Boundary conditions and proper-time contours for worldline path integrals

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Workshop on

First Quantisation for Physics in Strong Fields

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- Wavefunctions in Lorentzian path integral formalism

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- Simple examples for illustration

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- Reconciling Euclidean intuitions with Lorentzian path integrals
- Simple examples for illustration and the tough ones in 'future directions'!

Wavefunction

- Solutions to 'wave equations'

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$$\boxed{\Psi(x) = \int_{\text{Initial Condition}}^{X(\text{final})=x} \mathrm{d}[X]\,\mathrm{d}[\text{lag. mult.}]\,\mathrm{d}[\text{ghost}]\, e^{i(\text{Action})}}$$

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- Here[1]
  1. $X=$ dynamical variable
  2. Lagrangian multipliers $\Rightarrow$ constraints $+$ gauge-fixing
  3. Ghost $\Rightarrow$ invariant measure for gauge-fixed sub-space
  4. Action$=$ $S + S_{gauge-fixing} + S_{ghost}$

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1. Quantum Mechanics

Stationary states in Q.M.

$$\Psi_E(x) \stackrel{?}{=} \int_{\text{in.cond.}}^{q(1)=x} \int \mathcal{D}[q]\mathcal{D}[\mathcal{N}]\mathcal{D}[\Pi]\mathcal{D}[\text{ghosts}] \exp\left[iS_{\text{tot.}}\right]$$

$$S_{\text{tot.}} = \int_0^1 \left(\frac{m}{2\mathcal{N}} \dot{q}^2 - \mathcal{N} V(x) + E\mathcal{N} + \Pi\dot{\mathcal{N}} + L_{\text{ghost}} \right) dt$$

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- Also, evaluating the action at the above $\mathcal{N}(t)$ gives[2]

$$S = \pm \int \sqrt{2m(E - V)\dot{q}^2} \quad \Rightarrow \quad \text{Reparametrization invariance}$$

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- Fortunately, this can be reduced to a much simpler expression.

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Free particle: Q.M.

$$\Psi_E(x) = \int_{-\infty}^{\infty} d\mathcal{N} \int_{m\dot{q}(0)=p}^{q(1)=x} \int \mathcal{D}[q] \exp\left[iS\right]$$
$$\propto e^{ip\cdot x} \delta\left(\frac{p^2}{2m} - E\right)$$

Free particle: scalar QFT

$$\Psi_p(x) = \int_{-\infty}^{\infty} d\mathcal{N} \int_{m\dot{q}(0)=p}^{q(1)=x} \int \mathcal{D}[q] \exp\left[iS\right]$$
$$\propto \underbrace{e^{ip\cdot x}\delta\left(p^2 - m^2\right)}_{\text{mode-function with on-shell measure}}$$

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- Before going into the details, contrast the real-time approach with

$$\Psi_0(x) = \int_{q(-\infty)=0}^{q(0)=x} \mathcal{D}[q] \exp\left[-\int_{-\infty}^{0} L_{\mathrm{Eucl.}}[q,\dot{q}]d\tau\right]$$

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- This is equivalent to taking the limit $i\mathcal{N} \to \infty$, rather that integrating over $\mathcal{N}$.
- So the question arises: *How are the real-time and Euclidean approaches related?*

SHO: Initial condition

$$L = \frac{m}{2\mathcal{N}}\dot{q}^2 - \mathcal{N}\frac{m\omega^2}{2}q^2 + E\mathcal{N}$$

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$$q(1) = x \quad ; \quad i\frac{m}{\mathcal{N}}\dot{q}(1) + m\omega q(1) = m\omega x_0$$

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- Action is quadratic, so $\mathcal{D}[q]$ integral can be formally computed from the classical solution.
- Complex solutions: $q(t) = e^{-i\omega\mathcal{N}}\left[xe^{it\omega\mathcal{N}} - ix_0\sin(\mathcal{N}(t-1)\omega)\right]$

SHO: the propertime integral

$$\Psi_E(x) = \int_{-\infty}^{\infty} d\mathcal{N} e^{i\left(E-\frac{\omega}{2}\right)\mathcal{N}} \underbrace{\exp\left[-\frac{m\omega}{2}\left(x - x_0 e^{-i\omega\mathcal{N}}\right)^2 + \frac{1}{2}x_0^2 e^{-2i\omega\mathcal{N}}\right]}_{\text{Periodic}}$$

- Integrand is periodic for $E = E_n = \omega\left(\frac{1}{2} + n\right)$

SHO: the propertime integral

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- Integrand is periodic for $E = E_n = \omega\left(\frac{1}{2} + n\right)$
- Motivates the change of variable

$$z = e^{-i\omega\mathcal{N}} \Rightarrow d\mathcal{N} = \frac{i\omega^{-1} dz}{z}$$

Back to ground-state

- For $n = 0$, we get

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- For the SHO, the connection between real-time and Euclidean approaches is the Cauchy's residue theorem!
- Also, not that for a general n, the propertime integral generates the correct $\Psi_{E_n}(x)$ in the form of Cauchy's integral formula.

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- Bound state wave functions in QFT?

2. Schwinger effect

The Schwinger effect: intro

- The phenomenon of decay of QED vacuum in the presence of a constant external electromagnetic field via pair production.[3]

[3] F. Sauter(1931),W. Heisenberg and H. Euler (1936),J. Schwinger(1951)

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- Robust non-perturbative prediction in QED.

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The Schwinger effect: intro

- The phenomenon of decay of QED vacuum in the presence of a constant external electromagnetic field via pair production.[3]
- Robust non-perturbative prediction in QED.
- In this talk, we restrict to scalar QED.

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Worldline for Schwinger

- Worldline with periodic solution $\rightarrow$ vacuum persistence[Affleck, Alvarez, and Manton (1982), Dunne and Schubert (2005),Dunne, Wang, Gies and Schubert (2006),...]

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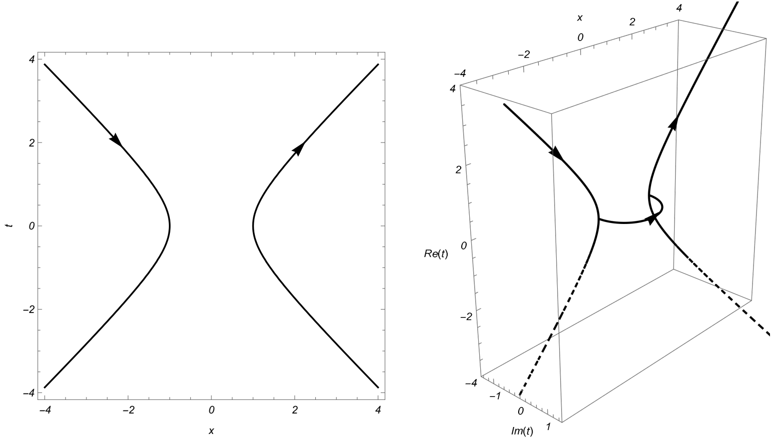
- Open worldline $\rightarrow$ particle spectrum[Barut and Duru (1990), Srinivasan and Padmanabhan (1990), K.R (2021), Esposti and Torgrimsson (2022)]

Heuristic picture[4]



[4]Srinivasan and Padmanabhan (1999)

Mode functions for Schwinger effect

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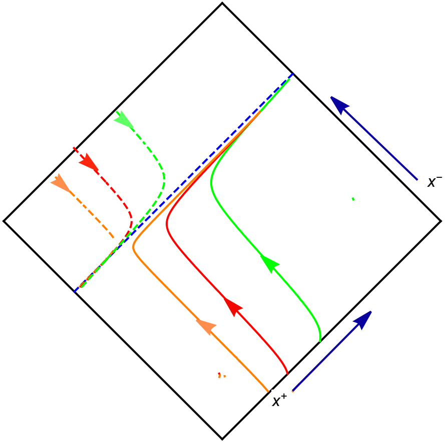
- In most frequently used gauges, these mode functions involve parabolic cylinder functions.
- However, in a class of gauges, including the so-called lightfront gauge below, the solutions are simple functions.

$$eA_\mu = (eA_+, eA_-, eA_\perp) = (eEx^-, 0, 0) \quad ; \quad eE > 0$$
$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \quad ; \quad x^\perp = (x^2, x^3)$$

Classical physics

Particle and antiparticle solutions for fixed p.



KG equation and the solutions

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$$\phi_p^+(x) = e^{-ip_+x^+ - ip_\perp x^\perp} \underbrace{e^{-\left(\frac{1}{2} - i\frac{m^2+p_\perp^2}{2eE}\right)\log\left(1 - eE\frac{x^-}{p_+}\right)}}_{\sum f_n(p;x)(eE)^n}$$

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- Note the 'horizon' at $x^- = p^+/(eE)$.

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$$\text{Opt. 2:} \quad e^{-ip_+x^+ - ip_\perp x^\perp} e^{-\left(\frac{1}{2} - i\frac{m^2+p_\perp^2}{2eE}\right)\log\left(1-\frac{x^-}{x_h^-}+i0^+\right)}$$

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- Option 1: Consistent with classical intuition.
- Option 2: positive energy w.r.t x^-.

First quantization

$$\phi_p^+(x) = \int_{-\infty}^{\infty} d\mathcal{N} \int_{\Pi(0)=p}^{q(1)=x} \mathcal{D}[q] e^{i\mathcal{S}_{\mathrm{WL}}[q,\mathcal{N}]} \quad ; \left(\text{Gauge fixed and exorcised}\right)$$

$$\mathcal{S}_{\mathrm{WL}} = -\int_0^1 \left(\frac{\dot{x}^2}{4\mathcal{N}} + \mathcal{N} eA \cdot \dot{x}\right) d\tau \quad \underbrace{-p \cdot q(0)}_{\text{Boundary term}}$$

Action

- Thanks to the quadratic action:

$$\phi_p^+(x) = \int_{-\infty}^{\infty} d\mathcal{N} \quad \underbrace{e^{eE\mathcal{N}}}_{\text{fluctuation factor}} \quad \times \quad \underbrace{e^{i\mathcal{S}_{\text{cl.}}}}_{\text{semiclassical=exact}}$$

$$\mathcal{S}_{\text{cl.}} = \frac{p_- \left(e^{2eE\mathcal{N}} - 1\right)\left(p_+ - eEx^-\right)}{eE} - p \cdot x - (p_\perp^2 + m^2)\mathcal{N}$$

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$$\left(q_{\text{cl.}}^-(\tau) - \frac{p_+}{eE}\right) = \left(x^- - \frac{p_+}{eE}\right) e^{2eE(\mathcal{N}-\tau)}$$

$$q_{\text{cl.}}^+(\tau) = x^+ + \left(\frac{e^{2eE\tau} - e^{2eE\mathcal{N}}}{eE}\right) p_-$$

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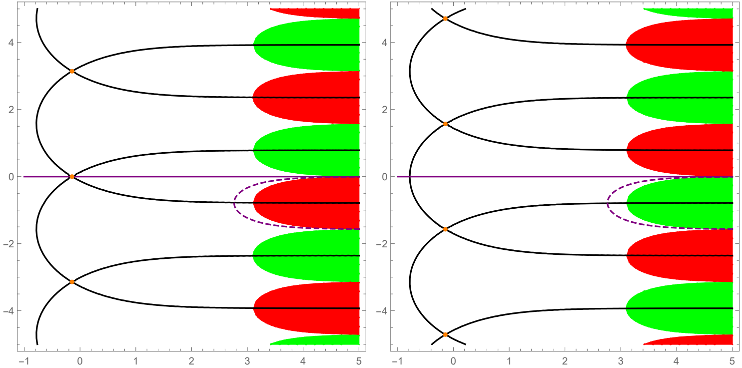
- Note: This is a sum of both *space-like* and time-like worldlines!
- On-shell solution is described by the saddle-point: $\partial_{\mathcal{N}_s}\mathcal{S}_{\text{cl.}} = 0$

Picard–Lefschetz



(a) Real saddle point exists.

(b) Real saddle point doesn't exist.

The mode function

- Case 1: $x^- < x_h^-$, Shift $\mathcal{N} = \mathcal{N}_s + \bar{\mathcal{N}}$

$$\phi_p^+(x) = e^{-ip_+x^+ - ip_\perp x^\perp} e^{-\left(\frac{1}{2} - i\frac{m^2+p_\perp^2}{2eE}\right)\log\left(1 - eE\frac{x^-}{p_+}\right)} \times \underbrace{\int_{-\infty}^{\infty} d\bar{\mathcal{N}}\ \text{function}(\bar{\mathcal{N}})}_{\text{constant}}$$

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$$\underbrace{\int_{-\infty}^{\infty} d\bar{\mathcal{N}}\ \text{function}(\bar{\mathcal{N}})}_{\text{constant}}$$

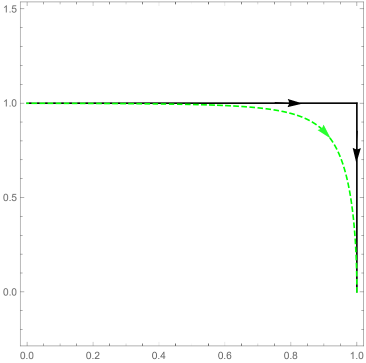
- Case 2: $x^- > x_h^-$, Shift $\mathcal{N} \to \bar{\mathcal{N}} + \underbrace{\Re[\mathcal{N}_s] - i\pi/(2eE)}_{\text{instanton!}}$

$$\phi_p^+(x) = e^{-\frac{p_\perp^2+m^2}{2eE}} e^{-i\frac{\pi}{2}} e^{-ip_+x^+ - ip_\perp x^\perp} e^{-\left(\frac{1}{2} - i\frac{m^2+p_\perp^2}{2eE}\right)\log\left(eE\frac{x^-}{p_+} - 1\right)} \times$$
$$\underbrace{\int_{-\infty}^{\infty} d\bar{\mathcal{N}}\ \text{function}(\bar{\mathcal{N}})}_{\text{constant}}$$

- Real time + complex $\mathcal{N}$+gauge transformation = Complex time!

$$\left(q^{-}_{\text{cl.}}(\tau) - \frac{p_+}{eE}\right) = \left(x^{-} - \frac{p_+}{eE}\right) e^{2eE(\mathcal{N}_s - \tau)} \quad ; \quad 0 < \tau < 1$$

$$\left(q^{-}_{\text{cl.}}(\bar{\tau}) - \frac{p_+}{eE}\right) = \left(x^{-} - \frac{p_+}{eE}\right) e^{2eE\mathcal{N}_s - eE\bar{\tau}} \quad ; \quad \mathcal{N}_s\tau = \bar{\tau} \in \mathbb{C}$$



Back to $+i0^+$ vs $\Theta(\ldots)$

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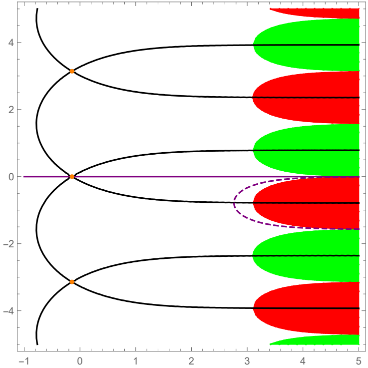
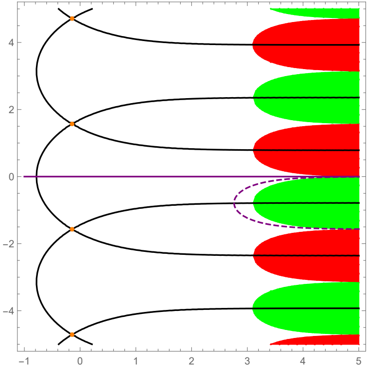
- The *real-time* worldline approach naturally leads to positive-energy solution, hence the $+i0^+$ prescription for the $\log\left(1-\frac{x^-}{x_h^-}\right)$
- How to get the other solution, with $\Theta(x_h^- - x^-)$?

Picard–Lefschetz(again)



(a) Real saddle point exists.

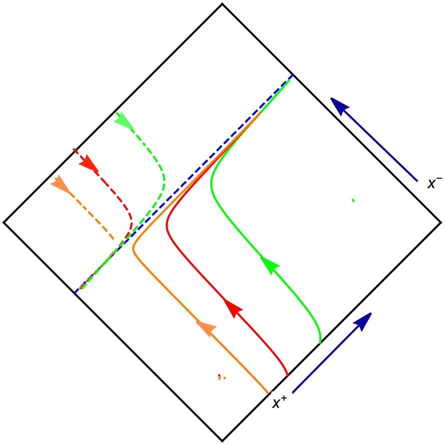


(b) Real saddle point doesn't exist.

The classical-like solution

$$\boxed{\int_{\infty}^{\infty - i\frac{\pi}{2eE}} d\mathcal{N} \int_{\Pi(0)=p}^{z(1)=x} \mathcal{D}[q] e^{i\mathcal{S}_{\mathrm{WL}}[q,\mathcal{N}]} \propto \phi_p^{+}(x)\Theta(x_h^{-} - x^{-})}$$

More



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  2. Horizon: Schwinger and Hawking effect connection; W.L. approach?

$$\psi_E(v) \sim e^{-iEv} e^{4GMi \log\left(\frac{v_h - v}{\mu}\right)}$$

THANK YOU!