Flying focus, self-duality and scattering

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First Quantization for Physics in Strong Fields

27 February 2025

work with Anton Ilderton `2501.06109 & wip`

Motivation

Traditional strong-field setups tend to face similar challenges:

(beyond computational complexity)

High multiplicity/loop processes

- can dominate 'lower-order' in strong fields, resummation required [Nishikov-Ritus, Narozhny, Ritus, Morozov, Di Piazza, Mironov-Meuren-Fedotov, Torgrimsson,...]

Depletion/backreaction effects

- background field is fixed, backreaction only perturbative

Focusing/spatial inhomogeneity

- physical fields have non-trivial profiles & dynamics in transverse plane

Many approaches to dealing with these issues...

...few analytic results

A notable exception is *worldline approaches*, where all-multiplicity master formulae can be derived

[Edwards-Schubert, Schubert-Shaisultanov, Ahmadiniaz-Lopez-Lopez-Schubert, Copinger-Edwards-Ilderton-Rajeev,...]

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A notable exception is *worldline approaches*, where all-multiplicity master formulae can be derived

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Spoiler Alert: not gonna to tell you anything about worldline (but will mention potential connections later)!

Today

Try to convince you that there are certain toy scenarios where *all three* of these issues (high-multiplicity, depletion, focusing) can be addressed at once!

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Start with a chat about focused fields

Focused fields

Plane wave strong fields do not feature spatial inhomogeneity

$$\text{Recall} \quad A^{\mathrm{PW}} = x^{\perp}\, \dot{\mathsf{a}}_{\perp}(x^{-})\, \mathrm{d}x^{-}$$

Think about a laser, for example: these things are typically *focused* in some sense

$\Rightarrow$ important consequences for allowed kinematics (e.g., photon emission spectra)

Want background fields with focusing features

Flying focus

Consider wave eq in Minkowski, lightfront coordinates
$\mathrm{d}s^2 = 2(\mathrm{d}x^- \, \mathrm{d}x^+ - |\mathrm{d}z|^2)$

Flying focus solutions of massless wave equation [Bateman, Brittingham, Sezginer, Hillion]

$$\Phi(x) = \frac{f(\sigma)}{1 + \mathrm{i}\, k\, x^+}\,, \qquad \sigma := x^- - \mathrm{i}\, k \frac{|z|^2}{1 + \mathrm{i}\, k\, x^+}$$

for $k \geq 0$ and $f(\sigma)$ arbitrary.

Properties

These are *complex* solutions to the wave eq – typically, consider $\mathrm{Re}(\Phi)$ as the 'physical' field

To see what's going on, take

$$f(\sigma) = \mathrm{e}^{-\mathrm{i}\omega\sigma}\,, \qquad \textit{flying focus beam}$$

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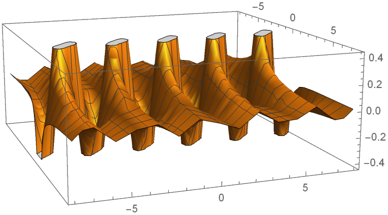
Then at $(z, \bar{z}) = (0, 0)$

$$\mathrm{Re}(\Phi)(x^+, x^-) = \frac{\cos(\omega\, x^-)}{1 + k^2\,(x^+)^2} - \frac{k\, x^+\, \sin(\omega\, x^-)}{1 + k^2\,(x^+)^2}$$



Focal width $\sim 1/\sqrt{\omega\, k}$

FF Maxwell fields

Many ways to lift Φ to a Maxwell field [Sezginer, Ramsey-et al., Formanek-et al.,...]

Anything with $F_{ab} \propto \Phi$ will have desired properties

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Anything with $F_{ab} \propto \Phi$ will have desired properties

Great, but there is bad news:

- no examples where background-coupled eoms can be solved explicitly
- typical studies are numerical or perturbative in nature

Claim: we can cook up a FF Maxwell field for which background-coupled eoms *can* be solved exactly (at least, for massless fields)

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But first, some notation...

2-spinor variables

Given any 4-vector v^μ, contract w/ Pauli matrices

$$v^{\alpha\dot{\alpha}} := v^\mu \, \sigma_\mu^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} v^0 + v^3 & v^1 - \mathrm{i}\, v^2 \\ v^1 + \mathrm{i}\, v^2 & v^0 - v^3 \end{pmatrix}$$

Manifesting local isomorphism
$\mathrm{Spin}(4, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$

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Raise and lower spinor indices as:

$$a^\alpha = \epsilon^{\alpha\beta} \, a_\beta \,, \quad a_\beta = a^\alpha \, \epsilon_{\alpha\beta} \,, \text{etc., for} \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{\alpha\beta}$$

Spinor conventions

Use lightfront variables

$$x^{\alpha\dot{\alpha}} = \begin{pmatrix} x^+ & \bar{z} \\ z & x^- \end{pmatrix}$$

Spinor dyad: $o_\alpha = (1,0) = \bar{o}_{\dot{\alpha}}$, $\iota_\alpha = (0,1) = \bar{\iota}_{\dot{\alpha}}$
Normalized: $\langle \iota\, o \rangle = 1 = [\bar{\iota}\, \bar{o}]$

So

$$\mathrm{d}s^2 = \epsilon_{\alpha\beta}\, \epsilon_{\dot{\alpha}\dot{\beta}}\, \mathrm{d}x^{\alpha\dot{\alpha}}\, \mathrm{d}x^{\beta\dot{\beta}}$$

$x^+ = x^{\alpha\dot{\alpha}}\, o_\alpha\, \bar{o}_{\dot{\alpha}}$, $z = x^{\alpha\dot{\alpha}}\, \iota_\alpha\, \bar{o}_{\dot{\alpha}}$, etc.

FF gauge field

Let $\mathcal{F}(\sigma) := \int^{\sigma} \mathrm{d}s\, f(s)$, and defined $\mathbb{C}$-gauge potential

$$A_{\alpha\dot{\alpha}} = \frac{\mathcal{F}(\sigma)}{1 + \mathrm{i}\, k\, x^{+}}\, o_{\alpha} \left(\bar{\iota}_{\dot{\alpha}} - \frac{\mathrm{i}\, k\, \bar{z}}{1 + \mathrm{i}\, k\, x^{+}}\, \bar{o}_{\dot{\alpha}} \right) = \frac{\mathcal{F}(\sigma)}{1 + \mathrm{i}\, k\, x^{+}}\, o_{\alpha}\, \tilde{s}_{\dot{\alpha}}$$

for

$$\tilde{s}_{\dot{\alpha}} := \bar{\iota}_{\dot{\alpha}} - \frac{\mathrm{i}\, k\, \bar{z}}{1 + \mathrm{i}\, k\, x^{+}}\, \bar{o}_{\dot{\alpha}}$$

Can always make a **real** gauge field:

$$A^{\mathbb{R}}_{\alpha\dot{\alpha}} = \frac{\mathcal{F}(\sigma)}{1 + \mathrm{i}\, k\, x^{+}}\, o_{\alpha}\, \tilde{s}_{\dot{\alpha}} + \frac{\overline{\mathcal{F}(\sigma)}}{1 - \mathrm{i}\, k\, x^{+}}\, s_{\alpha}\, \bar{o}_{\dot{\alpha}}$$

for

$$s_{\alpha} = \iota_{\alpha} + \frac{\mathrm{i}\, k\, z}{1 - \mathrm{i}\, k\, x^{+}}\, o_{\alpha}$$

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for

$$s_{\alpha} = \iota_{\alpha} + \frac{\mathrm{i}\, k\, z}{1 - \mathrm{i}\, k\, x^{+}}\, o_{\alpha}$$

Easy to show that $F^{\mathbb{R}}_{ab} \propto \mathrm{Re}(\Phi)$, but...

The bad news:

Eoms for charged fields are hopeless (analytically)

Strange idea

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Time for a quick digression...

Coherent states & backgrounds

Let

- $|\mathrm{in}\rangle$, $\langle\mathrm{out}|$ be collections of particles
- α, β be incoming/outgoing coherent states of photons
- $S[A]$ the QED S-matrix in a background field A

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- $S[A]$ the QED S-matrix in a background field A

Fact: [Kibble, Frantz]

$$\langle\mathrm{out};\,\beta|S|\alpha;\,\mathrm{in}\rangle = \langle\mathrm{out}|S[A]|\mathrm{in}\rangle$$

for

$$A_\mu(x) = \int \frac{\mathrm{d}^4\ell}{(2\pi)^3}\,\delta(\ell^2)\,\Theta(\ell_0)\,\left(\varepsilon^h_\mu\,\alpha_h(\ell)\,\mathrm{e}^{-\mathrm{i}\ell\cdot x} + \bar{\varepsilon}^h_\mu\,\bar{\beta}_h(\ell)\,\mathrm{e}^{\mathrm{i}\ell\cdot x}\right)$$

When $\alpha = \beta$, A_μ is real-valued and we have the standard background field S-matrix

$\alpha \neq \beta \Rightarrow$ backreaction on/depletion of initial coherent state

- backreaction/depletion modeled by *complex* backgrounds

[Zwanziger, Endlich-Penco, Ilderton-Seipt, Ekman-Ilderton, Aoude-Ochirov]

Upshot

Complex gauge fields can still encode real physics when they are coherent states

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OK, back to our complex FF gauge field

$$A_{\alpha\dot{\alpha}} = \frac{\mathcal{F}(\sigma)}{1 + \mathrm{i}\, k\, x^+}\, o_\alpha\, \tilde{s}_{\dot{\alpha}}$$

Spinorial field strengths

For any gauge potential $A_{\alpha\dot{\alpha}}$, the field strength can be decomposed as [Penrose]

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \partial_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}} - \partial_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\,\tilde{F}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}\,\hat{F}_{\alpha\beta}$$

Here $\tilde{F}_{\dot{\alpha}\dot{\beta}} = \tilde{F}_{(\dot{\alpha}\dot{\beta})}$, $\hat{F}_{\alpha\beta} = \hat{F}_{(\alpha\beta)}$

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Here $\tilde{F}_{\dot{\alpha}\dot{\beta}} = \tilde{F}_{(\dot{\alpha}\dot{\beta})}$, $\hat{F}_{\alpha\beta} = \hat{F}_{(\alpha\beta)}$

- $\tilde{F}_{\dot{\alpha}\dot{\beta}} \leftrightarrow$ *self-dual* part of F
- $\hat{F}_{\alpha\beta} \leftrightarrow$ *anti-self-dual* part of F

Self-dual FF

For the complex FF gauge field $A_{\alpha\dot{\alpha}}$, find

$$F = \mathrm{d}A = \tilde{F}_{\dot{\alpha}\dot{\beta}}\, \mathrm{d}x^{\alpha\dot{\alpha}} \wedge \mathrm{d}x_{\alpha}{}^{\dot{\beta}}\,, \qquad \tilde{F}_{\dot{\alpha}\dot{\beta}} = \frac{\Phi}{2}\, \tilde{s}_{\dot{\alpha}}\, \tilde{s}_{\dot{\beta}}$$

Self-dual FF

For the complex FF gauge field $A_{\alpha\dot{\alpha}}$, find

$$F = \mathrm{d}A = \tilde{F}_{\dot{\alpha}\dot{\beta}}\,\mathrm{d}x^{\alpha\dot{\alpha}} \wedge \mathrm{d}x_{\alpha}{}^{\dot{\beta}}\,, \qquad \tilde{F}_{\dot{\alpha}\dot{\beta}} = \frac{\Phi}{2}\,\tilde{s}_{\dot{\alpha}}\,\tilde{s}_{\dot{\beta}}$$

Upshot: this is a complex, *self-dual* flying focus Maxwell field!

(has appeared before in other guises [Hillion,...])

Who cares?

All FF gauge fields built from Φ *complex* in first instance (as Φ is complex)

Background-coupled equations typically intractable PDEs (contrast with plane waves)

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All FF gauge fields built from Φ *complex* in first instance (as Φ is complex)

Background-coupled equations typically intractable PDEs (contrast with plane waves)

Big difference in this case: *self-duality*

Self-duality as depletion

Recall that complex coherent states encode depletion
Taking $\beta = 0 \leftrightarrow$ total depletion of initial coherent state

SD background $\leftrightarrow$ total depletion of $\alpha_+ \neq 0$, $\alpha_- = 0$

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SD background $\leftrightarrow$ total depletion of $\alpha_+ \neq 0$, $\alpha_- = 0$

SD flying focus field

$$\alpha_1(\ell) = \frac{4\pi}{k}\,\mathcal{F}(\ell_-)\,\exp\left(-\frac{\ell_\perp^2}{2k\,\ell_-}\right)$$

Upshot

SD FF solution:

- complex Maxwell field with spatial inhomogenity, and
- can be viewed as model for total beam depletion

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OK, but we *really* want to solve background-coupled eoms and compute scattering amplitudes...

...*twistor theory* to the rescue!

Twistor theory

Broadly speaking: mathematical framework which translates

- physical data on spacetime into
- geometric data on complex projective variety, $\mathbb{PT} \subset \mathbb{CP}^3$

[Penrose, Ward, Atiyah-Hitchin-Singer, Sparling, Woodhouse, LeBrun,...]

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What does this have to do with SD FF?

First key fact:

Theorem (Ward 1977)

$\exists$ a 1:1-corresp between SD gauge fields on $\mathbb{M}$ and holomorphic vector bundles $E \to \mathbb{PT}$ (+ technical conditions)

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In real money: SD $A_{\alpha\dot{\alpha}}(x)$ encoded in a partial connection

$$\bar{D} = \bar{\partial} + \mathsf{a}\,, \quad \mathsf{a} \in \Omega^{0,1}(\mathbb{PT}, \mathrm{End}E)\,, \qquad \bar{\partial}\mathsf{a} + [\mathsf{a}, \mathsf{a}] = 0$$

Corollary: SD equations are classically integrable!

`[Belavin-Zakharov]`

Upshot

SD FF field has a twistor description in terms of an integrable differential operator $\bar{D} = \bar{\partial} + \mathsf{a}$

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Aside: others have used integrability of SD sector of QED/QCD

[Dunne-Schubert, TA-Mason-Sharma, TA-Ilderton-MacLeod, TA-Bogna-Mason-Sharma, Garner-Paquette, Dixon-Morales, Bittleston-Costello-Zeng,...]

Second key fact:

Theorem (Ward-Wells 1991)

$\exists$ an isomorphism between:
i.) helicity h linear fields coupled to SD background, and
ii.) $H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}(2h-2) \otimes \mathrm{End}\, E)$

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In real money: can solve massless, charged eoms with data on $\mathbb{PT}$ holomorphic wrt $\bar{D}$.

Upshot

Find suitable analytic data on $\mathbb{PT}$

$$\Leftrightarrow$$

Get *exact* solutions to background-coupled eoms in SD FF!

What do we mean by 'suitable' data?

- specified by on-shell momentum $p_{\alpha\dot{\alpha}}$, $p^2 = 0$
- reduce to SD PW Volkov solns when $k \to 0$

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Time for a bit more spinor yoga...

On-shell kinematics

Easy to see that $p^2 \propto \det(p^{\alpha\dot{\alpha}})$, so $p^2 = 0 \Leftrightarrow p^{\alpha\dot{\alpha}} = \kappa^{\alpha}\,\tilde{\kappa}^{\dot{\alpha}}$

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Helpful notation: $\epsilon_{\beta\alpha} \, \kappa_i^\alpha \, \kappa_j^\beta := \langle i \, j \rangle \,, \quad \epsilon_{\dot{\beta}\dot{\alpha}} \, \bar{\kappa}_i^{\dot{\alpha}} \, \bar{\kappa}_j^{\dot{\beta}} := [i \, j]$

e.g., Mandelstam invariants $s_{ij} = (p_i + p_j)^2 = 2 \, \langle i \, j \rangle \, [i \, j]$

Charged scalar

Consider $(\partial - \mathrm{i}\, eA)^2 \phi = 0$

Find *exact* solution

$$\phi(x) = \frac{\mathrm{e}^{-\mathrm{i}\, S_p(x)}}{1 + \mathrm{i}\, k\, x^+}\,, \qquad S_p(x) := \frac{p \cdot x - \mathrm{i}\, k\, p_-\, x^2}{1 + \mathrm{i}\, k\, x^+} - \mathrm{i}\, e\, \frac{\mathsf{p}}{p_+}\, \mathcal{F}(\sigma)$$

for $\mathsf{p} \equiv p_z$

$S_p(x)$ solves Hamilton-Jacobi equation $(\partial S_p + e\, A)^2 = 0$

Gluon wavefunctions

Embed $A_{\alpha\dot{\alpha}} \hookrightarrow$ Cartan of $\mathfrak{g}$

Can solve for background-coupled gluon wavefunctions:

$$a^{(+)}_{\alpha\dot{\alpha}}(x) = \mathsf{T}\,\frac{o_\alpha\,\tilde{K}_{\dot{\alpha}}(x)}{\langle o\,\kappa\rangle\,(1+\mathrm{i}\,k\,x^+)}\,\mathrm{e}^{-\mathrm{i}\,S_p(x)}$$

for T a generator of $\mathfrak{g}$ and

$$\tilde{K}_{\dot{\alpha}} = \tilde{\kappa}_{\dot{\alpha}} - \mathrm{i}\,e\,\frac{f(\sigma)}{\langle\kappa\,\iota\rangle}\,\bar{\iota}_{\dot{\alpha}} - \frac{\mathrm{i}\,k}{1+\mathrm{i}\,k\,x^+}\left(\langle o|x|\tilde{\kappa}] - \mathrm{i}\,e\,\frac{\bar{z}\,f(\sigma)}{\langle\kappa\,\iota\rangle}\right)\bar{o}_{\dot{\alpha}}$$

Similar story for neg. helicity gluons

Summary so far...

Can solve background-coupled, massless eoms in the SD flying focus field

These are the *only* known exact solutions in a FF background!

They have some interesting properties:

- almost WKB-exact
- both $\tilde{\kappa}_{\dot{\alpha}}$ *and* κ_{α} effectively dressed
- neg. chirality dressing controlled by k parameter of background

Great, but can we do any new computations?

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YES!

Two examples:

1. Non-linear, ultrarelativistic Compton scattering in QED
2. MHV gluon scattering in pure Yang-Mills

NLC scattering

Massless scalar QED, $e^- \to e^- + \gamma^+$ in SD FF background

$$\mathcal{M}(p \to p' + \ell) = \int \mathrm{d}^4 x \, \bar{\varepsilon}_+^{\mu} \left(\bar{\phi}_{p'} \, D_\mu \phi_p - \phi_p \, D_\mu \bar{\phi}_{p'} \right) \mathrm{e}^{\mathrm{i}\ell \cdot x}$$

for

$$\bar{\phi}_{p'} = \frac{\mathrm{e}^{-\mathrm{i}\, S_{-p'}(x)}}{1 + \mathrm{i}\, k\, x^+}$$

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for

$$\bar{\phi}_{p'} = \frac{\mathrm{e}^{-\mathrm{i} \, S_{-p'}(x)}}{1 + \mathrm{i} \, k \, x^+}$$

Can perform $\mathrm{d}x^+$ and $\mathrm{d}^2 x^\perp$ integrals by contour deformations

Leaves

$$\mathcal{M} = \frac{\mathrm{i}\, e}{2\pi\, k}\, \delta\!\left(p'_+ - p_+ - \frac{(p_\perp - q_\perp)^2}{2\ell_-} \right) \exp\!\left[-\frac{(p' + \ell - p)_\perp^2}{2\, k\, \ell_-} \right]$$
$$\times \left(\frac{\mathsf{p}\, p'_+ - \mathsf{p}'\, p_+}{(\mathsf{p} - \mathsf{p}')^2} \right) \int \mathrm{d}x^-\, \mathrm{e}^{\mathrm{i}\, \tilde{S}(x^-)}$$

where

$$\tilde{S}(x^-) := (\ell + p' - p)_-\, x^- + e \left(\frac{\mathsf{p}'}{p'_+} - \frac{\mathsf{p}}{p_+} \right) \mathcal{F}(x^-)$$

In the case of the FF beam, where

$$\mathcal{F}(\sigma) = \frac{E_0}{\omega}\,\mathrm{e}^{-\mathrm{i}\,\omega\,\sigma}$$

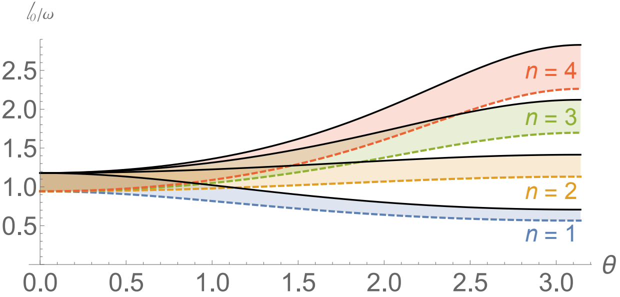
final integral can be performed exactly too!

$$\int \mathrm{d}x^-\,\mathrm{e}^{\mathrm{i}\,\tilde{S}(x^-)} = \sum_{n=0}^{\infty} \frac{2\pi}{n!}\,\delta((\ell + p' - p)_- - n\,\omega)\left(\frac{e\,E_0}{\omega^2}\left[\frac{\mathrm{p}'}{p'_+} - \frac{\mathrm{p}}{p_+}\right]\right)^n$$

cf., 'harmonic spectrum' in monochromatic PW

[Berestetskii-Lifshitz-Pitaevskii]

For instance, when selectron scatters into kinematic region forbidden in PWs, find spread of emitted photon frequencies



where θ is scalar scattering angle
solid lines are PW spectra

MHV gluon scattering

Second example, $g^- \to g^+ + (n-2)\, g^-$ in YM

Find even simpler result:

$$\mathcal{A}_n^{\mathrm{MHV}} = \frac{\langle r\, s\rangle^4}{\langle 1\, 2\rangle \cdots \langle n\, 1\rangle} \int \frac{\mathrm{d}^4 x}{(1 + \mathrm{i}\, k\, x^+)^4} \prod_{i=1}^{n} \mathrm{e}^{-\mathrm{i}\, S_i(x)}$$

$$= (2\pi)^3\, \delta\left(\sum_{i=1}^{n} p_{i\,+}\right) \delta^2\left(\sum_{i=1}^{n} p_{i\,\perp}\right) \frac{\langle r\, s\rangle^4}{\langle 1\, 2\rangle \cdots \langle n\, 1\rangle} \times \int \mathrm{d}x^- \prod_{i=1}^{n} \mathrm{e}^{-\mathrm{i}\, \tilde{S}_i(x^-)}$$

for $\tilde{S}_i(x^-) := p_{i\,-} x^- - \mathrm{i}\, e_i\, \frac{\mathrm{p}_i}{p_{i\,+}}\, \mathcal{F}(x^-)$

Again, can do final integral for FF beam:

$$\int \mathrm{d}x^- \prod_{i=1}^{n} \exp\left[-\mathrm{i}\left(p_{i\,-}\,x^- + \frac{e_i\,\mathrm{p}_i}{\omega\,p_{i\,+}}\,\mathrm{e}^{-\mathrm{i}\,\omega\,x^-}\right)\right]$$

$$= \sum_{q=0}^{\infty} \frac{2\pi}{q!}\,\delta\left(\sum_{i=1}^{n} p_{i\,-} - q\,\omega\right)\left(\frac{E_0}{\omega^2}\sum_{j=1}^{n}\frac{e_j\,\mathrm{p}_j}{p_{j\,+}}\right)^q$$

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This formula holds for *any* n!

Take home message

Using self-duality, able to:

- describe gauge field w/ spatial inhomogeneity
- solve background-coupled eoms exactly
- obtain analytic formulae for amplitudes
- go to high-multiplicity for some processes
- view as model for total depletion

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- view as model for total depletion

Of course, plenty of room for improvement!

- focus moving at speed of light
- total depletion
- massless probes only
- ...

Worldline challenge

Is it possible to recover all-multiplicity results from worldline master formulae when background is self-dual?

Last but not least...

THANK YOU

Anton, James, Karthik and Patrick!