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A massive NPR scheme for heavy quark observables

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NPR

physical observable | continuum limit | lattice QCD

$$\langle \mathcal{O} \rangle_{\text{cont}}^{\overline{\text{MS}}}(\mu) \qquad \lim_{a\to 0} \qquad \langle \mathcal{O} \rangle_{\text{lat}}^{\text{bare}}(am)$$

NPR

physical observable | continuum limit | lattice QCD

$$\langle \mathcal{O} \rangle^{\overline{\mathrm{MS}}}_{\mathrm{cont}}(\mu) \quad = \quad R^{\overline{\mathrm{MS}}\leftarrow S}_{\mathcal{O}}\left(\mu\right) \quad \lim_{a\to 0} \quad Z^{S}_{\mathcal{O}}(am, a\mu) \quad \langle \mathcal{O} \rangle^{\mathrm{bare}}_{\mathrm{lat}}(am)$$

matching | renormalisation

NPR

physical observable — continuum limit — lattice QCD

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matching — renormalisation

non-perturbative renormalisation

- Rome-Southampton method [Martinelli et al NPB 445 (1995)]

  regularisation-independent (RI) momentum-subtraction (MOM) scheme

$$Z_{\mathcal{O}}(\mu) \langle p \,|\, \mathcal{O}^{\text{bare}}_{\text{lat}} \,|\, p \rangle_{p^2 = \mu^2} \equiv \langle p \,|\, \mathcal{O}_{\text{tree}} \,|\, p \rangle$$

NPR

physical observable | continuum limit | lattice QCD

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- Variations: RI/MOM', **RI/SMOM**, RI/IMOM… [Sturm et al PRD 80 (2009), Garron et al PRD 108 (2023)]
- Other NPR schemes: [Lüscher et al NPB 384 (1992), Sint NPB 421 (1994), Tomii et al PRD 94 (2016)] …

Heavy quarks

- Charm and bottom physics

Heavy quarks

- Charm and bottom physics
- **Discretisation effects!**

$$\begin{aligned}\langle \mathcal{O} \rangle^S_{\mathrm{lat}}(am, a\mu) &= Z^S_{\mathcal{O}}(a\mu)\, \langle \mathcal{O} \rangle^{\mathrm{bare}}_{\mathrm{lat}}(am) \\ &= \langle \mathcal{O} \rangle^S_{\mathrm{cont}}(m, \mu) \Big[1 + \hat{\delta}\left(am, a\mu\right) \Big]\end{aligned}$$

Heavy quarks

- Charm and bottom physics
- **Discretisation effects!**

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$$= \langle \mathcal{O} \rangle^{S}_{\text{cont}}(m, \mu) \left[1 + \hat{\delta}\left(am, a\mu\right) \right]$$

what we want	what we need	what we have
$\lim_{a \to 0} \hat{\delta} \leq O(a^2)$	$m \ll \mu \ll a^{-1}$	$m_c, m_b \approx a^{-1}$

Heavy quarks

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- **Discretisation effects!**

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- RI/MOM or SMOM are *massless* schemes: $Z^{S}_{\mathcal{O}}(a\mu)$ defined for $am \ll 1$

STRATEGY: define a scheme S where $Z^{S}_{\mathcal{O}}$ absorbs $\hat{\delta}$ in $\lim a \to 0$

RI/mSMOM [Boyle et al PRD 95 (2017)]

- Extension of RI/SMOM for fermion bilinears
  - ✓ Ward identities satisfied
  - ✓ Z-factors in continuum limit similar to $\overline{\mathrm{MS}}$
  - ✓ Valid beyond the regime $am \ll 1$

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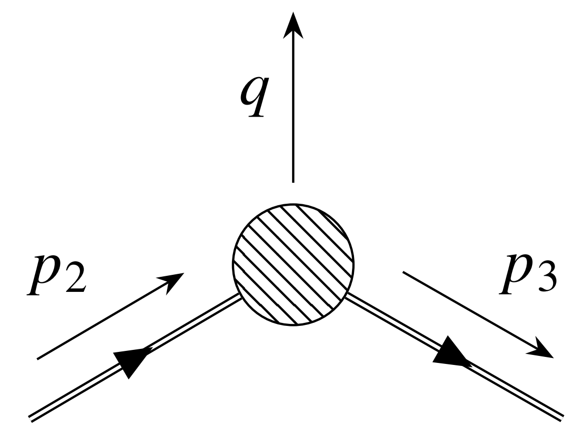


- NPR conditions imposed at some finite value of the renormalised mass

RI/SMOM: $$\lim_{m_R \to 0} \frac{Z_V}{Z_q} \frac{1}{q^2} \mathrm{Tr}\left[\left(q \cdot \Lambda_V \right) \not{q} \right]_{q^2=\mu^2} = 12, \qquad Z_V = Z_V(a\mu)$$

RI/mSMOM: $$\lim_{m_R \to \overline{m}_R} \frac{Z_V}{Z_q} \frac{1}{q^2} \mathrm{Tr}\left[\left(q \cdot \Lambda_V \right) \not{q} \right]_{q^2=\mu^2} = 12, \qquad Z_V = Z_V(a\mu, a\overline{m})$$

◉ Can **tune** $\overline{m}_R$ to a value where $Z_{\mathcal{O}}\langle \mathcal{O} \rangle$ has milder a-dependence

Numerical implementation
renormalised charm quark mass

- In RI/mSMOM scheme

$$m_{c,R}(\mu, \overline{m}_R) = \lim_{a\to 0} Z_m(a\mu, a\overline{m})\, m_c^{\mathrm{bare}}$$

- Using 6 RBC/UKQCD domain wall fermion ensembles
  - 3 lattice spacings: coarse (C), medium (M), fine (F): 0.11 - 0.08 fm
  - Möbius (M) and Shamir (S) kernels

[PRD84(2011), PRD93(2016), PRD110(2024)]

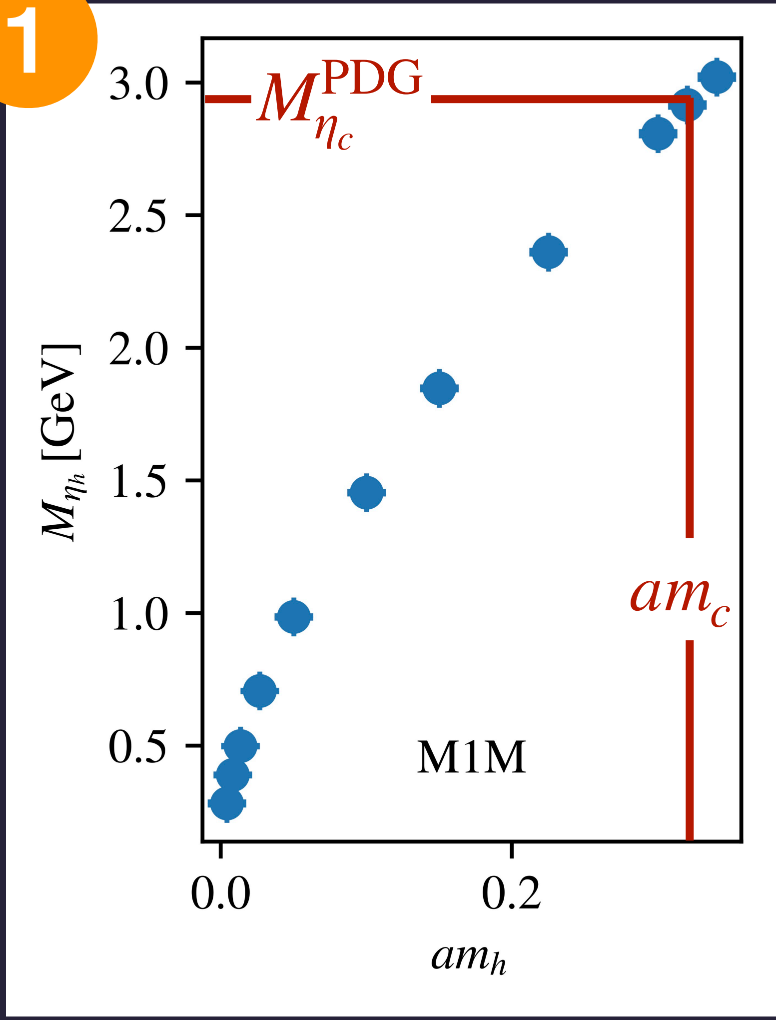
name	L/a	T/a	a^{-1}[GeV]	m_π[MeV]	am_l	am_s
C1M	24	64	1.7295(38)	276	0.005	0.0362
C1S	24	64	1.7848(50)	340	0.005	0.04
M1M	32	64	2.3586(70)	286	0.004	0.02661
M1S	32	64	2.3833(86)	304	0.004	0.03
F1M	48	96	2.708(10)	232	0.002144	0.02144
F1S	48	96	2.785(11)	267	0.002144	0.02144

Ingredients
renormalised charm quark mass

- In RI/mSMOM scheme

$$m_{c,R}(\mu, \overline{m}_R) = \lim_{a\to 0} Z_m(a\mu, a\overline{m})\, m_c$$

1



Ingredients

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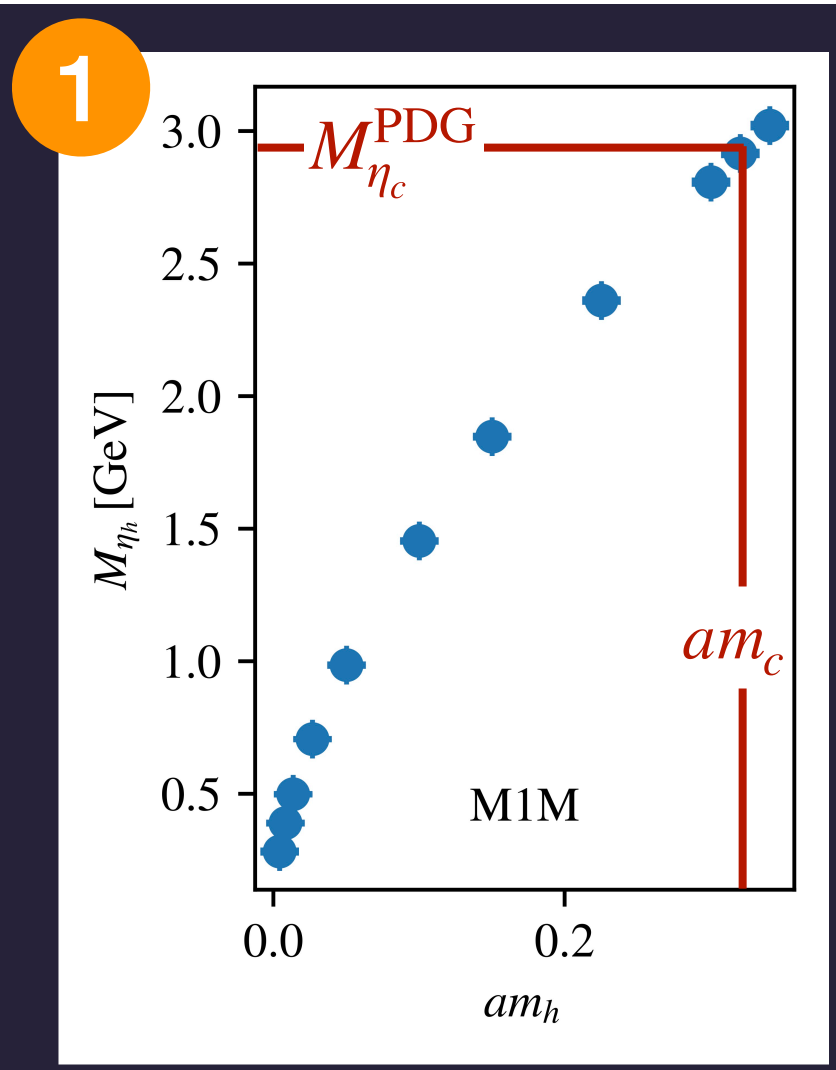

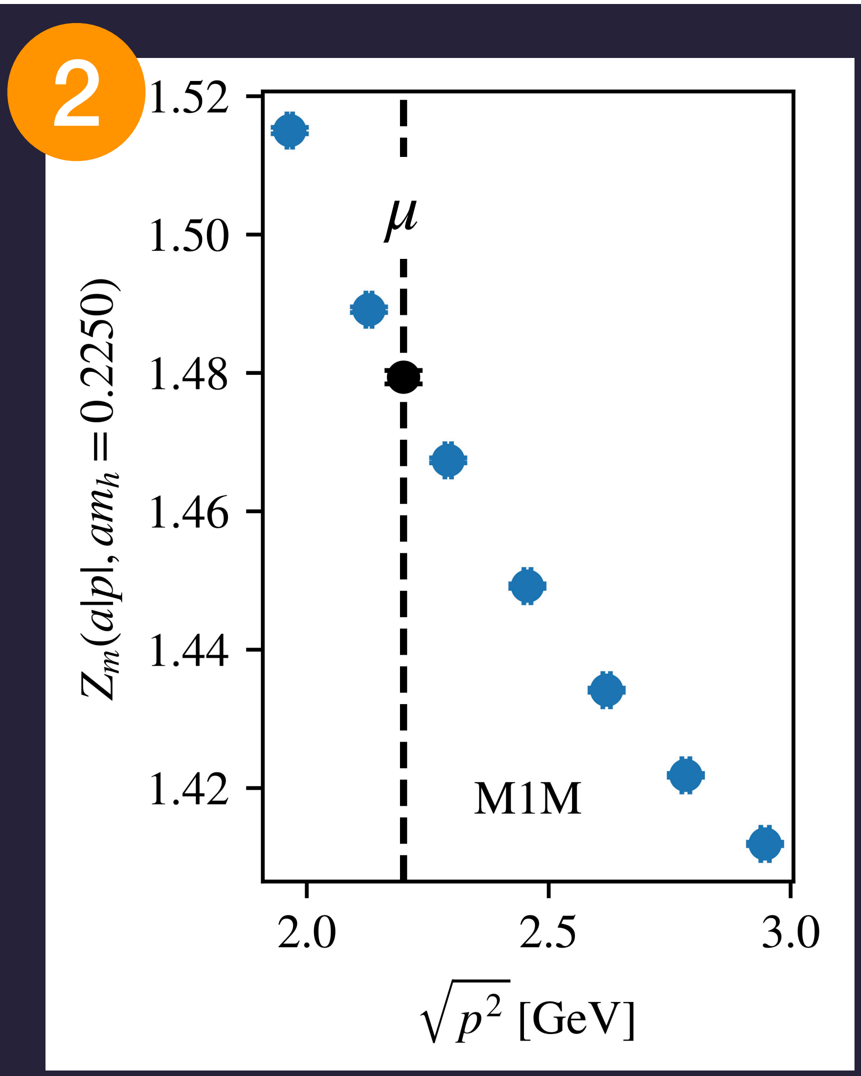

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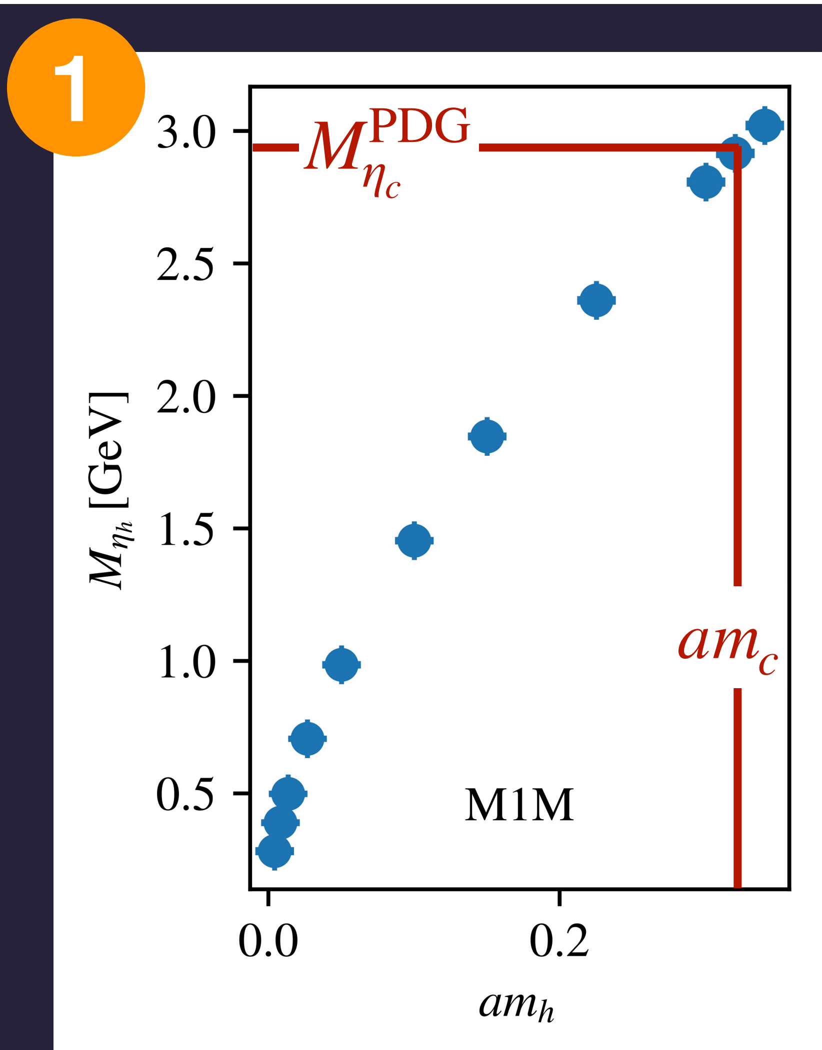

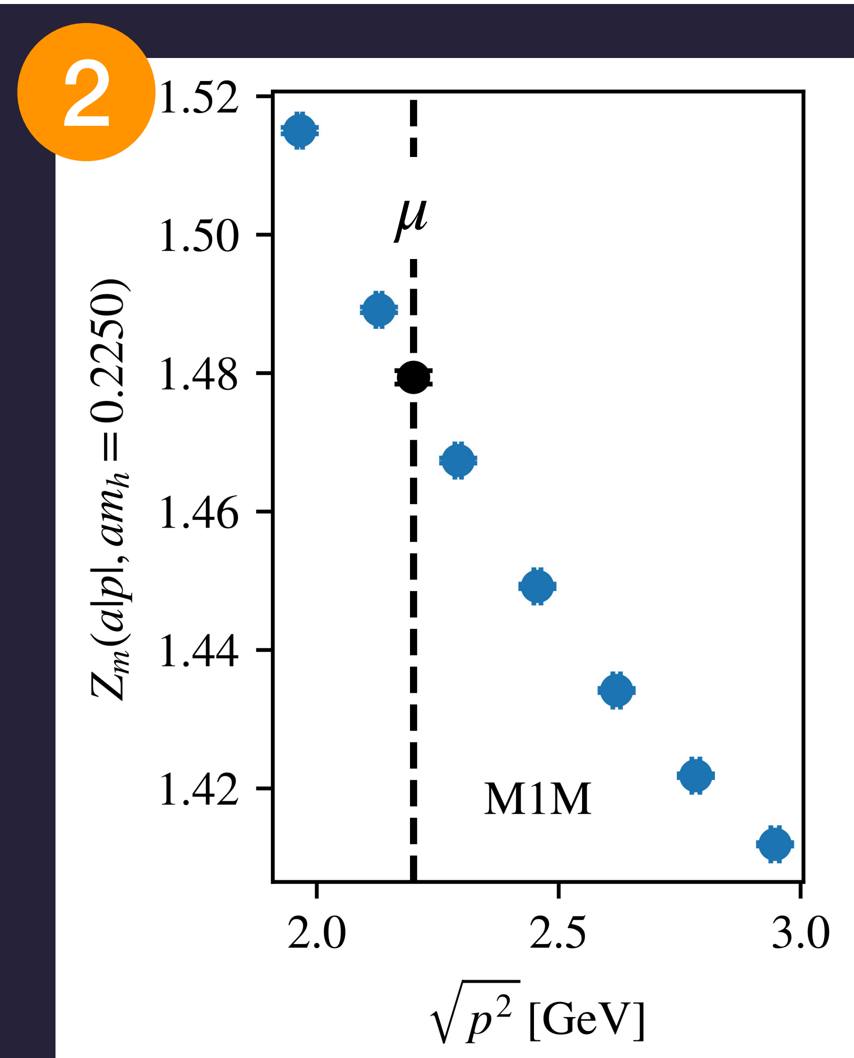

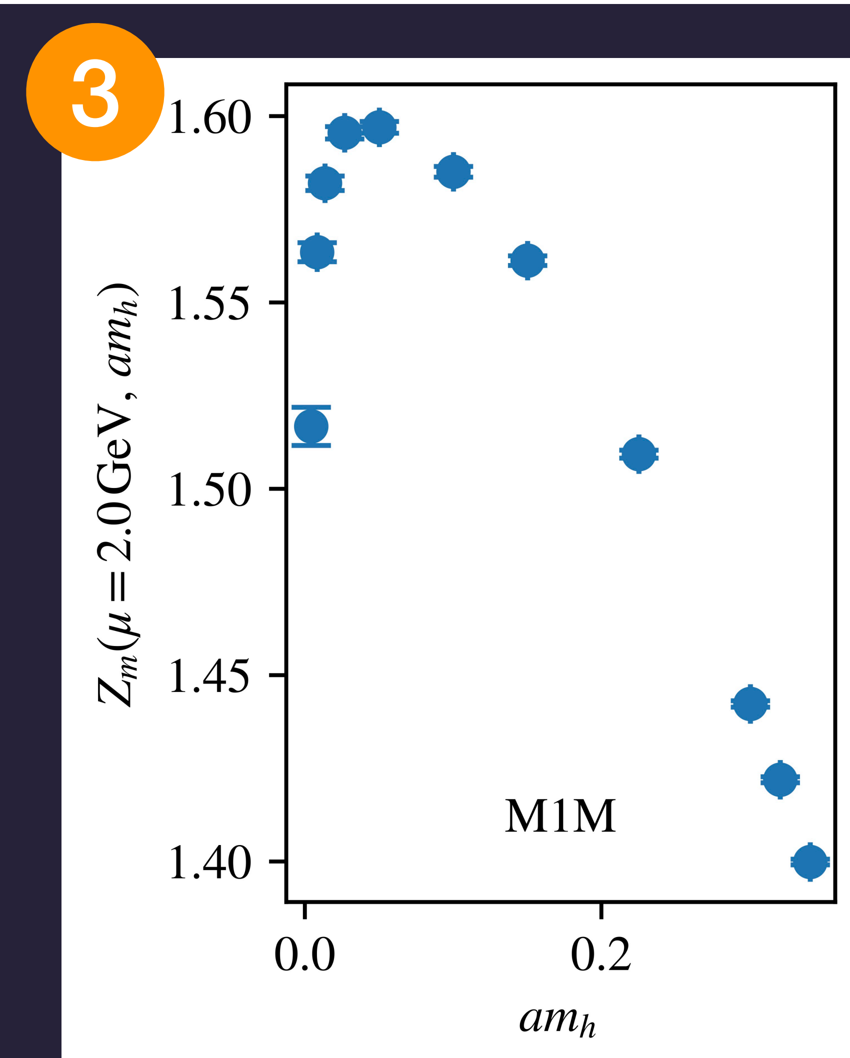

Ingredients
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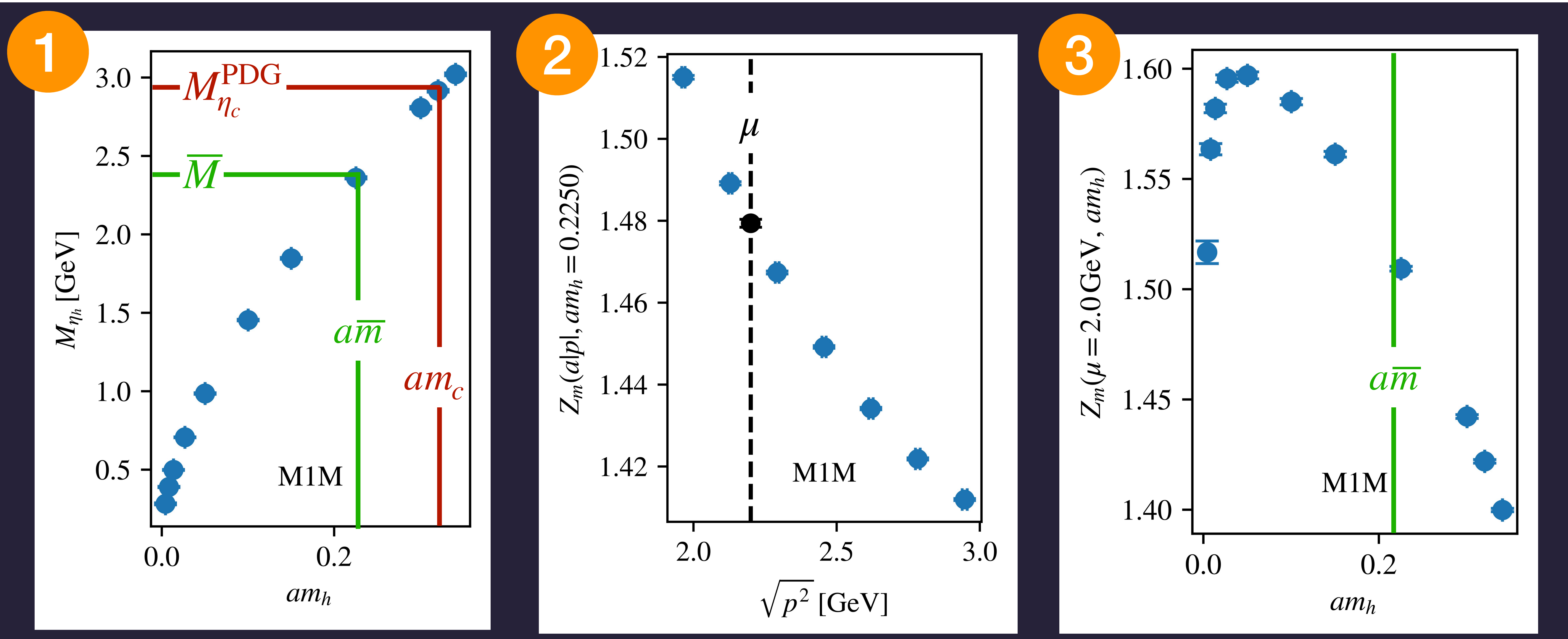
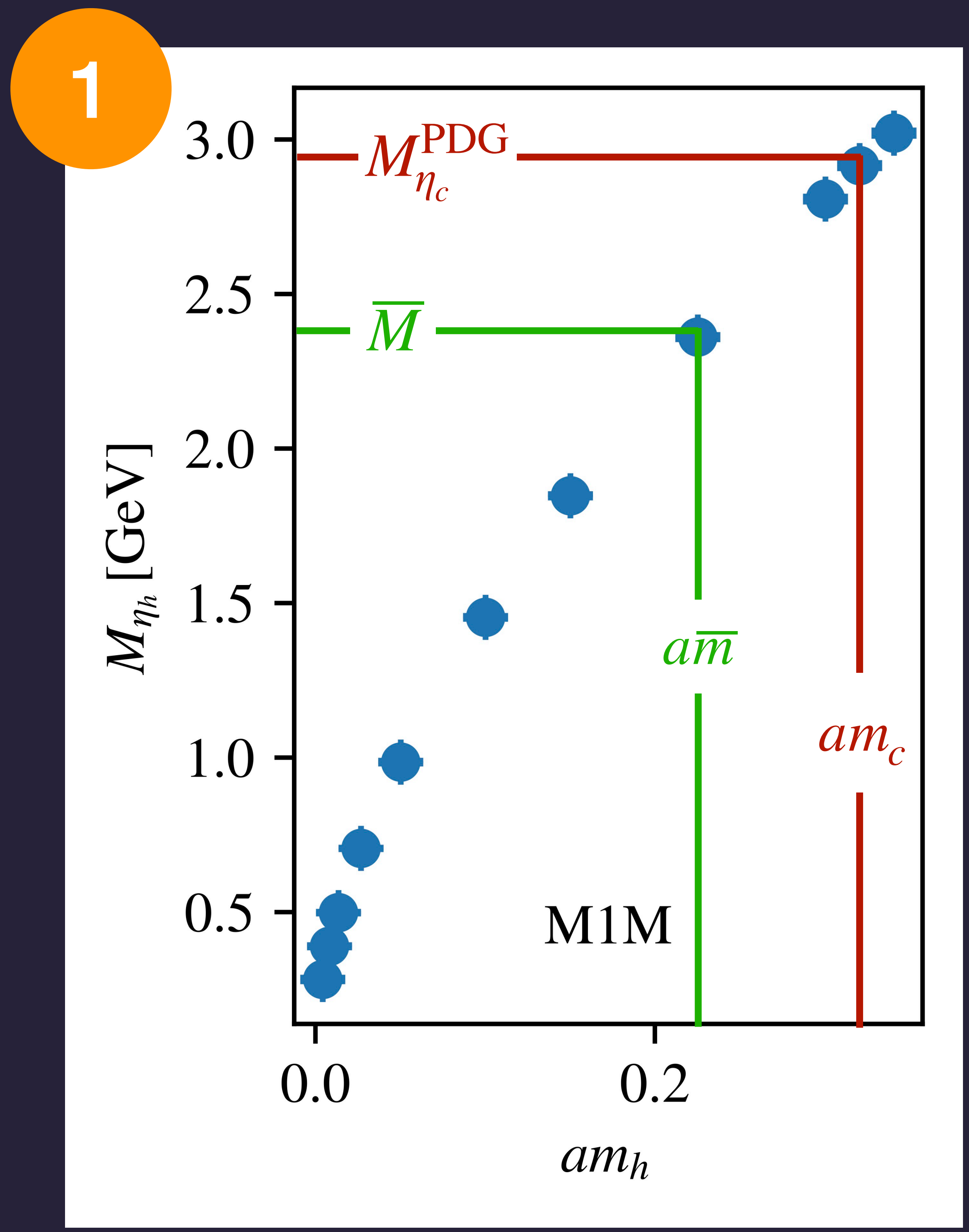


Reference scales

$$m_{c,R}(\mu, \overline{m}_R) = \lim_{a \to 0} Z_m(a\mu, a\overline{m})\, m_c$$



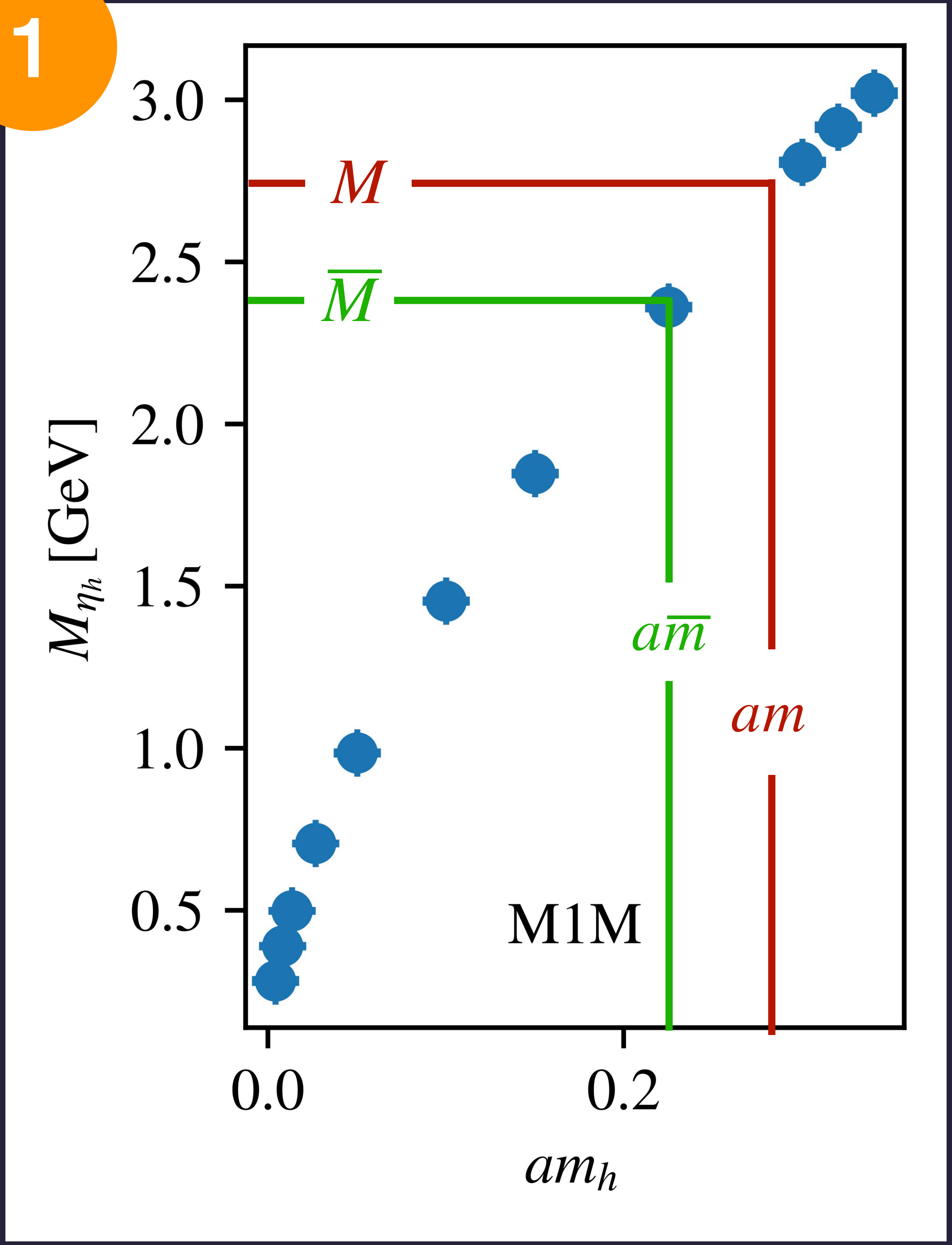
Reference scales

$$m_R(\mu, \overline{m}_R) = \lim_{a\to 0} Z_m(a\mu, a\overline{m})\, m$$

choose $\overline{M}$

choose M

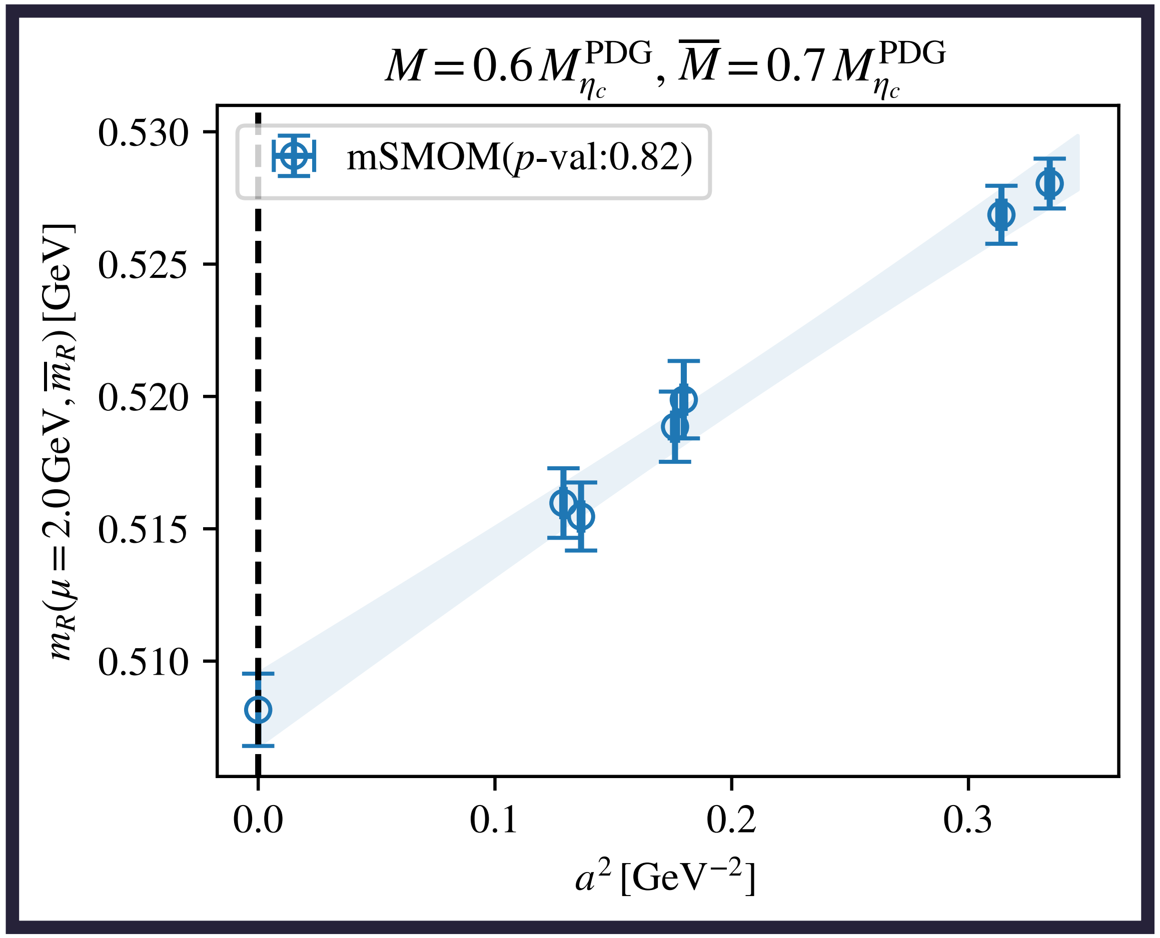
1



Continuum extrapolation
renormalised charm quark mass

Step 1. Choose $(M, \mu, \overline{M})$

$$m_R(\mu, \overline{m}_R) = \lim_{a \to 0} Z_m(a\mu, a\overline{m})\, m$$



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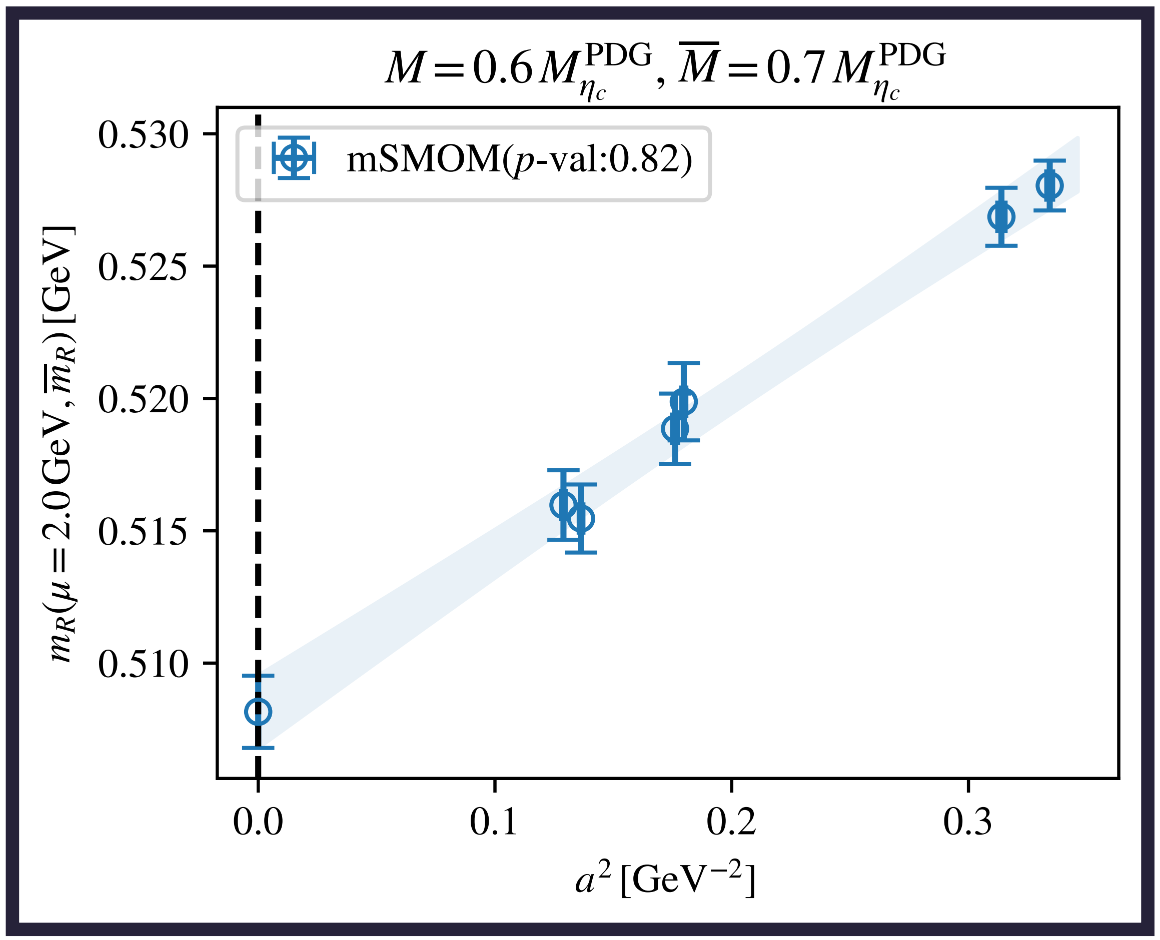
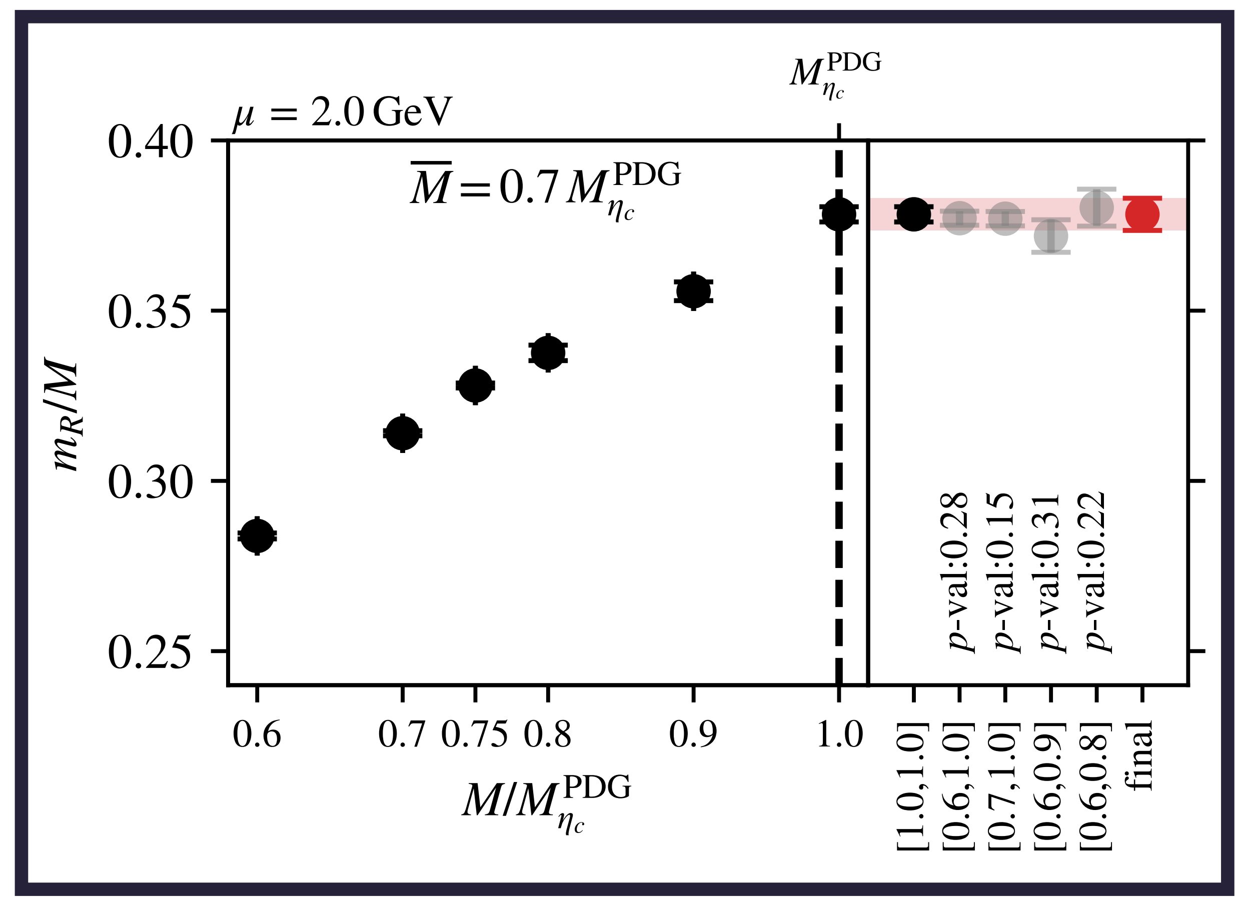
Step 1. Choose $(M, \mu, \overline{M})$

$$m_R(\mu, \overline{m}_R) = \lim_{a\to 0} Z_m(a\mu, a\overline{m})\, m$$

Step 2. Extrapolate to physical charm scale

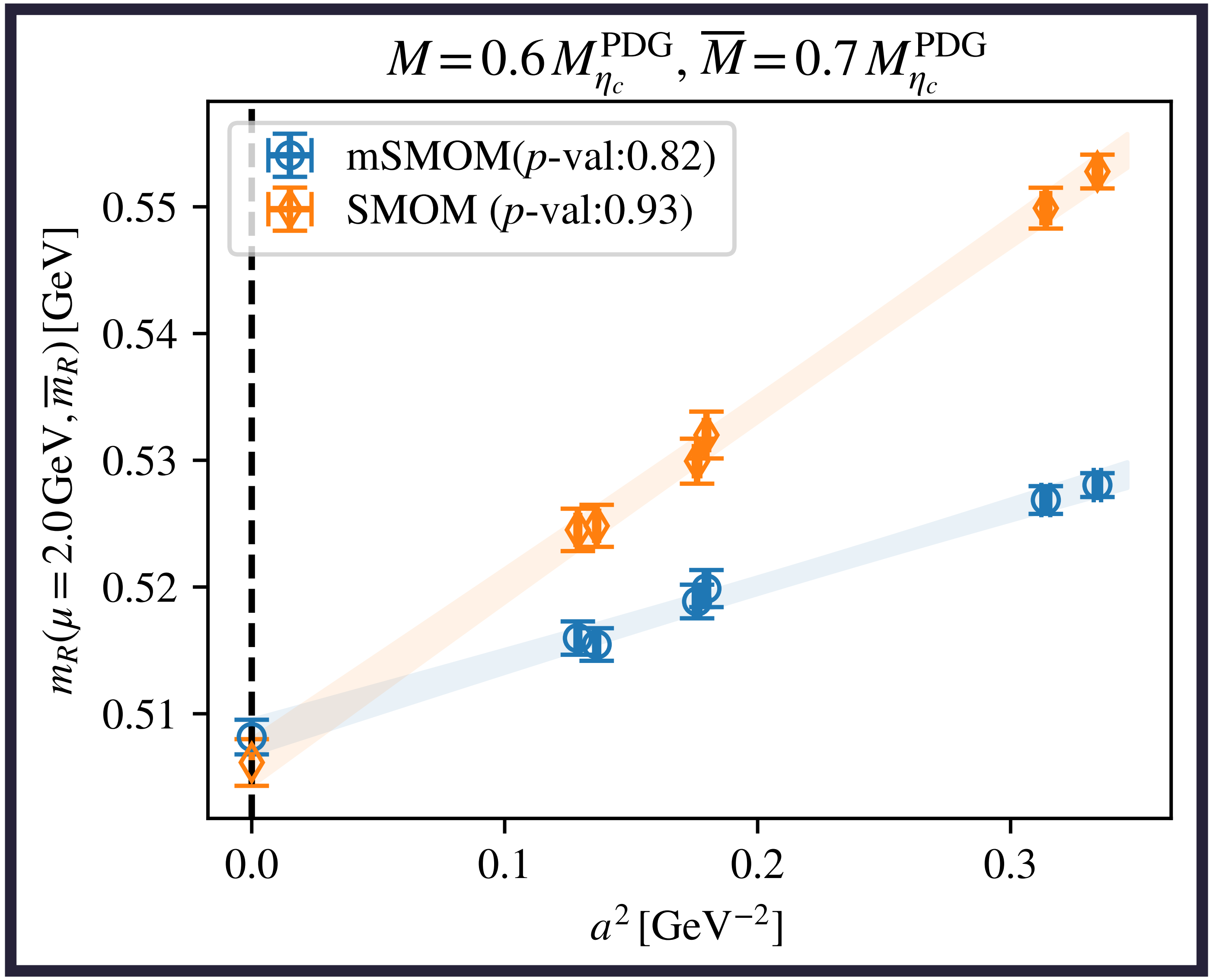
$$m_{c,R}(\mu, \overline{m}_R) = \lim_{M\to M_{\eta_c}^{\mathrm{PDG}}} m_R(\mu, \overline{m})$$





Absorption of cutoff effects

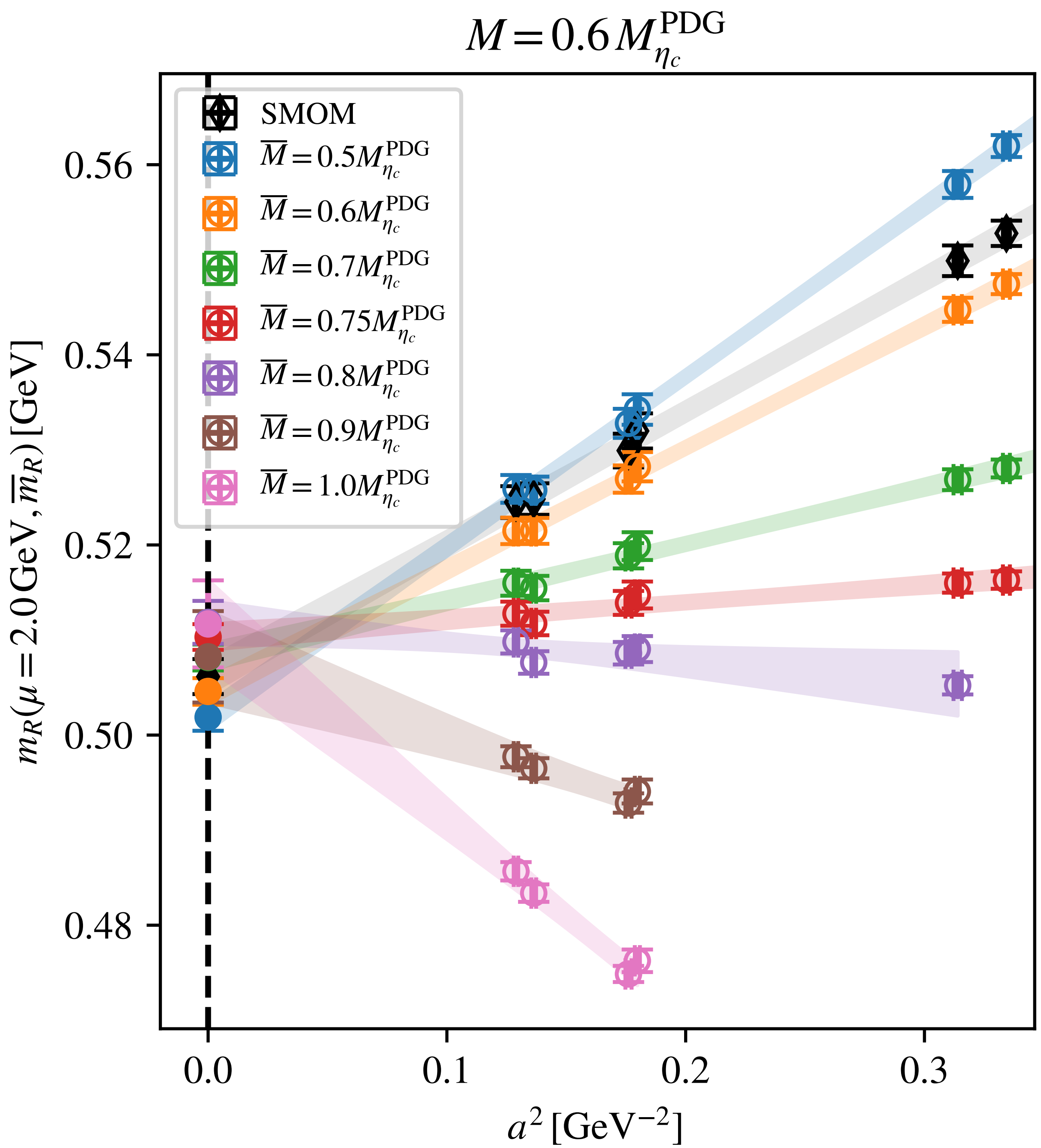
SMOM (massless) vs mSMOM (massive)



A flatter approach to the continuum using the massive scheme

Absorption of cutoff effects

tuning mSMOM reference mass using $\overline{M}$



$\overline{M}$ can be varied to find the flattest continuum approach

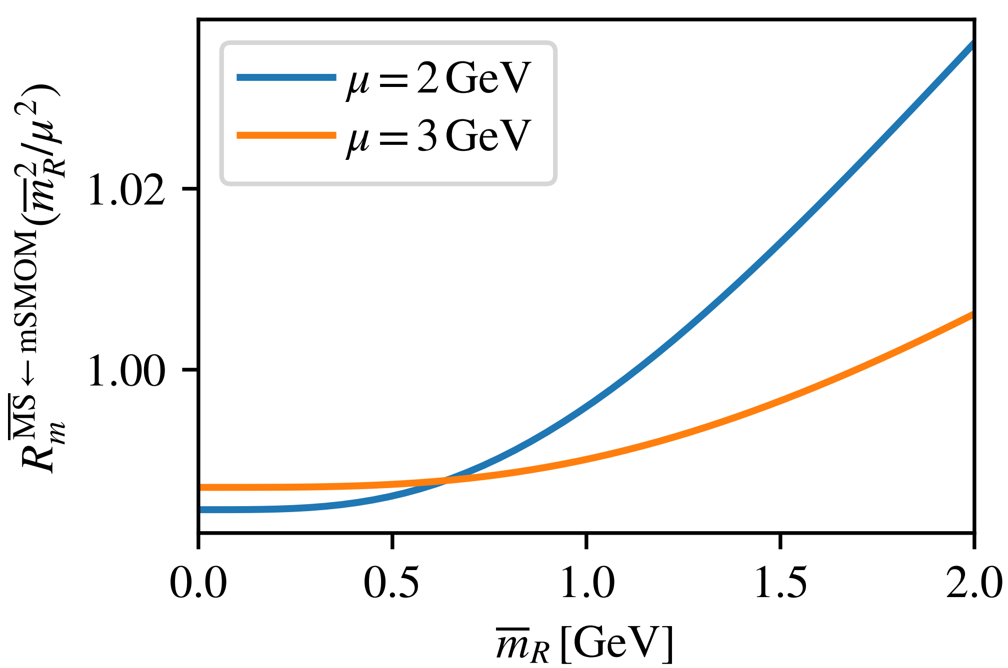
Matching to $\overline{\mathrm{MS}}$ renormalised charm quark mass

Step 3: Perturbative matching to $\overline{\mathrm{MS}}$ using $(\mu, \overline{m}_R)$

$$m_{c,R}^{\overline{\mathrm{MS}}}(\mu) = R_m^{\overline{\mathrm{MS}}\leftarrow\mathrm{mSMOM}}\left(\frac{\overline{m}_R^2}{\mu^2}\right) m_{c,R}^{\mathrm{mSMOM}}(\mu, \overline{m}_R)$$

- Conversion factors computed to 1-loop in Landau gauge: $(u = \overline{m}_R^2/\mu^2)$

$$R_m^{\overline{\mathrm{MS}}\leftarrow\mathrm{mSMOM}}(u) = 1 + \frac{\alpha}{4\pi}C_F\left[-4 + \frac{3}{2}C_0(u) + 3\ln(1+u) - 3u\ln\left(\frac{u}{1+u}\right)\right]$$



Matching to $\overline{\mathrm{MS}}$
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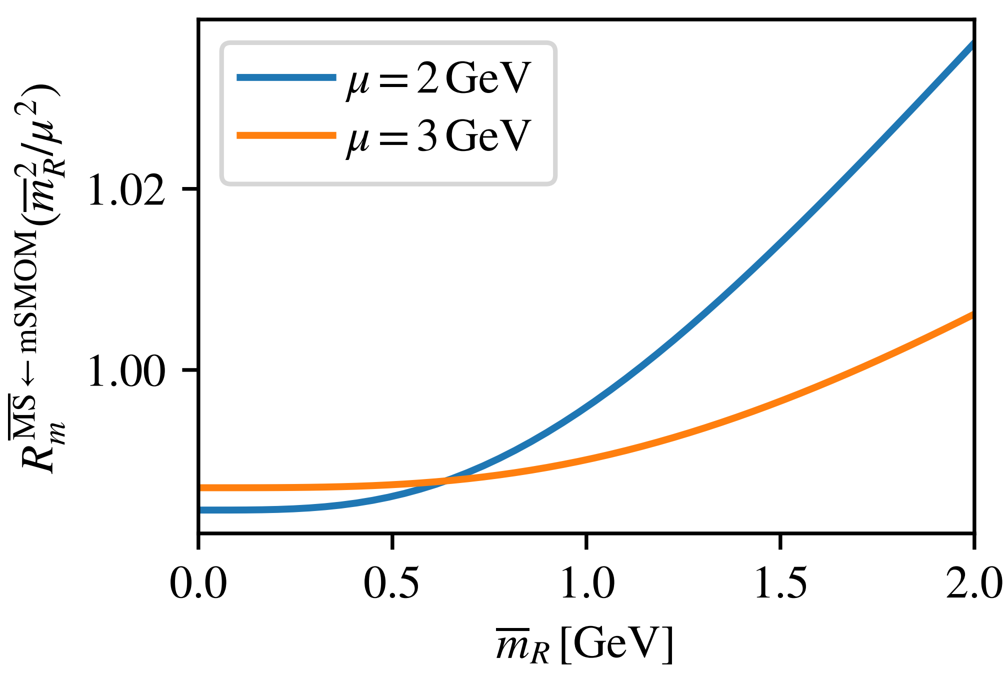
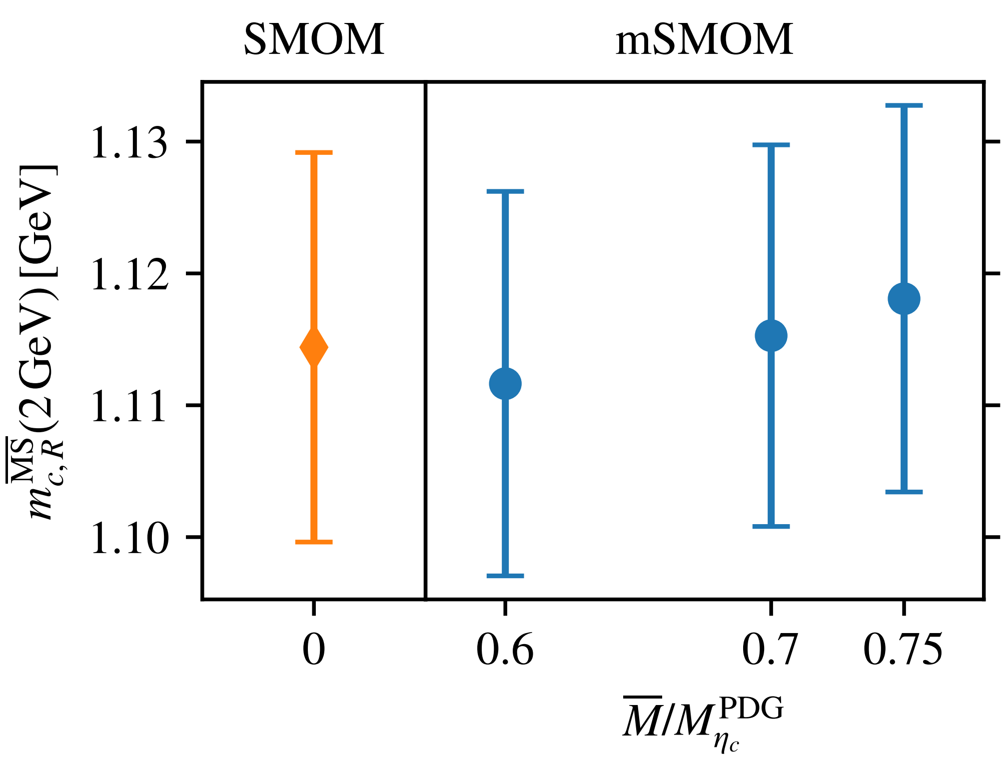




Final results

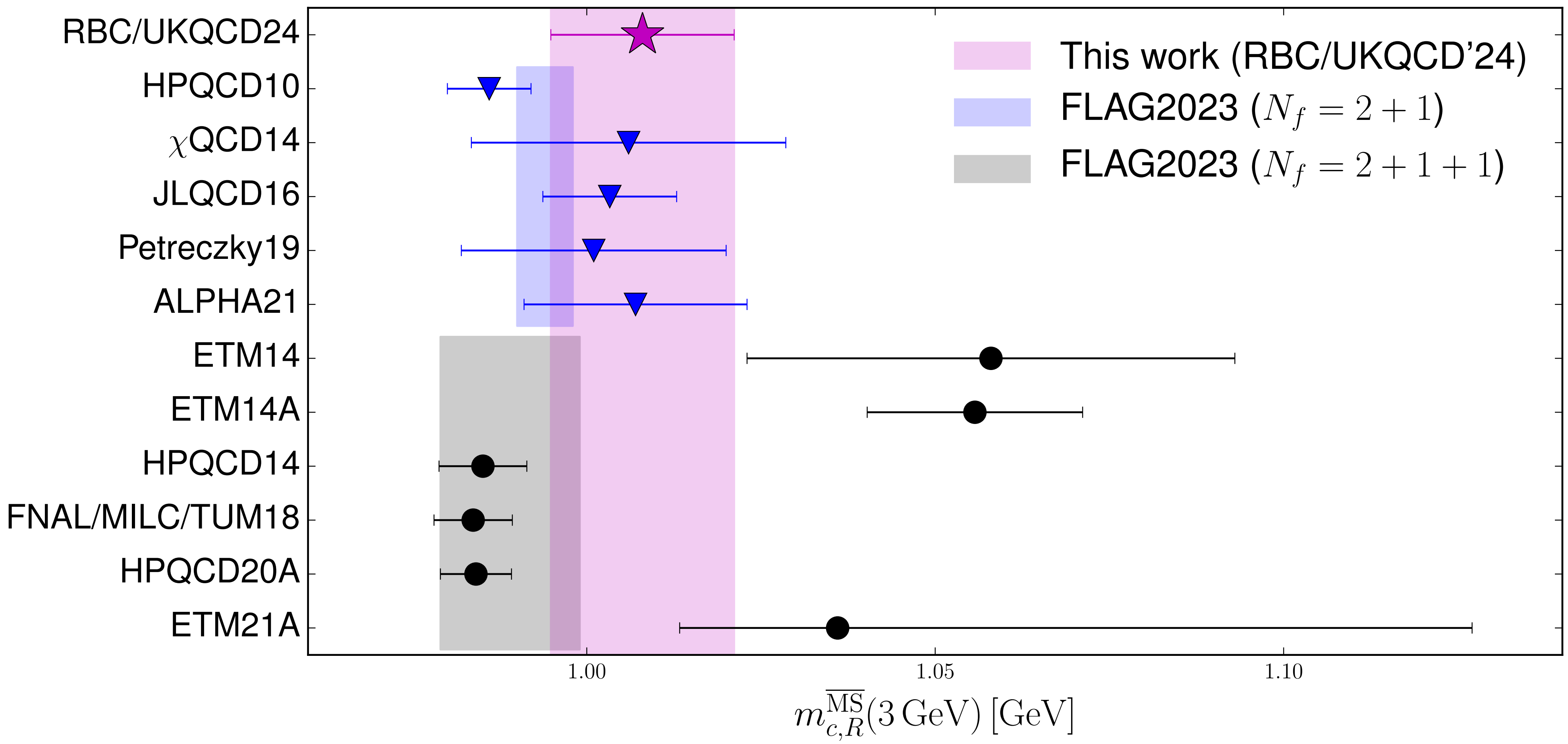
renormalised charm quark mass

✓ Full error budget with statistical + systematic + PT matching error

$$m_{c,R}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 1.115(7)(12)(4)\,\mathrm{GeV}$$

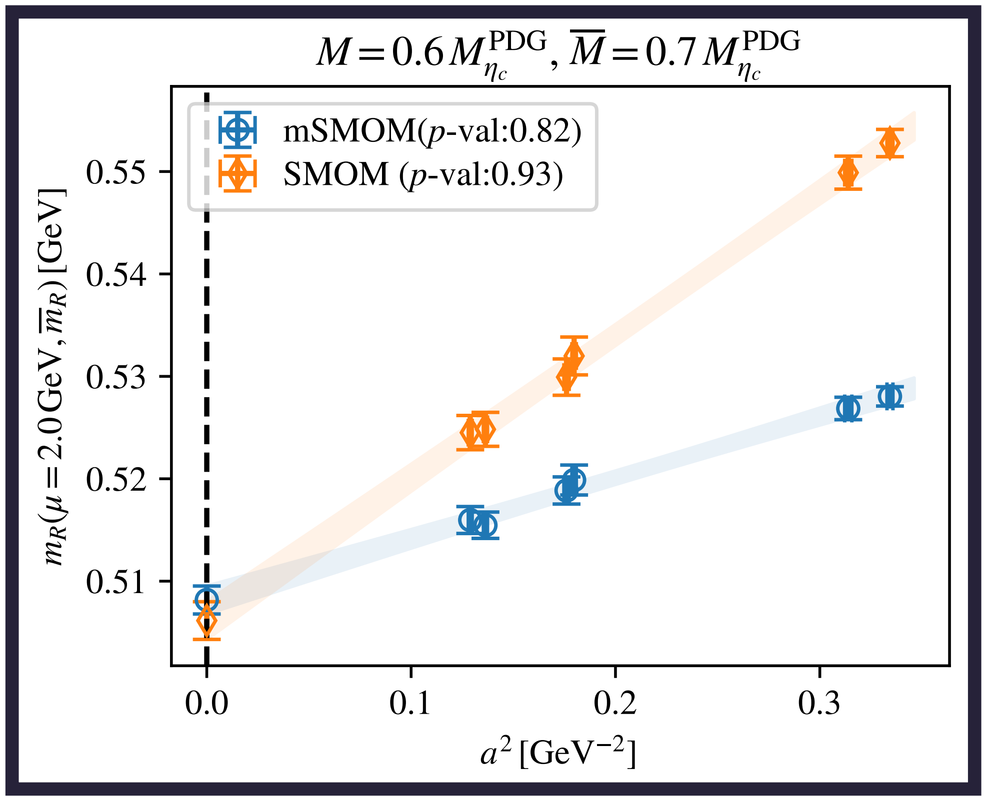
$$m_{c,R}^{\overline{\mathrm{MS}}}(3\,\mathrm{GeV}) = 1.008(6)(11)(4)\,\mathrm{GeV}$$

$$m_{c,R}^{\overline{\mathrm{MS}}}(m_{c,R}^{\overline{\mathrm{MS}}}) = 1.292(5)(10)(4)\,\mathrm{GeV}$$

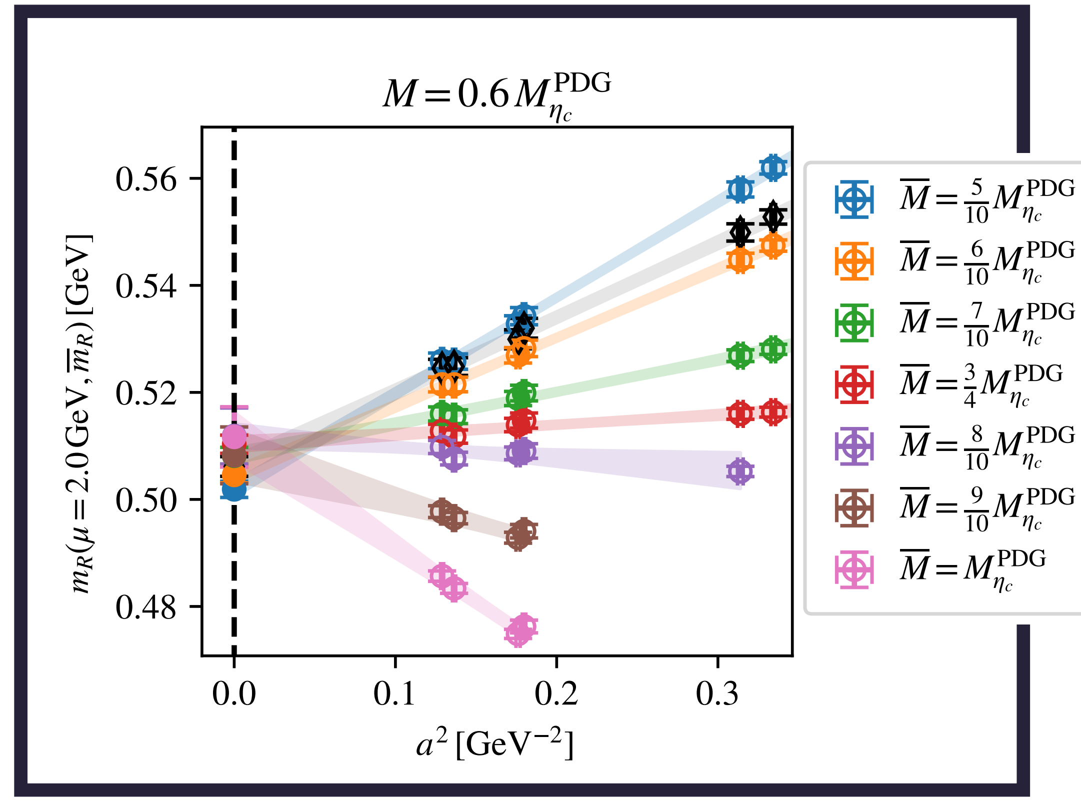


Main takeaways and outlook

- A massive NPR scheme: RI/mSMOM, test case: charm quark mass



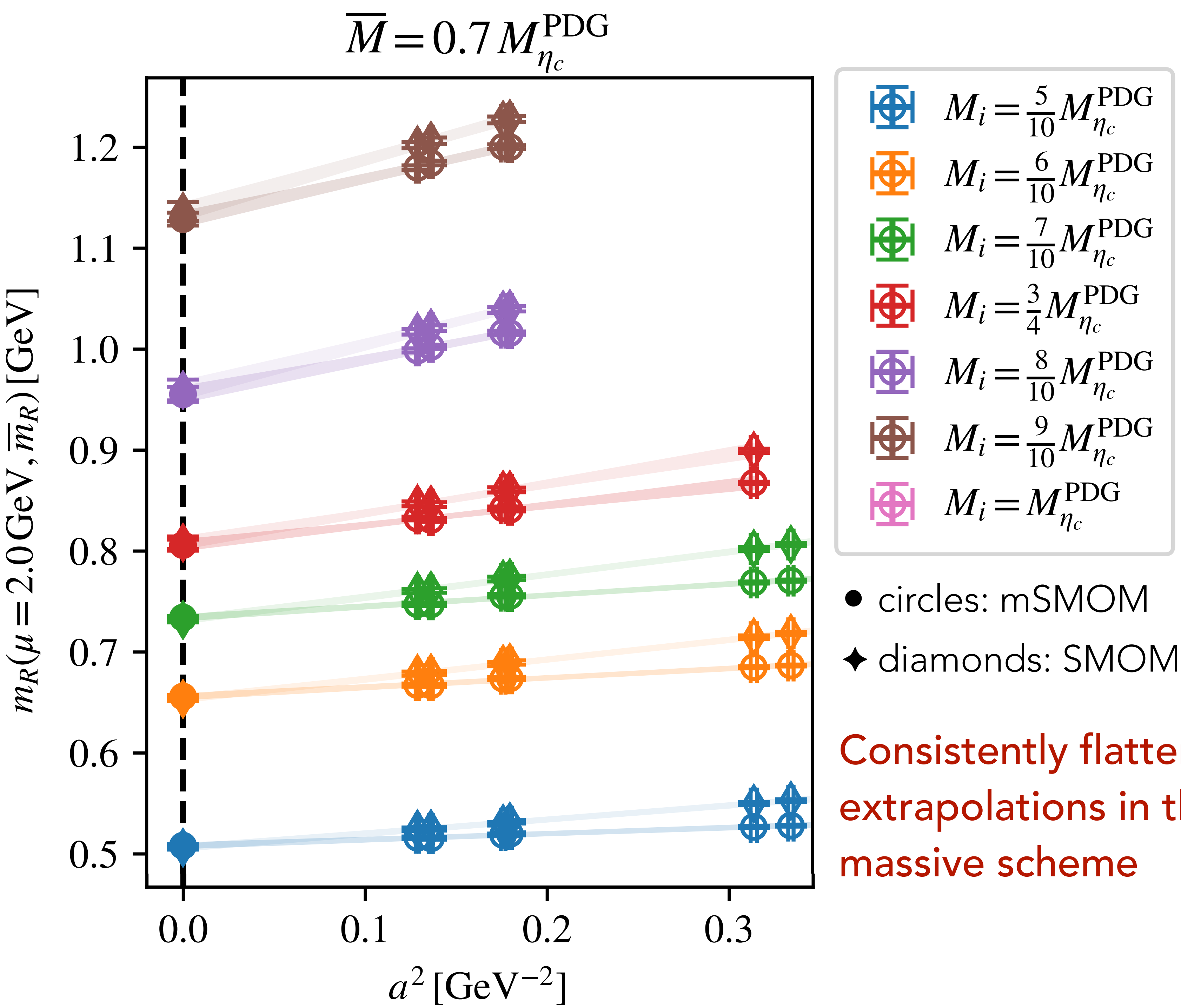
Milder continuum extrapolation using the massive scheme



Can tune $\overline{m}_R$ to find flattest approach to the continuum (observable-dependent)

- Study other bilinear operators
- Expand RI/mSMOM scheme for four-quark vertices

Backup: variations with reference mass M_i



- circles: mSMOM
- ✦ diamonds: SMOM

Consistently flatter extrapolations in the massive scheme

Backup: pion mass dependence

name	L/a	T/a	a^{-1}[GeV]	M_π[MeV]	am_l	am_s
C1M	24	64	1.7295(38)	276	0.005	0.0362
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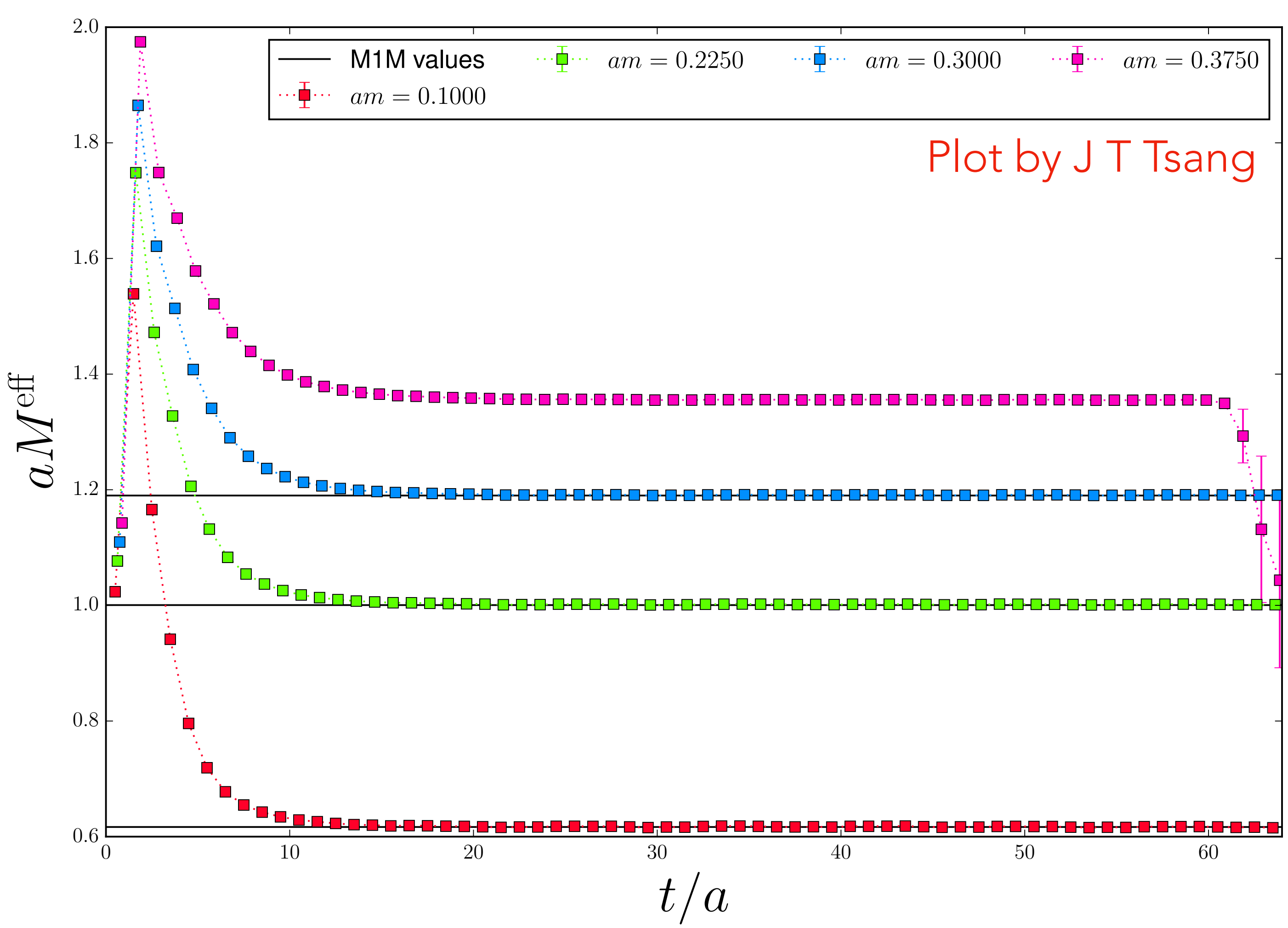


- Comparison of aM_{η_h} values on M1M ensemble to those on M0M ensemble (physical point) - good agreement!
- Pion mass dependence (from sea effects) expected to be low/negligible

Backup: mSMOM renormalisation conditions

$$Z_q: \quad \lim_{\boxed{m_R \to \overline{m}_R}} \frac{1}{12p^2} \mathrm{Tr}\left[-iS_R(p)^{-1} \not{p}\right]\Big|_{p^2=\mu^2} = 1,$$

$$Z_m: \quad \lim_{m_R \to \overline{m}_R} \frac{1}{12m_R} \left\{ \mathrm{Tr}\left[S_R(p)^{-1}\right]\Big|_{p^2=\mu^2} + \frac{1}{2}\mathrm{Tr}\left[\left(iq \cdot \Lambda_{\mathrm{A},R}\right)\gamma_5\right]\Big|_{\mathrm{sym}} \right\} = 1,$$

$$Z_V: \quad \lim_{m_R \to \overline{m}_R} \frac{1}{12q^2} \mathrm{Tr}\left[\left(q \cdot \Lambda_{\mathrm{V},R}\right) \not{q}\right]\Big|_{\mathrm{sym}} = 1,$$

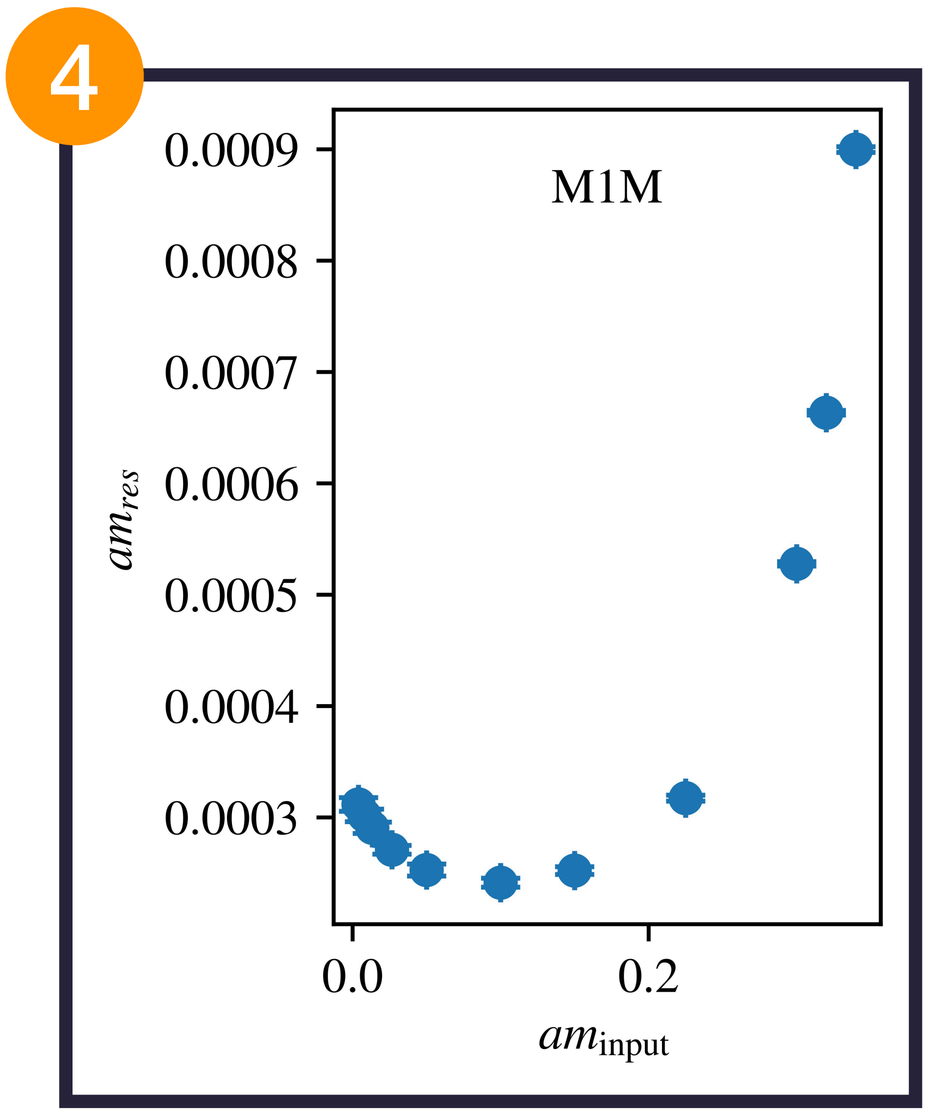
$$Z_A: \quad \lim_{m_R \to \overline{m}_R} \frac{1}{12q^2} \mathrm{Tr}\left[\left(q \cdot \Lambda_{\mathrm{A},R} + \boxed{2m_R \Lambda_{P,R}}\right)\gamma_5 \not{q}\right]\Big|_{\mathrm{sym}} = 1,$$

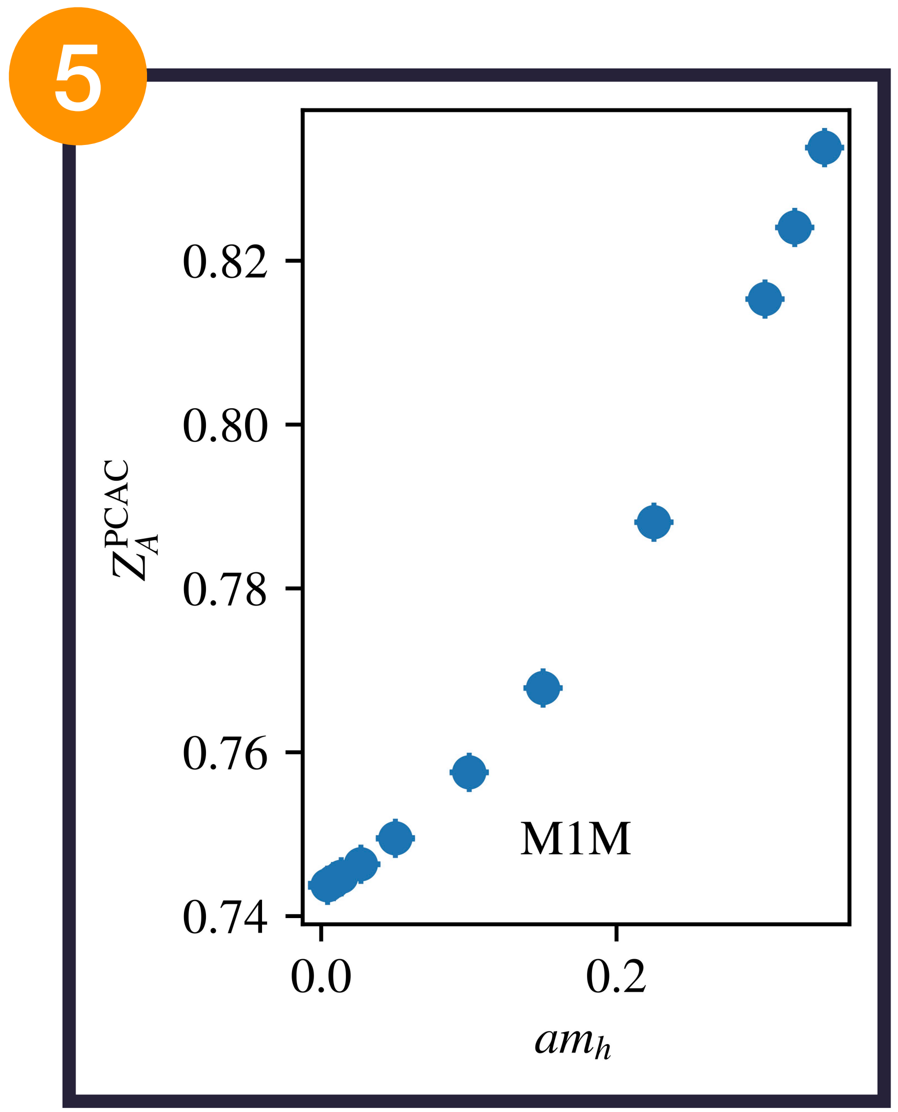
$$Z_P: \quad \lim_{m_R \to \overline{m}_R} \frac{1}{12} \mathrm{Tr}\left[\Lambda_{\mathrm{P},R}\gamma_5\right]\Big|_{\mathrm{sym}} = 1,$$

$$Z_S: \quad \lim_{m_R \to \overline{m}_R} \left\{ \frac{1}{12}\mathrm{Tr}\left[\Lambda_{\mathrm{S},R}\right] + \boxed{\frac{1}{6q^2}\mathrm{Tr}\left[2m_R \Lambda_{P,R}\gamma_5 \not{q}\right] \Big\}}\right.\Big|_{\mathrm{sym}} = 1.$$

Backup: other ingredients

- For DWFs, $am_h = am_{\text{input}} + am_{\text{res}}$, using plateau of $am_{\text{res}}^{\text{eff}}(t) = \dfrac{\langle PJ_{5q}\rangle(t)}{\langle PP\rangle(t)}$
- Set $Z_A = Z_A^{\text{PCAC}}$, using plateau of $Z_A^{\text{eff}}(t) = \dfrac{1}{2}\left[\dfrac{C(t+\frac{1}{2}) + C(t-\frac{1}{2})}{2L(t)} + \dfrac{2C(t+\frac{1}{2})}{L(t)+L(t+1)}\right]$

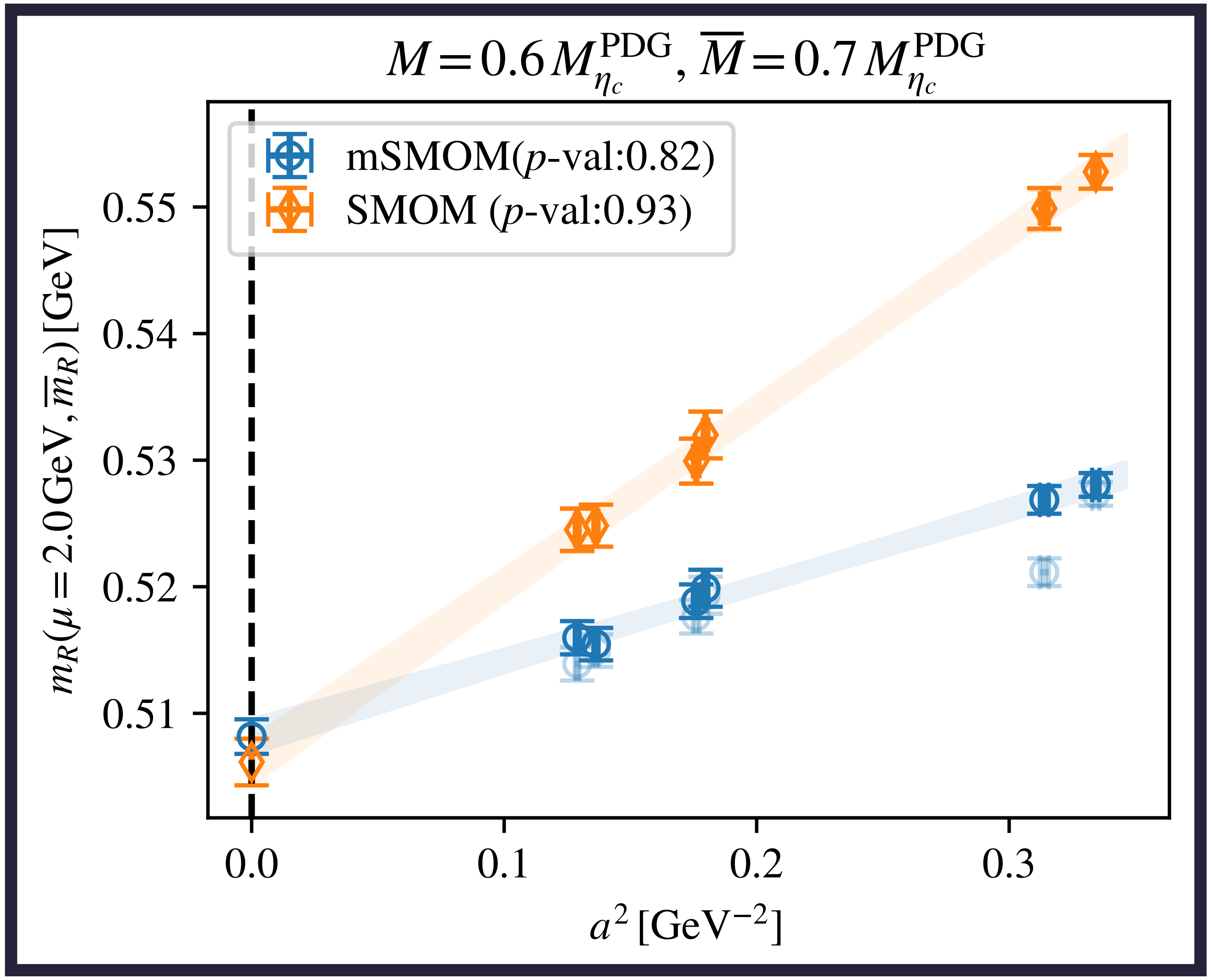




Backup: continuum extrapolation: am_{res}

Fit ansatz:

$$m(a) = m(0)\left[1 + \alpha a^2 + \beta am_{\text{res}}(M)\right]$$

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