdot(F+f)\cdot\dot{q} - \frac{e}{T} x_-\cdot(F+f)\cdot q)}$$
$$=(2\pi)^4\delta^4(p+p') \int_0^\infty dT\, (4\pi T)^{-2} \mathrm{e}^{-i(m^2 - p'^2)T}$$
$$\int_{\mathrm{DBC}} \mathcal{D}q\, \mathrm{e}^{-i\int_0^T d\tau\, [\frac{\dot{q}^2}{4} + \frac{e}{2} q\cdot(F+f)\cdot\dot{q} + 2ie\, p'\cdot(F+f)\cdot q + \frac{e^2}{T}\int_0^T d\tau'\, q(\tau)\cdot(F+f)^2\cdot q(\tau')]}$$

- New *non-local* operator for momentum space propagator, but still with DBCs in coordinate space.

- We require the non-local Green function, w/ $\langle\tau|\hat{\mathrm{F}}^2|\tau'\rangle := F^2$,

$$\Xi(\tau,\tau') = \langle\tau|\frac{1}{\hat{\partial}_\tau^2 - 2eF\hat{\partial}_\tau - \frac{4e^2}{T}\hat{\mathrm{F}}^2}|\tau'\rangle$$

that satisfies

$$\left(\partial_\tau^2 - 2eF\partial_\tau\right)\Xi(\tau,\tau') - \frac{4e^2}{T}F^2\int d\tilde{\tau}\,\Xi(\tilde{\tau},\tau') = \delta(\tau-\tau')$$

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- Solution is provided in terms of DBC Green function

$$\Xi(\tau, \tau') = \Delta(\tau, \tau') - \frac{4}{T^3} \underset{\smile}{\Delta}^{\circ}(\tau) \cdot Z \cdot \tanh(Z) \cdot {}^{\circ}\underset{\smile}{\Delta}(\tau')$$

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- Must expand (e.g. for determinant)

$$\frac{\mathrm{Det}^{-\frac{1}{2}}\left[\hat{\partial}_\tau^2 - 2e(F+f)\hat{\partial}_\tau - \frac{4e^2}{T}(\hat{\mathrm{F}}+\hat{\mathrm{f}})^2\right]}{\mathrm{Det}^{-\frac{1}{2}}\left[\hat{\partial}_\tau^2 - 2eF\hat{\partial}_\tau - \frac{4e^2}{T}\hat{\mathrm{F}}^2\right]}$$

$$= \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\ln\left[1 - 2e\left(\hat{\partial}_\tau^2 - 2eF\hat{\partial}_\tau - \frac{4e^2}{T}\hat{\mathrm{F}}^2\right)^{-1}\cdot\left(F\hat{\partial}_\tau + \frac{2e}{T}(\hat{\mathrm{F}}\cdot\hat{\mathrm{f}} + \hat{\mathrm{f}}\cdot\hat{\mathrm{F}} + \hat{\mathrm{f}}^2)\right)\right]}$$

Conclusions

Low-energy expansion

1. Treat either the loop or the line in Fock-Schwinger gauge.
2. All external photons captured with **constant field**! Tremendous simplification!
3. For constant background field:
   - Functional expansion about **worldline Green functions**
   - Evaluate parameter integrals / matrix expansion
4. Scalars or spinors
5. Novel momentum space Green function expansion

Thank you for your time and attention!