Tadpole contribution to magnetic photon-graviton conversion

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First Quantisation for Physics in Strong Fields
Higgs Center (Edinburgh), Feb 24-27 (2025)

INTRODUCTION

In the presence of external electromagnetic fileds,

- Many quantum processes exist which are forbidden in vacuum such as: photon-photon scattering, pair prodcution, etc.
- In particular, transitions between bosons of different spins becomes possible.
- One of such process is the well-studied axion-photon mixing in a magnetic field (Raffelt and Stodolsky (1988), Sikivie (1983)).
- Similarly, photon-graviton mixing is possible in an external field (Gertsenshtein (1962), Raffelt and Stodolsky (1988)).

Einstein-Maxwell theory contains a tree level vertex in the coppling $h_{\mu\nu}T^{\mu\nu}$:

$$h_{\mu\nu} : \text{graviton field} \quad , \quad T^{\mu\nu} = F^{\mu\alpha}F^{\nu}_{\alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu}$$

Taking $F^{\mu\nu} = F^{\mu\nu}_{\text{ext}} + f^{\mu\nu}$ (external field + the photon field) the photon-graviton conversion in a constant EM field

$$\frac{\kappa}{2}h_{\mu\nu}\left(F^{\mu\alpha}_{\text{ext}}f^{\nu}_{\alpha} + f^{\mu}_{\alpha}F^{\nu\alpha}_{\text{ext}}\right) - \frac{\kappa}{4}h^{\mu}_{\mu}F^{\alpha\beta}_{\text{ext}}f_{\alpha\beta}$$

Due to smallness of the coupling $\kappa = \sqrt{16\pi G_N}$ this process has attracted less attention than the axion-photon coupling.
Nevertheless, its relevance for astrophysics has been studied:

- Photon-graviton conversion near a plusar was studied $\rightarrow$ very small transition rate (Raffelt and Stodolsky (1988)).

- It has been suggested that the same conversion due to a primordial magnetic field could be responsible for the observed anisotropy of the cosmic microwave background (Magueijo (1994), Chen (1995)).

- Renewed interest by the models with large extra dimensions, additional massive Klauza-Klein gravitons which might enhance this conversion (Arkani-Hamed et al (1998-9)).

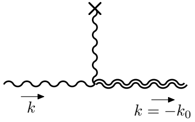
- Possible laboratory experiments are discussed in: Long et al (1994), Deffayet et al (2000)).

In momentum space, this vertex becomes

$$\Gamma^{(\mathrm{tree})}(k,\varepsilon;F)=\epsilon_{\mu\nu}\,\varepsilon_{\alpha}\,\Pi^{\mu\nu,\alpha}_{(\mathrm{tree})}(k;F)\quad,\quad \Pi^{\mu\nu,\alpha}_{(\mathrm{tree})}(k;F)=-\frac{i\kappa}{2}C^{\mu\nu,\alpha} \tag{1}$$

with

$$C^{\mu\nu,\alpha}=F^{\mu\alpha}k^{\nu}+F^{\nu\alpha}k^{\mu}-(F\cdot k)^{\mu}\delta^{\nu\alpha}-(F\cdot k)^{\nu}\delta^{\mu\alpha}+(F\cdot k)^{\alpha}\delta^{\mu\nu} \tag{2}$$



In the presence of magnetic field graviton can couple to photon, in a similar way as axion with two crutial differences:

- Gravitons are massless
- While axions couple only to one polarization, gravitons couple to both polarizations with equal strength.

EXPERIMENTAL ESTIMATES

Long et al (1994) considered the photon-graviton conversion in a homogeneous electric field of a flat condensor (of volume V and in x direction) and obtained:

$$\frac{d\sigma^E}{d\Omega}|_{\theta\sim 0} = \frac{\kappa^2 E^2}{16(2\pi)^2}\omega^2 V^2 \tag{3}$$

and in a static magnetic field of a selenoid (with radius R and length L and in z direction):

$$\frac{d\sigma^B}{d\Omega}|_{\theta=\frac{\pi}{2}} = \frac{\kappa^2 R^2 B^2}{16\omega^2}\sin^2\left(\frac{\omega L}{2}\right) J_1^2(R\omega) \quad , \quad J_1 \rightarrow \text{Spherical Bessel function} \tag{4}$$

For the latter case

$$\frac{d\sigma^B}{d\Omega} \approx 1.3 \times 10^{-15} cm^2 \tag{5}$$

for $V = 10^6 cm^3$, $B = 10^6 G = 10T$, $\omega = 12.4 eV$.

SELECTION RULES

Physical polarizations

- Photon: $\varepsilon_{\perp}^{\alpha}$, $\varepsilon_{\parallel}^{\alpha}$
- Graviton: $\epsilon^{\oplus\mu\nu} = \varepsilon^{\perp\mu}\varepsilon^{\perp\nu} - \varepsilon^{\parallel\mu}\varepsilon^{\parallel\nu}$, $\epsilon^{\otimes\mu\nu} = \varepsilon^{\perp\mu}\varepsilon^{\parallel\nu} + \varepsilon^{\parallel\mu}\varepsilon^{\perp\nu}$

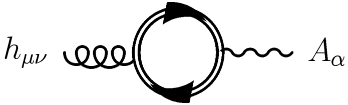
CP invariance:

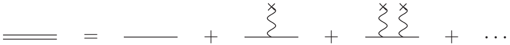
- C: $\mathbf{E} \to -\mathbf{E}$, $\mathbf{B} \to -\mathbf{B}$
- P: $\mathbf{E} \to -\mathbf{E}$, $\mathbf{B} \to +\mathbf{B}$
- CP invariance: $\mathbf{E} \to +\mathbf{E}$, $\mathbf{B} \to -\mathbf{B}$

Therefore

- Photon polarizations: $\varepsilon^{\perp}$ is CP-odd , $\varepsilon^{\parallel}$ is CP-even
- Graviton polarizations: $\epsilon^{\oplus}$ is CP-even , $\epsilon^{\otimes}$ is CP-odd
- For a purely magnetic field $\epsilon^{\oplus}$ couples only to $\varepsilon^{\perp}$ and $\epsilon^{\otimes}$ only to $\varepsilon^{\parallel}$
- For a purely electric field $\epsilon^{\oplus}$ couples only to $\varepsilon^{\parallel}$ and $\epsilon^{\otimes}$ only to $\varepsilon^{\perp}$

ONE-LOOP





The photon-graviton conversion at one loop was done using worldline method

- Bastianelli and Schubert (2005)
  For arbitrary constant fields and for charged particles with spin $0, 1/2$.
  $\Rightarrow$ They obtained a compact two parameter integral representation

- Bastianelli, Nucamendi and Schubert (2007)
  $\Rightarrow$ studied the structure and magnetiude for various polarization components.

MIXED EM-GRAVITY AMPLITUDES IN THE WL

The effective action can be represented by (ξ parametrizes an additional coupling to the scalar curvature R.)

$$\Gamma[g,A] = -\int_0^\infty \frac{dT}{T} \int_{\mathrm{PBC}} \mathcal{D}x \, e^{-S[x;g,A]}$$

$$S[x;g;A] = \int_0^T d\tau \left(\frac{1}{4}\dot{x}^\mu \dot{x}^\nu + ieA_\mu(x)\dot{x}^\mu + \xi R(x) + m^2 \right) \qquad (6)$$

External fields are specialized to plane waves:

$$V^A_{\mathrm{scal}}[k,\varepsilon] = (-ie)\varepsilon_\mu \int_0^T d\tau \dot{x}^\mu \, e^{ik\cdot x(\tau)} \ , \ \ x^\mu(\tau) = x_0^\mu + q^\mu(\tau)$$

$$x_0^\mu \equiv \frac{1}{T}\int_0^T d\tau x^\mu(\tau) \ , \ \ \int \mathcal{D}x(\tau) = \int d^D x_0 \int Dq(\tau)$$

$$\int d^D x_0 \rightarrow (2\pi)^D \delta^D(\cdots) \ , \ \ \int Dq(\tau) \rightarrow \text{Gaussian} \rightarrow \text{Wick contractions}$$

$$\langle q^\mu(\tau_i) q^\nu(\tau_j)\rangle = -\delta^{\mu\nu} G_B(\tau_i,\tau_j) \quad , \quad G_{Bij} = |\tau_i-\tau_j| - \frac{(\tau_i-\tau_j)^2}{T}$$
$$\dot G_{Bij} = \mathrm{sign}(\tau_i-\tau_j) - 2\frac{\tau_i-\tau_j}{T} \quad , \quad \ddot G_{Bij} = 2\delta(\tau_i-\tau_j) - \frac{2}{T}$$

The inclusion of a constant EM field $F_{\mu\nu}$ we use Fock-Schwinger gauge centered at x_0: $A_\mu(x) = \frac{1}{2} F_{\nu\mu} q^\nu$. Therefore an additional term in the worldline Lagrangian:

$$\Delta L = \frac{1}{2} i e q^\mu F_{\mu\nu} \dot q^\nu$$

Only quadratically in q^μ it can be taken into account $\to$ correlator:

$$\langle q_i^\mu q_j^\nu\rangle = -\mathcal{G}_{Bij}^{\mu\nu}$$

$$\mathcal{G}_{Bij} = \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}} e^{-i\mathcal{Z}\dot G_{Bij}} + i\mathcal{Z}\dot G_{Bij} - 1\right) = \mathcal{G}_{Bji}^T \quad , \quad \mathcal{Z}_{\mu\nu} = eTF_{\mu\nu}$$
$$\dot{\mathcal{G}}_{Bij} = \frac{i}{\mathcal{Z}}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}} e^{-i\mathcal{Z}\dot G_{Bij}} - 1\right) = -\dot{\mathcal{G}}_{Bji}^T \quad , \quad \ddot{\mathcal{G}}_{Bij} = 2\delta_{ij} - \frac{2}{T}\frac{\mathcal{Z}}{\sin\mathcal{Z}} e^{-i\mathcal{Z}\dot G_{Bij}}$$

The path integral determinant becomes

$$\int Dq(\tau)\exp\left[-\int^T d\tau\left(\frac{1}{4}\dot x^2 + \frac{1}{2} i e\, x\cdot F\cdot \dot x\right)\right] = (4\pi T)^{-\frac{D}{2}} \det{}^{-\frac{1}{2}}\left[\frac{\sin\mathcal{Z}}{\mathcal{Z}}\right]$$

- In flat space, the Gaussian path integration provides well-defined parameter integral representations.
- In the presence of gravity, more precise worldline regularizations are needed, and it's useful to exponentiate the path integral measure using ghost fields.

The covariant measure is of the form:

$$\mathcal{D}x = Dx \prod_{0\le\tau<T} \sqrt{\det g_{\mu\nu}(x(\tau))} \quad , \quad Dx = \prod_{\tau} d^D x(\tau) \text{ : standard measure}$$

It can be represented more conveniently by introducing commuting a^μ and anticommuting b^μ, c^μ ghosts

$$\mathcal{D}x = Dx \prod_{0\le\tau<T} \sqrt{\det g_{\mu\nu}(x(\tau))} = Dx \int_{\text{PBC}} DaDbDc\, e^{-S_{\text{gh}}[x,a,b,c]}$$

$$S_{\text{gh}}[x,a,b,c] = \int_0^T d\tau \frac{1}{4} g_{\mu\nu}(x)\,(a^\mu a^\nu + b^\mu c^\nu) \quad , \quad g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa\epsilon_{\mu\nu} e^{ik\cdot x(\tau)}$$

Vertex operator for the graviton coupled to the scalar loop ($\bar{\xi} = \xi - 1/4$)

$$V^h_{\text{scal}} = (-\frac{\kappa}{4})\epsilon_{\mu\nu} \int_0^t d\tau \Big(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu + 4\bar{\xi}(k^2\delta^{\mu\nu} - k^\mu k^\nu)\Big) e^{ik\cdot x} \qquad (7)$$

Wick contraction rules for the ghosts:

$$\langle a^\mu(\tau_1)a^\nu(\tau_2)\rangle = 2\delta(\tau_1-\tau_2)\delta^{\mu\nu} \quad , \quad \langle b^\mu(\tau_1)c^\nu(\tau_2)\rangle = -4\delta(\tau_1-\tau_2)\delta^{\mu\nu}$$

- In perturbative expansion around flat space various worldline Feynman diagrams are linearly and logaritmically UV divergnet
- Ghost correlators eliminate these divergences
- Finite ambiguities are left to be dealt with by regularization and renormelaization conditions
- These renormalization conditions produce a finite counterterm of the form which must be added to the action:

$$\Delta_{\mathrm{CT}} = \int_0^T d\tau 2V_{\mathrm{CT}}$$

Three regularization schemes (Bastianelli et al 1998,2000):

$$\begin{aligned} V_{\mathrm{MR}} &= -\frac{1}{8}R + \frac{1}{8}g^{\mu\nu}\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} \quad , \quad V_{\mathrm{DR}} = -\frac{1}{8}R \\ V_{\mathrm{TS}} &= -\frac{1}{8}R - \frac{1}{24}g^{\mu\nu}g^{\alpha\beta}g_{\lambda\rho}\Gamma^{\lambda}_{\mu\alpha}\Gamma^{\rho}_{\nu\beta} \end{aligned} \tag{8}$$

Linear order in graviton, dominant part is $-\frac{1}{8}R$ then all schemes are equivalent!

In the case of the spin $1/2$ particle the coupling to R is fixed by the Dirac equation and corresponds to $\bar{\xi} = 0$ (Bastianelli and Zirotti 2002)

Photon-graviton amplitude in a constant field/scalar loop:

$$
\begin{aligned}
\langle h(k_1)A(k_2)\rangle \quad &= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{PBC} Dx Da Db Dc\, V^h_{\text{scal}}[k_1,\epsilon^h] V^A_{\text{scal}}[k_2,\varepsilon^A] \\
&\quad \times \exp\left[-\int_0^T d\tau \left(\frac{1}{4}(\dot{x}^2 + a^2 + b\cdot c) + \frac{1}{2} i e x^\mu F_{\mu\nu}\dot{x}^\nu\right)\right] \qquad (9)
\end{aligned}
$$

After splitting $x(\tau) = x_0 + q(\tau)$, the path integral $\int Dq(\tau)$ is Gaussian and thus can be reduced to Wick contraction:

$$
\begin{aligned}
\langle h(k_1)A(k_2)\rangle \quad &= (2\pi)^D \delta(k_1+k_2) \int_0^\infty \frac{dT}{T} e^{-m^2 T} (4\pi T)^{-\frac{D}{2}} \det{}^{-\frac{1}{2}}\left[\frac{\sin \mathcal{Z}}{\mathcal{Z}}\right] \\
&\quad \times \left\langle V^h_{\text{scal}}[k_1,\epsilon^h] V^A_{\text{scal}}[k_2,\varepsilon^A]\right\rangle = (2\pi)^D \delta(k_1+k_2)\epsilon^h_{\mu\nu}\varepsilon^A_\alpha \Pi^{\mu\nu,\alpha}_{\text{scal}}(k)
\end{aligned}
$$

where

$k_1 \equiv k$ and we define

$$\Pi^{\mu\nu,\alpha}_{\rm scal}(k) = \frac{e\kappa}{4(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T} e^{-m^2T} T^{-\frac{D}{2}} \det{}^{-\frac{1}{2}}\left[\frac{\sin\mathcal{Z}}{\mathcal{Z}}\right] \times \int_0^T d\tau_1 \int_0^T d\tau_2 e^{-k\cdot\bar{\mathcal{G}}_{B12}\cdot k}\, I^{\mu\nu,\alpha}_{\rm scal} \tag{10}$$

$$\begin{aligned} I^{\mu\nu,\alpha}_{\rm scal} = & -\left(\ddot{\bar{\mathcal{G}}}{}^{\mu\nu}_{B11} - 2\delta_{11}\delta^{\mu\nu}\right)\left(k\cdot\dot{\bar{\mathcal{G}}}_{B12}\right)^{\alpha} - \left[\ddot{\bar{\mathcal{G}}}{}^{\mu\alpha}_{B12}\left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^{\nu} + (\mu\leftrightarrow\nu)\right] \\ & +\left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^{\mu}\left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^{\nu}\left(k\cdot\dot{\bar{\mathcal{G}}}_{B12}\right)^{\alpha} - 4\bar{\xi}(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\left(k\cdot\dot{\bar{\mathcal{G}}}_{B12}\right)^{\alpha} \end{aligned}$$

Some remarks:

- Same type of two parameter integral that appear in one-loop photon vacuum polarization in a constant field (Adler 1971; Ritus 1975; etc)
- Low-field limit $\Rightarrow$ nonzero amplitude for odd number of interactions
- Singular at $T = 0 \Rightarrow$ UV divergences $\Rightarrow$ in $D = 4$ some terms logarithmic divergence
- Those divergent terms involve the field only linearly, and thus easy to compute by expanding to the linear order in F.

In DR, the result of this divergnet term is written as

$$\Pi^{\mu\nu,\alpha}_{\mathrm{scal,div}}(k) = \frac{ie^2\kappa}{3(4\pi)^2}\frac{1}{D-4}\,C^{\mu\nu,\alpha} \tag{11}$$

where $C^{\mu\nu,\alpha}$ is the tree-level vertex:

$$C^{\mu\nu,\alpha} = F^{\mu\alpha}k^{\nu} + F^{\nu\alpha}k^{\mu} - (F\cdot k)^{\mu}\delta^{\nu\alpha} - (F\cdot k)^{\nu}\delta^{\mu\alpha} + (F\cdot k)^{\alpha}\delta^{\mu\nu} \tag{12}$$

which as expected is the momentum version of the tree-level interaction.

- Renormalization: by subtracting the amplitude at zero field and zero momentum limit $\Rightarrow$ Done under the T-integral leading to $\bar{\Pi}$:

$$\begin{aligned}\bar{\Pi}^{\mu\nu,\alpha}_{\mathrm{scal}}(k) \quad &= \quad \frac{e\kappa}{64\pi^2}\int_0^\infty \frac{dT}{T^3}e^{-m^2T}\Big\{\det{}^{-\frac{1}{2}}\left[\frac{\sin\mathcal{Z}}{\mathcal{Z}}\right] \\ &\times \int_0^T d\tau_1\int_0^T d\tau_2\, e^{-k\cdot\bar{\mathcal{G}}_{B12}\cdot k}\,I^{\mu\nu,\alpha}_{\mathrm{scal}} + \frac{2}{3}ieT^2C^{\mu\nu,\alpha}\Big\}\end{aligned} \tag{13}$$

Similar expression can be obtained for the spinor loop case!

In order to compare the one-loop and the tree-level contributions, we normalize the former by the latter:

$$\hat{\Pi}^{hA}_{\text{scal,spin}}(\hat{\omega}, \hat{B}, \hat{E}) \equiv \text{Re}\left(\frac{\bar{\Pi}^{hA}_{\text{scal,spin}}(\hat{\omega}, \hat{B}, \hat{E})}{-\frac{i}{2}\kappa C^{hA}}\right) \quad , \quad \hat{\omega} = \frac{\omega}{m}, \hat{B} = \frac{B}{B_{\text{cr}}}, \hat{E} = \frac{E}{E_{\text{cr}}}$$

$$(h = \oplus, \otimes) \; , \; (A = \perp, \parallel) \; , \; C^{hA} = \epsilon^{h}_{\mu\nu} C^{\mu\nu,\alpha} \varepsilon^{A}_{\alpha} \tag{14}$$

Special cases:

- Strightforward numerical evaluation for $\hat{\omega} < \hat{\omega}_{\text{cr}} = 2$ (after rotaiting to Euclidean time: $T = is$)
- Small B, arbitrary ω ($\rightarrow$ integrals over Airy functions)
- Large B limit:

$$\hat{\Pi}^{hA}_{\text{scal}}(\hat{\omega}, \hat{B}) \sim -\frac{\alpha}{12\pi} \ln \hat{B} \quad , \quad \hat{\Pi}^{hA}_{\text{spin}}(\hat{\omega}, \hat{B}) \sim -\frac{\alpha}{3\pi} \ln \hat{B}$$

- $\omega = 0$, arbitrary B limit:

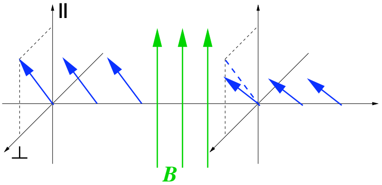
$$\hat{\Pi}^{\oplus\perp}_{\text{scal,spin}}(\hat{\omega} = 0, \hat{B}) = -\frac{2\pi\alpha}{m^4}\left(\frac{1}{\hat{B}}\frac{\partial}{\partial \hat{B}} + \frac{\partial^2}{\partial \hat{B}^2}\right)\mathcal{L}^{\text{EH}}_{\text{scal,spin}}(\hat{B})$$

$$\hat{\Pi}^{\otimes\parallel}_{\text{scal,spin}}(\hat{\omega} = 0, \hat{B}) = -\frac{4\pi\alpha}{m^4}\frac{1}{\hat{B}}\frac{\partial}{\partial \hat{B}}\mathcal{L}^{\text{EH}}_{\text{scal,spin}}(\hat{B})$$

where $\mathcal{L}^{\text{EH}}_{\text{scal,spin}}$ are the renormalized Euler-Heisenberg lagrangian.

DICHROISM STARTS AT ONE-LOOP

- For realistic parameters, one-loop corrections are small compared to tree-level amplitudes.
- There is a qualitative difference between tree-level and one-loop effects.
- Tree-level photon-graviton conversion does **not** lead to **dichroism** (rotation of the polarization vector of an EM wave), unlike the photon-axion case. This is because both polarization components have equal conversion rates at the tree level.
- **At the one-loop level, absorption coefficients become polarization-dependent:** $\gamma_{||} \neq \gamma_{\perp}$

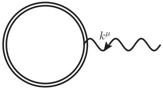


This is the leading contribution to magnetic dichroism in the standard model including Einstein-Maxwell theory! (M. Ahles et al 2009)

1PR CONTRIBUTIONS IN QED

In vacuum QED, there is, of course, no one-photon amplitude because of Furry's theorem.

In the presence of external fields, one-photon tadpole diagrams in general will be non-zero and can be important:

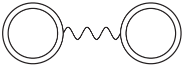


In both scalar and spinor QED, the one-loop one-photon amplitude in a constant field vanishes, since

- A constant field emits only photons with zero energy-momentum, thus there is a factor of $\delta(k)$.
- Because of gauge invaraince, this diagram in a momentum expansion starts with term linear in momentum.
- $\delta(k)k^\mu = 0$

If the tadpole vanishes, all diagrams containing it must vanish too!!!

HANDCUFF DIAGRAM AND THE TWO-LOOP EHL



- Dittrich and Reuter 1984:
  They argue that the handcuff diagram is zero because of Lorentz invariance.
- Fradkin, Gitman and Shvartsman 2011:
  They argue that in the constant and homogeneous external field combined with a plane wave, all diagrams containing tadpole, are equal to zero.

Tadpole contribution to the EHL:
Gies and Karbstein, JHEP 2017, looked at the problem again: such diagrams can give finite values beacuse of the infrared divergence of the connecting photon propagator.

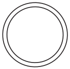
$$\int d^D k \, \delta^D(k) \frac{k^\mu k^\nu}{k^2} = \frac{\eta^{\mu\nu}}{D} \quad \Rightarrow \quad \mathcal{L}_{\mathrm{EH}}^{(2-\mathrm{loop})1\mathrm{PR}} = \frac{1}{2} \frac{\partial \mathcal{L}_{\mathrm{EH}}^{(1)}}{\partial F^{\mu\nu}} \frac{\partial \mathcal{L}_{\mathrm{EH}}^{(1)}}{\partial F_{\mu\nu}} \tag{15}$$

$$\mathcal{L}_{\text{EH}}^{(1)} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}e^{-m^2T}\left\{\frac{(eaT)(ebT)}{\tan(eaT)\tanh(\text{ebT})} - \frac{2}{3}(eT)^2\mathcal{F} - 1\right\} \quad (16)$$

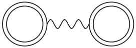
$$a = \left(\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F}\right)^{\frac{1}{2}} \quad , \quad b = \left(\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}\right)^{\frac{1}{2}}$$

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbf{B}^2-\mathbf{E}^2) \quad , \quad \mathcal{G} = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = -\mathbf{E}\cdot\mathbf{B} \quad (17)$$

where $\mathcal{L}_{\text{EH}}^{(1)}$ is the one-loop EH Lagrangian given by



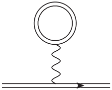
This fact, adds on the standard diagram for the two-loop EHL (studied by Ritus about 50 years ago!):

$\mathcal{L}_{\text{HE}}^{\text{2-loop}} =$  $+$ 

Karbstein in 2020

studied the strong magnetic field limit of the all-loop EHL $\Rightarrow$ beyond one-loop this limit is fully determined by 1PR contributions.

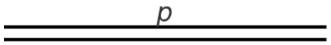
ONE-LOOP TADPOLE CONTRIBUTION TO THE PROPAGATOR



- For the electron propagator, a tadpole correction exists already at the the one-loop level.
- In scalar QED (Edwards and Schubert, 2017)
- In spinor QED (N.A, Bastianelli, Corradini, Edwards and Schubert, 2017)
- Constant crossed fields, constant magnetic fields, and plane waves (N. A, Edwards and Ilderton, 2019).
- Generalized the above Gies-Karbstein equation

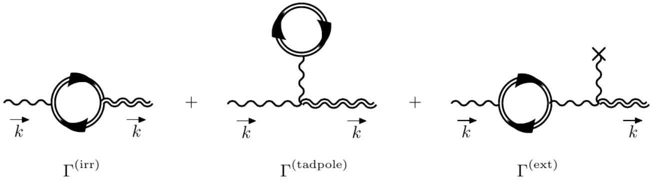
$$S^{(1)\mathrm{PR}}(p) = \frac{\partial S(p)}{\partial F_{\mu\nu}} \frac{\partial \mathcal{L}^{(1)}_{\mathrm{EH}}}{\partial F^{\mu\nu}} \tag{18}$$

where $S(p)$ is the dressed propagator:



BACK TO PHOTON-GRAVITON CONVERSION IN A CONSTANT FIELD

Photon-graviton diagram also has a previously overlooked tadpole contribution



Using the momentum integral

$$\int d^D k \, \delta^D(k) \frac{k^\mu k^\nu}{k^2} = \frac{\eta^{\mu\nu}}{D} \tag{19}$$

on the connecting propagator, we get

Scalar loop

$$\Gamma^{\text{tadpole}}_{\text{scal}}(k^\alpha, \varepsilon_\beta; \epsilon_{\mu\nu}; F_{\kappa\lambda}) = -i\frac{\alpha\kappa}{8\pi}\Big(\varepsilon\cdot F\cdot\epsilon\cdot k + \varepsilon\cdot\epsilon\cdot F\cdot k\Big) \times \int_0^\infty \frac{dz}{z}\, e^{-\frac{m^2}{eB}z}\, \frac{\coth(z) - (1/z)}{\sinh(z)} \tag{20}$$

Spinor loop

$$\Gamma^{\text{tadpole}}_{\text{spin}}(k^\alpha, \varepsilon_\beta; \epsilon_{\mu\nu}; F_{\kappa\lambda}) = i\frac{\alpha\kappa}{4\pi}\Big(\varepsilon\cdot F\cdot\epsilon\cdot k + \varepsilon\cdot\epsilon\cdot F\cdot k\Big) \times \int_0^\infty \frac{dz}{z}\, e^{-\frac{m^2}{eB}z}\, \frac{\coth(z) - \tanh(z) - (1/z)}{\tanh(z)} \tag{21}$$

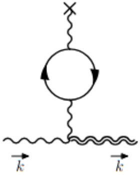
RENORMALIZATION

For $D = 4$ these contributions contain UV divergences, stemming from the terms linear in the field. Expanding out the tadpole in powers of the external field we see that the leading term, which is linear in the field is removed by photon wave function renormalization

$$\Gamma^{\text{tadpole}}_{\text{scal}}(k^{\alpha}, \varepsilon_{\beta}; \epsilon_{\mu\nu}; F_{\kappa\lambda}) = -i\frac{\alpha\kappa}{8\pi}\Big(\varepsilon\cdot F\cdot\epsilon\cdot k + \varepsilon\cdot\epsilon\cdot F\cdot k\Big) \times \int_0^{\infty}\frac{dz}{z}\, e^{-\frac{m^2}{eB}z}\,\Big(\frac{\coth(z)-(1/z)}{\sinh(z)} - \frac{1}{3}\Big) \tag{22}$$

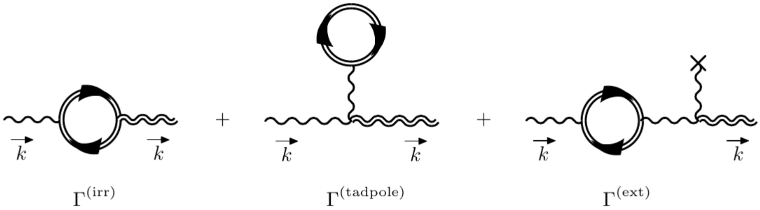
$$\Gamma^{\text{tadpole}}_{\text{spin}}(k^{\alpha}, \varepsilon_{\beta}; \epsilon_{\mu\nu}; F_{\kappa\lambda}) = i\frac{\alpha\kappa}{4\pi}\Big(\varepsilon\cdot F\cdot\epsilon\cdot k + \varepsilon\cdot\epsilon\cdot F\cdot k\Big) \times \int_0^{\infty}\frac{dz}{z}\, e^{-\frac{m^2}{eB}z}\,\Big(\frac{\coth(z)-\tanh(z)-(1/z)}{\tanh(z)} + \frac{2}{3}\Big) \tag{23}$$

However, note that, unlike the renormalization which we performed on the irrducible contribution, this one is nothing new; the contribution of the tadpole linear in the field corresponds to



which makes it clear that this subtraction is just a special case (the zero-momentum limit) of the QED photon wave function renormalization.

EXTRA DIAGAM



Like the tadpole contribution, it has an UV divergence from the field-independent part of the vacuum polarization tensor (fermion loop in vacuum)

$$\bar{\Gamma}^{\mathrm{ext}}_{\mathrm{spin}} \quad = \quad v_{F\beta}\Big(\bar{\Pi}^{\beta\alpha}(k,F) - \bar{\Pi}^{\beta\alpha}(k,0)\Big)\varepsilon_\alpha \quad , \quad v_F = -i\kappa\frac{\{\epsilon, F\}\cdot k}{k^2} \qquad (24)$$

$$\bar{\Pi}^{\beta\alpha}_{\mathrm{spin}}(k,F) = \frac{e^2}{8\pi^2}\int_0^\infty \frac{dT}{T}e^{-m^2T}\int_0^1 du\Big\{\det{}^{\frac{1}{2}}\Big[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\Big] I^{\beta\alpha}_{\mathrm{spin}}\, e^{-Tk\cdot\mathcal{G}_{B12}\cdot k} + \frac{2}{3}(\delta^{\beta\alpha}k^2 - k^\beta k^\alpha)\Big\}$$

WARD IDENTITY

Let us discuss how the Ward identities work for the photon-graviton amplitude in a constant field, returning to off-shell unrenormalized amplitudes. It is easy to show that Ward identity holds for each of the three contributios:

$$\Gamma_{\text{spin}}^{(\text{irr}),(\text{tadpole}),(\text{extr})}[-k,\epsilon;k,\delta\varepsilon;F]=0 \quad , \quad \delta\varepsilon^{\mu}=k^{\mu}(\text{from gauge invariance}) \tag{25}$$

Now the gravitational Ward identity:

$$\begin{aligned}
&\Gamma_{\text{spin}}^{(\text{irr})}[k_0,\delta\epsilon;k,\varepsilon;F]=\Gamma_{\text{spin}}[k_0+k,\tilde{\varepsilon};F]+\Gamma_{\text{spin}}[k_0,\tilde{\varepsilon}_F;k,\varepsilon;F]\\
&\delta\epsilon^{\mu\nu}=k_0^{\mu}\zeta_0+k_0^{\nu}\zeta_0^{\mu} \quad (\text{from diffeomorphism invariance})\\
&\tilde{\varepsilon}^{\mu}\equiv\kappa(f_i\cdot\zeta_0)^{\mu} \quad , \quad \tilde{\varepsilon}_F^{\mu}\equiv-\kappa(F\cdot\zeta_0)^{\mu}
\end{aligned} \tag{26}$$

RHS represents one- and two-photon amplitudes, the former vanishes after $k_0=-k$.

Now, using (N.A, Balli, Corradini, Dávila, Schubert (2019))

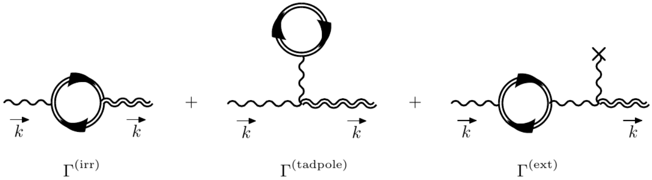
$$\delta v_F^\mu = -\tilde{\varepsilon}_F^\mu - i\kappa \frac{k_0 \cdot F \cdot \zeta_0}{k_0^2} k_0^\mu \tag{27}$$

We can combine the irreducible and the extra contributions and get the following on-shell Ward identity

$$\Gamma^{(\mathrm{irr})}_{\mathrm{spin}}[-k, \delta\epsilon; k, \varepsilon; F] + \Gamma^{(\mathrm{ext})}_{\mathrm{spin}}[-k, \delta\epsilon; k, \varepsilon; F] = 0 \tag{28}$$

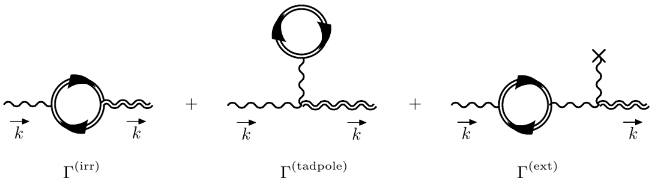
while the tadpole contribution on-shell becomes invariant by itself

$$\Gamma^{(\mathrm{tadpole})}_{\mathrm{spin}}[-k, \delta\epsilon; k, \varepsilon; F] = 0 \tag{29}$$



COMPARISON WITH THE MAIN DIAGRAM

These amplitudes are of the same structure that we obtained from the main diagram.

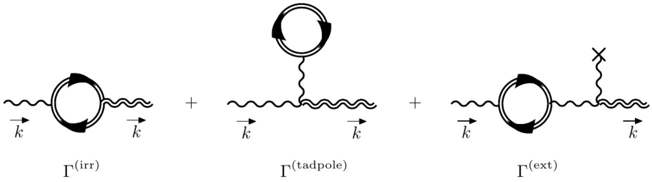


However, they do not contribute to dichroism discussed for the main diagram since the polarizations are still bound up in the tree-level vertex $(\varepsilon \cdot F \cdot \epsilon \cdot k + \varepsilon \cdot \epsilon \cdot F \cdot k)$.

This means that the primary source of dichroism continues to be the process described in the main diagram, and the additional reducible contributions do not introduce any competing effects.

Thus the analysis by Ahlers et al remains unaffected!

SUMMARY



- We have presented the first example of a Gies-Karbstein addentum in Einstein-Maxwell theory (see arXiv: 2122.01980 [hep-th]).
- To appear soon!!
- In the ultra strong-field limit, the tadpoles have been shown even to dominate the (multi-loop) effective action in QED (Karbstein 2019). It would be interesting to extend this analysis to the Einstein-Maxwell case.

QUESTIONS?

Thank you for your attention!