Numerical conformal bootstrap for the 3d Ising CFT

arXiv:2411.15300 with Chang, Dommes, Erramilli, Homrich, Liu, Mitchell, Poland, Simmons-Duffin

Petr Kravchuk

April 28 2025, UKLFT

King's College London

Outline

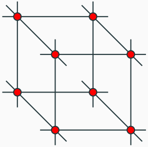
- Introduction: 3d Ising CFT & numerical bootstrap
- Setting up T, σ, ϵ in 3d Ising
- Results

3d Ising CFT

3d Ising CFT describes phase transitions associated with breaking of $\mathbb{Z}_2$ symmetry, e.g. in the 3d Ising model

$$H[\sigma] = J \sum_{\langle ij \rangle} (1 - \sigma_i \sigma_j) + h \sum_i \sigma_i, \quad Z = \sum_{\sigma_i = \pm 1} e^{-H[\sigma]/T}$$

Second-order phase transition at $T = T_c$ and $h = 0$. Conformal invariance believed to emerge at large distances.



3d Ising CFT

Why 3d Ising CFT?

- A very common universality class (from H_2O to QCD?).
- The simplest interacting quantum field theory in $d > 2$?
- Conformal bootstrap is very effective.
- Good testing ground for new methods.

3d Ising CFT

Conformal bootstrap works directly in the scaling limit.

3d Ising a unitary CFT with the symmetries:

1. connected conformal group $\widetilde{\mathrm{SO}}(2,3)$
2. space parity
3. $\mathbb{Z}_2$ global symmetry

Local operators are labelled by scaling dimension Δ and $[j^{\pm}, \pm]$.
Only two relevant scalars:

1. $\sigma \in [0^+, -], \quad \Delta_\sigma = \frac{1}{2} + \frac{\eta}{2} \approx 0.52$
2. $\epsilon \in [0^+, +], \quad \Delta_\epsilon = 3 - \frac{1}{\nu} \approx 1.41$

3d Ising CFT

3d Ising model/CFT has been studied using many methods: Monte-Carlo, High-T expansions, ERG, ϵ-expansion, skeleton expansions, fuzzy sphere...

The most precise values of $\Delta_\sigma, \Delta_\epsilon$ from non-bootstrap methods come from Monte-Carlo [Hasenbusch'21]

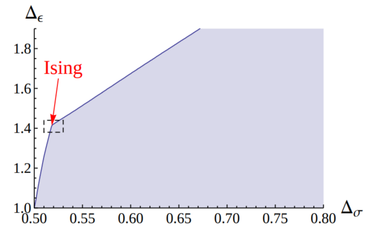
$$\Delta_\sigma = 0.518142(20), \quad \Delta_\epsilon = 1.41265(13)$$

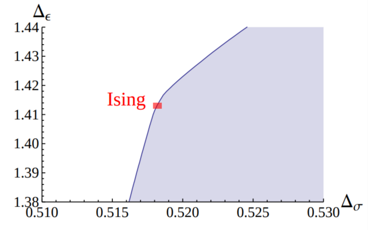
Recent fuzzy-sphere regularization seems to offer some advantages [Zhu, Han, Huffman, Hofman, He'22]

3d Ising CFT & conformal bootstrap

Conformal bootstrap constrains properties of CFTs based on crossing symmetry of four-point functions. [Rattazzi, Rychkov, Tonni, Vichi'08]

For $\langle\sigma\sigma\sigma\sigma\rangle$ one finds





[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi'12]

3d Ising & conformal bootstrap

Conformal invariance implies $\left(z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \right)$

$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle = \frac{g(z,\bar{z})}{x_{12}^{2\Delta_\sigma} x_{34}^{2\Delta_\sigma}}.$$

Using operator product expansion $\sigma \times \sigma = \sum_{\mathcal{O}} \lambda_{\sigma\sigma\mathcal{O}} \mathcal{O}$,

$$g(z,\bar{z}) = (z\bar{z})^{\Delta_\sigma} \sum_{\mathcal{O}} \lambda_{\sigma\sigma\mathcal{O}}^2 g_{\Delta_{\mathcal{O}}, j_{\mathcal{O}}}(z,\bar{z}).$$

The operators $\mathcal{O}$ have quantum numbers $[j^+, +]$, $j \in 2\mathbb{Z}$.

$$g_{\Delta,j}(z,\bar{z}) = \text{conformal block (computable function)}$$

3d Ising & conformal bootstrap

Permutation invariance under $x_1 \leftrightarrow x_3$ implies

$$g(z,\bar{z}) = g(1-z, 1-\bar{z}).$$

Together with $j \in 2\mathbb{Z}$ this implies full S_4 permutation symmetry.

$$\sum_{\mathcal{O}} \lambda^2_{\sigma\sigma\mathcal{O}} g_{\Delta_{\mathcal{O}},j_{\mathcal{O}}}(z,\bar{z}) = \sum_{\mathcal{O}} \lambda^2_{\sigma\sigma\mathcal{O}} g_{\Delta_{\mathcal{O}},j_{\mathcal{O}}}(1-z,1-\bar{z})$$

$$\sum_{\mathcal{O}} \lambda^2_{\sigma\sigma\mathcal{O}} F_{\Delta_{\mathcal{O}},j_{\mathcal{O}}}(z,\bar{z}) = 0.$$

At $z = \bar{z} = \frac{1}{2}$ and odd $m+n$, using $\lambda_{\sigma\sigma 1} = 1$

$$\partial^m \bar{\partial}^n F_{0,0} + \sum_{\mathcal{O}\neq 1} \lambda^2_{\sigma\sigma\mathcal{O}} \partial^m \bar{\partial}^n F_{\Delta_{\mathcal{O}},j_{\mathcal{O}}} = 0.$$

Crucially, $\lambda^2_{\sigma\sigma\mathcal{O}} \geq 0$ due to unitarity.

3d Ising & conformal bootstrap

$$\partial^m \bar{\partial}^n F_{0,0} + \sum_{\mathcal{O} \neq 1} \lambda^2_{\sigma\sigma\mathcal{O}} \partial^m \bar{\partial}^n F_{\Delta_\mathcal{O}, j_\mathcal{O}} = 0.$$

Toy model: for odd n,

$$(0-1)^n + \sum_\Delta \lambda^2_\Delta (\Delta - 1)^n = 0.$$

Suppose all $\Delta \geq 3$ in the sum. We prove it is impossible:

1. Take $-4\times(n=1$ equation) plus $n=3$ equation.
2. Obtain

$$P(0) + \sum_\Delta \lambda^2_\Delta P(\Delta) = 0,$$

$$P(\Delta) = -4(\Delta - 1) + (\Delta - 1)^3 = (\Delta + 1)(\Delta - 1)(\Delta - 3)$$

3. Contradiction from $P(0) = 3$ and $P(\Delta) \geq 0$ for $\Delta \geq 3$.

3d Ising & conformal bootstrap

In the real problem

$$\partial^m \bar{\partial}^n F_{0,0} + \sum_{\mathcal{O} \neq 1} \lambda^2_{\sigma\sigma\mathcal{O}} \partial^m \bar{\partial}^n F_{\Delta_{\mathcal{O}}, j_{\mathcal{O}}} = 0.$$

[Rattazzi, Rychkov, Tonni, Vichi'08]

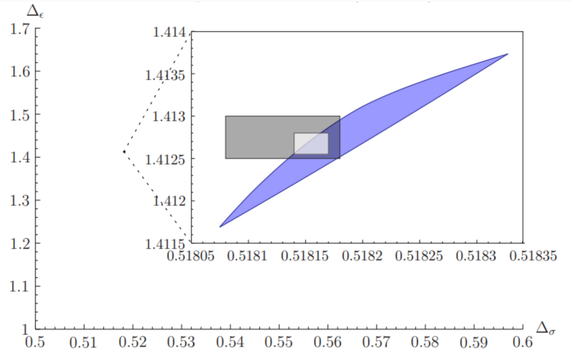
1. Take linear combinations with coefficients α_{mn}.
2. Cutoff at $m + n \leq \Lambda$. Typical $\Lambda \lesssim 50$.
3. Search for coefficients α numerically so as to exclude potential solutions.

3d Ising & conformal bootstrap

Studying four-point functions of σ and ϵ,

$$\langle\sigma\sigma\sigma\sigma\rangle, \quad \langle\epsilon\epsilon\epsilon\epsilon\rangle, \quad \langle\sigma\epsilon\sigma\epsilon\rangle,$$

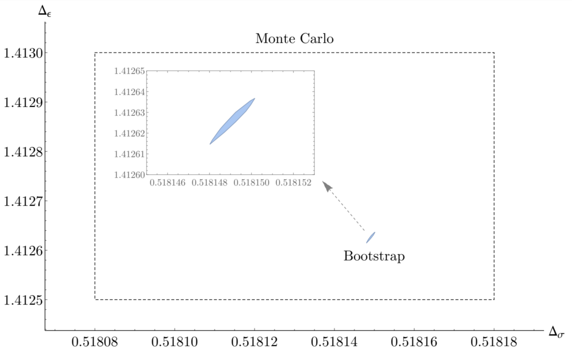
one finds ($\Lambda = 19$)



[Kos,Poland,Simmons-Duffin'14]

3d Ising & conformal bootstrap

[Simmons-Duffin'15] introduced bootstrap-tuned solver SDPB and [Kos,Poland,Simmons-Duffin,Vichi'16] compute at $\Lambda = 43$



State of the art prior to this work.

Stress-tensor bootstrap

What's new: add T to σ, ϵ

$$\langle\sigma\sigma\sigma\sigma\rangle, \quad \langle\epsilon\epsilon\epsilon\epsilon\rangle, \quad \langle\sigma\epsilon\sigma\epsilon\rangle$$
$$\langle T\sigma\sigma\epsilon\rangle, \quad \langle T\epsilon\epsilon\epsilon\rangle$$
$$\langle TT\sigma\sigma\rangle, \quad \langle TT\epsilon\epsilon\rangle$$
$$\langle TTT\epsilon\rangle$$
$$\langle TTTT\rangle$$

Stress-tensor bootstrap

$\{T, \sigma, \epsilon\}$ advantages:

1. Access to almost all symmetry sectors.
   (except $[0^-, -]$, $[1^-, -]$, $[1^-, +]$, $[1^+, +]$)
2. Access to more operators, OPE coefficients.
3. Improved precision.

$\{T, \sigma, \epsilon\}$ challenges:

1. Many essentially different conformal blocks $g_{\Delta,j}$ to compute.
2. Many more crossing equations to impose.
3. A large jump in computational complexity.

Computation of conformal blocks

We use a fully algorithmic approach based on Zamolodchikov-like recursion relations, schematically

$$g_{\Delta,j}(z,\bar{z}) = g_{\infty,j}(z,\bar{z}) + \sum_i \frac{1}{\Delta - \Delta_i} r^{n_i} g_{\Delta_i+n_i,j_i}(z,\bar{z})$$

[Zamolodchikov'87] [PK'17] [Erramilli,Iliesiu,PK'19]

Implemented in `blocks_3d` software.

Can compute completely general 3d conformal blocks.

[Erramilli,Iliesiu,PK,Landry,Poland,Simmons-Duffin'20]

Managing crossing equations

The full $\{T, \sigma, \epsilon\}$ involves many more crossing equations.

Low-level manual implementation is error-prone and not practical.

We use the `hyperion` family of `Haskell` packages.
Designed to handle very general systems of crossing equations given only a high-level description.

[Simmons-Duffin, PK, Erramilli, Liu, ...]

Managing data

`hyperion` also handles the generation and bookkeeping of the conformal blocks, other intermediate data, and the final results.

	fresh build	using pre-built $\langle TTTT \rangle$ blocks	during OPE scan
`Block3d`	629	79	0
`CompositeBlock`	10538	4576	0
`PartChunk`	7926	7926	53
`Part`	363	363	1
total time	4.8 hours	1.4 hours	13 seconds

build times using 2048 compute cores

The amount of data that needs to be generated required us to develop a custom scheduling algorithm to distribute tasks to allocated compute cores.

Computational efficiency

This work used `SDPB 3.0.0` [Simmons-Duffin, Landry, Dommes, ...]

1. Arbitrary-precision semidefinite program solver
2. Highly-optimized for numerical bootstrap applications
3. Scales well to > 10 compute nodes using MPI

Computations

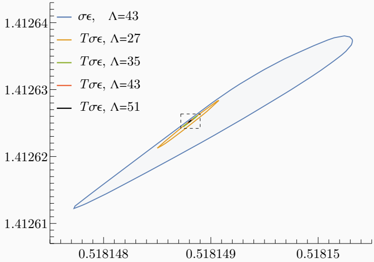
- DiRAC Memory Intensive service Cosma8 at Durham University
- Expanse cluster at the San Diego Supercomputing Center (SDSC)
- CREATE High Performance Cluster at King's College London.
- Resnick High Performance Computing Center, Caltech
- Yale Grace computing cluster

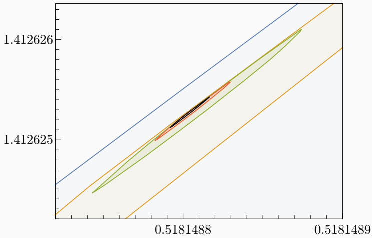
O(10) MCPUhrs

Results: $\Delta_\sigma, \Delta_\epsilon$

Method	Δ_σ	Δ_ϵ
Monte-Carlo	$0.518142(20)$	$1.41265(13)$
$\{\sigma, e\}@\Lambda = 43$	$0.5181489(10)$	$1.412625(10)$
$\{T, \sigma, \epsilon\}@\Lambda = 51$	$0.51814804(13)$	$1.41262527(16)$

$\Lambda = 51$ is preliminary





Gap assumptions: $\Delta_{\text{other scalar}} \geq 3$, $\Delta_{T'} \geq 4$, $\tau_{\text{gap}} \geq 10^{-6}$.

Results: OPE coefficients

3-point functions between σ, ϵ, T enter into the crossing equations and can be bounded (not as rigorous as $\Delta_\sigma, \Delta_\epsilon$)

	$T\sigma\epsilon$ (this work)	prev. bootstrap	monte carlo	fuzzy sphere
$\lambda_{\sigma\sigma\epsilon}$	1.05185373(11)	1.0518537(41)	1.051(1)	1.0539(18)
$\lambda_{\epsilon\epsilon\epsilon}$	1.53244304(58)	1.532435(19)	1.533(5)	1.5441(23)
c_T/c_B	0.946538675(42)	0.946534(11)	0.952(29)	0.955(21)
$\lambda_{TT\epsilon}$	0.95331513(42)	0.958(7)	—	0.9162(73)
n_B	0.933444559(75)	0.933(4)	—	—
n_F	0.013094116(33)	0.014(4)	—	—

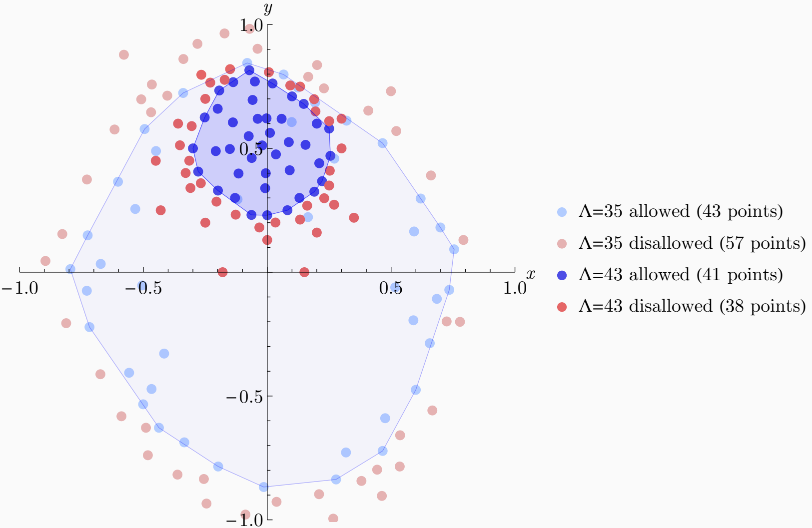
$$\langle TT \rangle = c_T \times (\text{standard})$$

$$\langle TTT \rangle = n_B \langle TTT \rangle_B + n_F \langle TTT \rangle_F$$

$$c_T/c_B = n_B + n_F$$

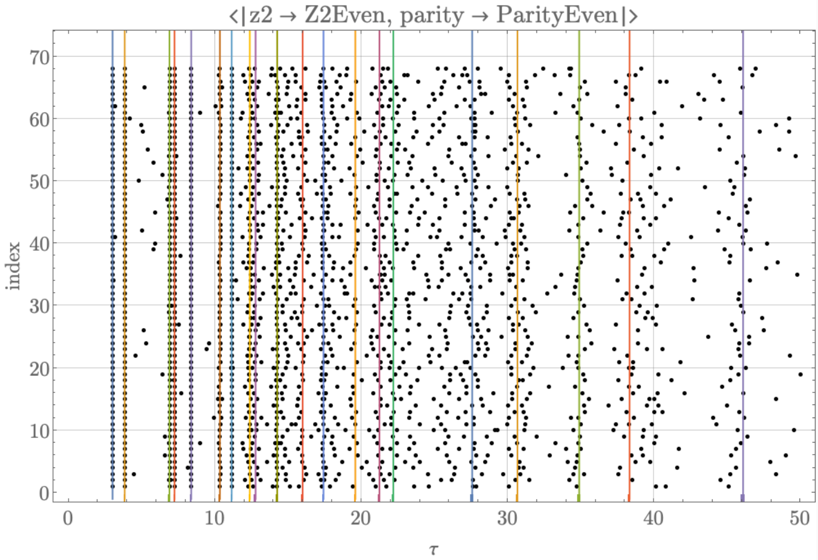
Results: spectrum extraction

Affine transformation of the $\Lambda = 34, 43$ islands



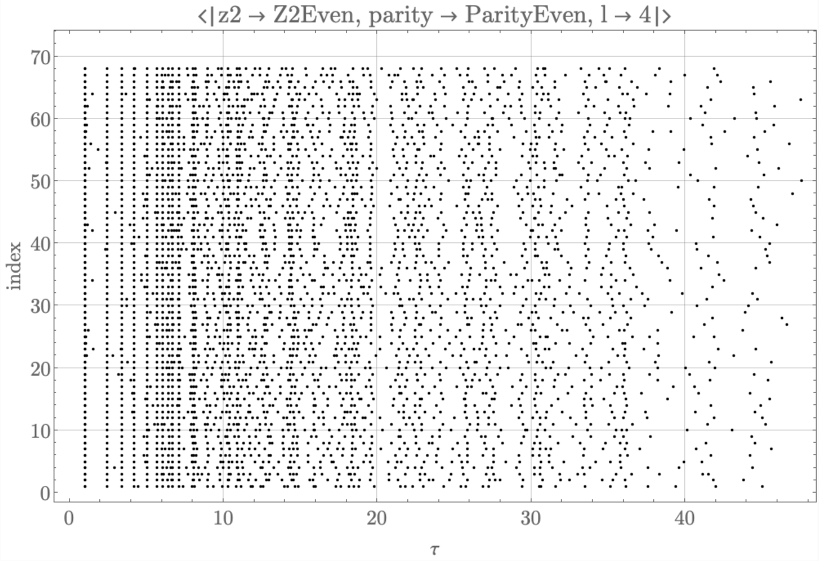
Results: spectrum extraction

$$[0^+, +], \quad \tau = \Delta - j$$



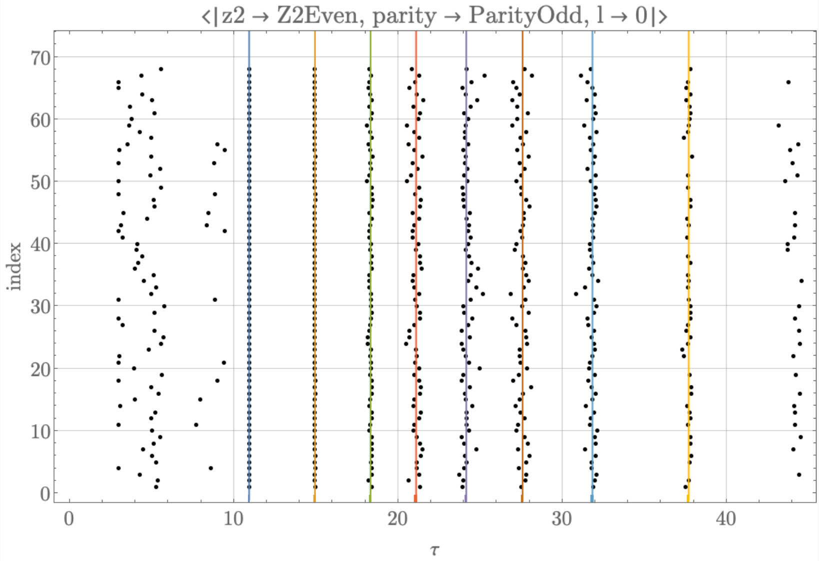
Results: spectrum extraction

$$[4^+, +], \quad \tau = \Delta - j$$



Results: spectrum extraction

$$[0^-, +], \quad \tau = \Delta - j$$



Summary & future directions

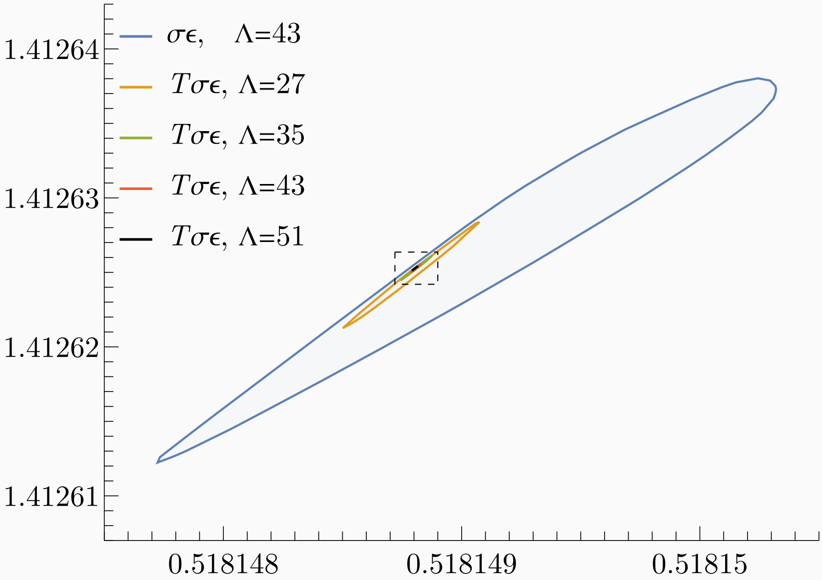
Summary:

- We have been able to study $\{T, \sigma, \epsilon\}$ system at very high Λ.
- High-precision determinations of $\Delta_\sigma, \Delta_\epsilon$ and $\{T, \sigma, \epsilon\}$ three-point functions.
- Analysis of extremal spectra in progress
- Possible thanks to substantially improved numerical algorithms for SDPs, conformal blocks, and software infrastructure.
- Requires O(10 MCPUhrs)

Future:

- Natural extensions to $O(N)$ models and other CFTs in 3d
- Bootstrap of 4d CFTs

Zoom-out plot



Zoom-in plot