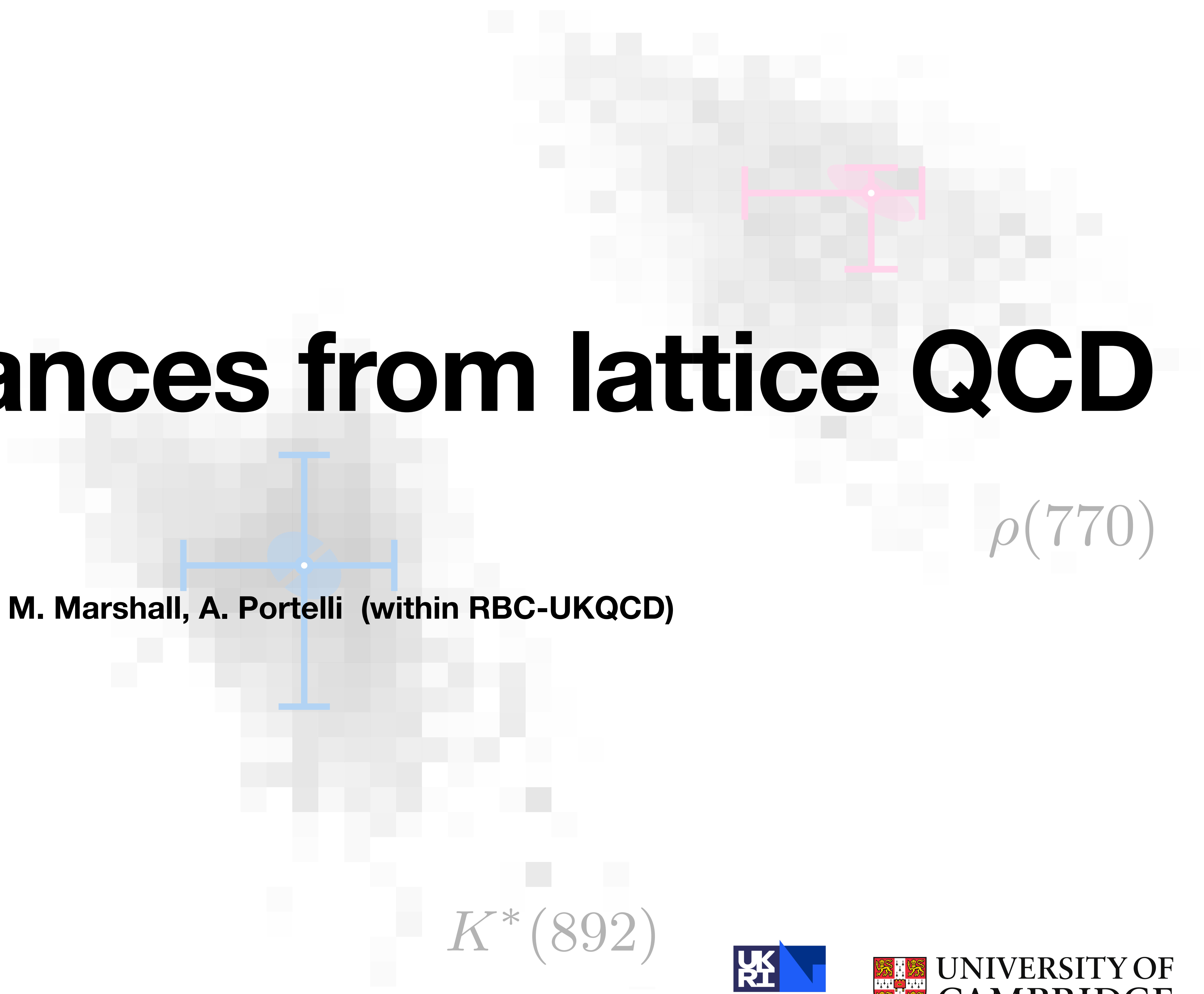
UKLFT
Annual
Meeting

Hadronic resonances from lattice QCD



Nelson Pitanga Lachini

in collaboration with:
P. Boyle, F. Erben, V. Gülpers, M. T. Hansen, F. Joswig, M. Marshall, A. Portelli (within RBC-UKQCD)

PhysRevD.111.054510
PhysRevLett.134.111901

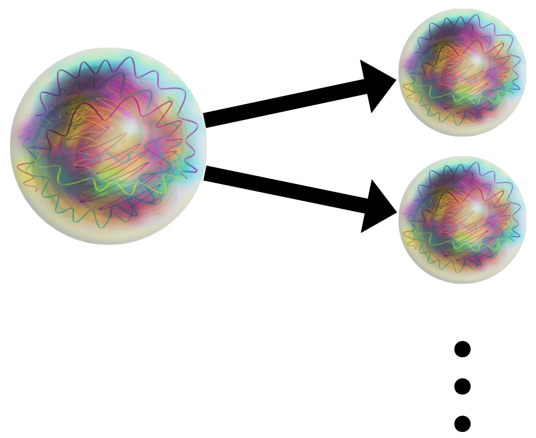
np612@cam.ac.uk
28th April 2025

Science and Technology Facilities Council

UNIVERSITY OF CAMBRIDGE

Hadronic Resonances

SM ⊃ QCD: hadrons, most ***resonances***

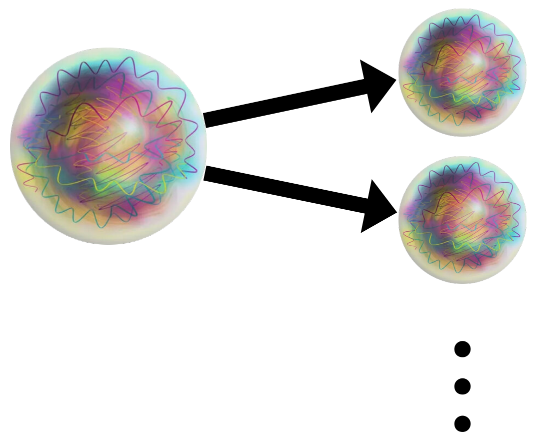


Hadronic Resonances

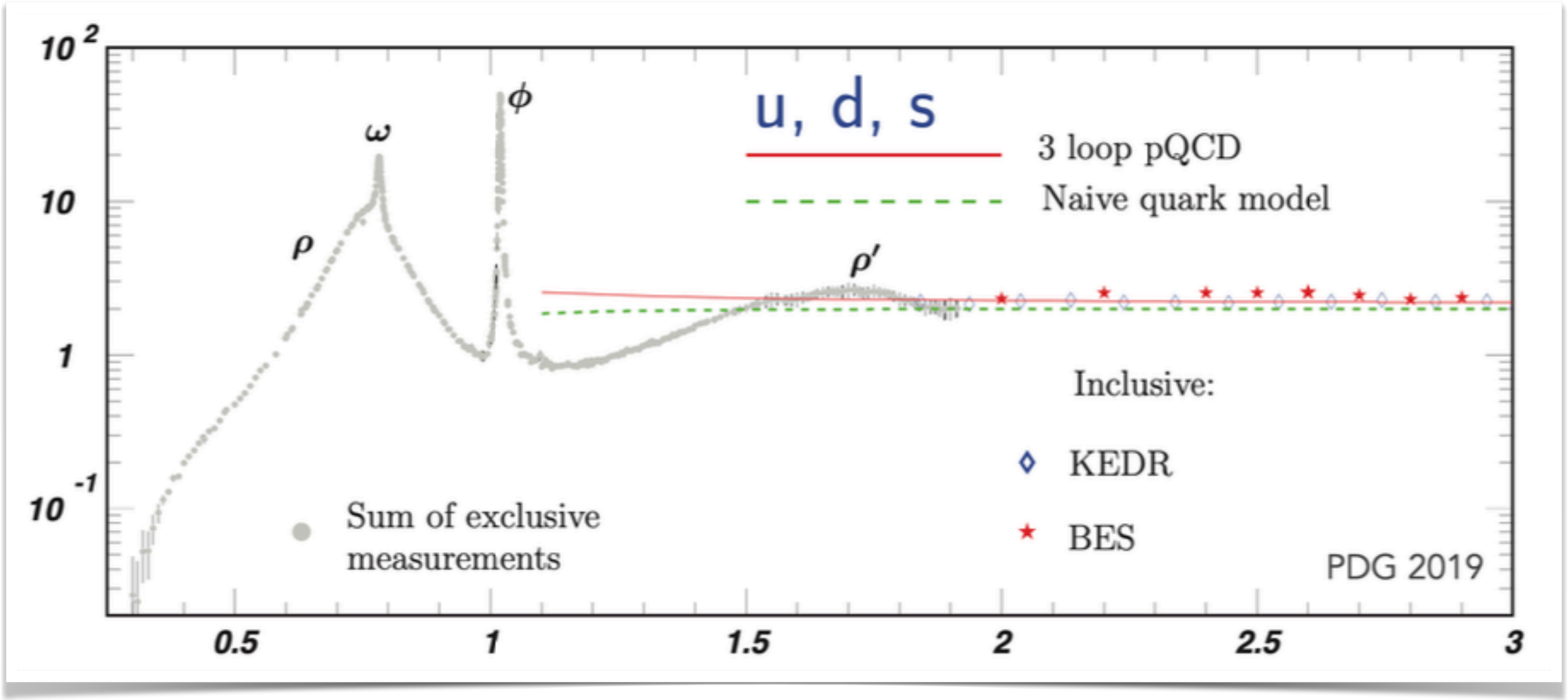
SM ⊃ QCD: hadrons, most ***resonances***

Phenomenological description:

- cross-section "enhancements"
- process dependent

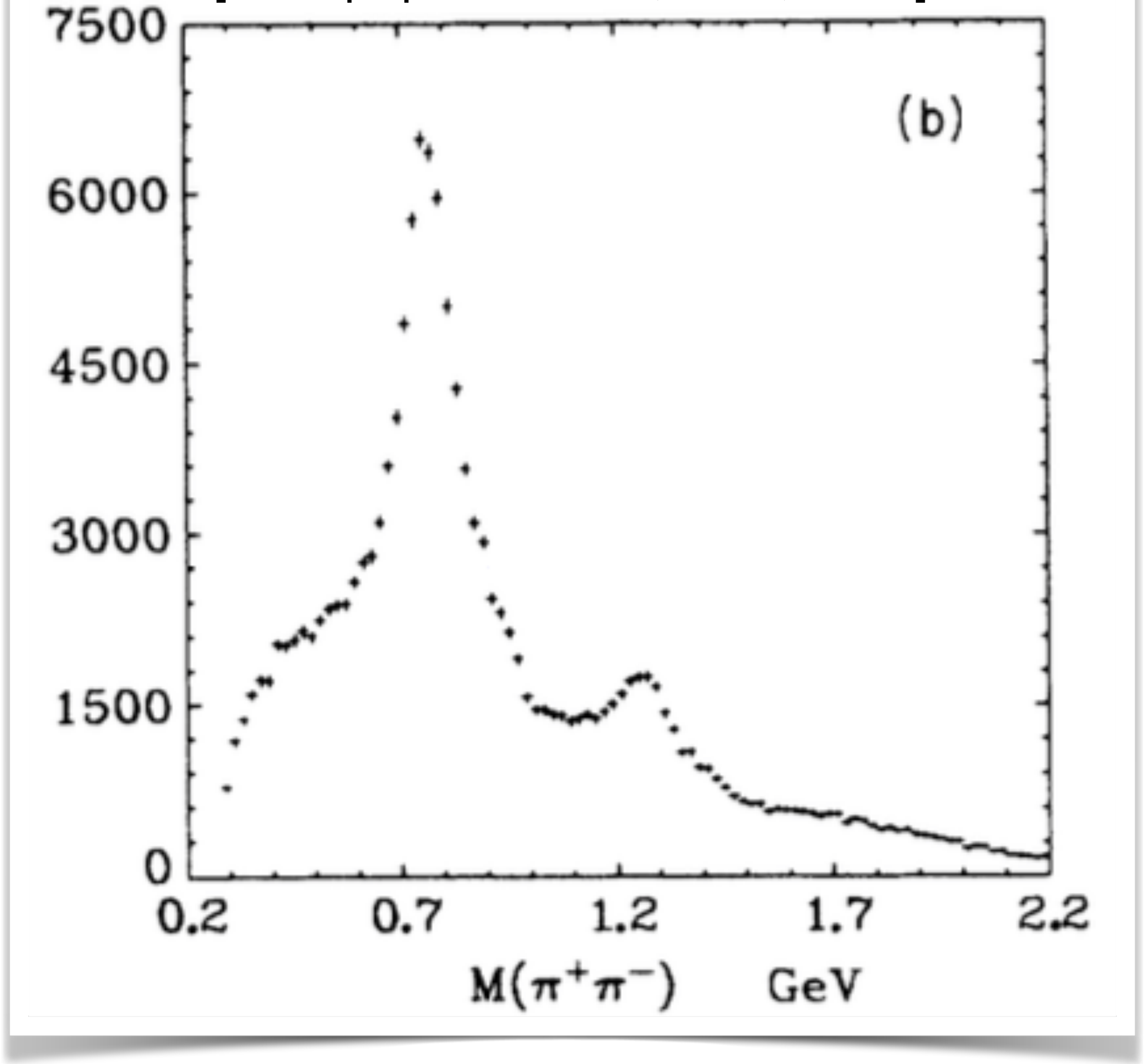


$e^+e^- \to \pi\pi \; (\to \rho \to \pi\pi)$



$\pi p \to \Delta\pi\pi \; (\to \rho \to \pi\pi)$

[Protopopescu et al, PRD, 1973]



Hadronic Resonances

SM ⊃ QCD: hadrons, most ***resonances***

eg, Breit-Wigner

$$\sigma \propto \frac{1}{(s - m_{bw}^2)^2 + \Gamma_{bw}^2 m_{bw}^2}$$

Phenomenological description:

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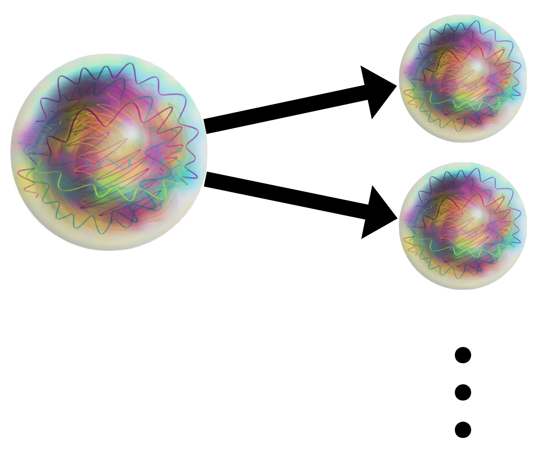

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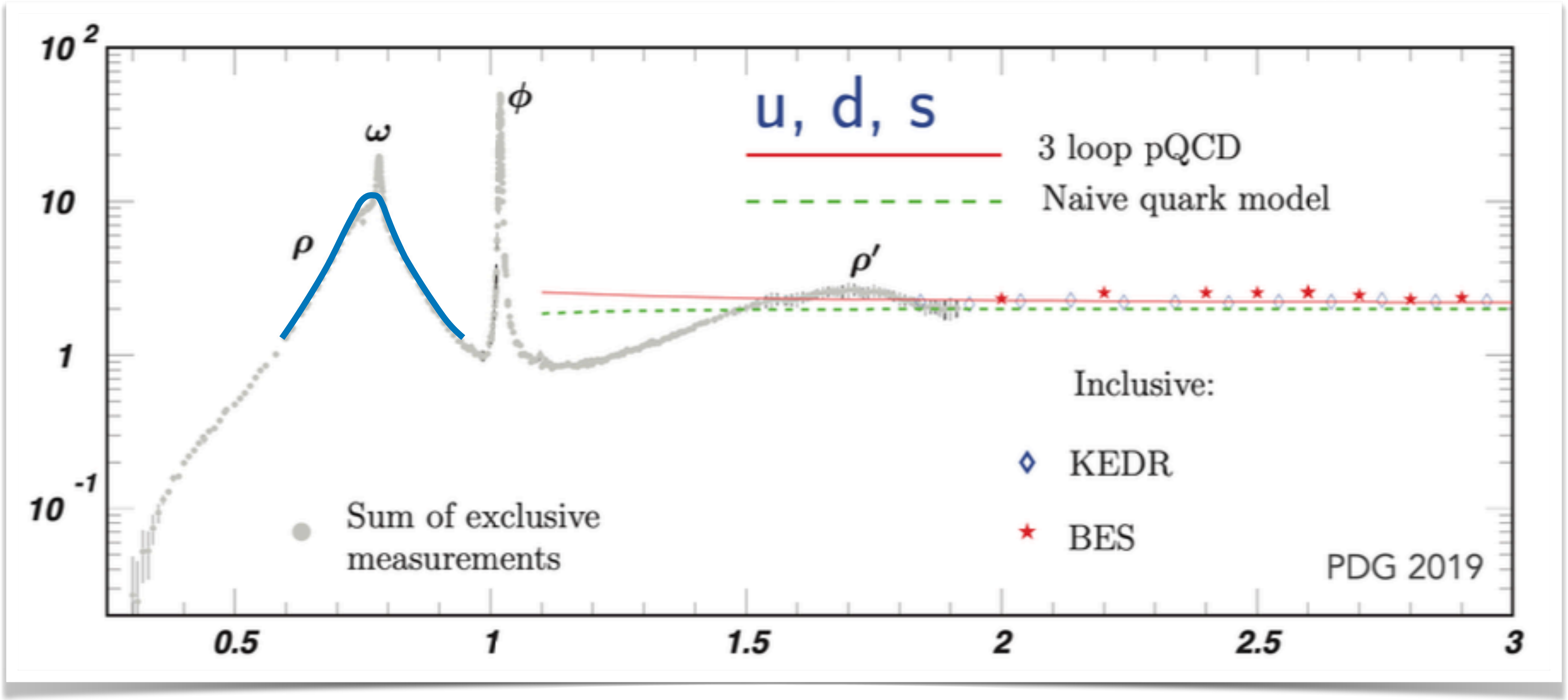

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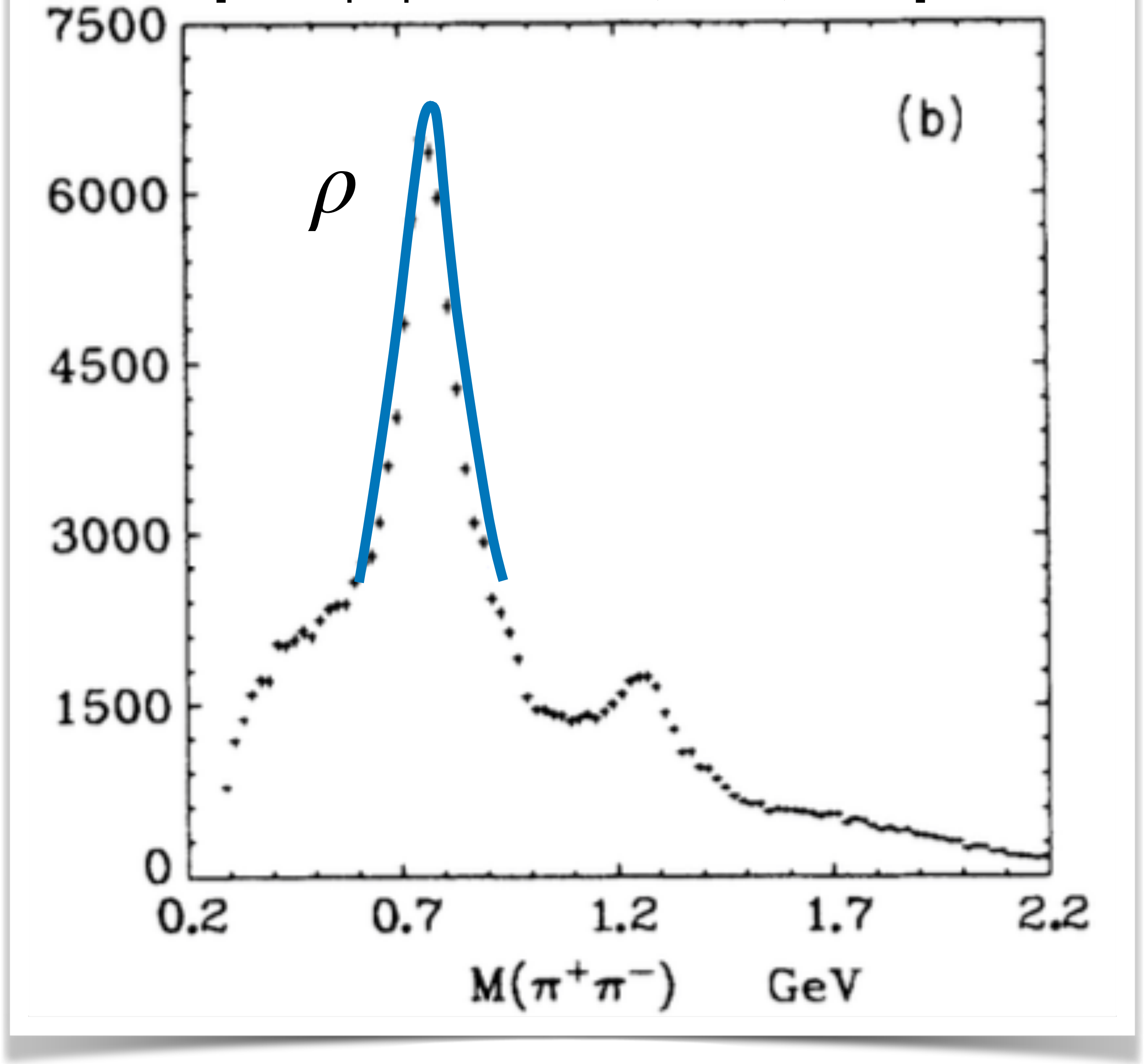


(Beyond) Standard Model

Experiments

(Beyond) Standard Model

Experiments

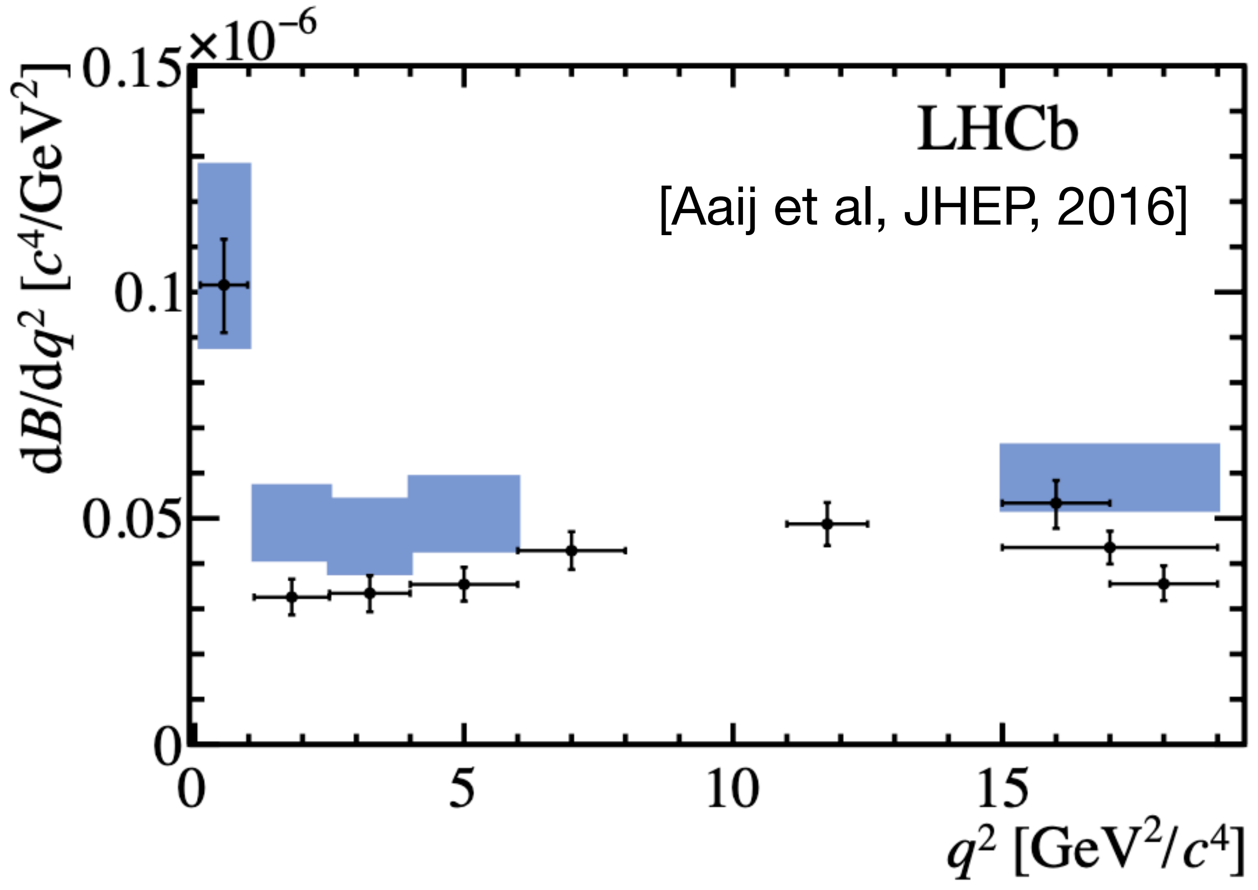
LFUV

$B \to K^* \ell^+ \ell^-$

LHCb [Aaij et al, JHEP, 2016] [Aaij et al, Nature, 2022] [Aaij et al, PRD,2023]

$B \to \rho \ell \nu$

Belle II [Abudinén, 2206.05946v4, 2020] [Bernlochner, PRD, 2014]



CP violation in strange, charm

$K \to \pi\pi \quad (\sigma?)$

LHCb [Aaij et al, PRL, 2019]

$D \to \pi\pi, K\bar{K} \quad (f_0(1710)?)$

BaBar [Aubert et al, PRL, 2008]

Muon $g-2$

Muon g-2 [Aguillard et al, PRD, 2024]

$e^+e^- \to \rho \to \pi\pi$

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⋮

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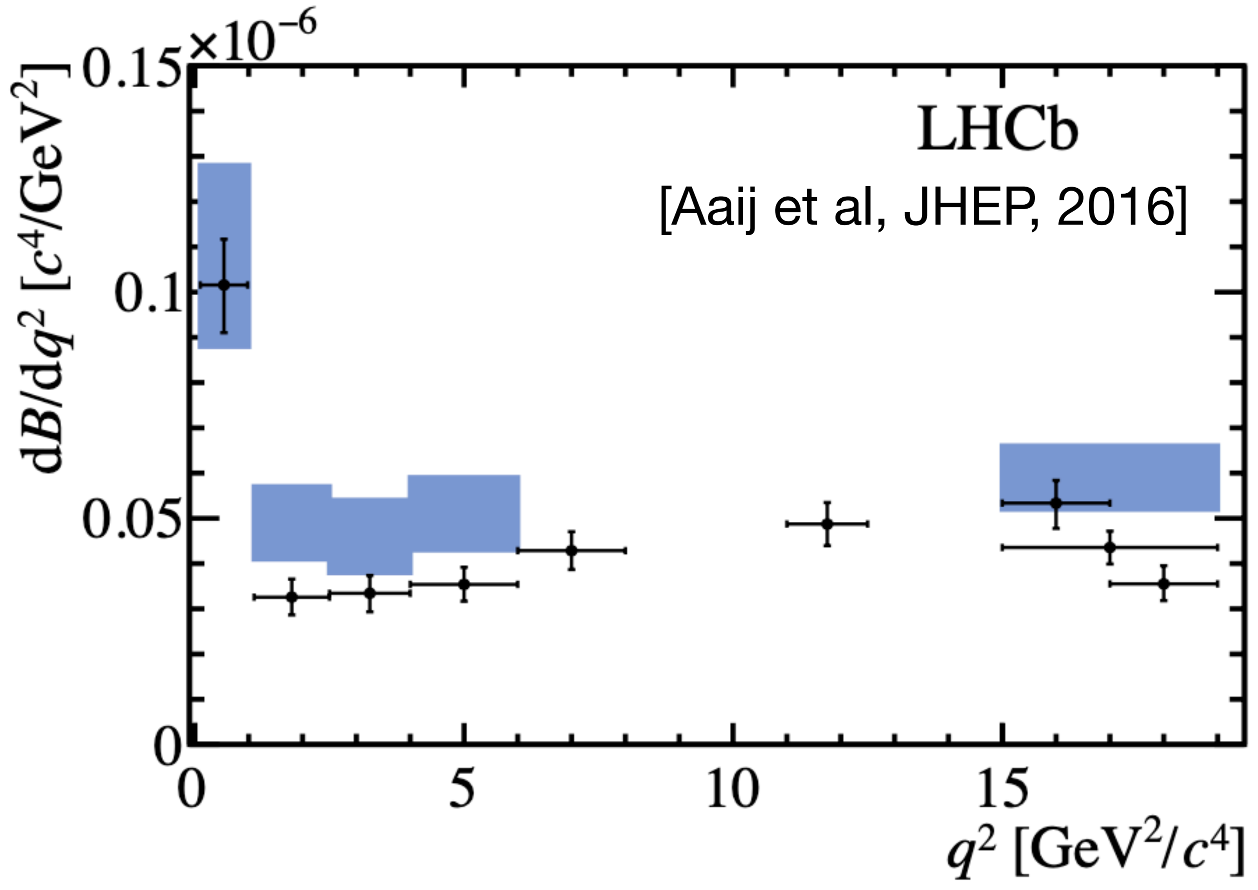

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⋮

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nonperturbative phenomena

[Gurbernari et al, PRL, 2019]

[Schacht & Soni, PRB, 2022]

⋮

Muon $g-2$

$e^+e^- \to \rho \to \pi\pi$

Muon g-2 [Aguillard et al, PRD, 2024]

→control the QCD side

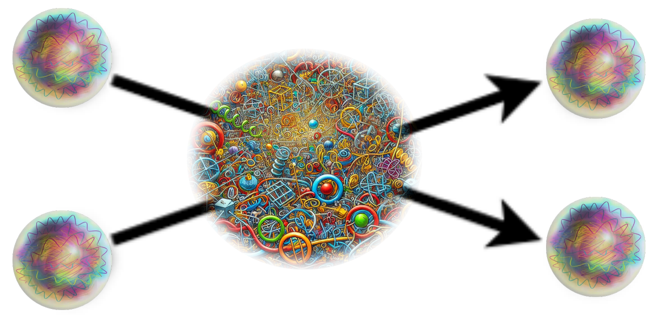
→this work: $\rho \to \pi\pi, \quad K^* \to K\pi$

Phase Shift

unitarity & symmetry

$$S_\ell(E_{cm}) = e^{2i\delta_\ell(E_{cm})}$$

partial waves



scattering amplitude

$$T_\ell = \left[S - \mathbf{1}\right]_\ell$$
$$= (\cot\delta_\ell - i)^{-1}$$

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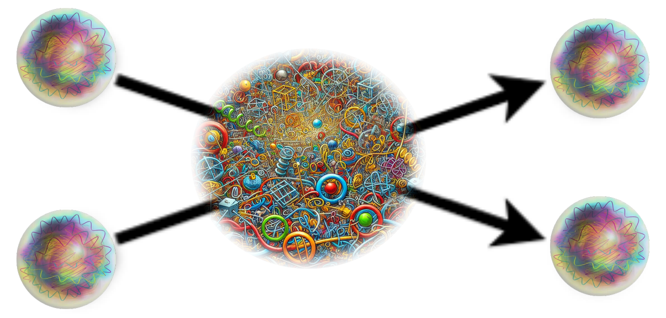

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$$T_\ell = \left[S - \mathbf{1}\right]_\ell = (\cot\delta_\ell - i)^{-1}$$

Poles

$$T_\ell(E_{\text{cm}}) \to T_\ell\left(\sqrt{s}\right), \quad \sqrt{s} \text{ complex}$$

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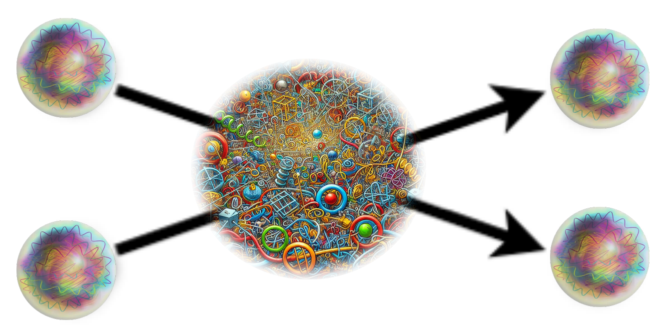

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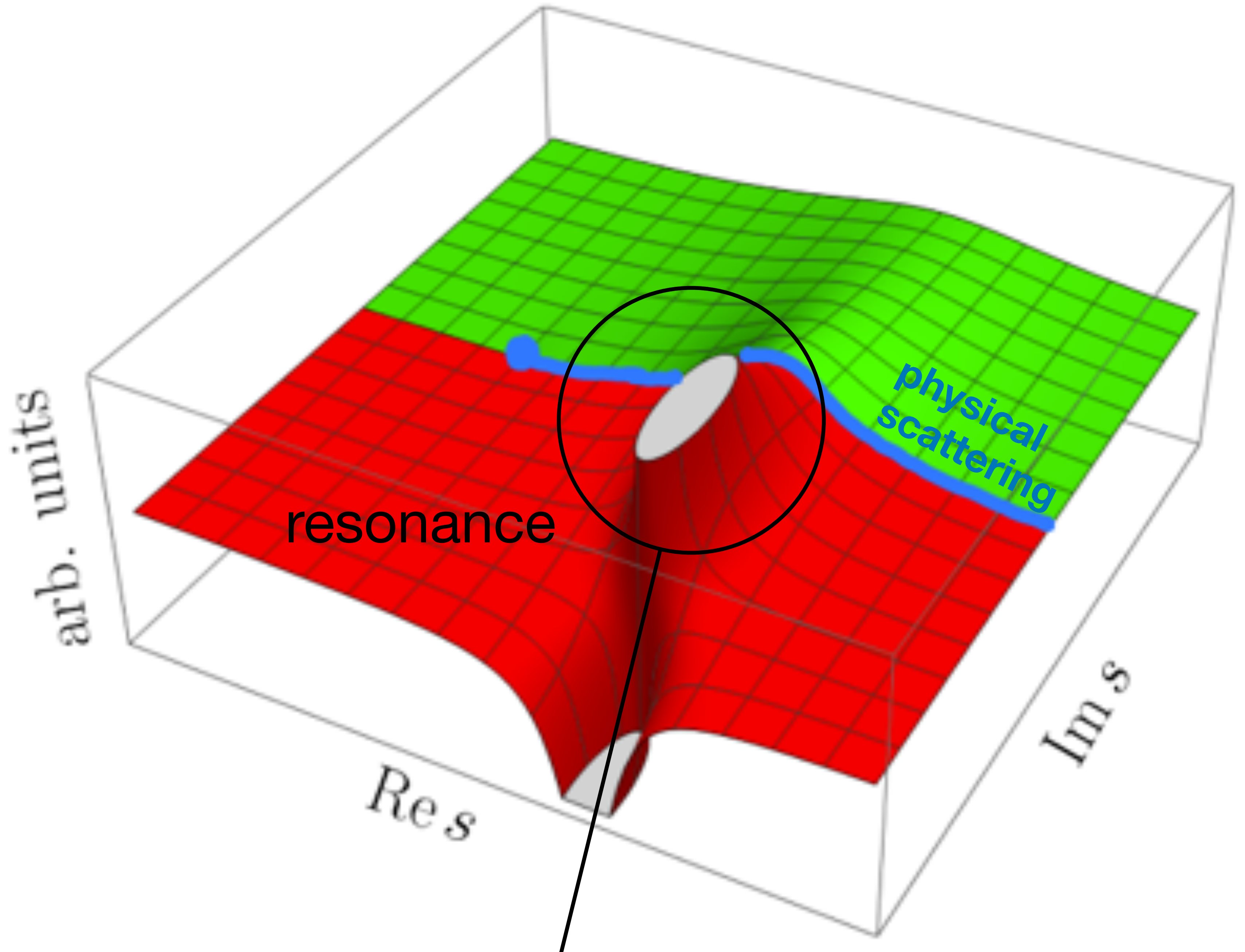
Poles

$$T_\ell(E_{\text{cm}}) \to T_\ell(\sqrt{s}), \quad \sqrt{s} \text{ complex}$$

resonance pole: typically above E_{thr}, with Im $\sqrt{s} \neq 0$ (unitarity) and on **sheet-II** (causality)



$$\sqrt{s_{\text{res}}} = M - \frac{i}{2}\Gamma$$

Quantisation Condition (QC)

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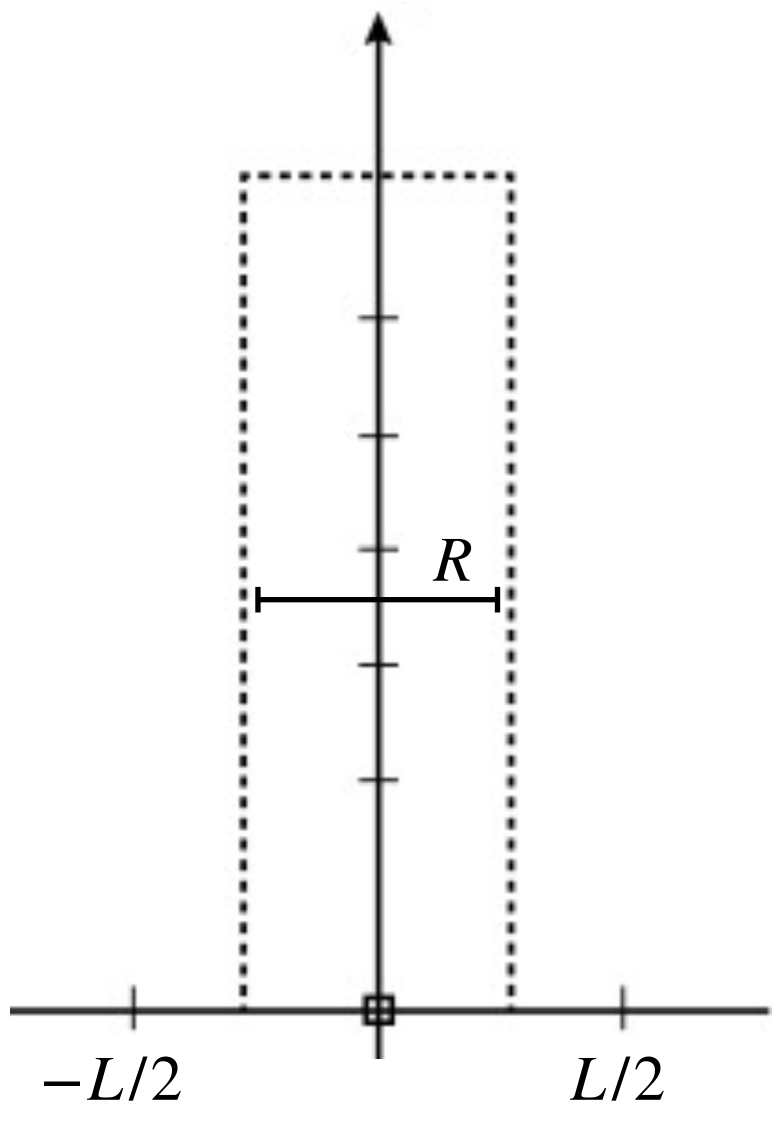
Confined barrier

$$V(x)\begin{cases} = 0, & |x| > R \\ > 0, & |x| < R \end{cases}$$

$$R < L$$



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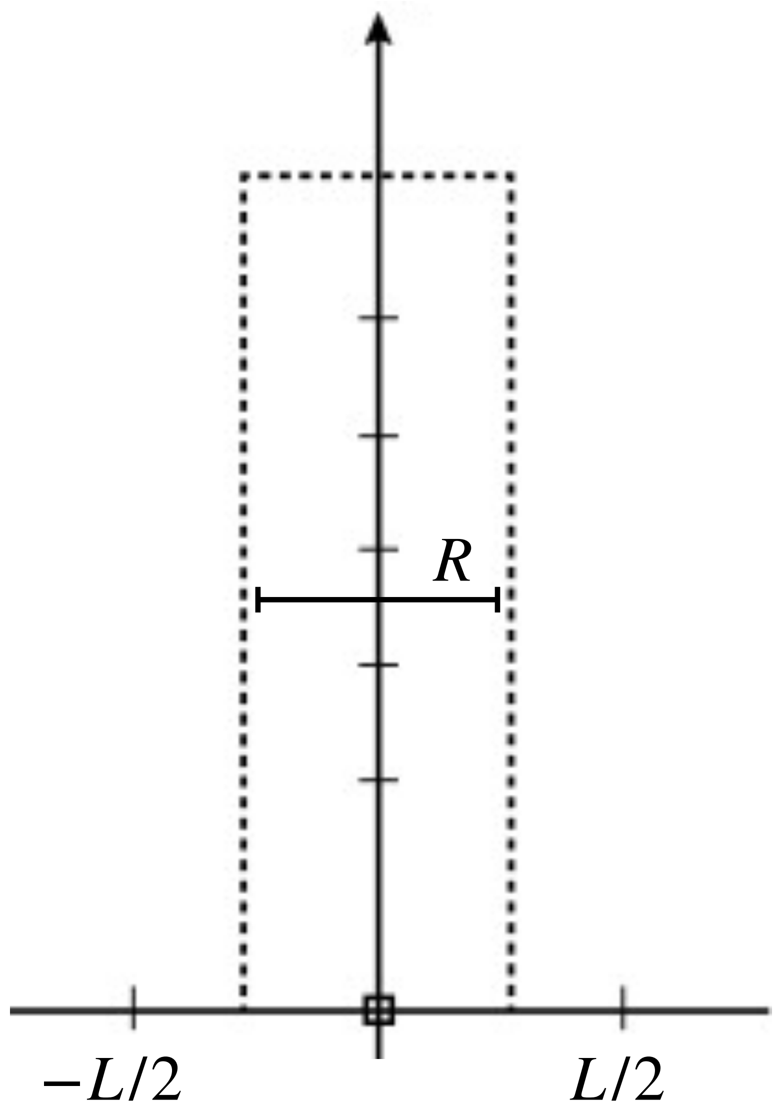
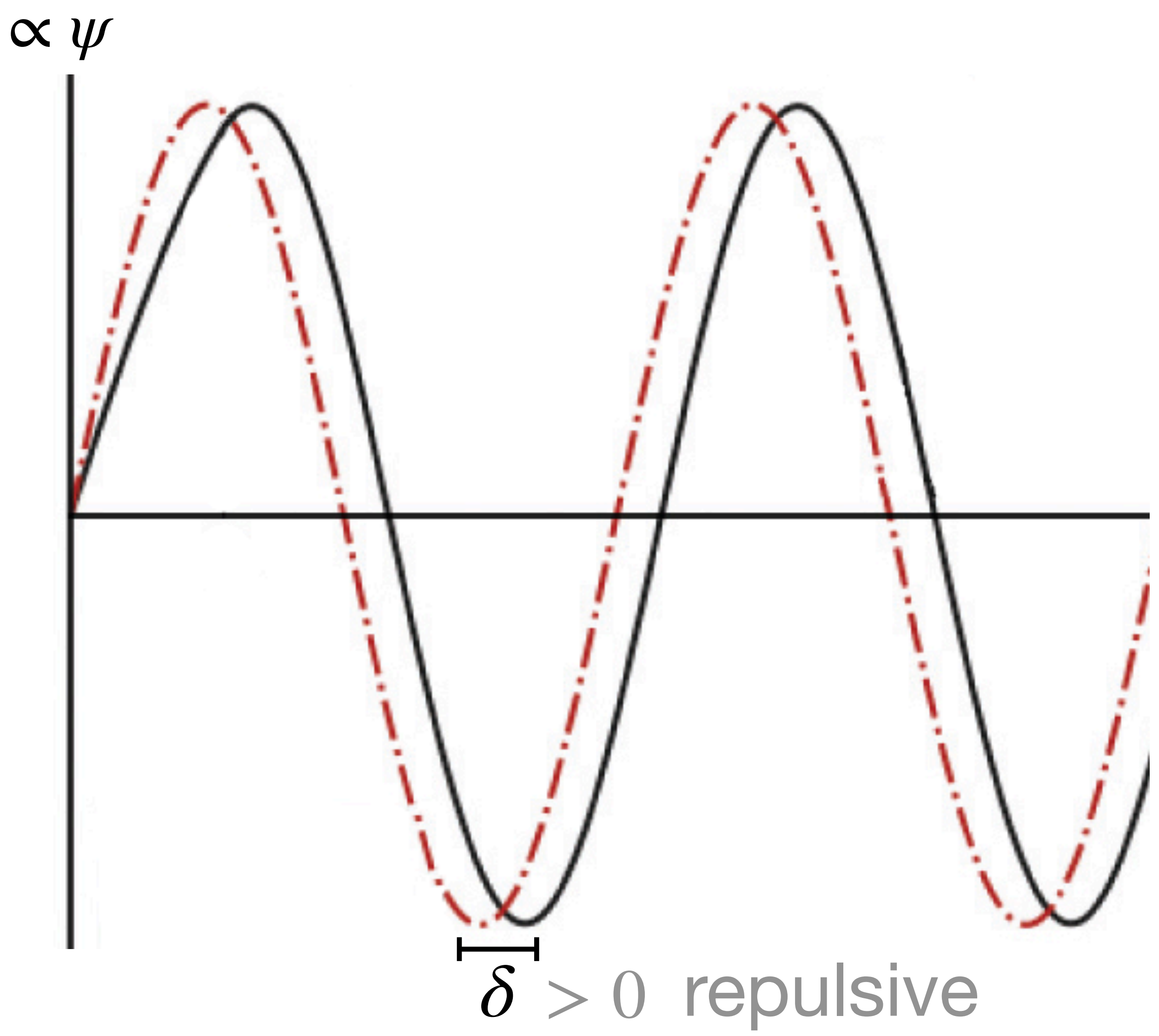


Phase shift

$$in : \psi(x) \rightarrow out : \psi(x - \delta)$$



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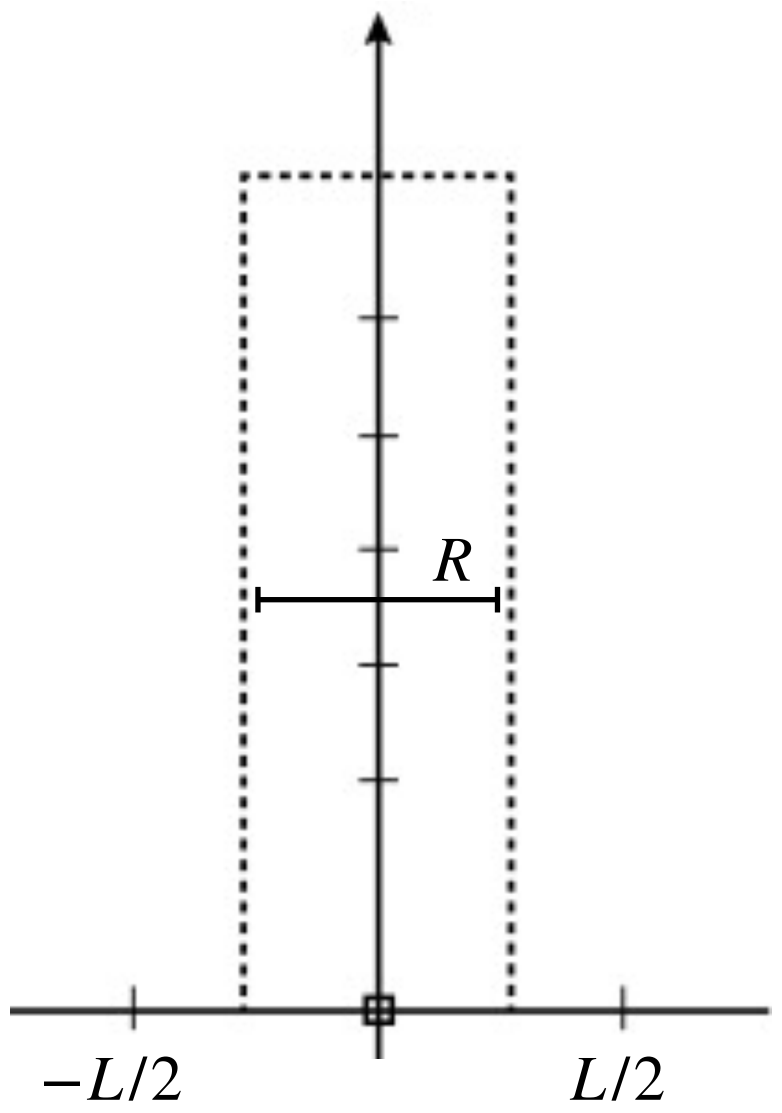
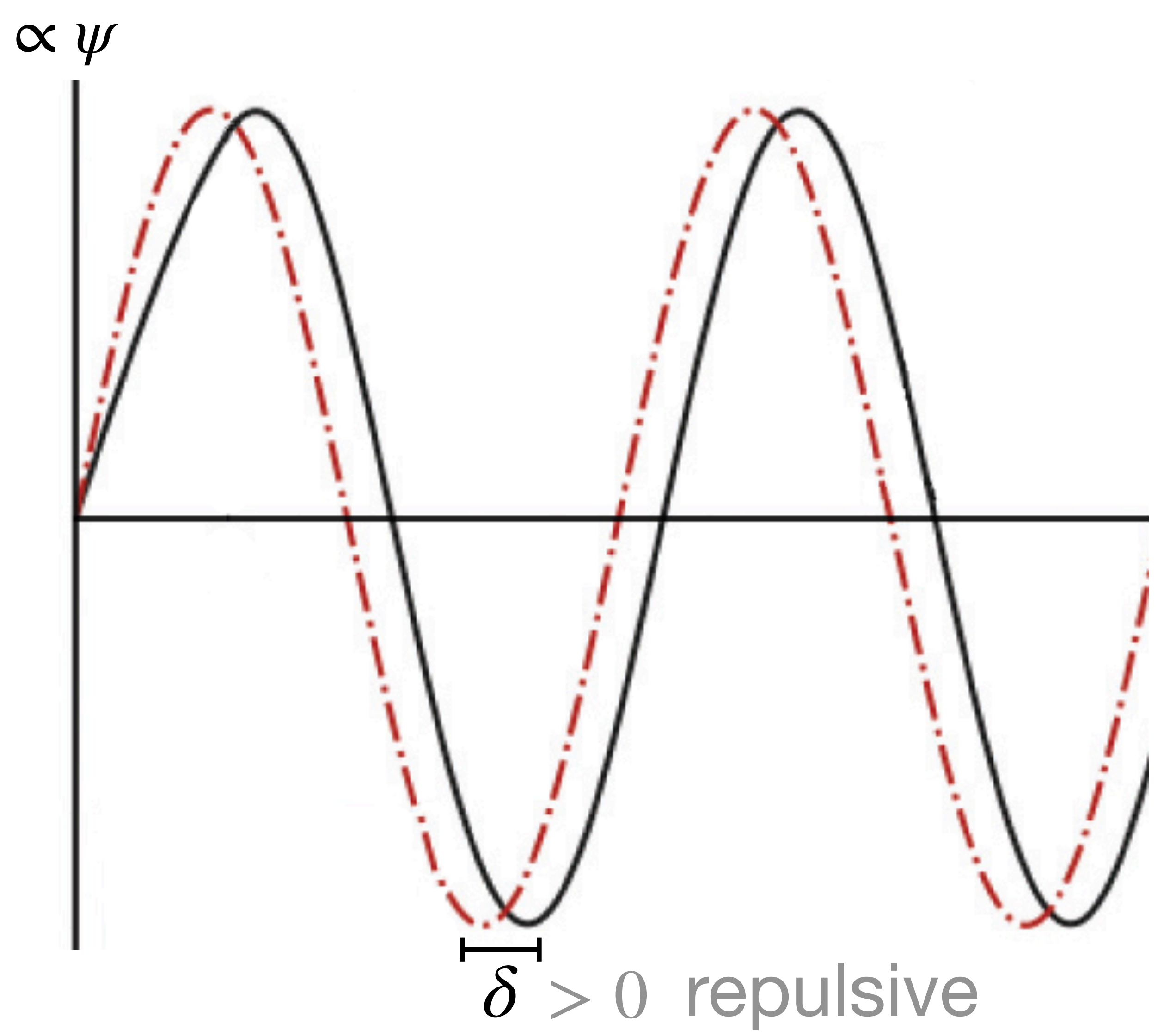


Phase shift $in : \psi(x) \rightarrow out : \psi(x - \delta)$

Periodicity $\psi(x) = \psi(x + L)$



$$\delta(k) = n\pi - kL/2, \quad n \in \mathbb{Z}$$

$$E \propto k^2 \leftrightarrow \delta(k)$$

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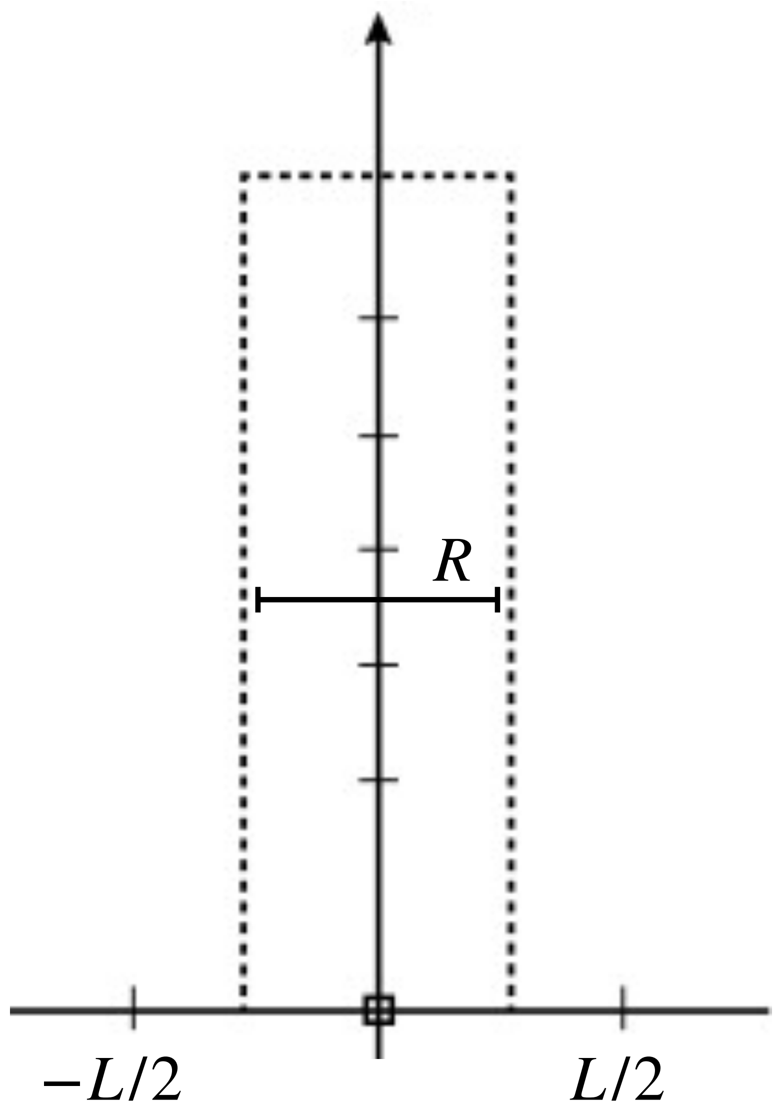

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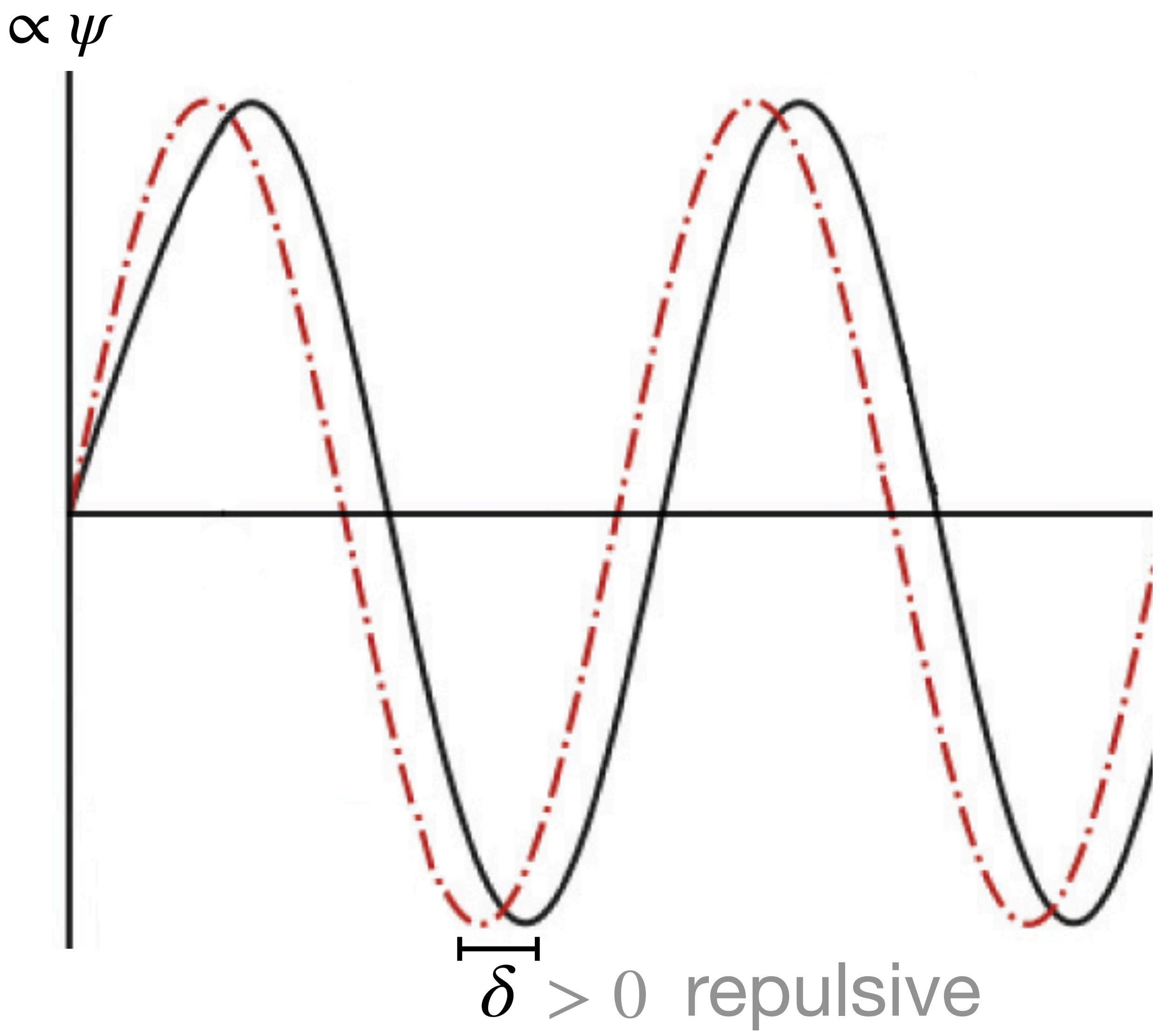

Periodic 3d:

$$\delta\big(E_{\mathrm{cm}}(L)\big) = n\pi - \phi\big(E_{\mathrm{cm}}(L), L\big), \quad n \in \mathbb{Z}$$

[Lüscher, 1986]
[Lüscher, 1991]

→driven by $\mathcal{O}(L^{-b})$, neglects $\mathcal{O}(e^{-mL})$

generalised to multiple channels, spin,…

[Rummukainen & Gottlieb, 1995] [Kim & Sachrajda & Sharpe, 2005] [Hansen & Sharpe, 2012] [Leskovec & Prelovsek, 2012] [Fu, 2012] [Briceno, 2014]…

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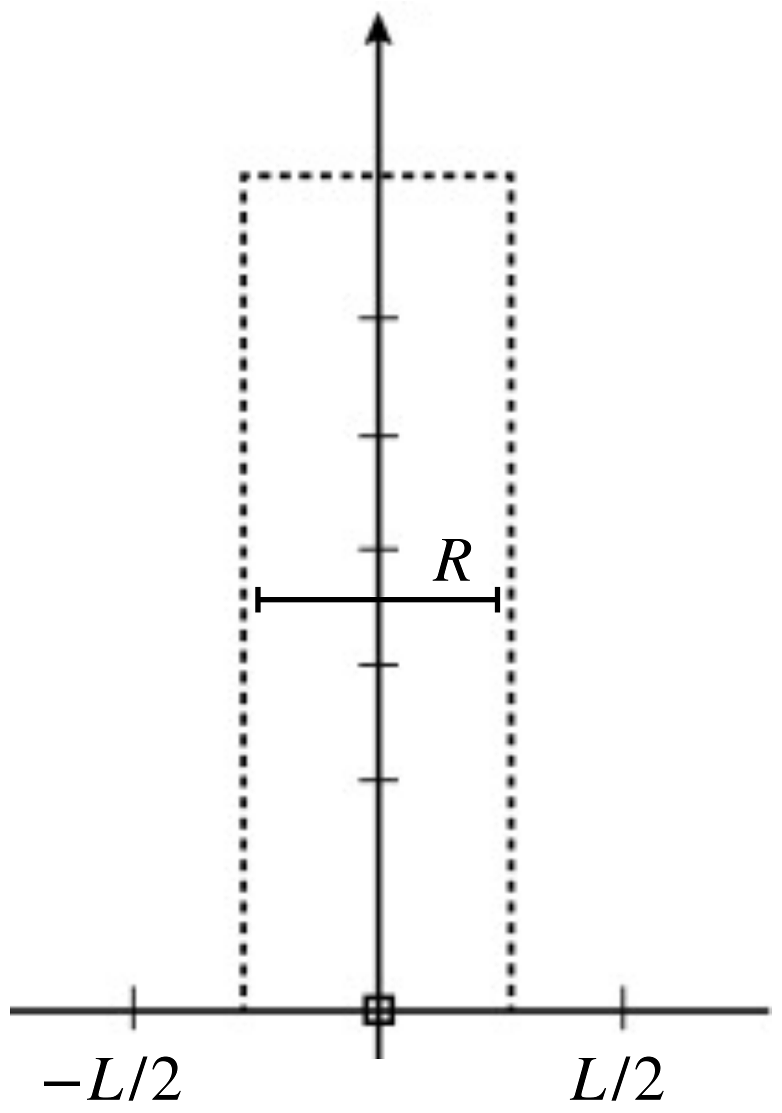

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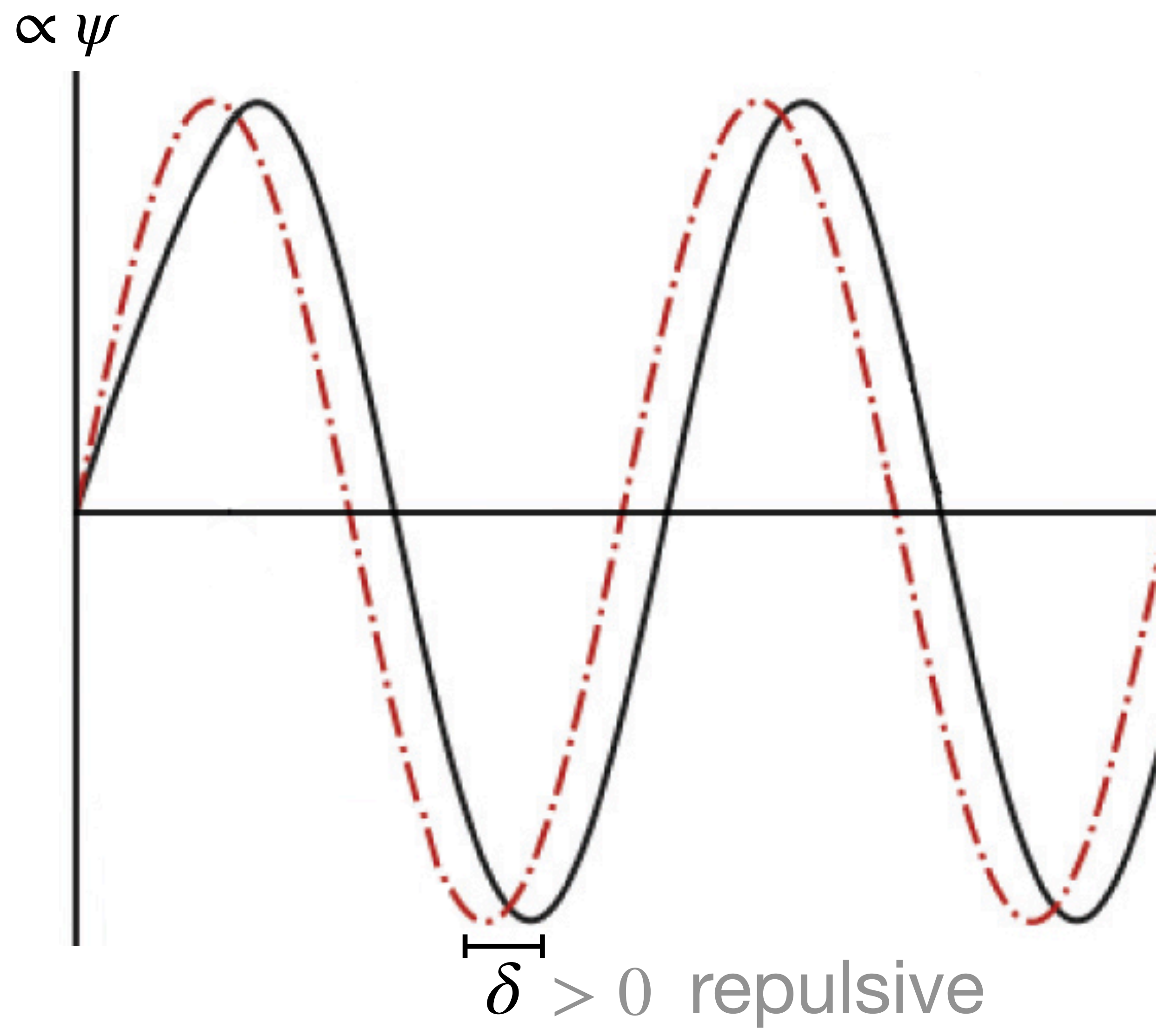

Periodic 3d:

at lattice $E_{\text{cm}}(L)$ known function

$$\delta\left(E_{\text{cm}}(L)\right) = n\pi - \phi\left(E_{\text{cm}}(L), L\right), \quad n \in \mathbb{Z}$$

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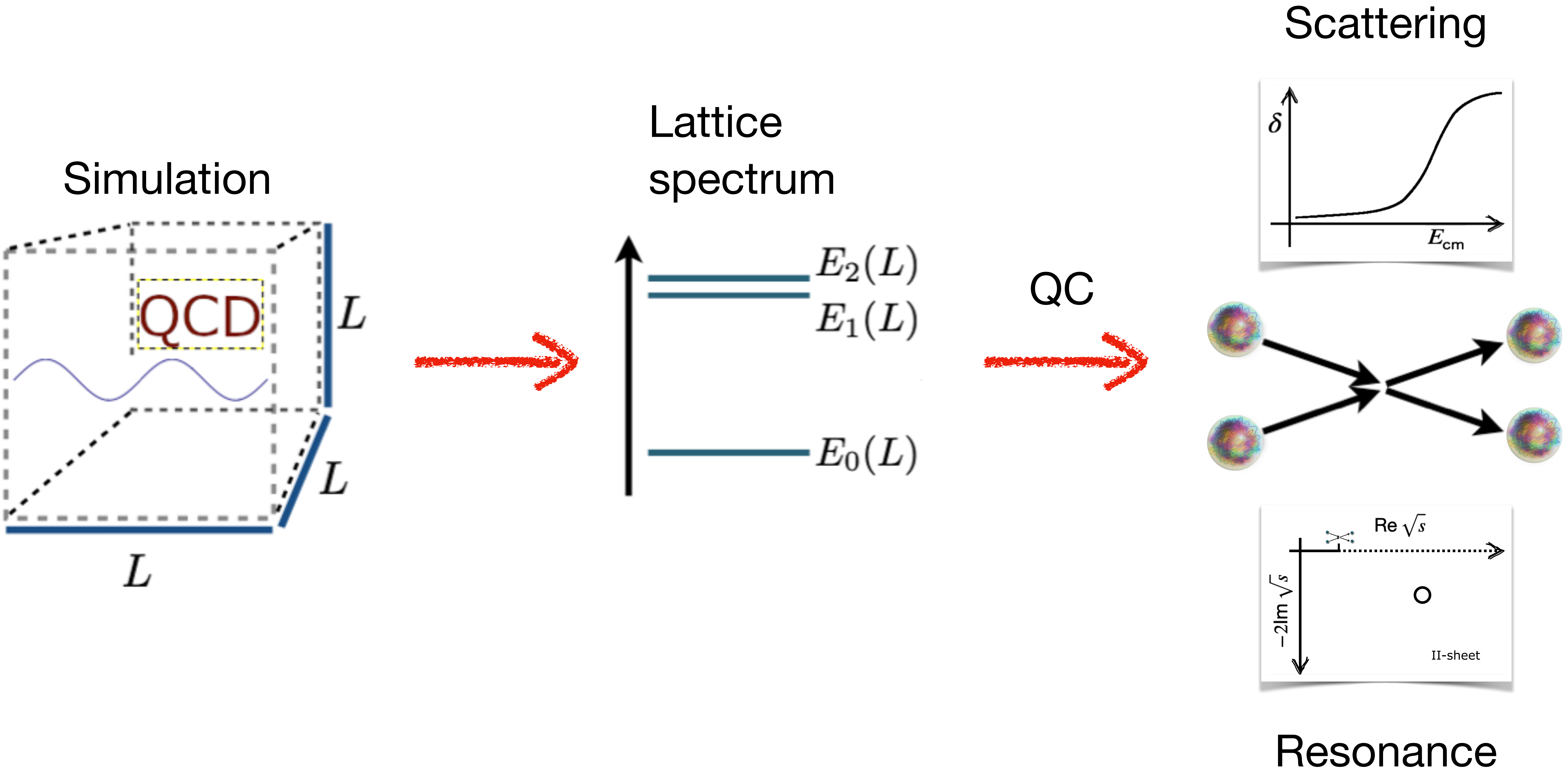
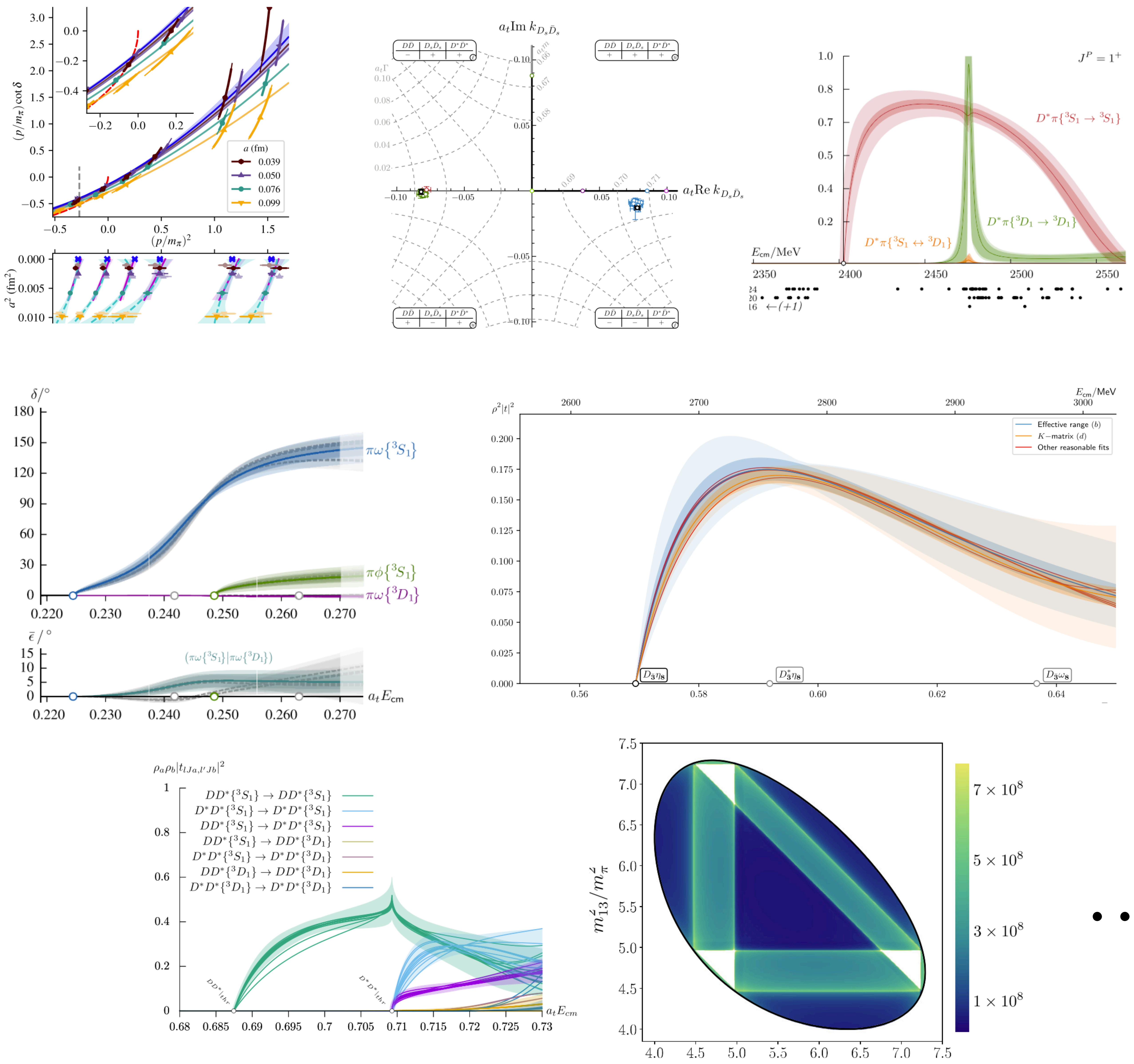
Method


Simulation
QCD
L
L
L
Lattice spectrum
$E_2(L)$
$E_1(L)$
$E_0(L)$
QC
Scattering
δ
E_{cm}
Re $\sqrt{s}$
-2Im $\sqrt{s}$
II-sheet
Resonance

Scattering from lattice QCD

[Briceño, Dudek, Young - RevModPhys, 2018] [Mai et al, PhysRep, 2023]

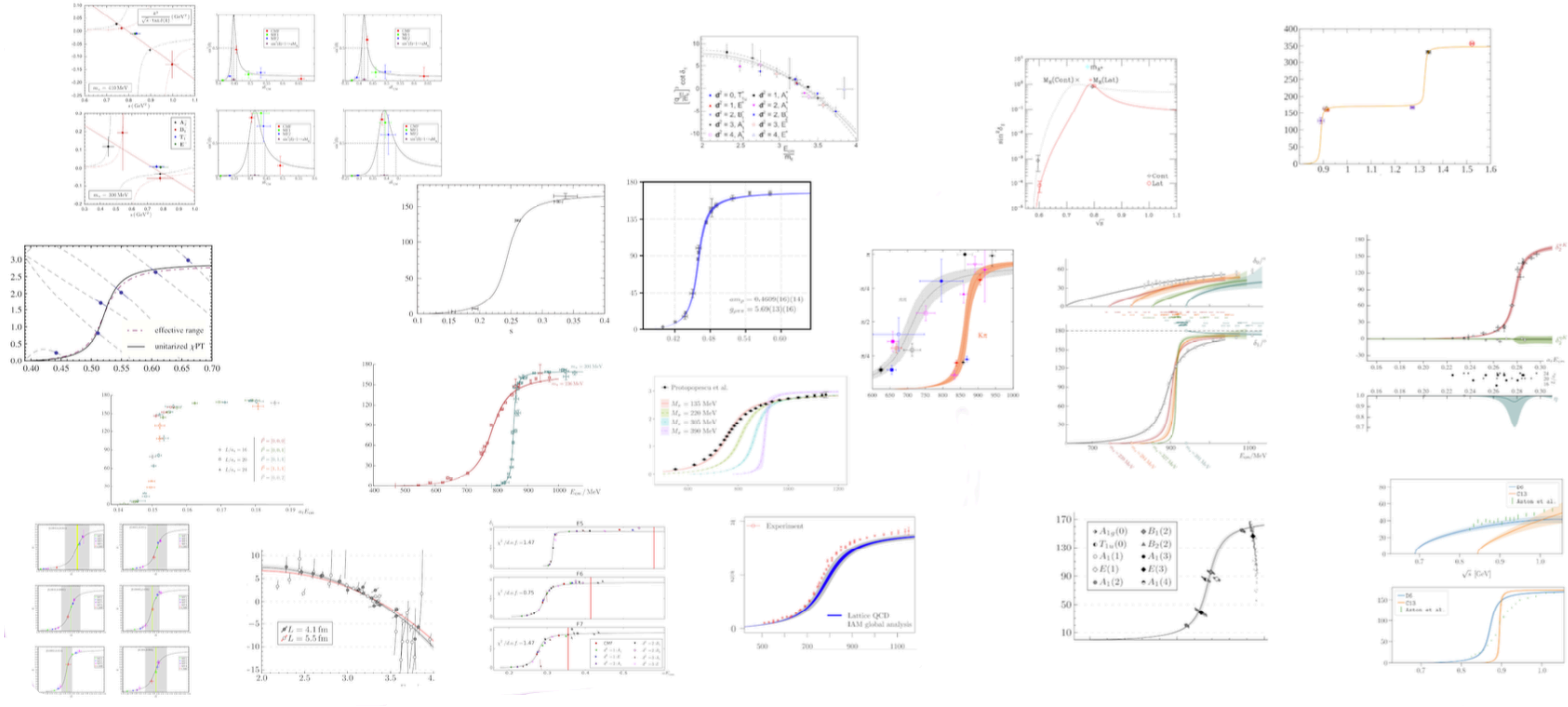
- baryons [Green et al, PRL, 2021] [Bulava et al, PRD, 2024]
- hidden-charm [Wilson et al, PRL & PRD, 2021] [Prelovsek et al, JHEP, 2021]
- charm-light [Yeo et al, JHEP, 2024] [Gayer et al, JHEP, 2021] [Lang & Wilson, PRL, 2021] [Mohler et al, PRD, 2013]
- doubly-charm [Whyte et al, PRD, 2025]
- exotics, hybrids [Woss et al, PRD, 2021] [Woss et al, PRD, 2019]
- multiple-channels [Dudek et al, PRL, 2014] [Wilson et al, PRD, 2015]
- three-body [Hansen et al, PRL, 2021] [Mai et al, PRL, 2021]



. . .

Scattering from lattice QCD

[Fischer et al - PLB, 2021][Rendon et al - PRD, 2021] [Wilson et al - PRL, 2019] [Bali et al - PRD, 2016] [Bret et al - Nuc.Phys.B, 2018] [Aoki et al - PRD, 2011] [Feng et al - PRD, 2011] [Lang et al - PRD, 2011] [Pelissier et al - PRD, 2013] [Dudek et al - PRD, 2013] [Bulava et al - Nuc.Phys.B, 2016] [Fu et al - PRD, 2016] [Andersen et al - Nuc.Phys.B, 2019] [Erben et al - PRD, 2020] [Alexandrou et al - PRD, 2017]



$\pi\pi \to \rho \to \pi\pi$ and $K\pi \to K^* \to K\pi$

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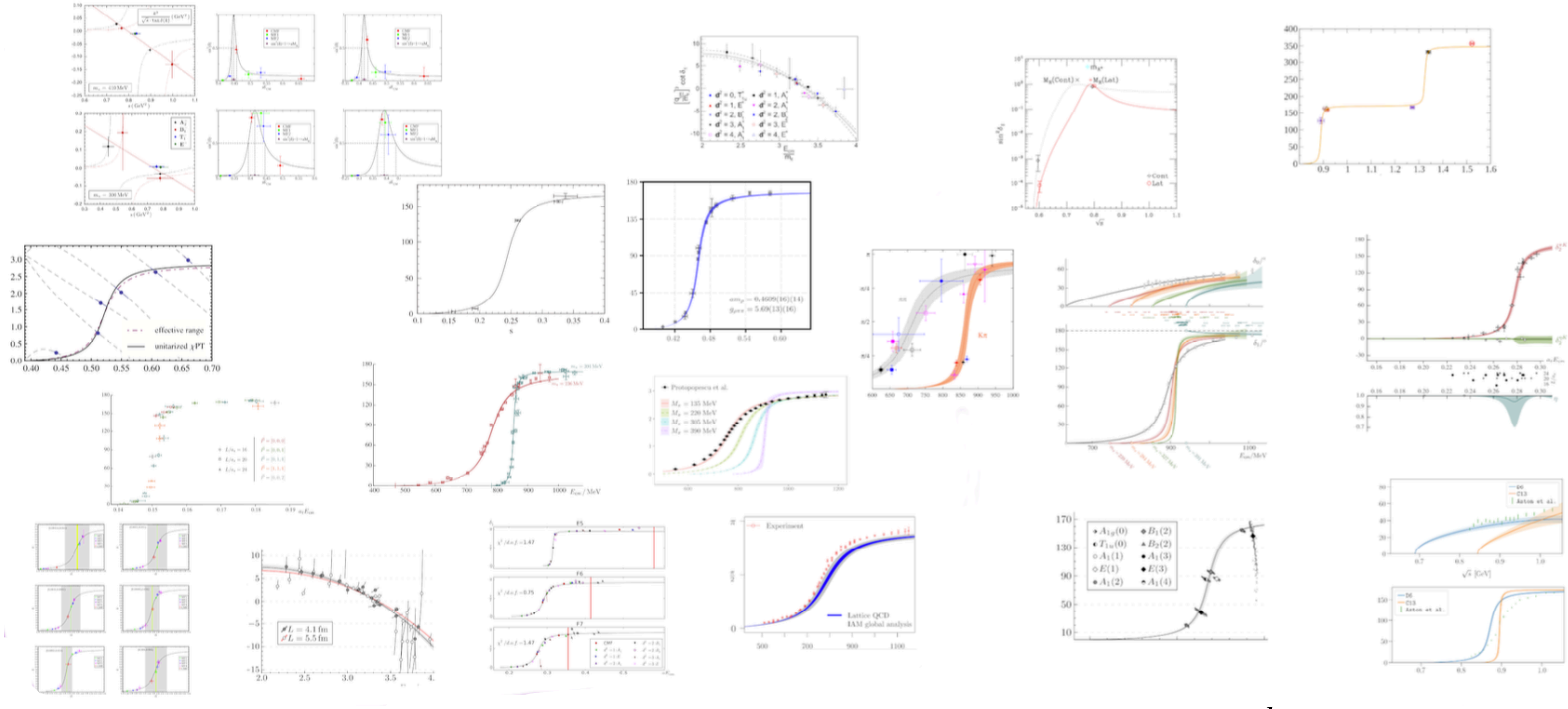

$\pi\pi \to \rho \to \pi\pi$ and $K\pi \to K^* \to K\pi$

having $m_\pi \approx m_\pi^{phys} \approx 139$ MeV important for precision!

Physical m_π determination: ρ and K^*

Main decay products ($J = \ell = 1$)

[PDG, 2024]

$$K^*(892) \to K\pi, K\gamma, K\pi\pi, \ldots$$

$$\rho(770) \to \pi\pi, \pi\gamma, 4\pi, \ldots$$

PHYSICAL REVIEW LETTERS **134,** 111901 (2025)

Light and Strange Vector Resonances from Lattice QCD at Physical Quark Masses

Peter Boyle,[1,2] Felix Erben,[3,2] Vera Gülpers,[2] Maxwell T. Hansen,[2] Fabian Joswig,[2] Michael Marshall,[2] Nelson Pitanga Lachini,[4,2,*] and Antonin Portelli[2,3,5]

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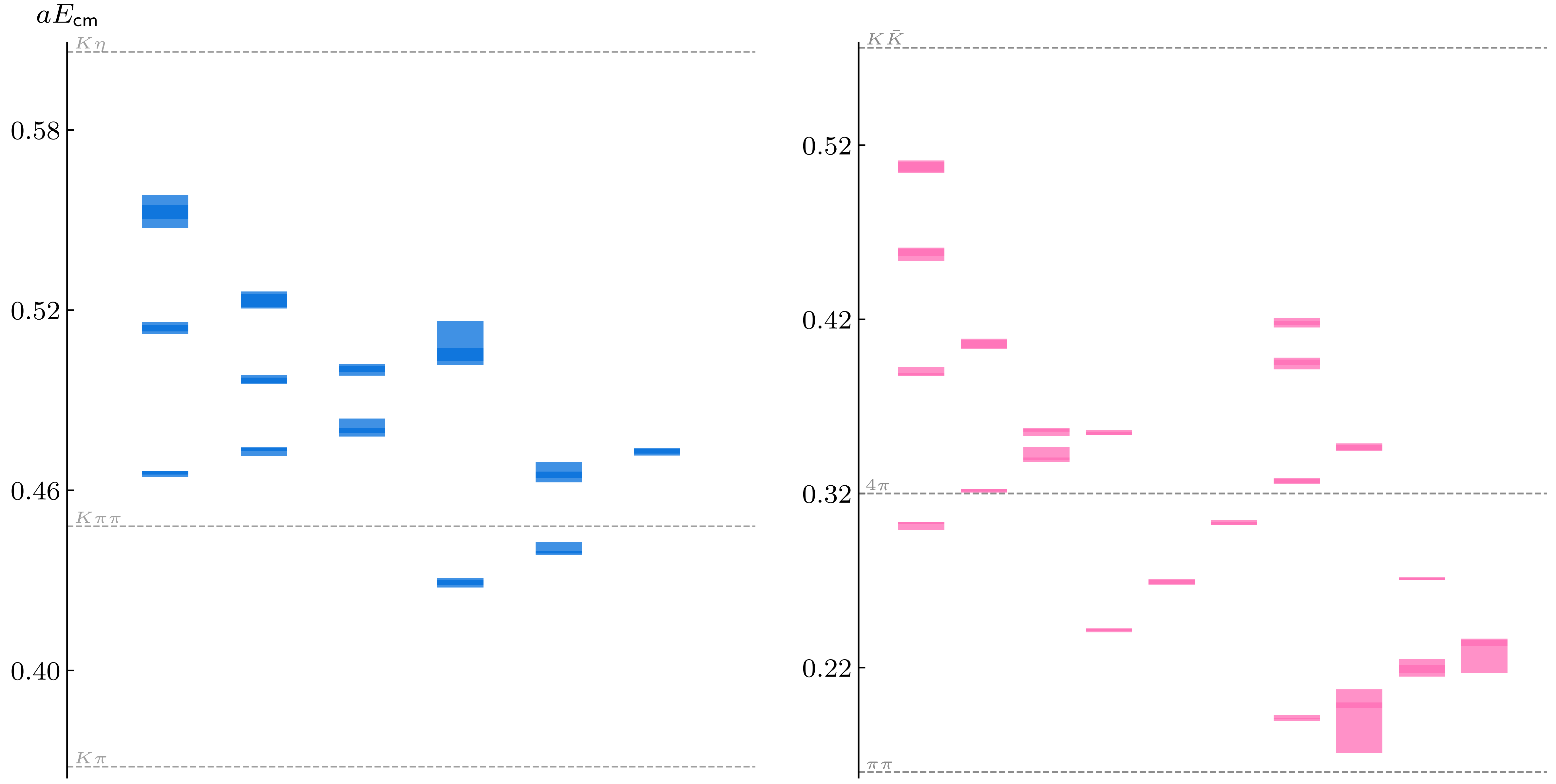
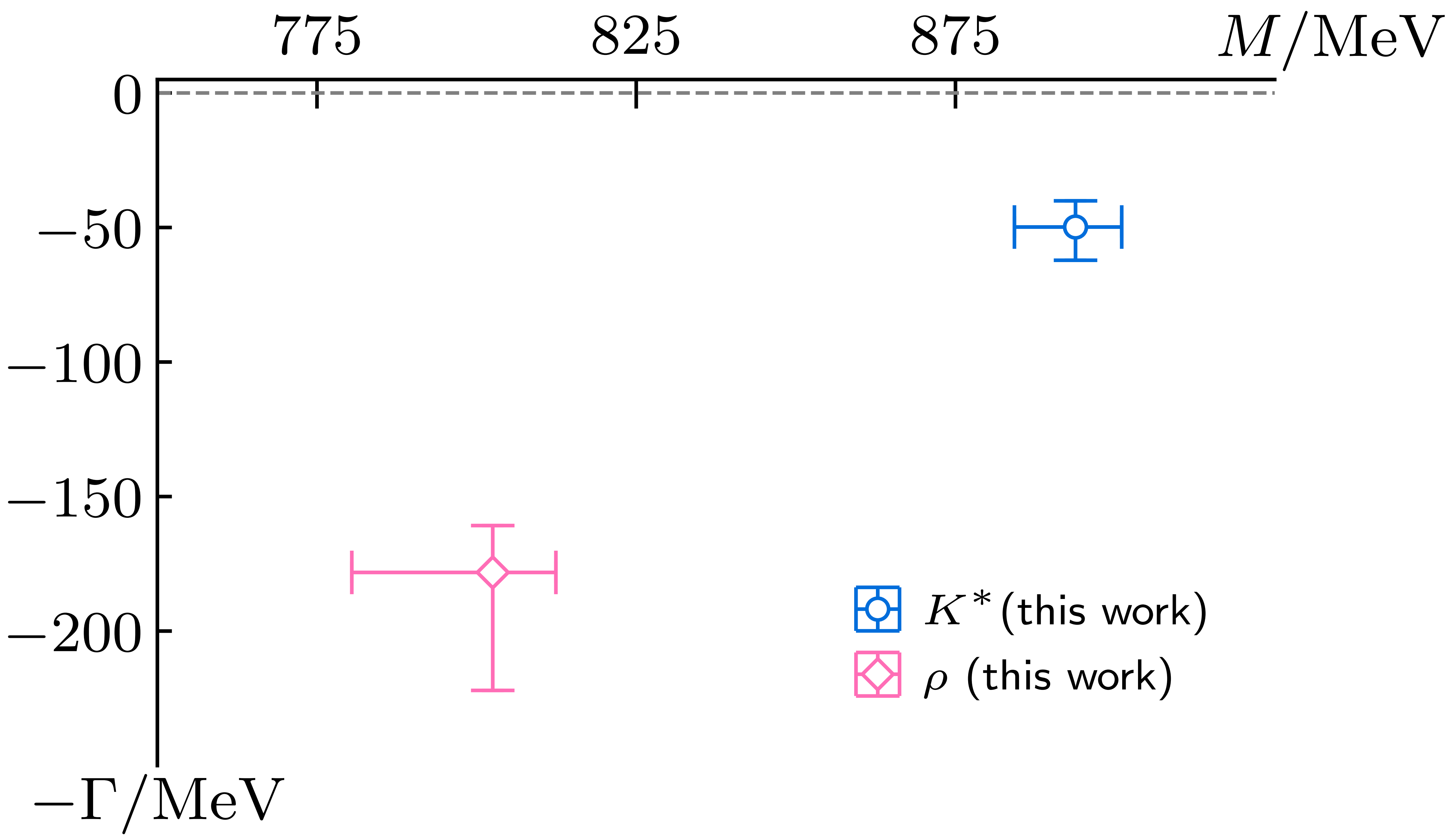
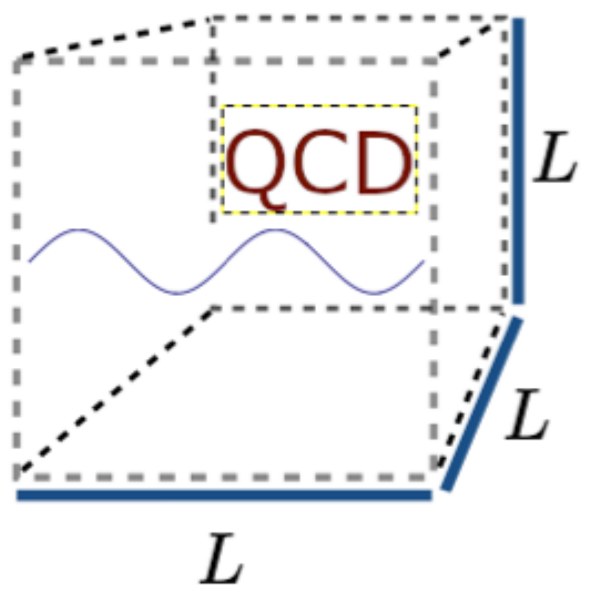
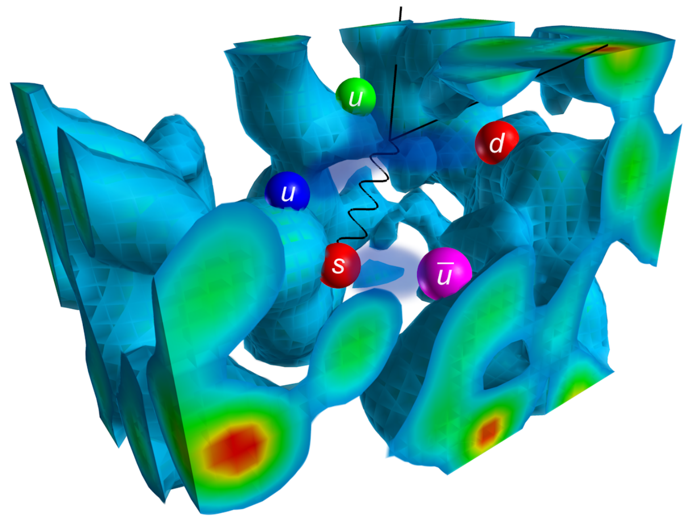






[physics.adelaide.edu.au/theory/staff/leinweber/VisualQCD/Nobel]

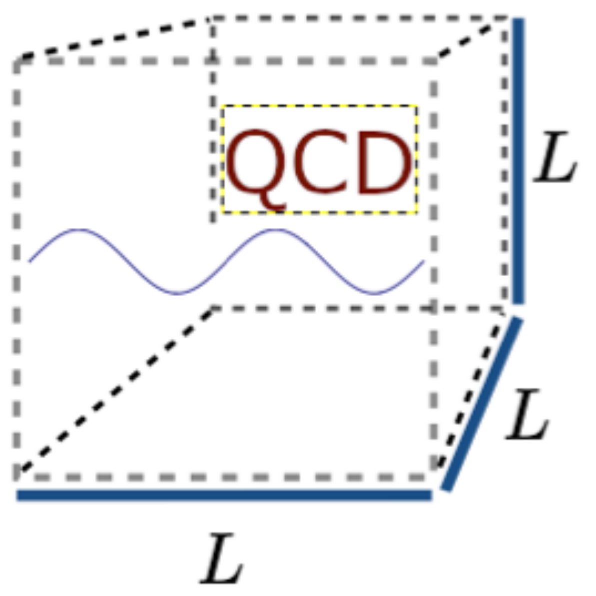
$90\times$ 

RBC-UKQCD lattice

volume	$48^3 \times 96$
a	≈ 0.114 fm
L	≈ 5.5 fm
$m_\pi L$	≈ 3.8
$\boldsymbol{m_\pi}$ m_K	$\approx$ **139** MeV ≈ 499 MeV

[Blum et al, PRD, 2016]

$$N_f = 2+1 \begin{cases} m_u = m_d \\ m_s \end{cases}$$



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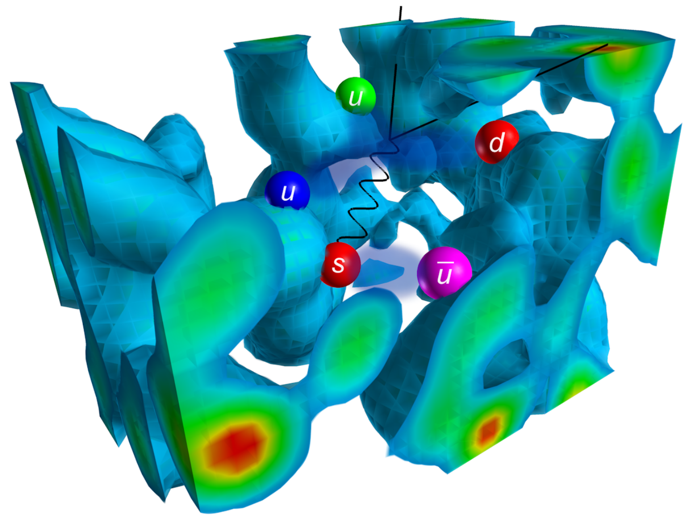
$90 \times$ 

"raw" observables

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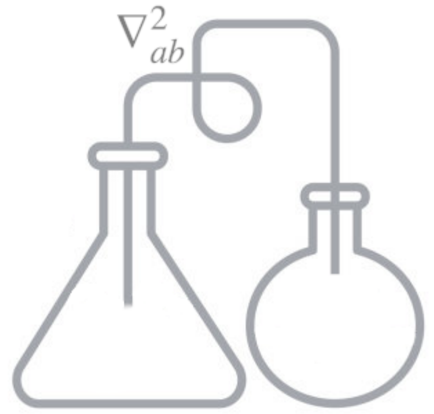
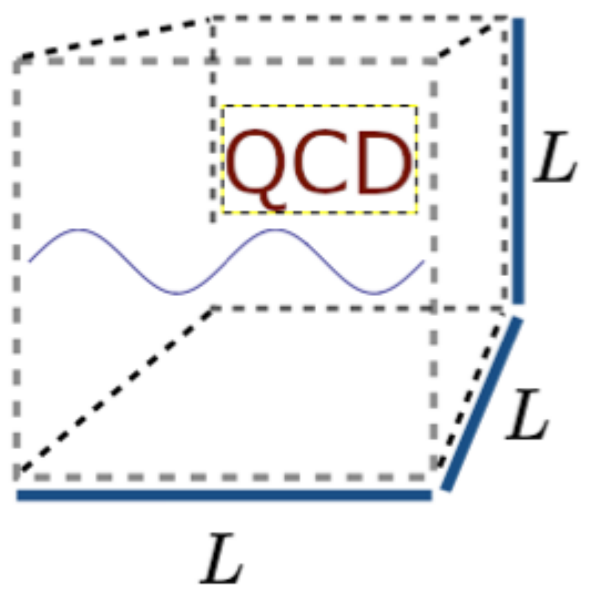


[Peardon et al, PRD, 2009]
[Morningstar et al, PRD, 2011]

Distillation: sources built from covariant *Laplacian*



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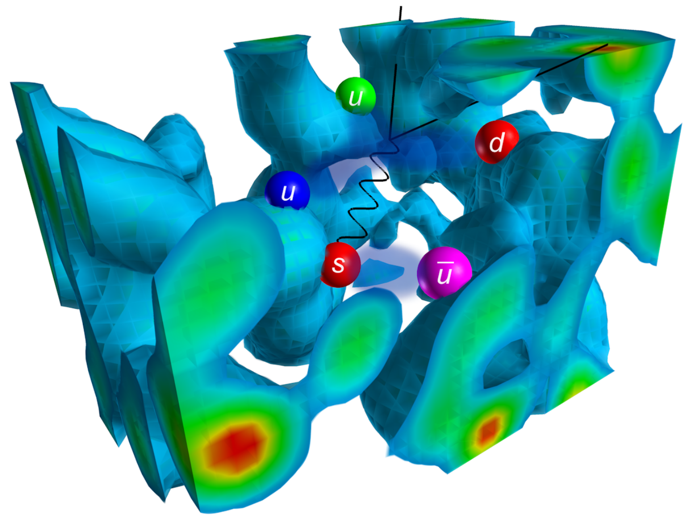
$90 \times$ 

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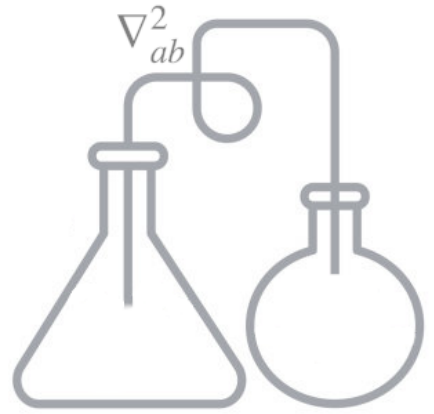

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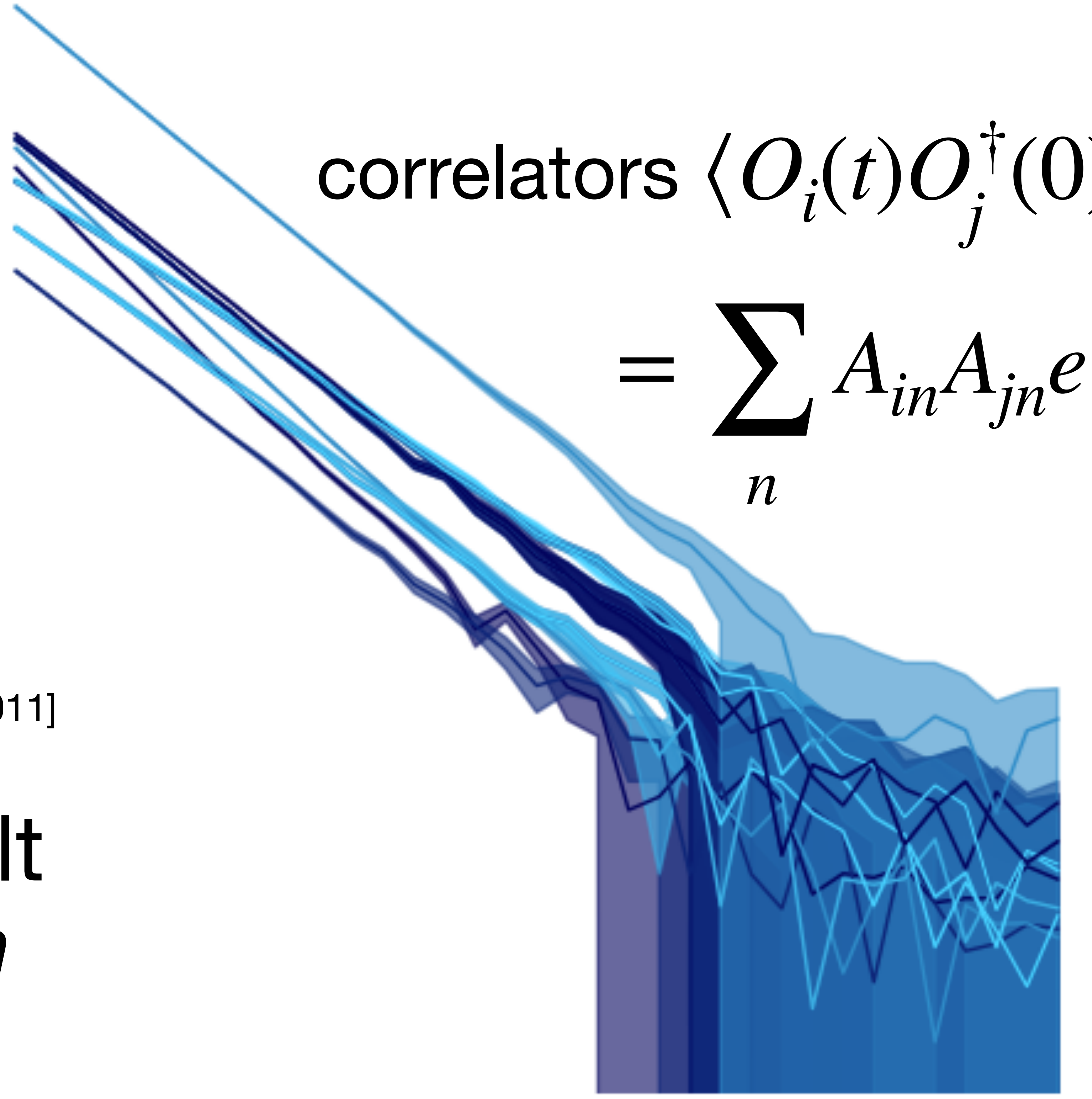
Distillation: sources built from covariant *Laplacian*

correlators $\langle O_i(t) O_j^\dagger(0) \rangle$

$$= \sum_n A_{in} A_{jn} e^{-tE_n}$$



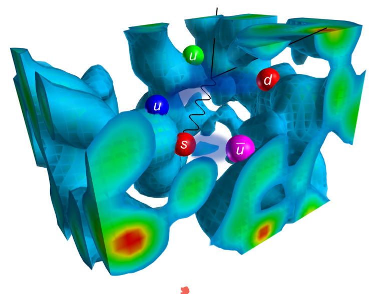
Implementation

Implementation

Open-source and free software

- *Grid*: data parallel C++ lattice library GRID
- *Hadrons*: workflow management for lattice simulations Hadrons

Implementation



workflow example

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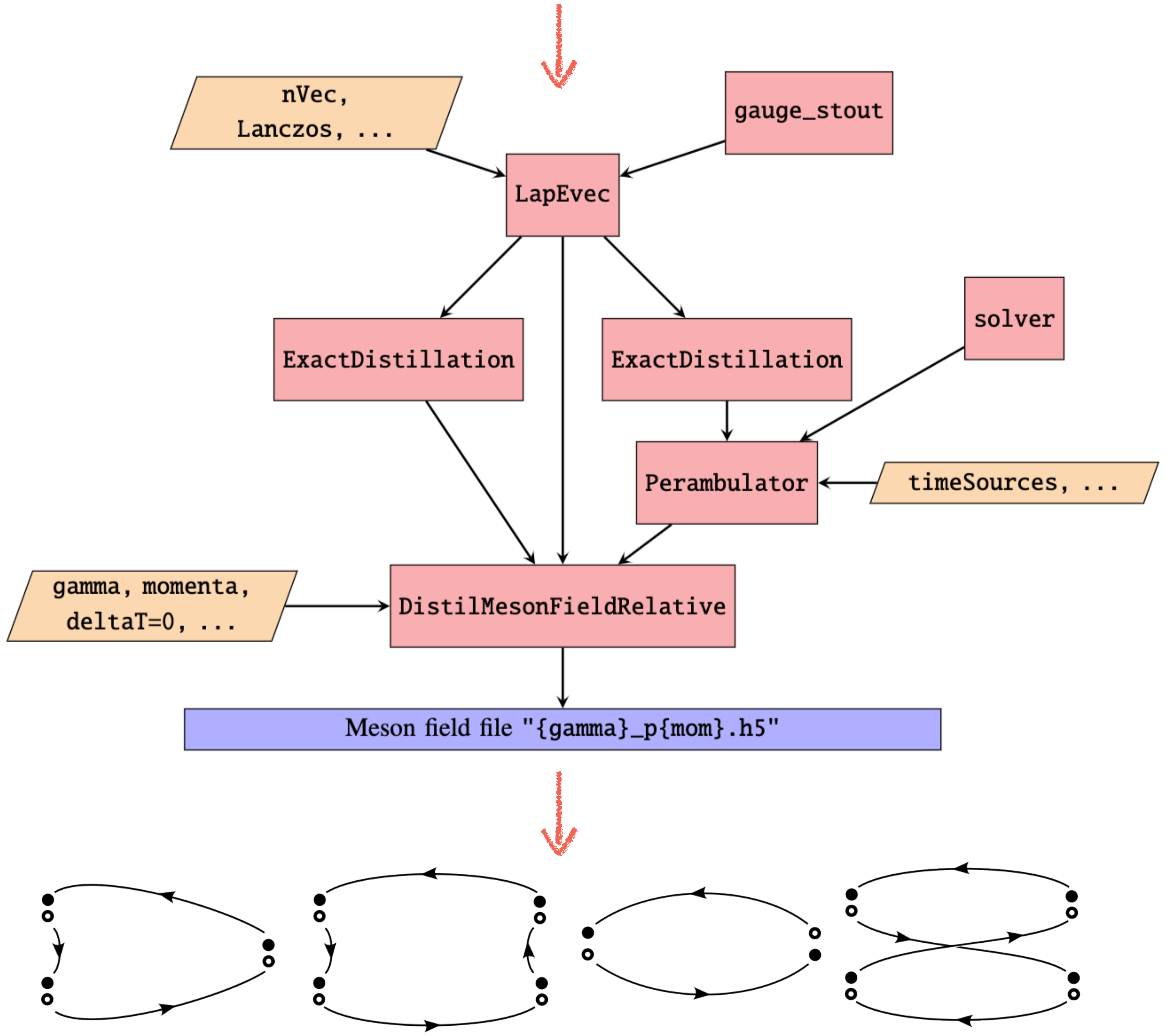

Distillation within *Grid* and *Hadrons*

- agnostic to action
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[dirac.ac.uk/extreme-scaling-edinburgh]

- 2 DiRAC machines, same high-level code
- 'raw' correlators publicly shared

[repository.cern/records/vy9x7-bzn92]

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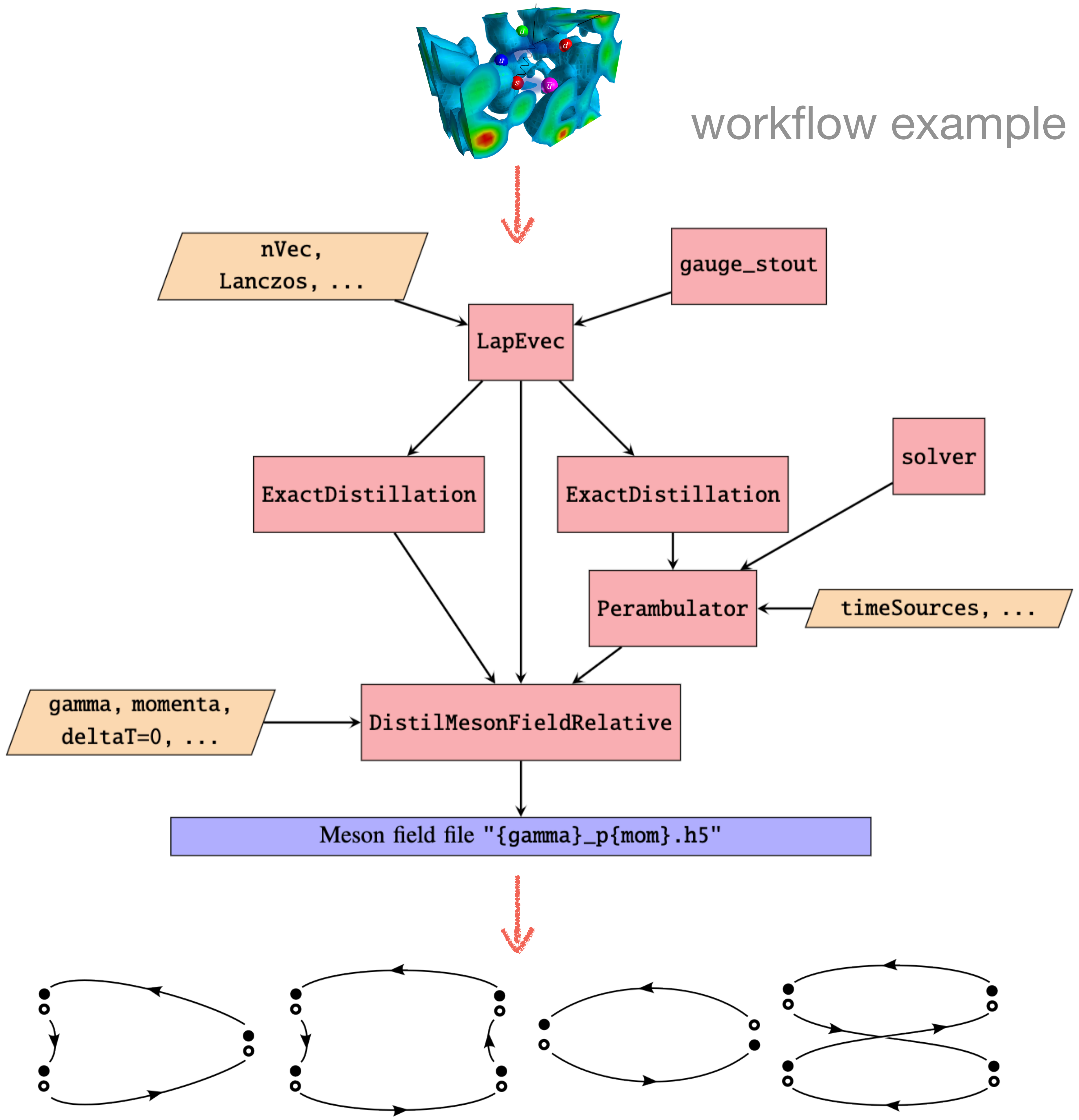

Tesseract (CPU)

Tursa (GPU)

Implementation

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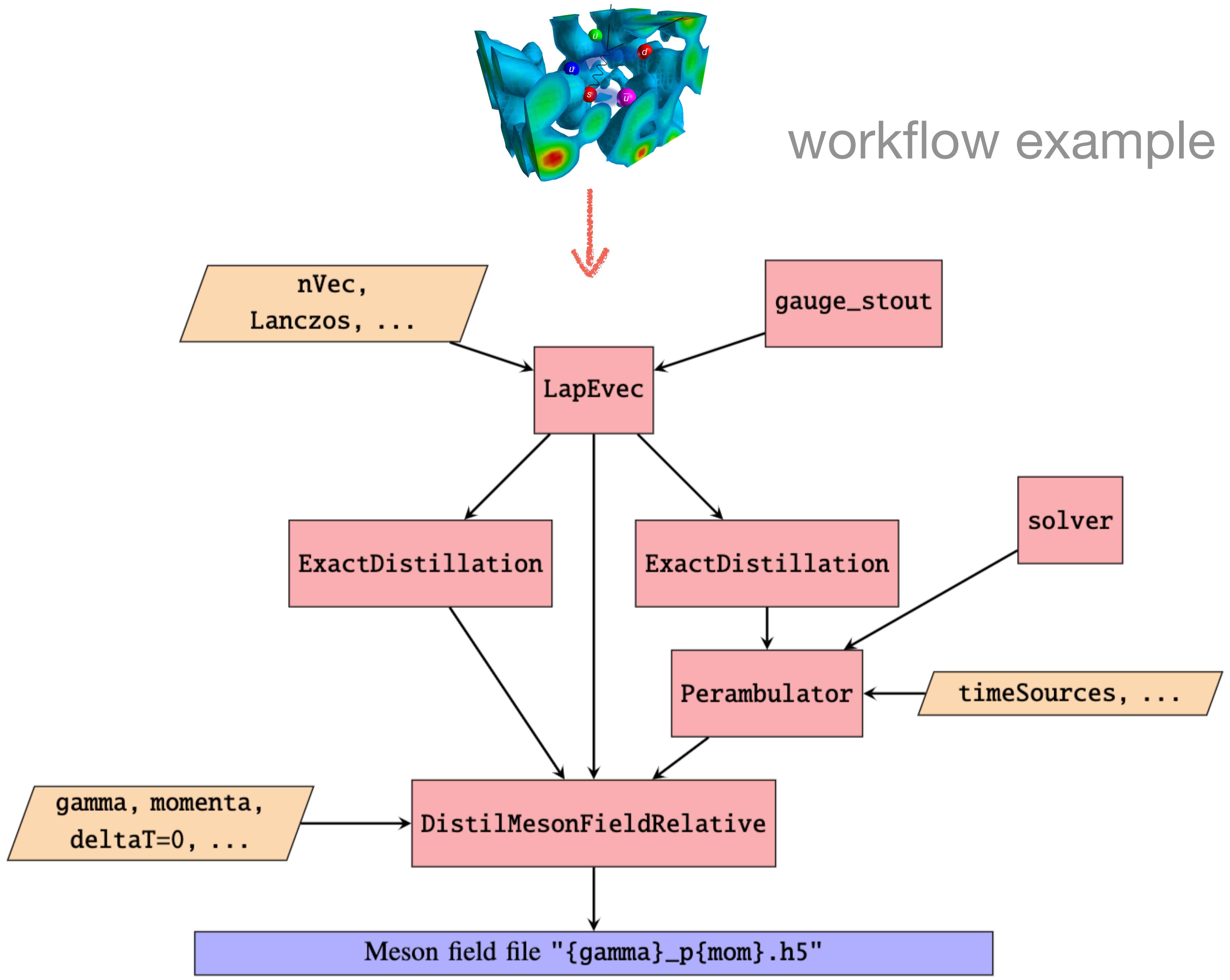
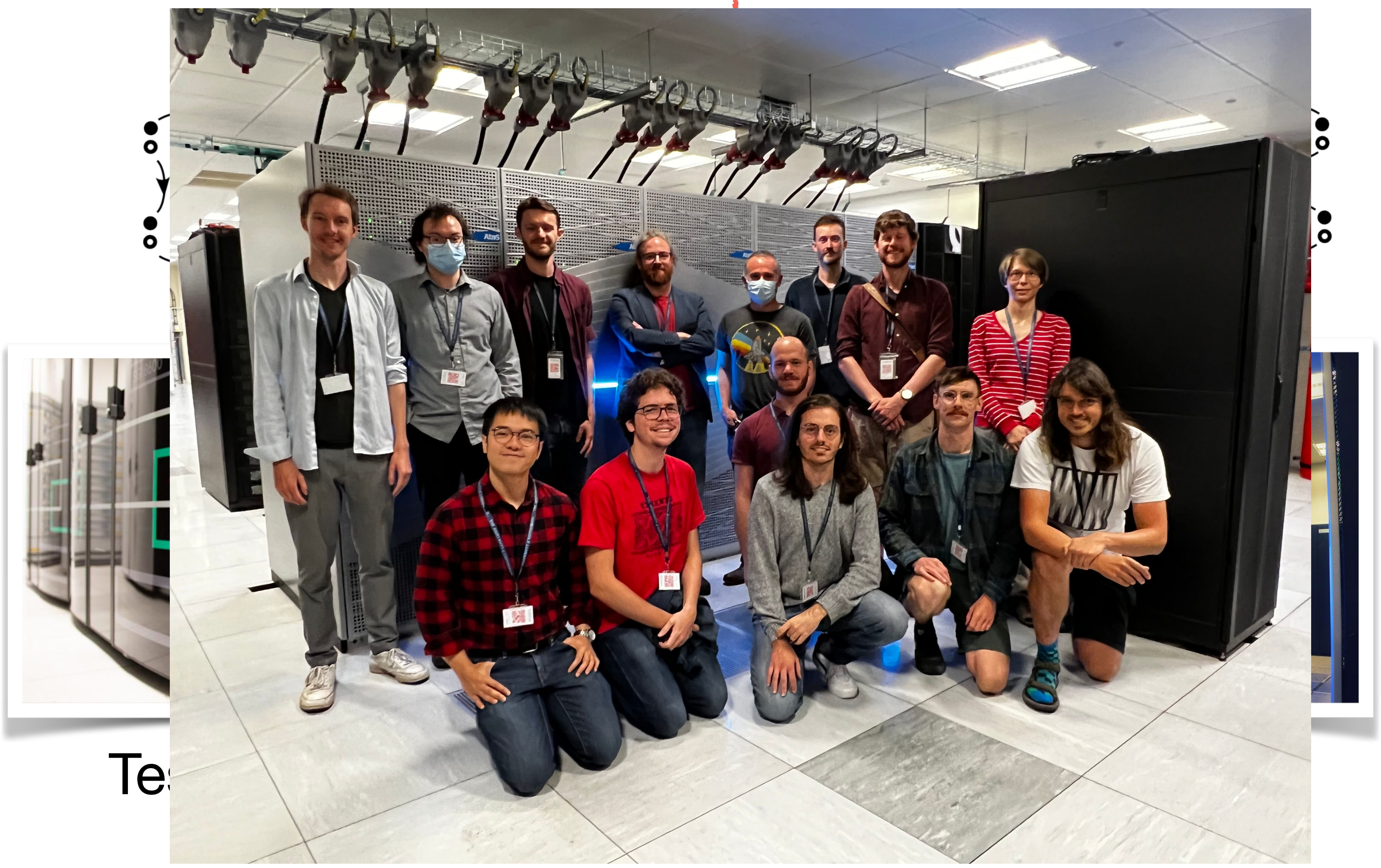
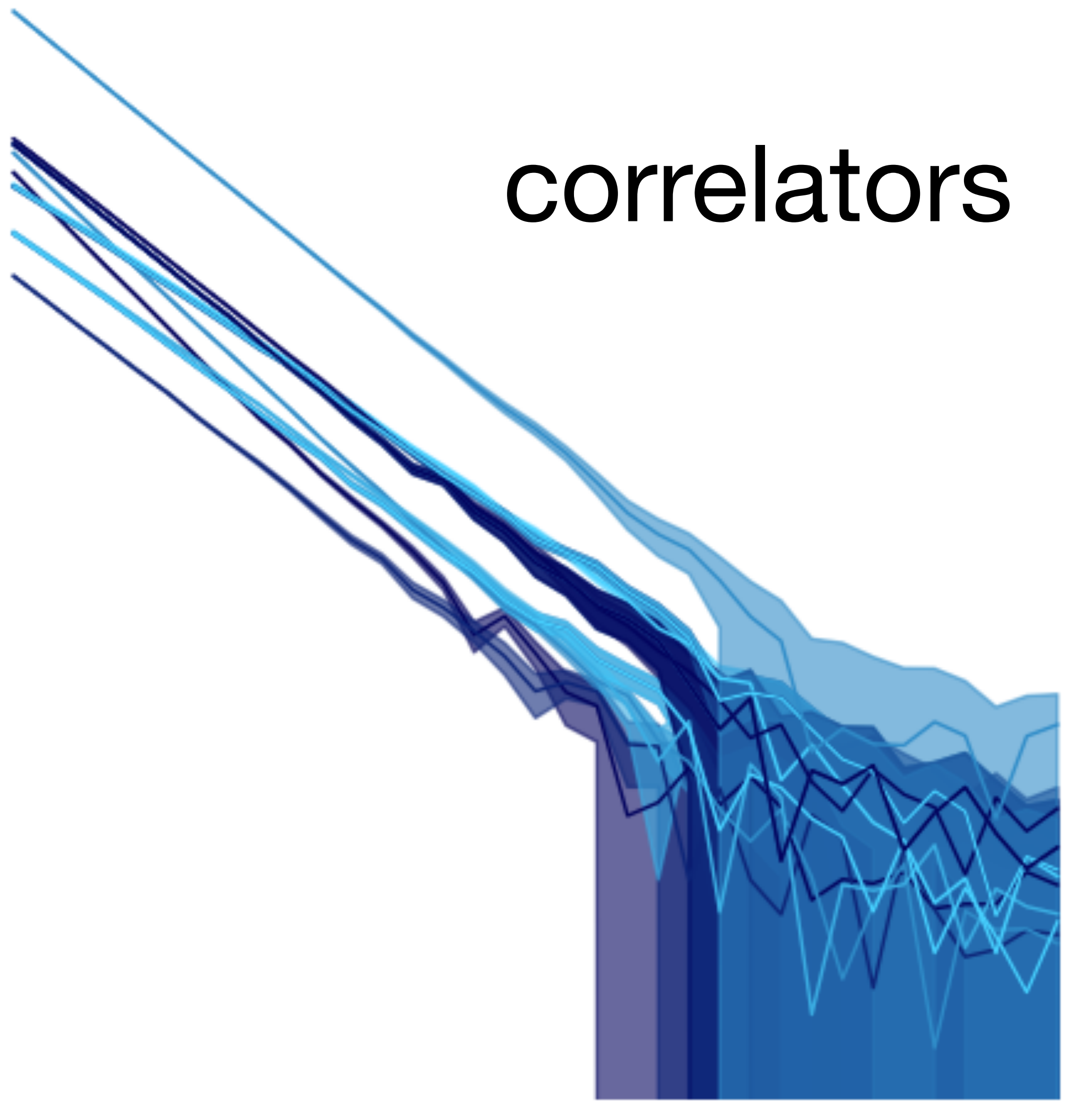




Generalised Eigenvalue Problem (GEVP)

$\{O_i\} \to \{\Omega_i\}$ such that $\langle 0 | \Omega_i | n \rangle \approx \delta_{ni}$?



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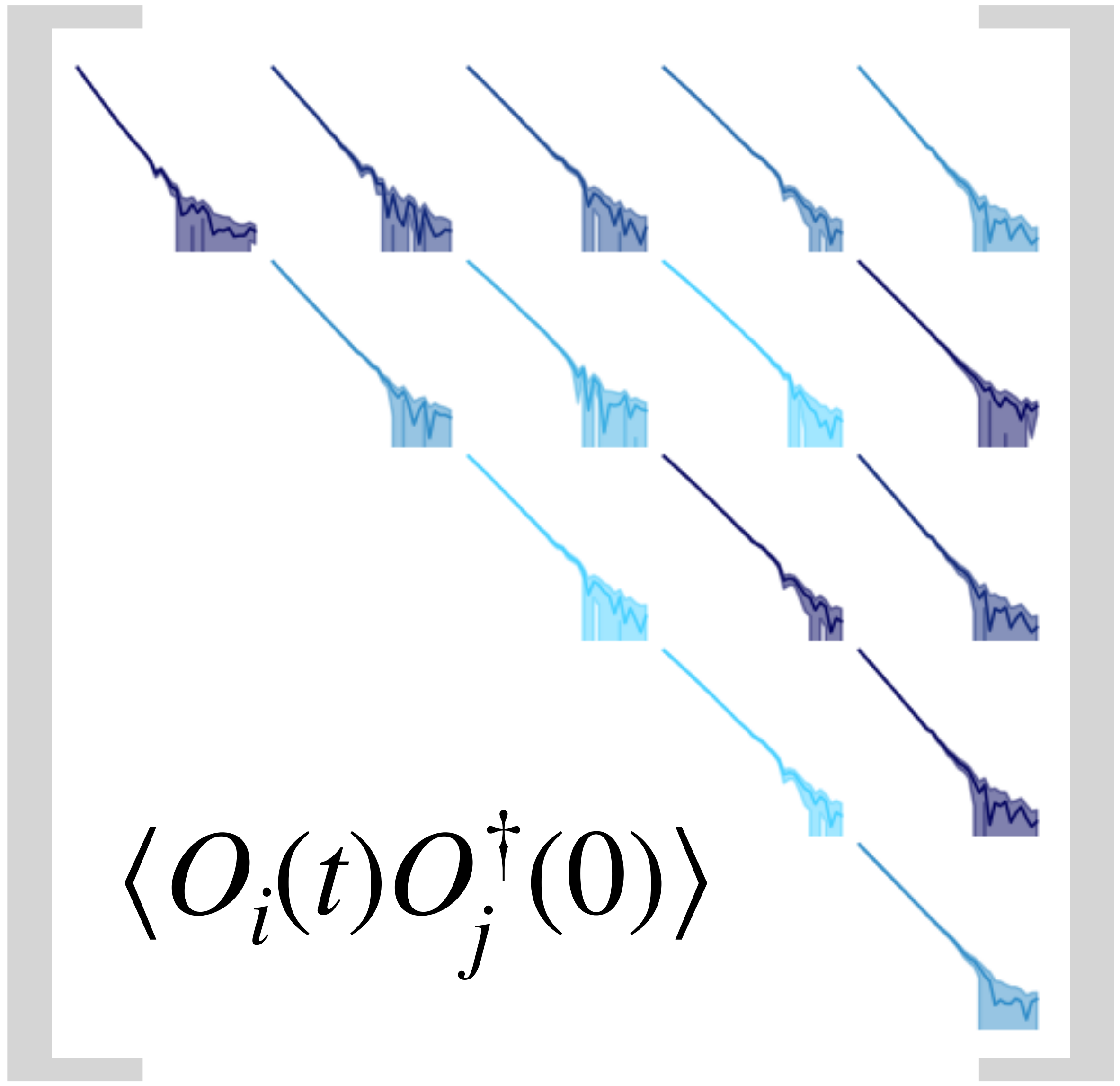

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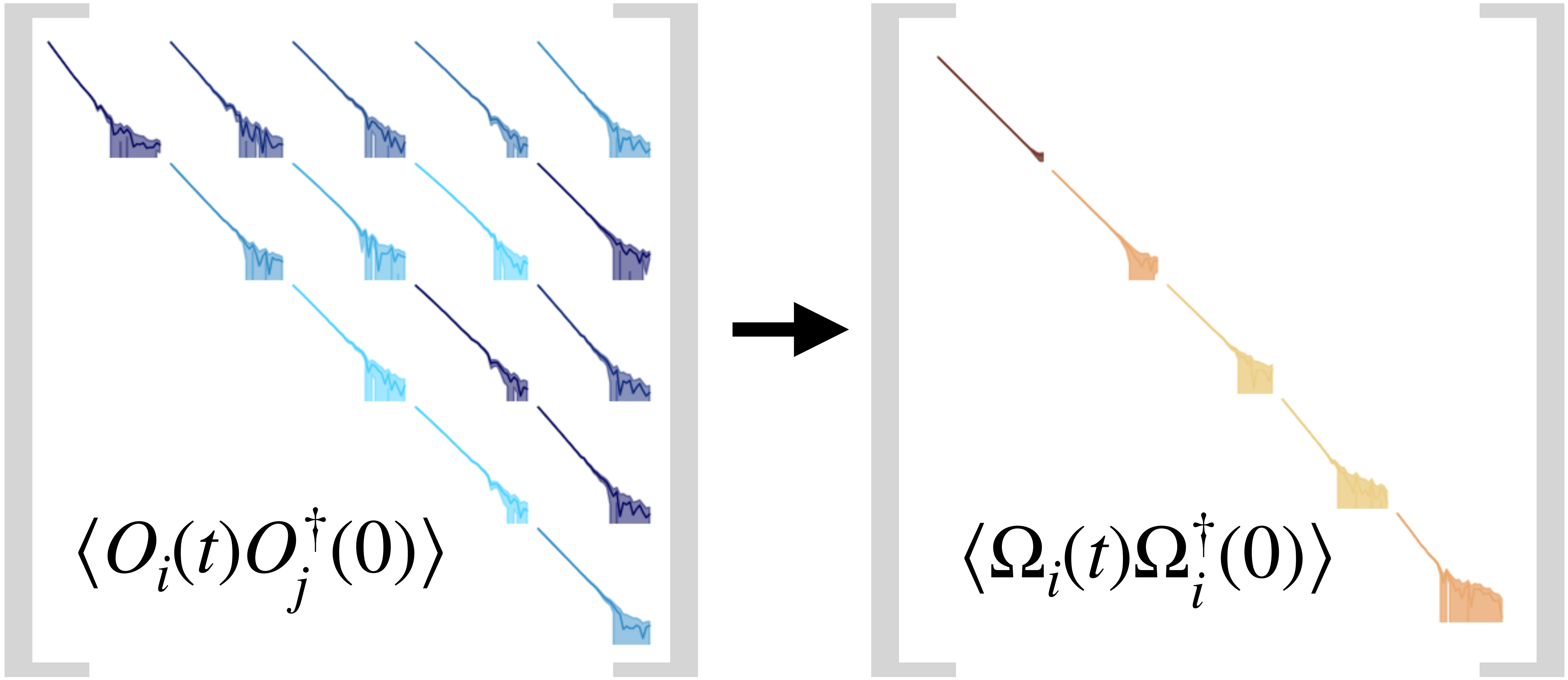

$$\lambda^n(t) \xrightarrow{t \gg 1} \approx Z_n e^{-E^n t}$$

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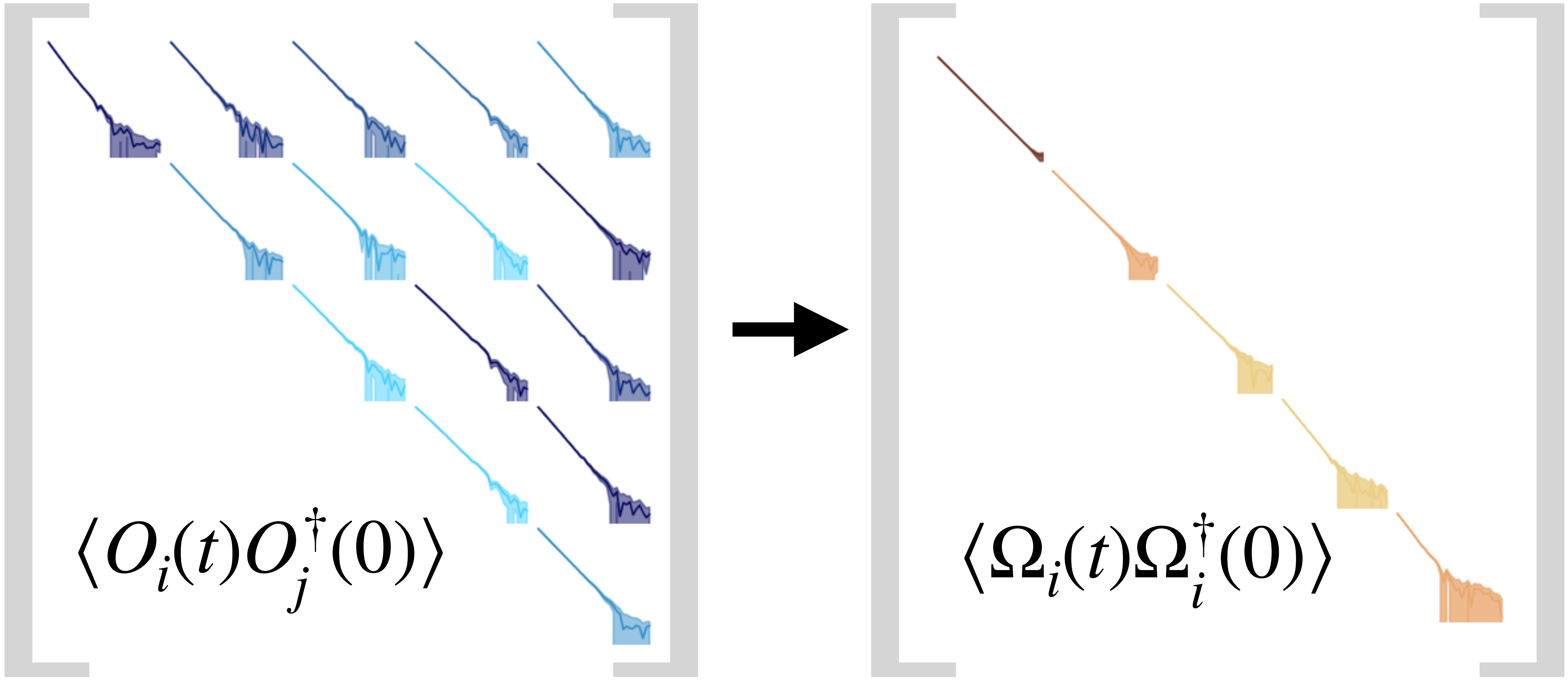
$$m_{\text{eff}}(t) = \log \frac{\lambda^n(t)}{\lambda^n(t+1)} \rightarrow E^n$$



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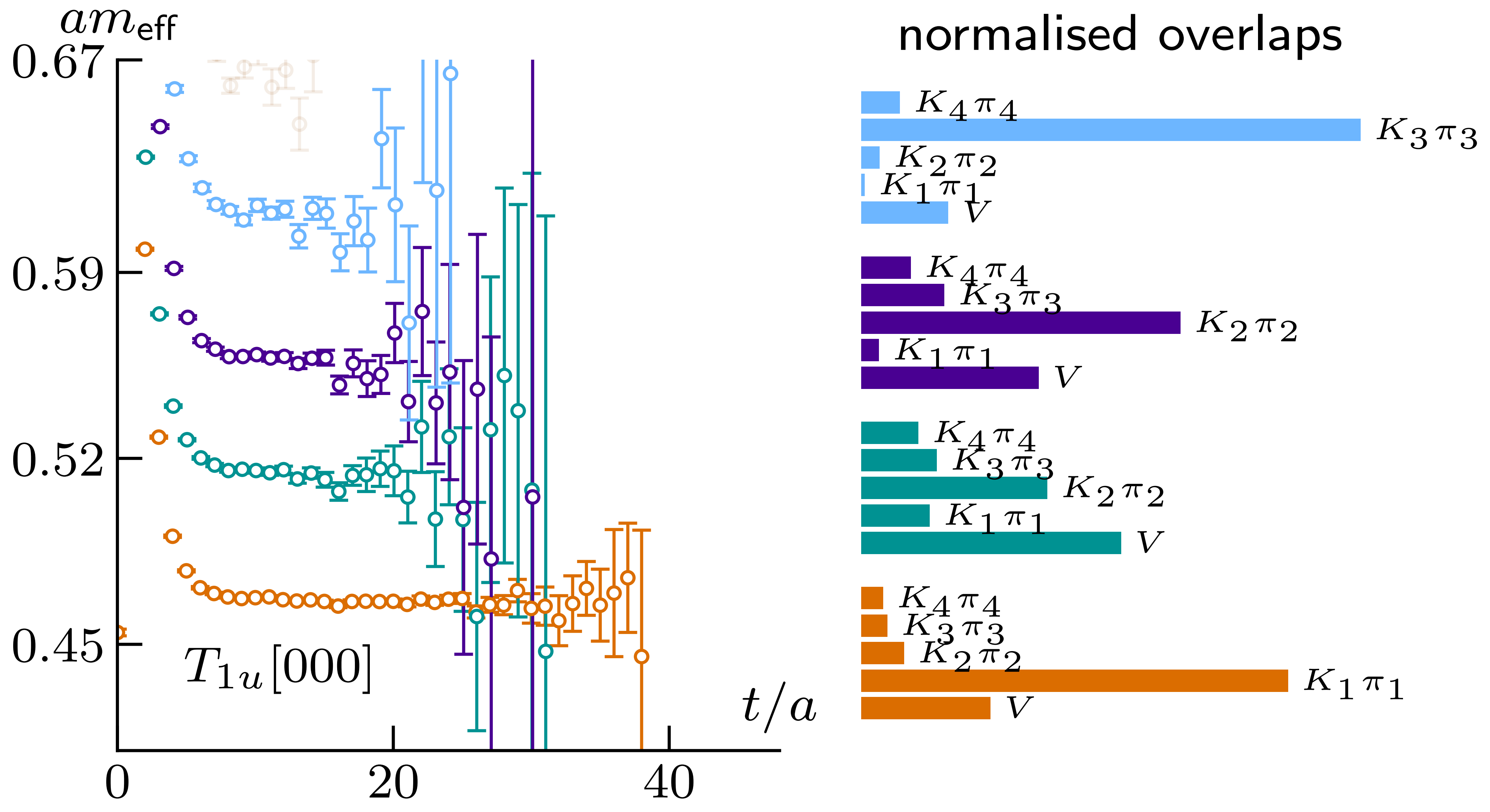

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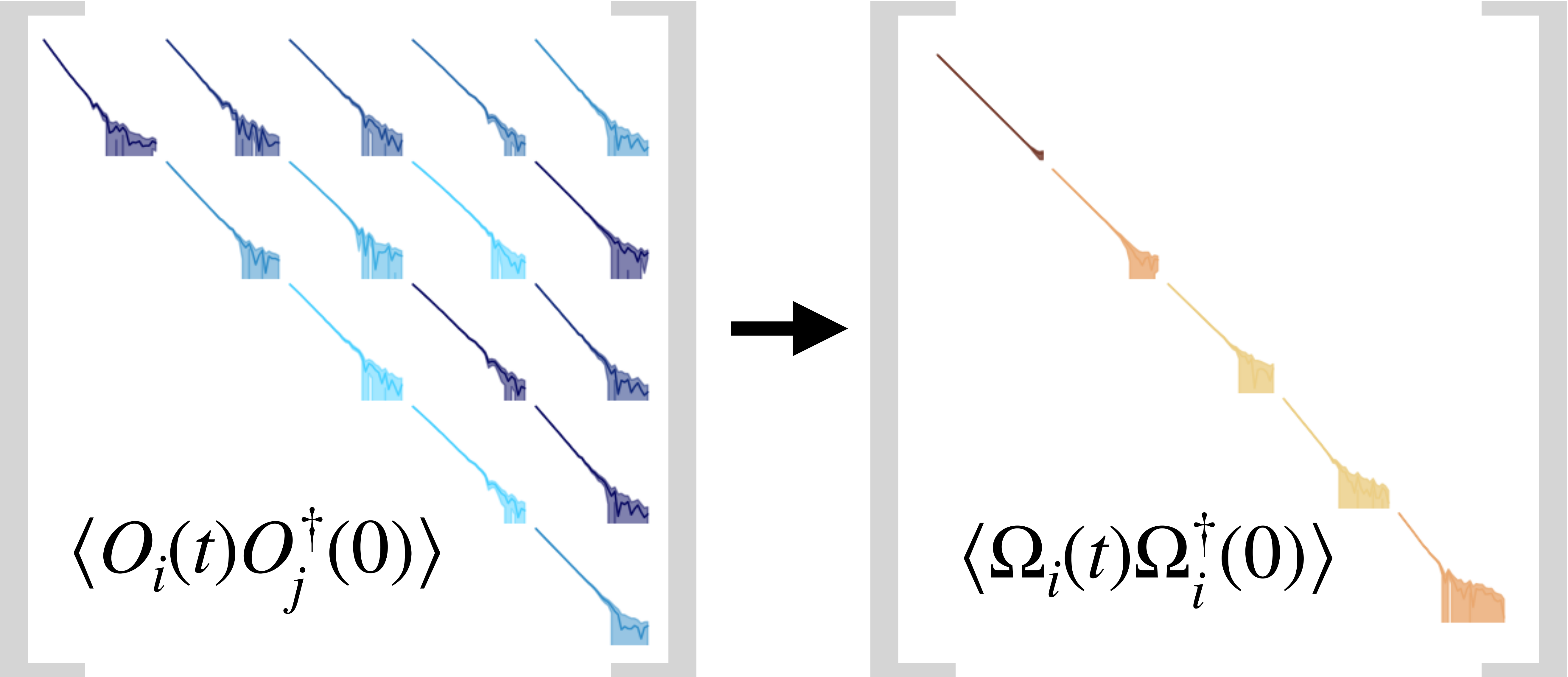

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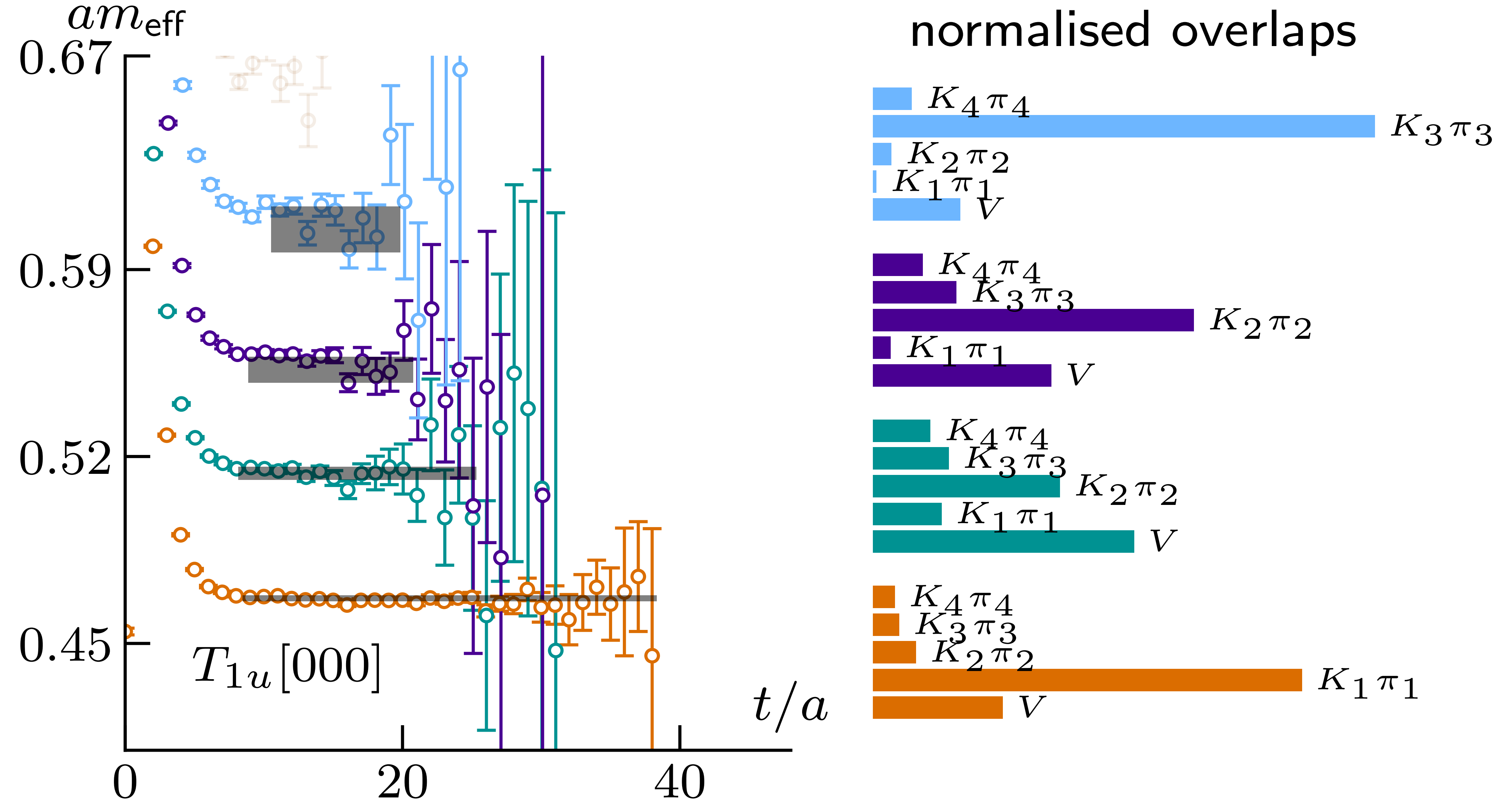
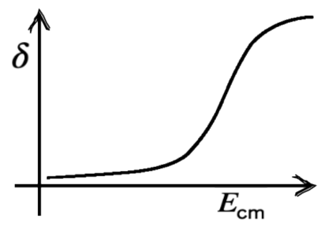


fit via correlated χ^2: $\lambda_n^{\text{mod}}(t) = Z_n^{\text{mod}} e^{-tE_{\text{mod}}^n}$

Phase-shift model

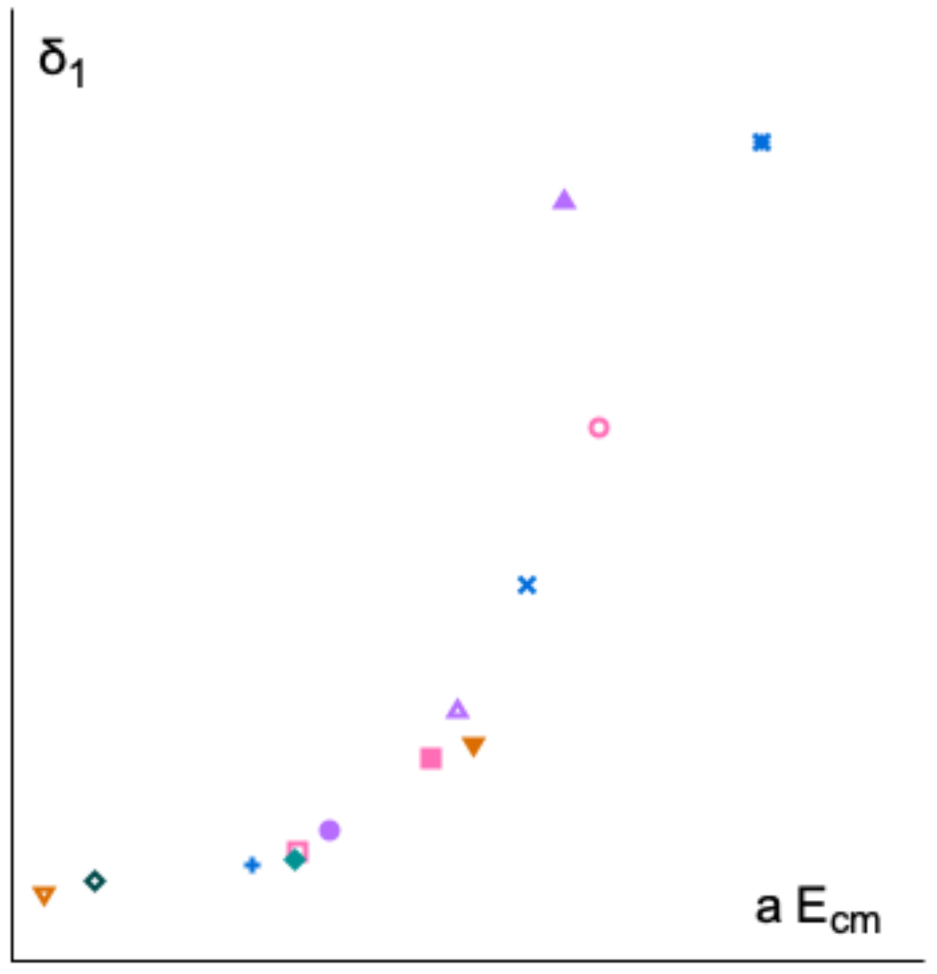


QC reminder:

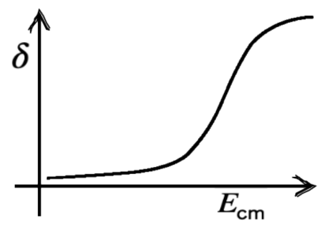
$$\delta_1\big(E_{\mathrm{cm}}(L)\big) = n\pi - \phi^{\Lambda}\big(E_{\mathrm{cm}}(L), L\big), \quad n \in \mathbb{Z}$$

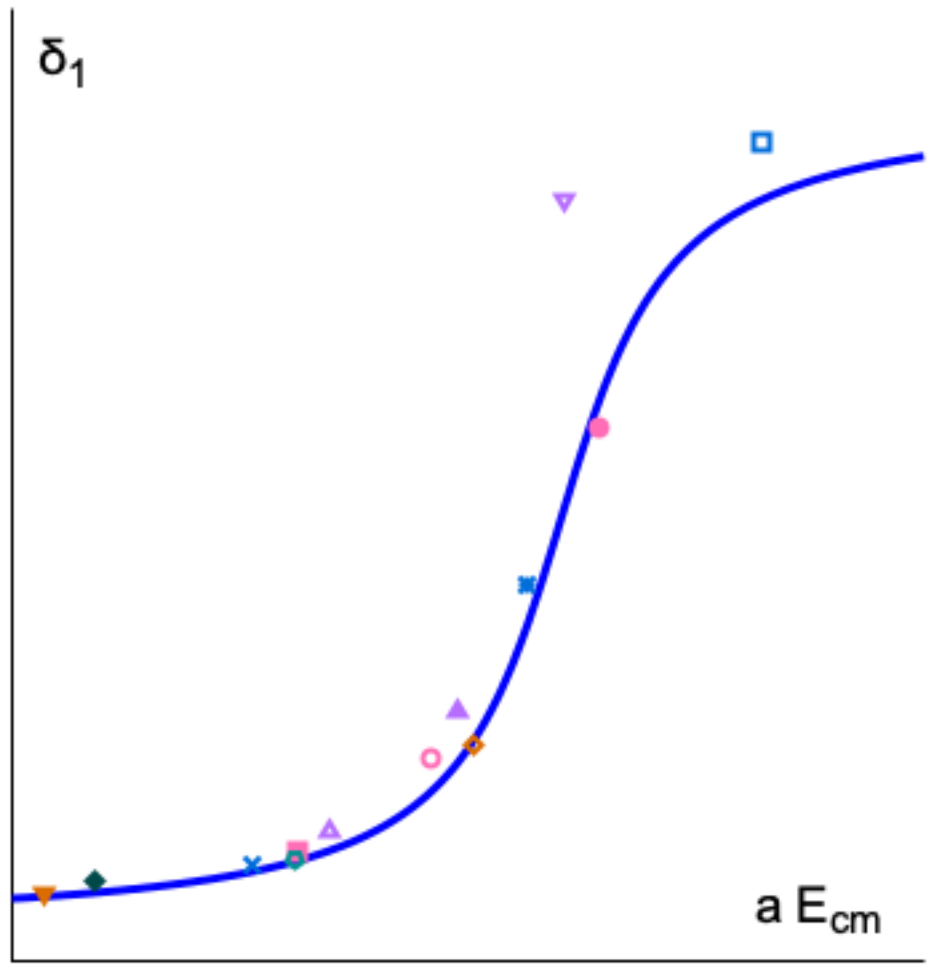
$(i) \equiv (n, \text{irrep } \Lambda, \text{flavour})$

Allows computation of $\delta_1(E_{\mathrm{cm}}^{(i)})$, but poles inaccessible



Phase-shift model





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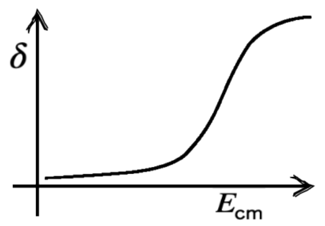
Instead, find $\delta^{\rm mod}$ with parameters $\alpha^{\rm mod}$ through a fit

Phase-shift model 

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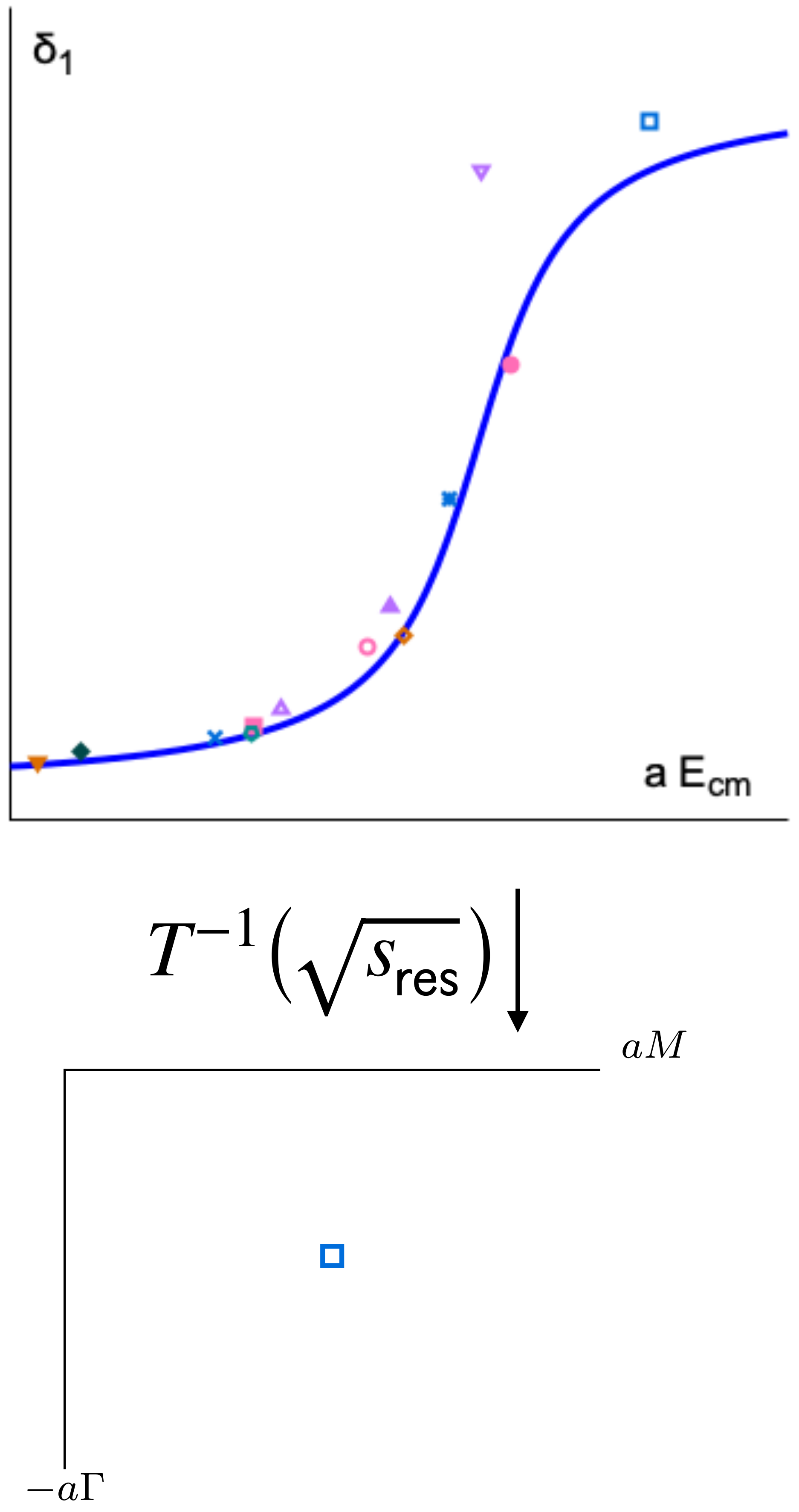
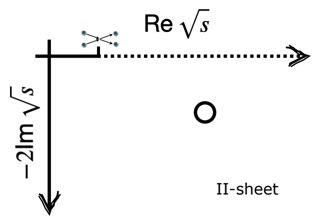


Resonance Pole 

Substitute and analytically-continue

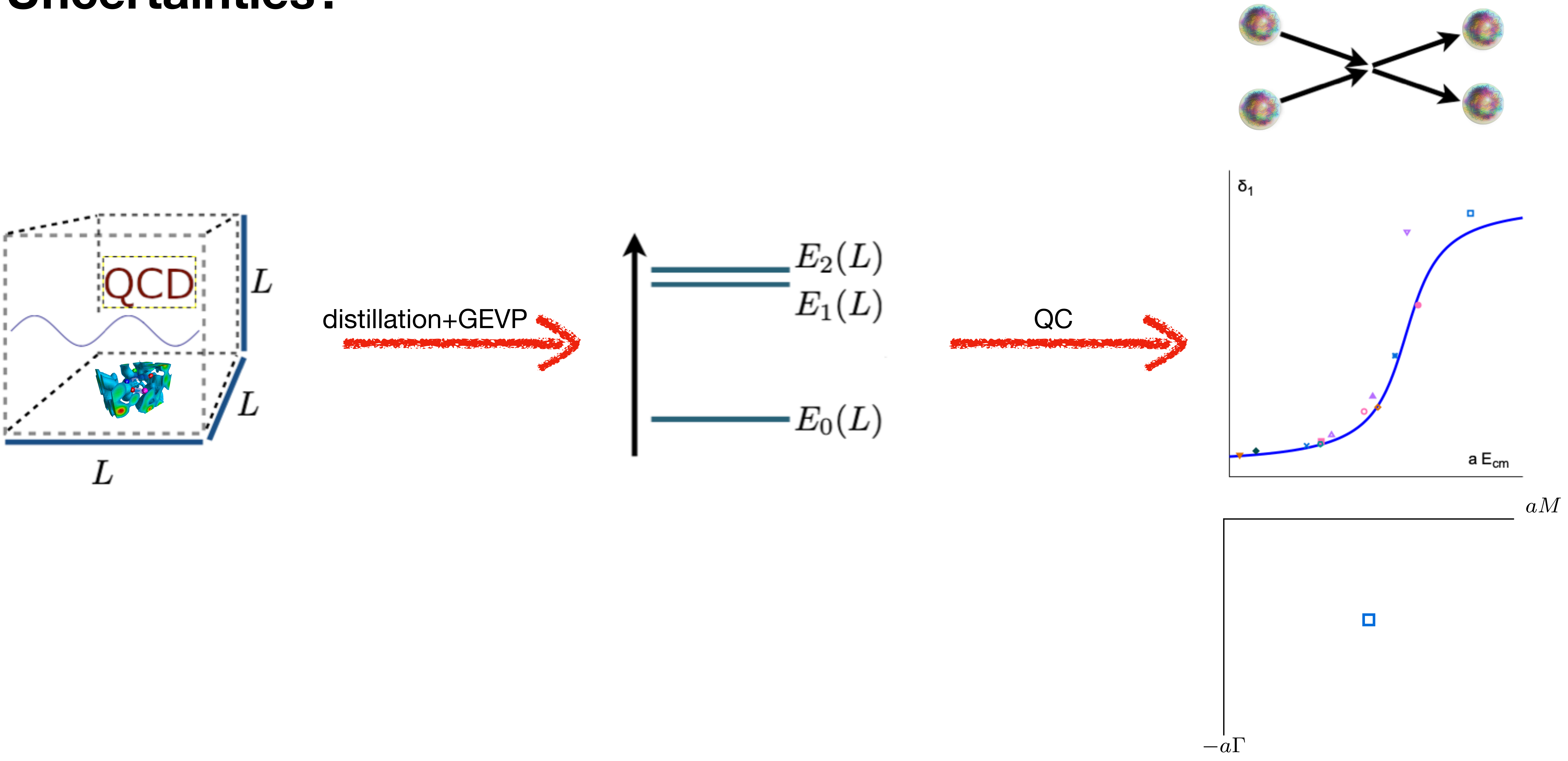
$$T^{\text{mod}}(\sqrt{s}) = \frac{1}{\cot\delta^{\text{mod}}(\sqrt{s}) - i}$$

find T_1^{mod} complex pole

⬍

find root $\cot\delta^{\text{mod}} - i$

Uncertainties?



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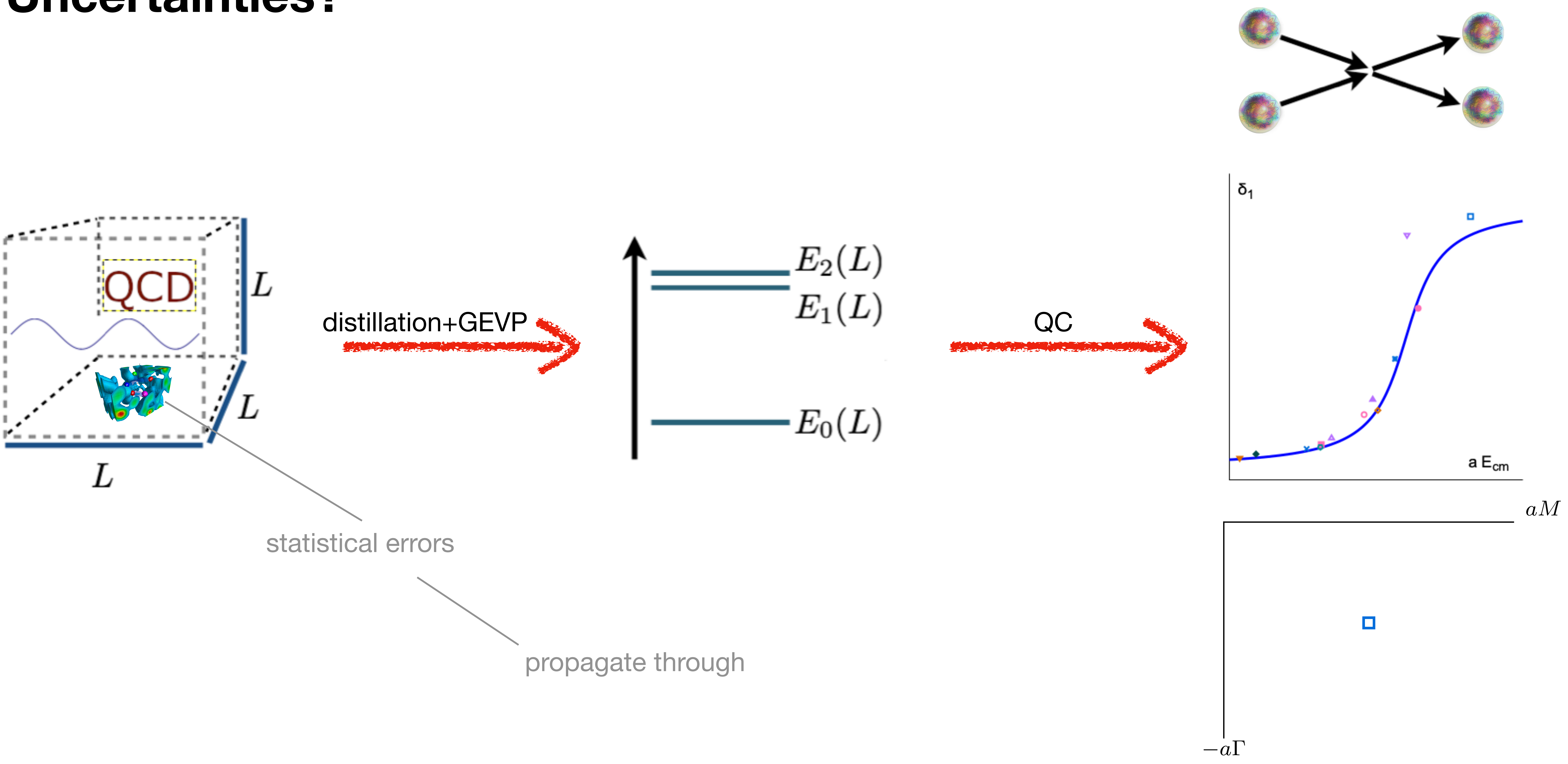
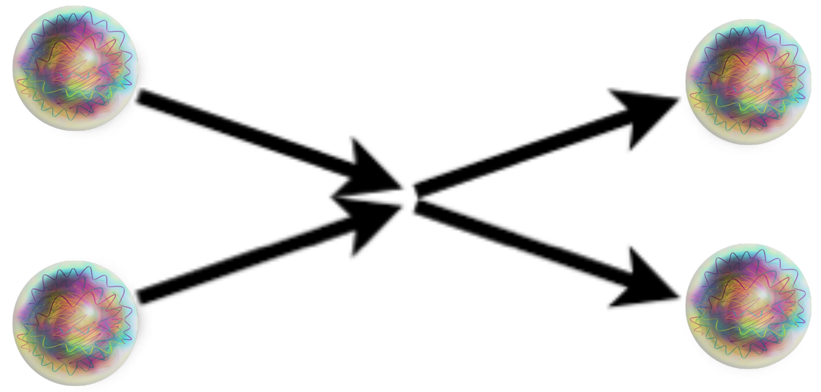
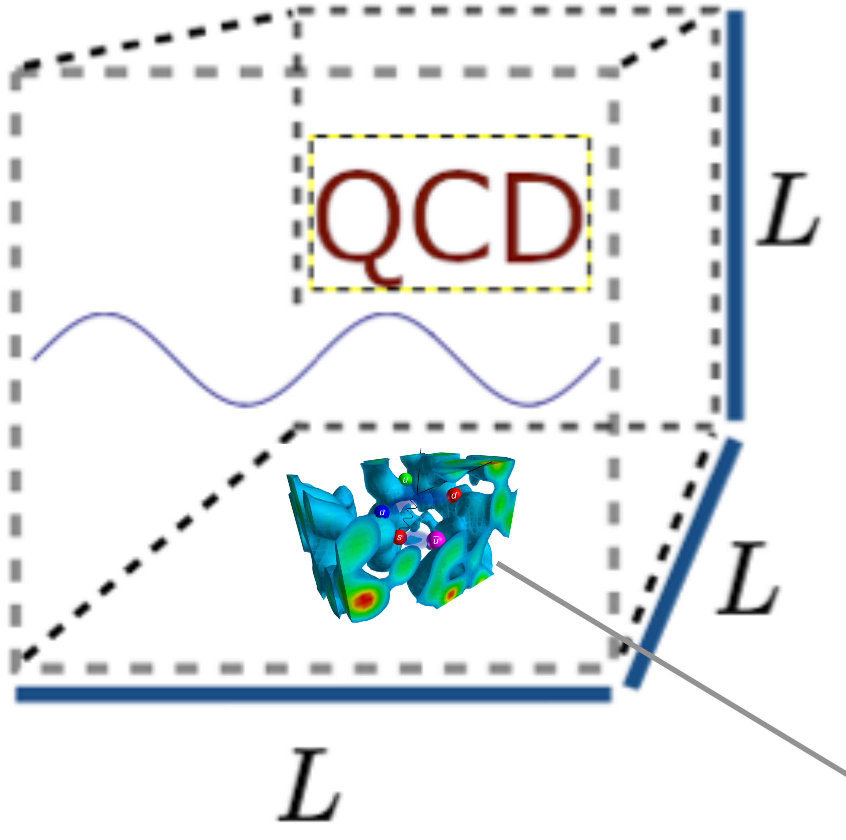


Uncertainties?

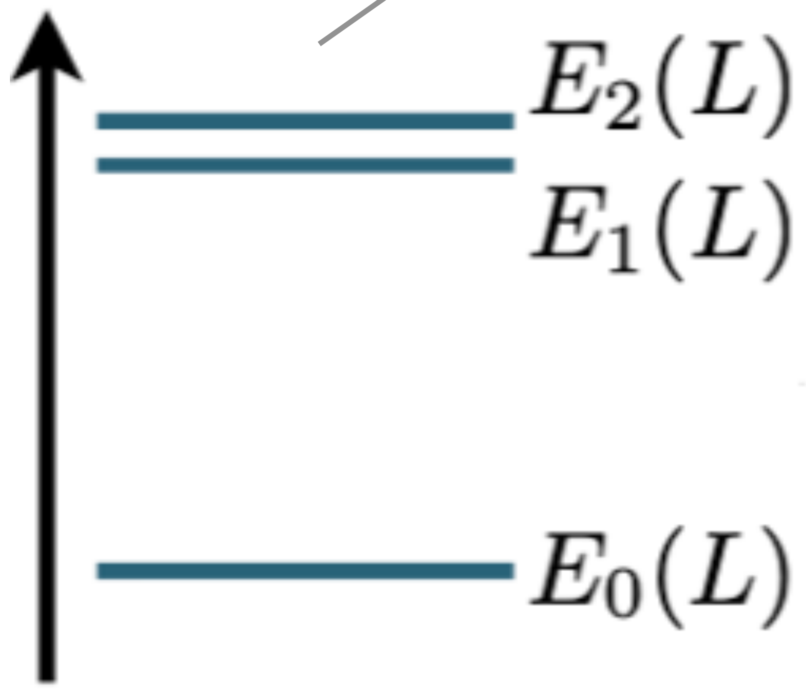
model average: $w_{\mathrm{t}}(\text{choice}) \propto e^{-\mathrm{AIC}_{\mathrm{tot}}(\text{choice})/2}$

systematical errors

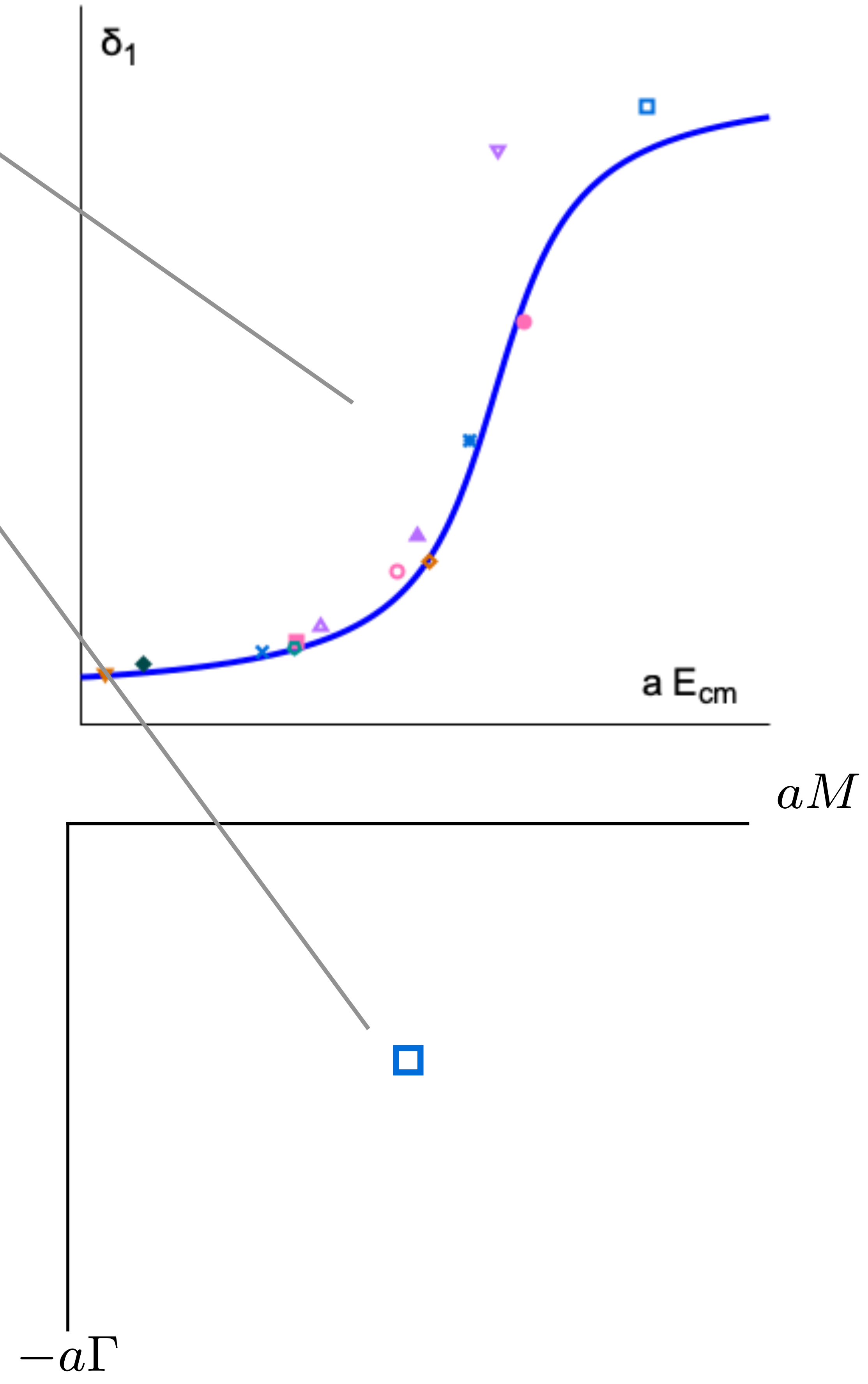




distillation+GEVP



QC



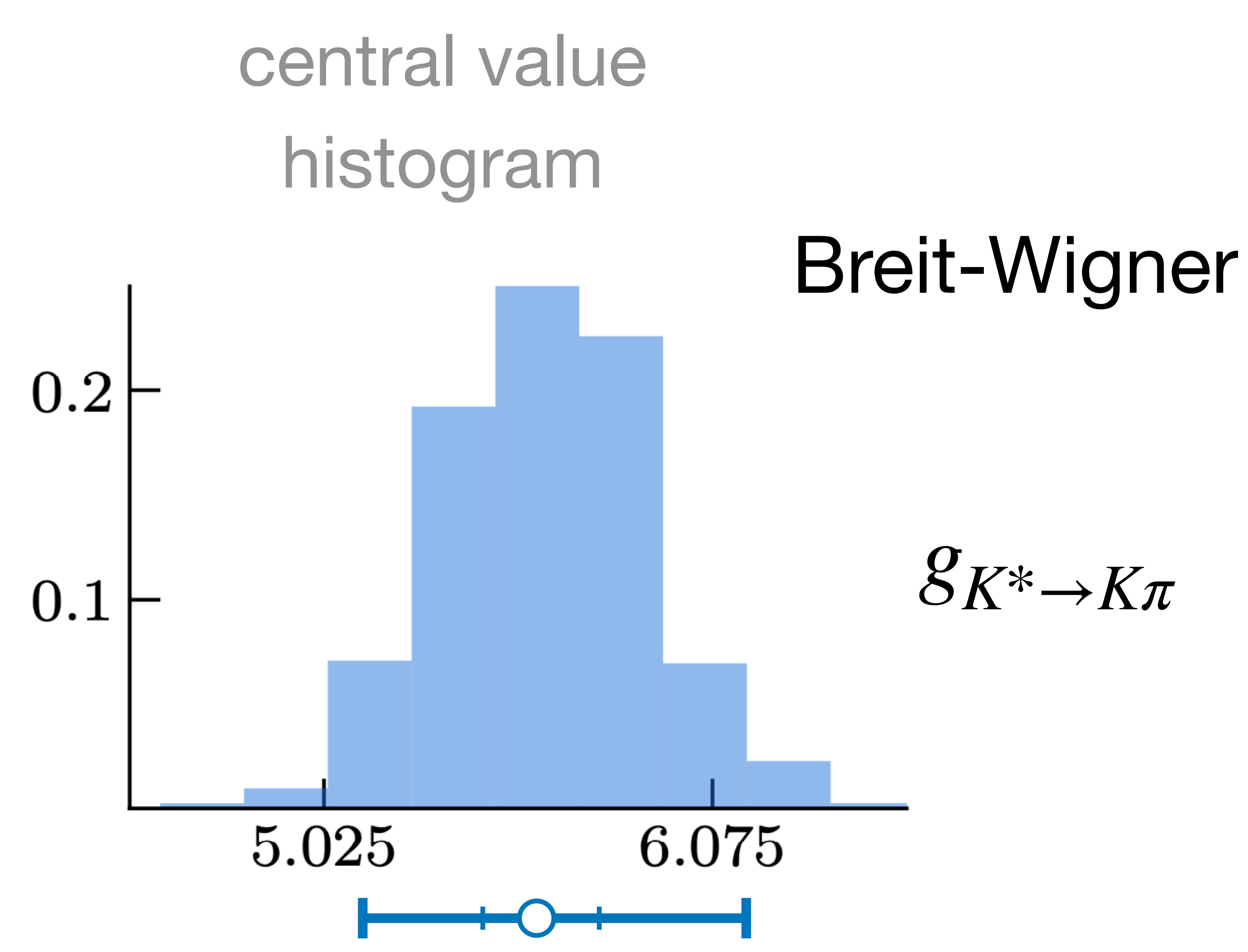
statistical errors

propagate through

Result prescription

- data-driven systematic: weighted $95\ \%$ confidence interval of (central) weighted mean
- statistical: fluctuation of above over replicas

50000 fit-ranges $\times$ 2000 replicas



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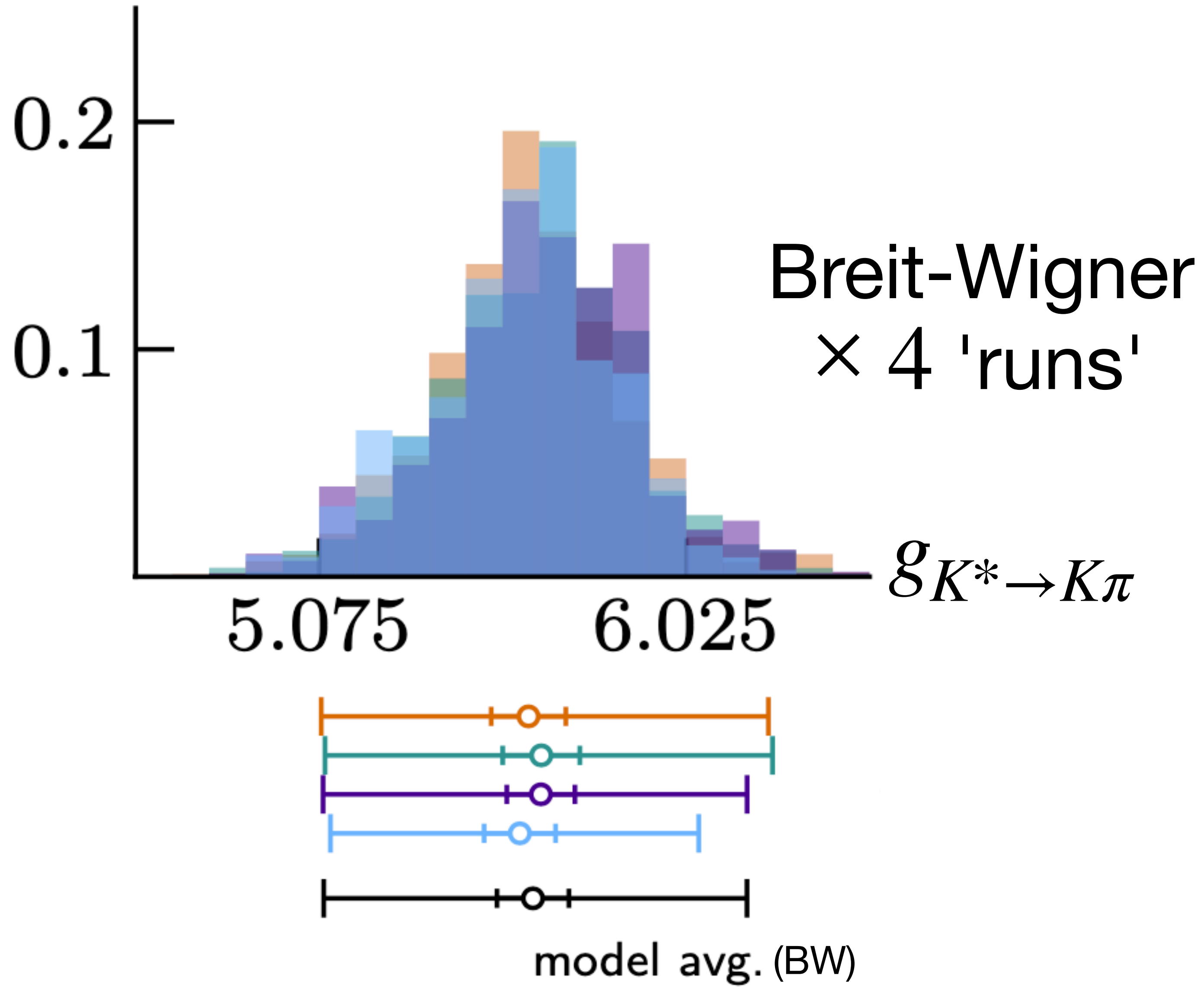

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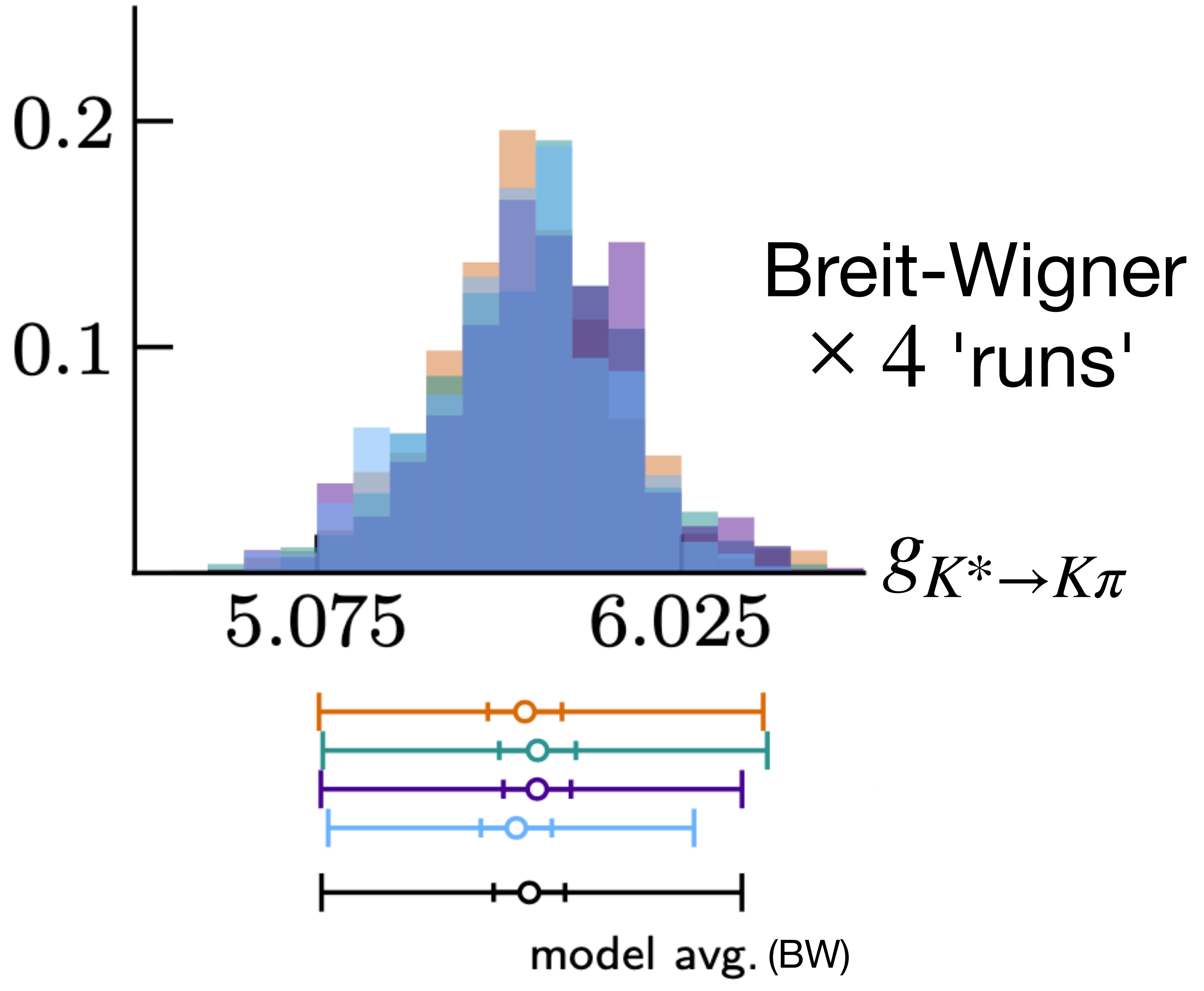

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extended model average: $\sum_{\text{mod}} \sum_{\text{cuts}} \sum_{\text{fit ranges}}$

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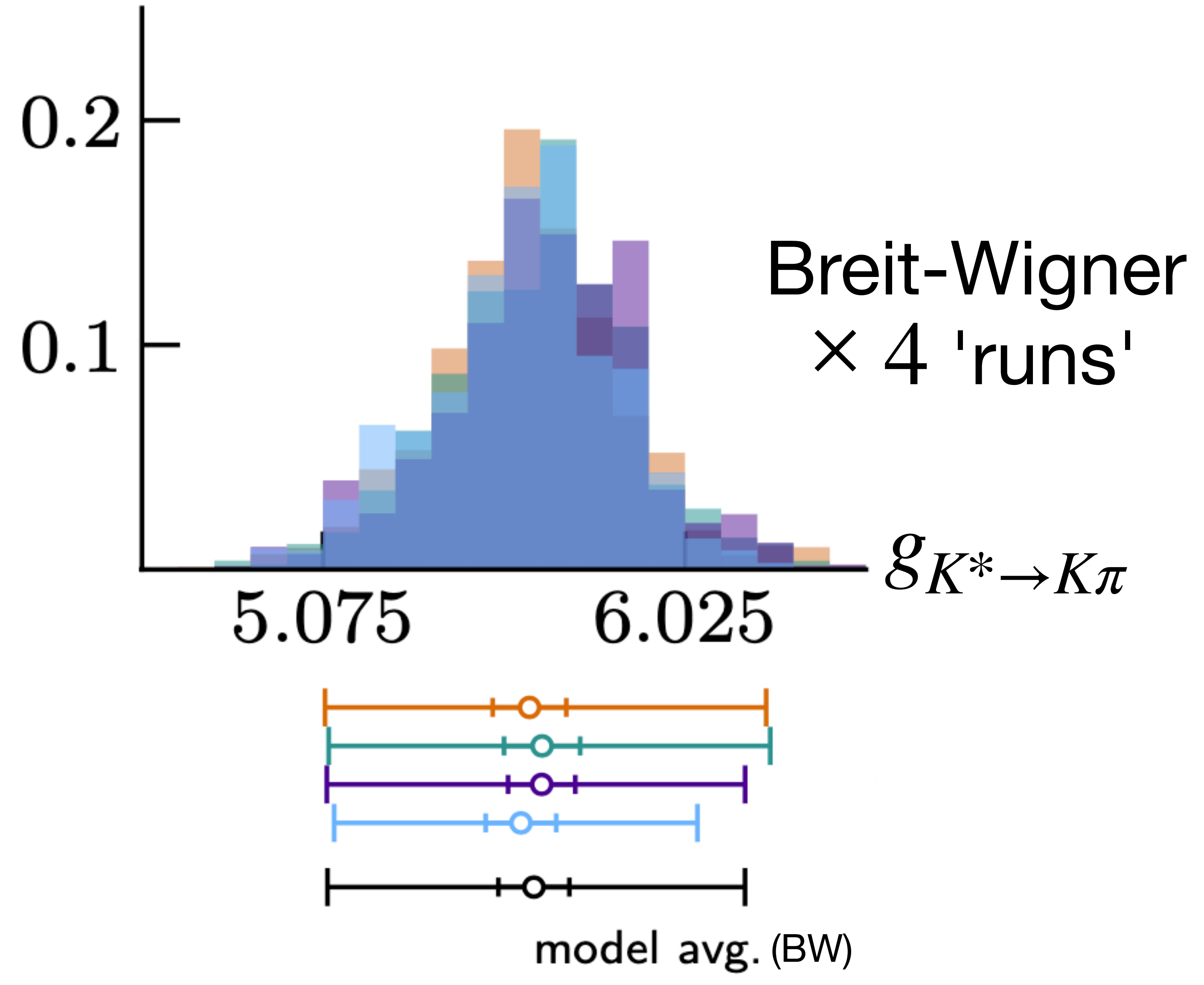

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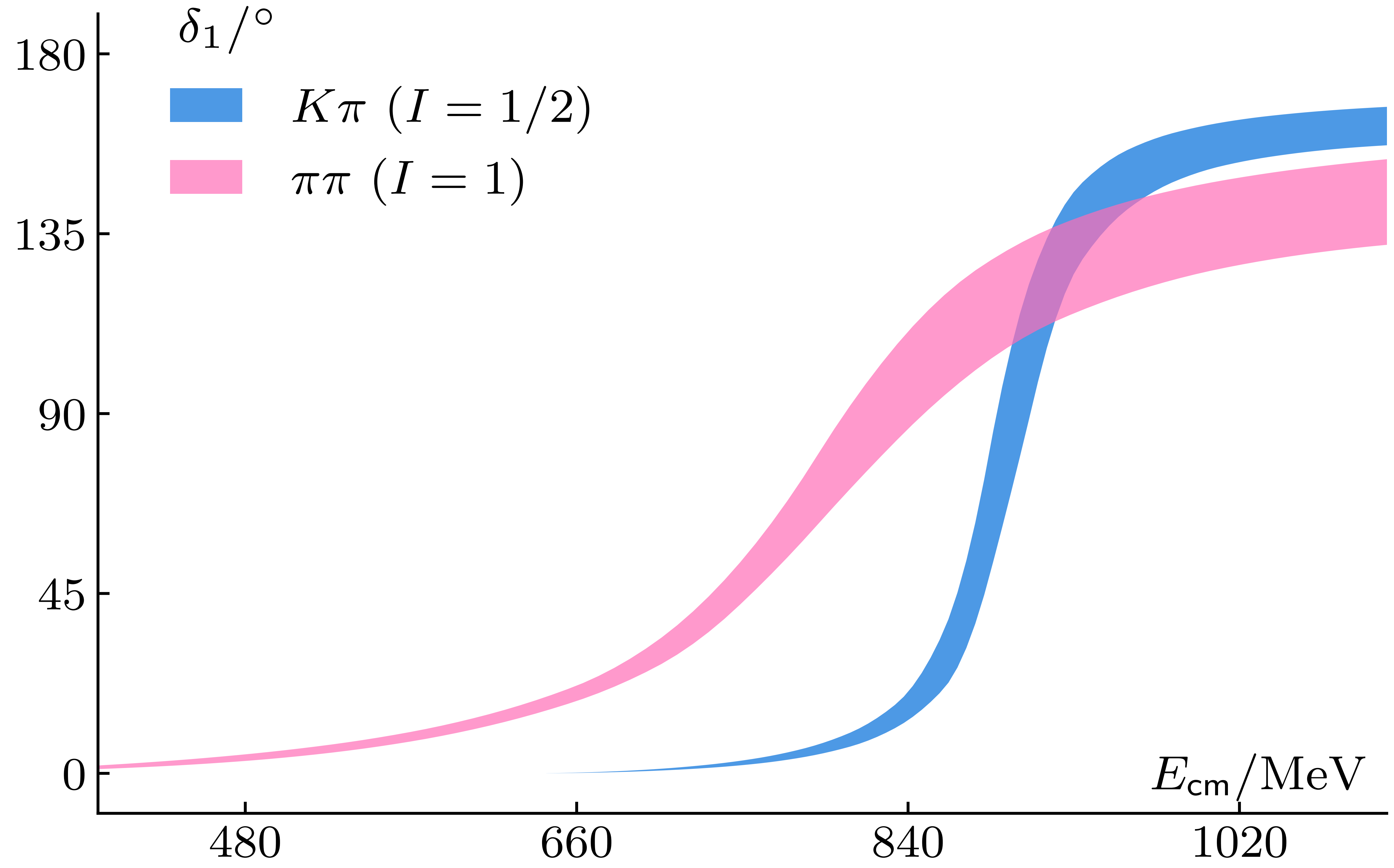

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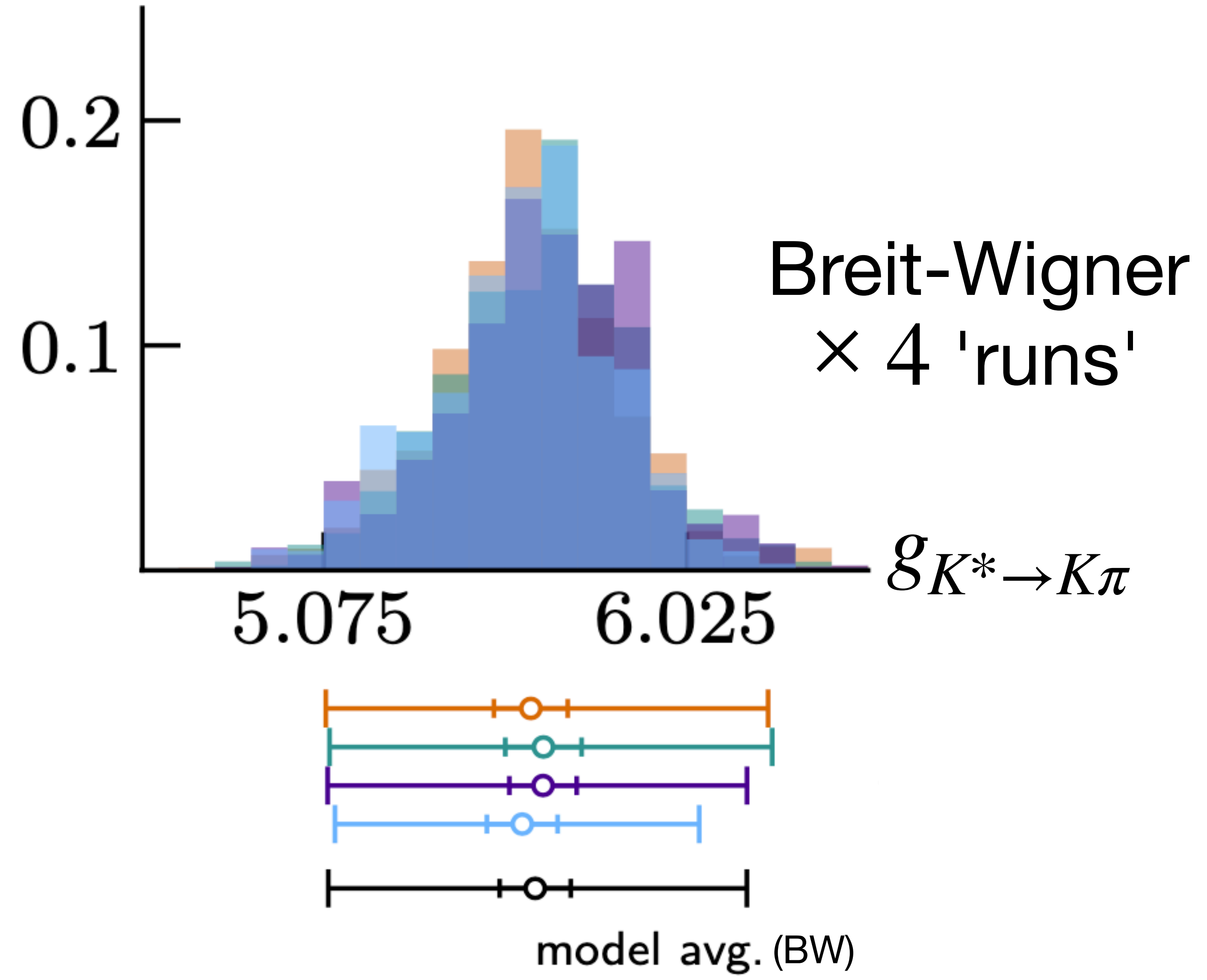

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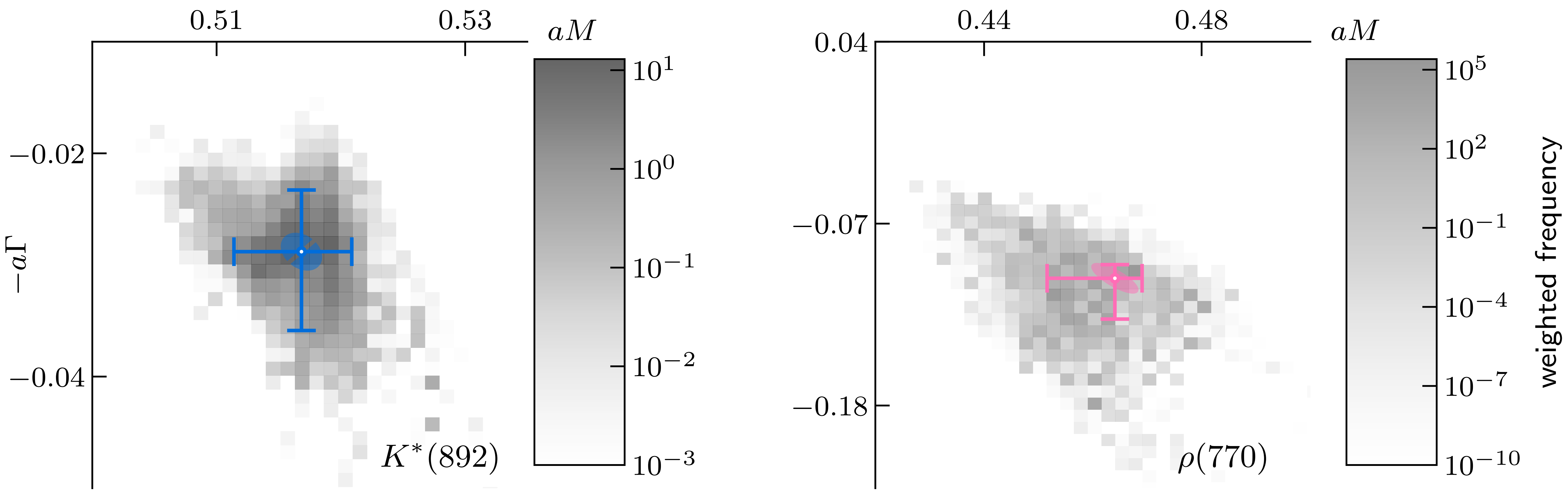

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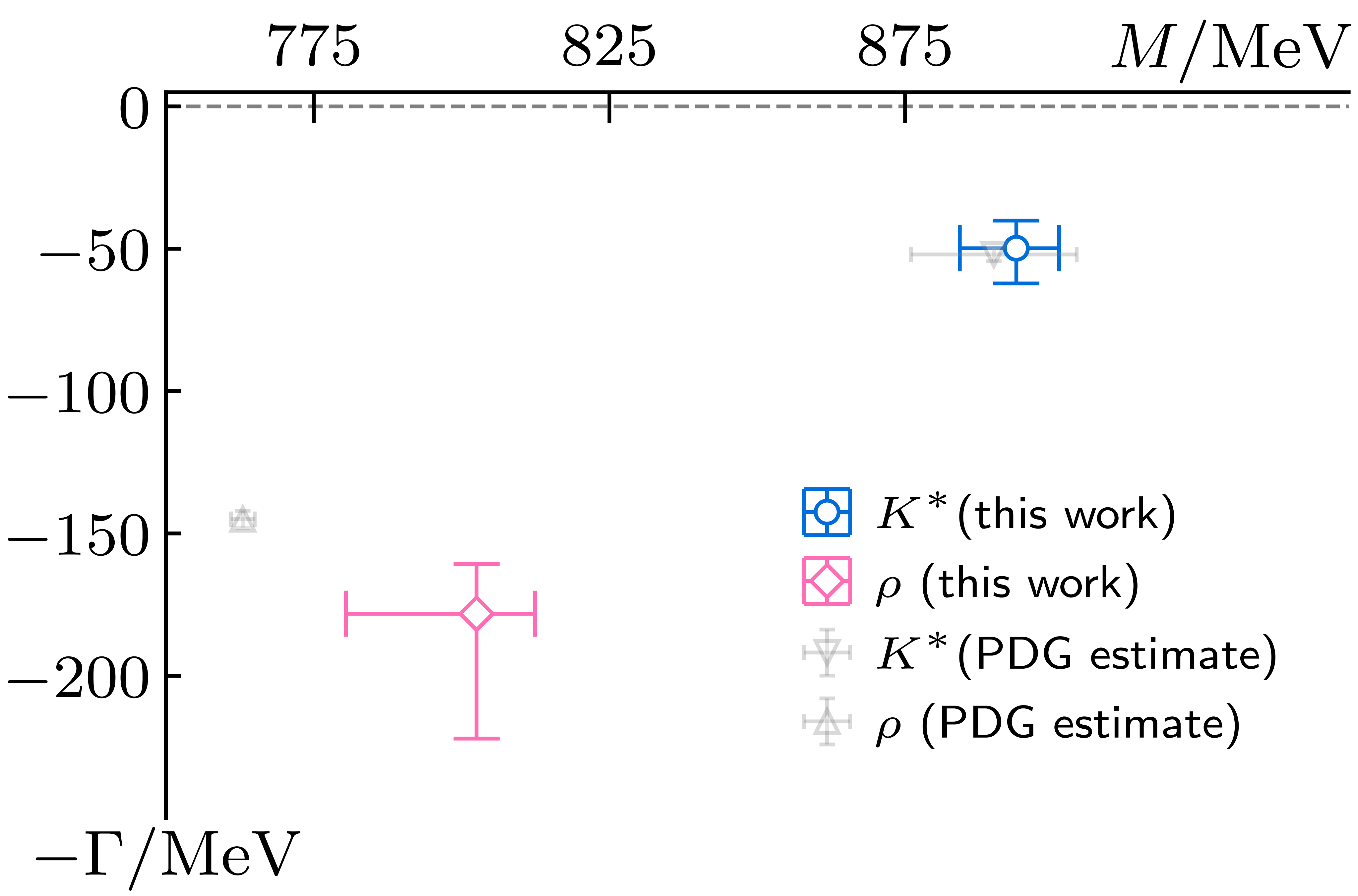

Statistical and data-driven systematic (quadrature in plot)

$$K^*(892)\begin{cases} M &= 893(2)(8)\ \mathrm{MeV} \\ \Gamma &= 51(2)(11)\ \mathrm{MeV}\end{cases}$$

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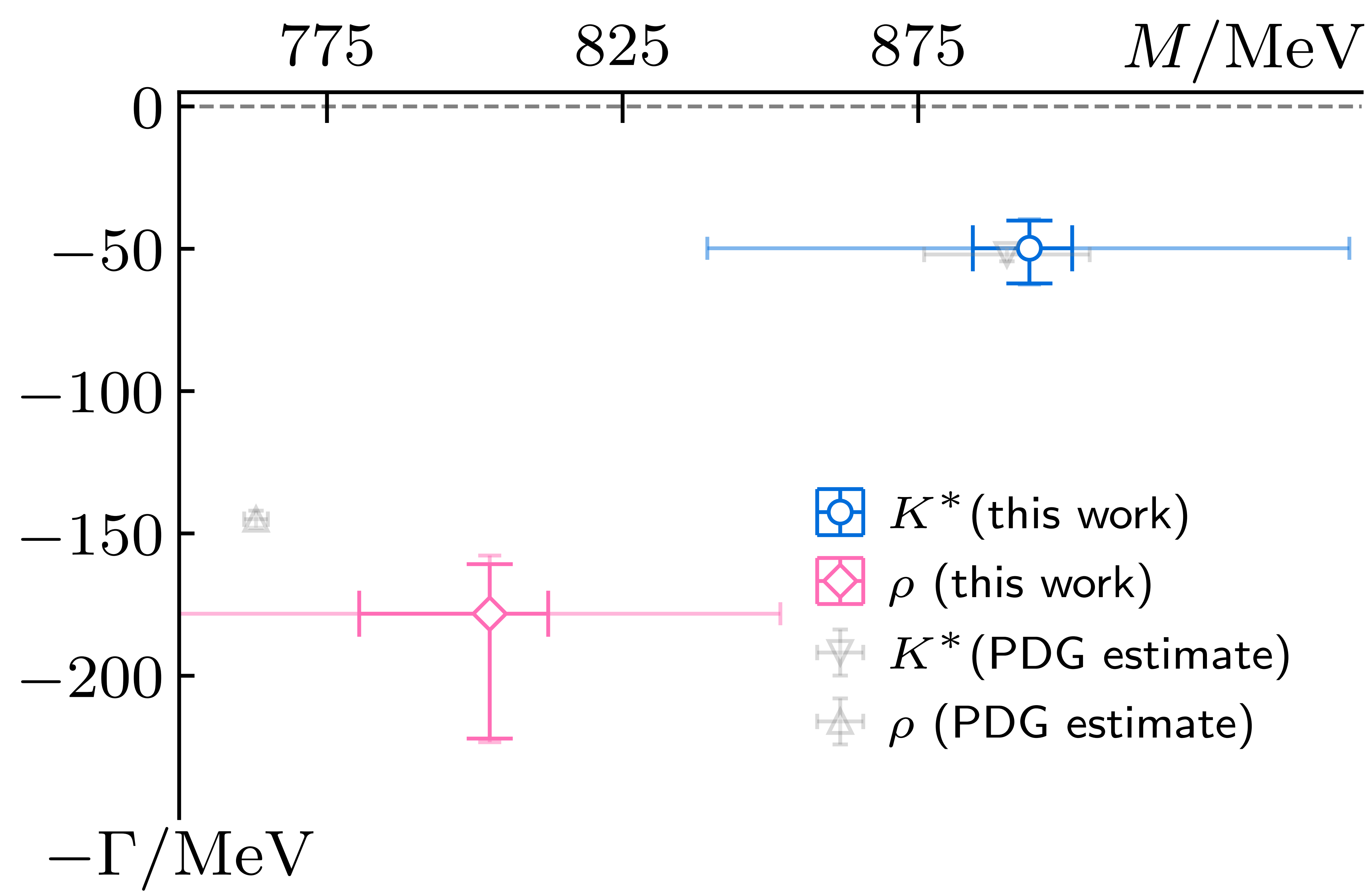

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- assume $(a\Lambda_{QCD}) \approx 5\,\%$ conservative *discretisation* uncertainty + other estimated extra systematics $\sim 6\,\%$ total

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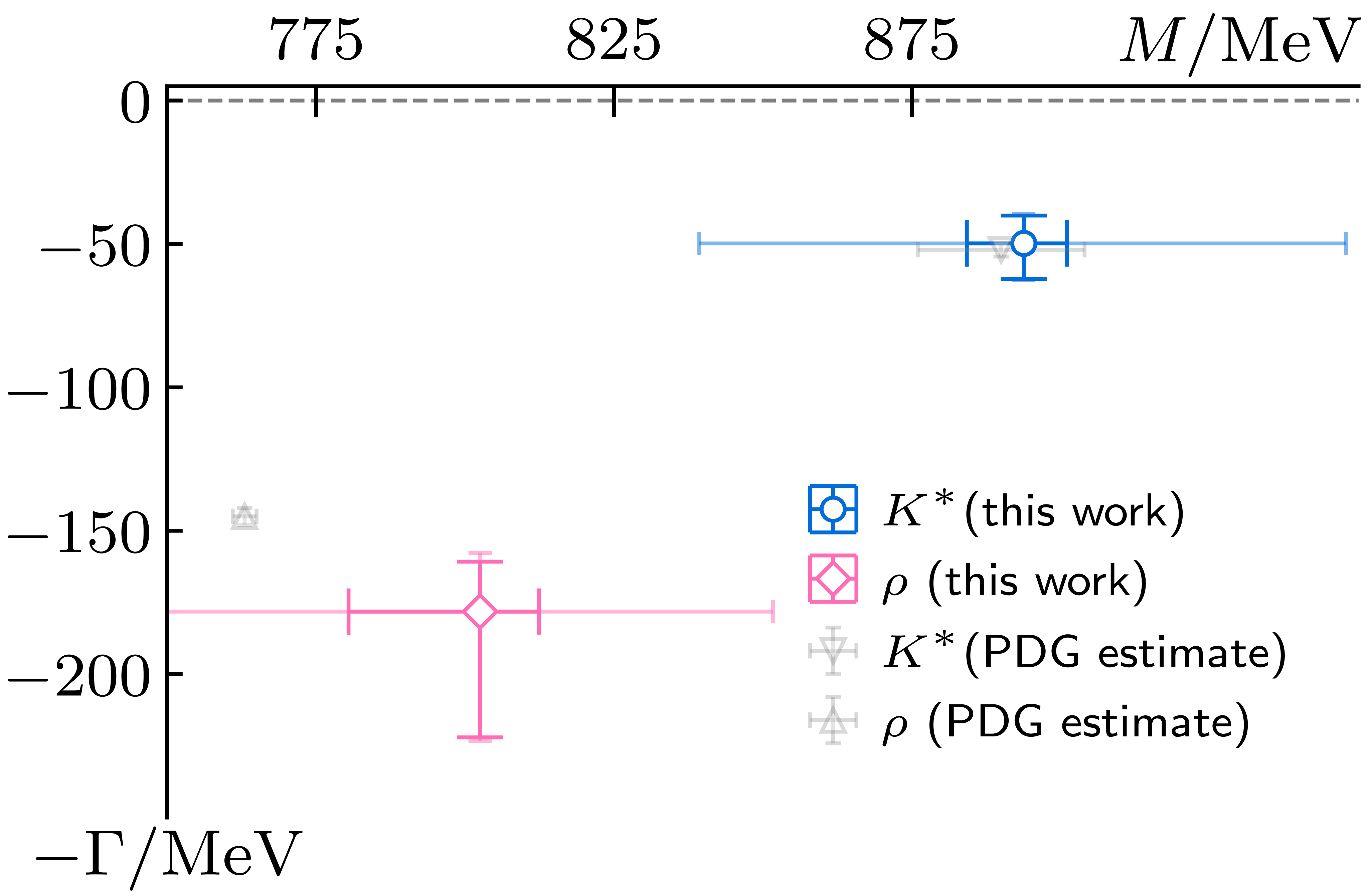

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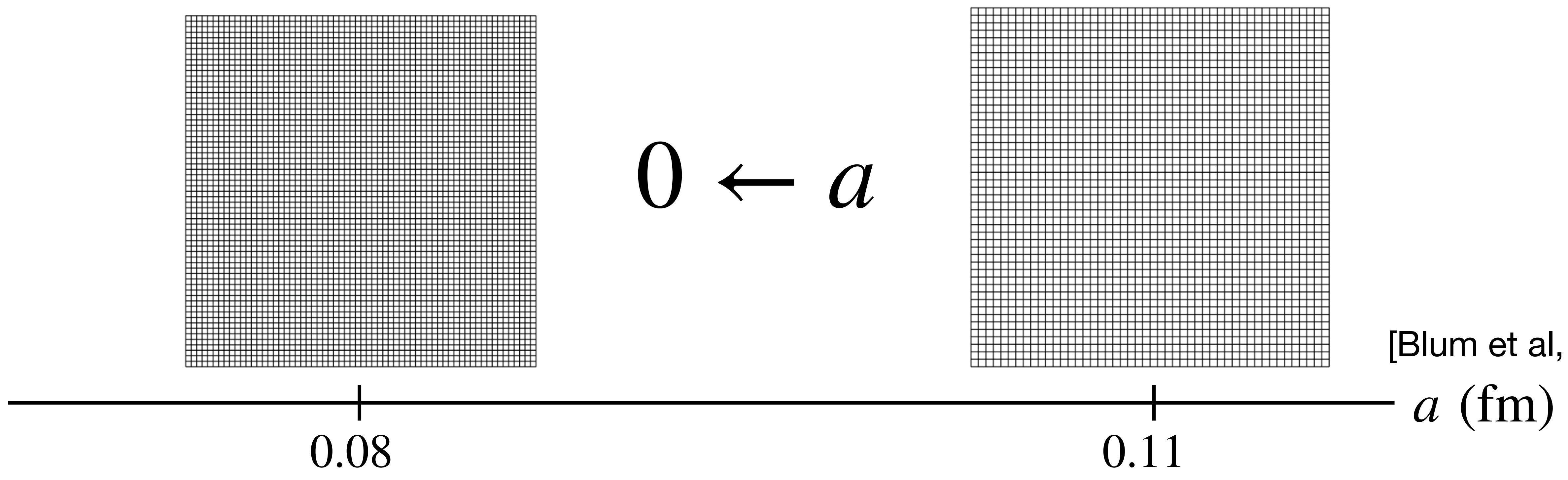
next frontier:
continuum limit

[Green et al, PRL, 2021]
[Peterken & Hansen, 2408.07062, 2024]



[Blum et al, PRD, 2016]

Conclusions

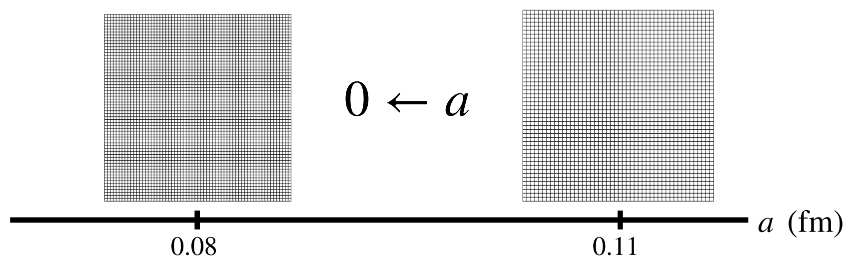
PhysRevD.111.054510
PhysRevLett.134.111901

$K^*(892)$ and $\rho(770)$ at $m_\pi \approx 139$ MeV from Lattice QCD

Data-driven systematic via sampling method of lattice energies

Important towards precision

- continuum limit



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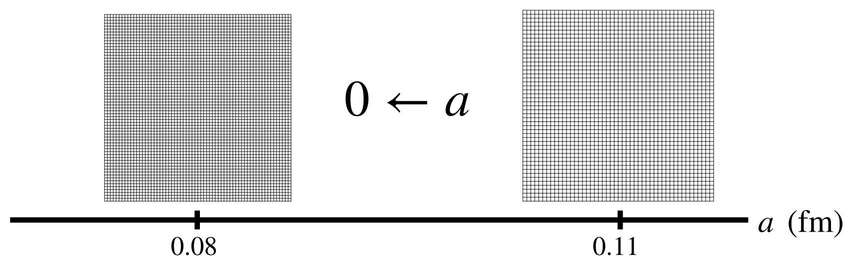

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- scalars $\sigma, f_0 \to \pi\pi\ 0^+(0^{++}),\ \ \kappa \to K\pi\ 1/2(0^-)$

dp393

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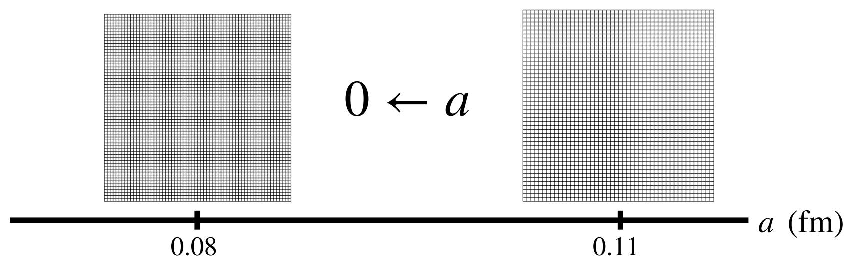

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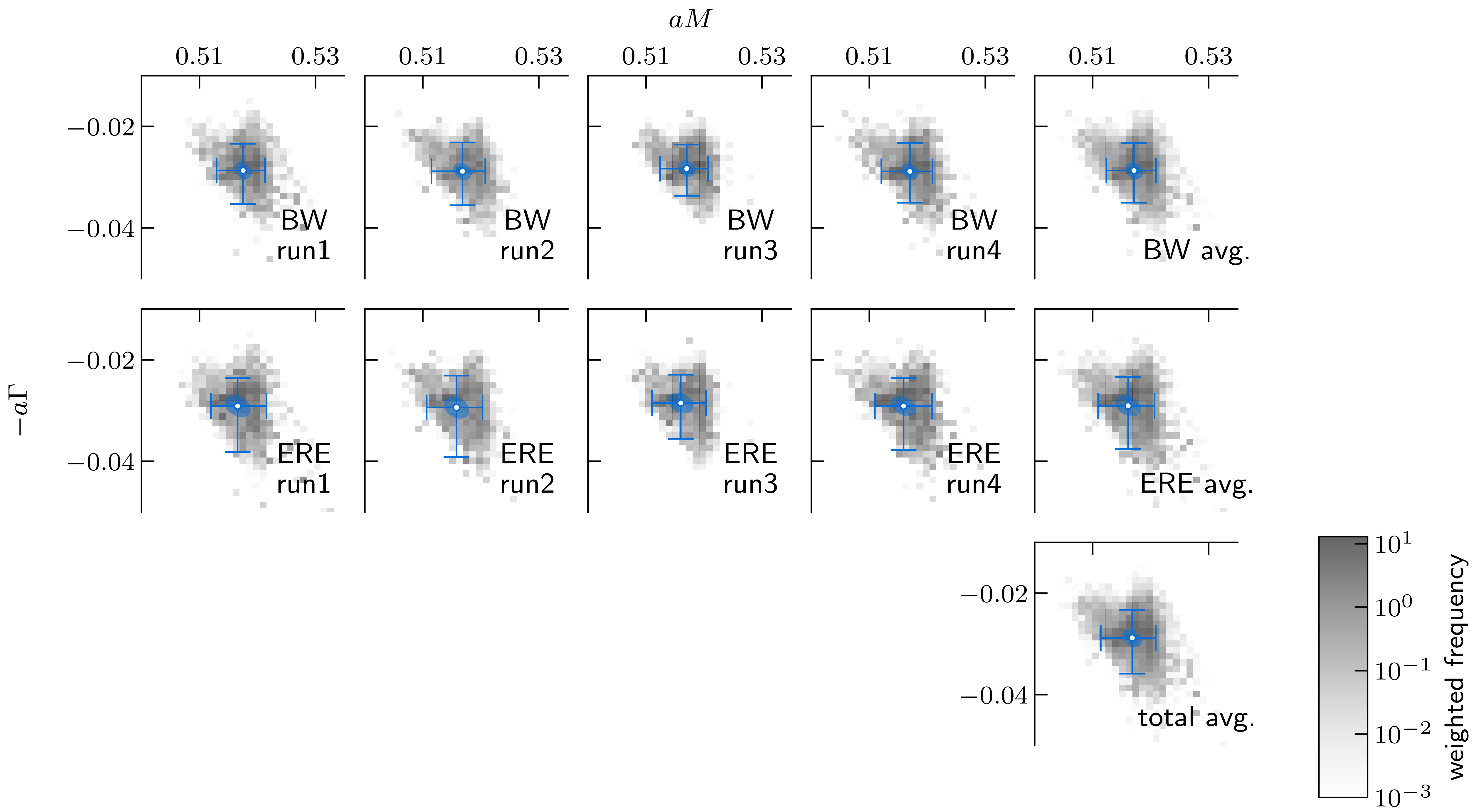
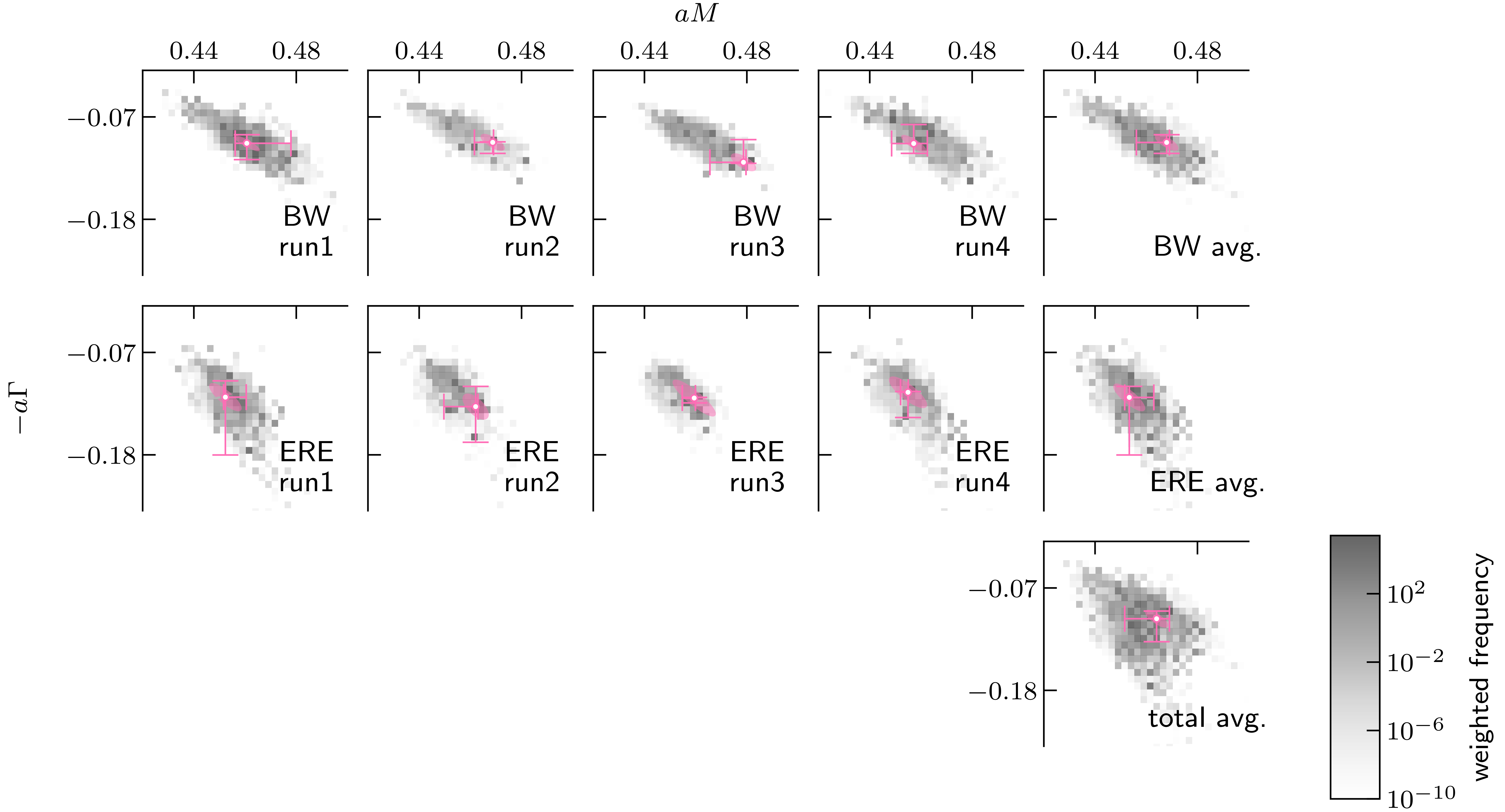
Thanks for the attention!

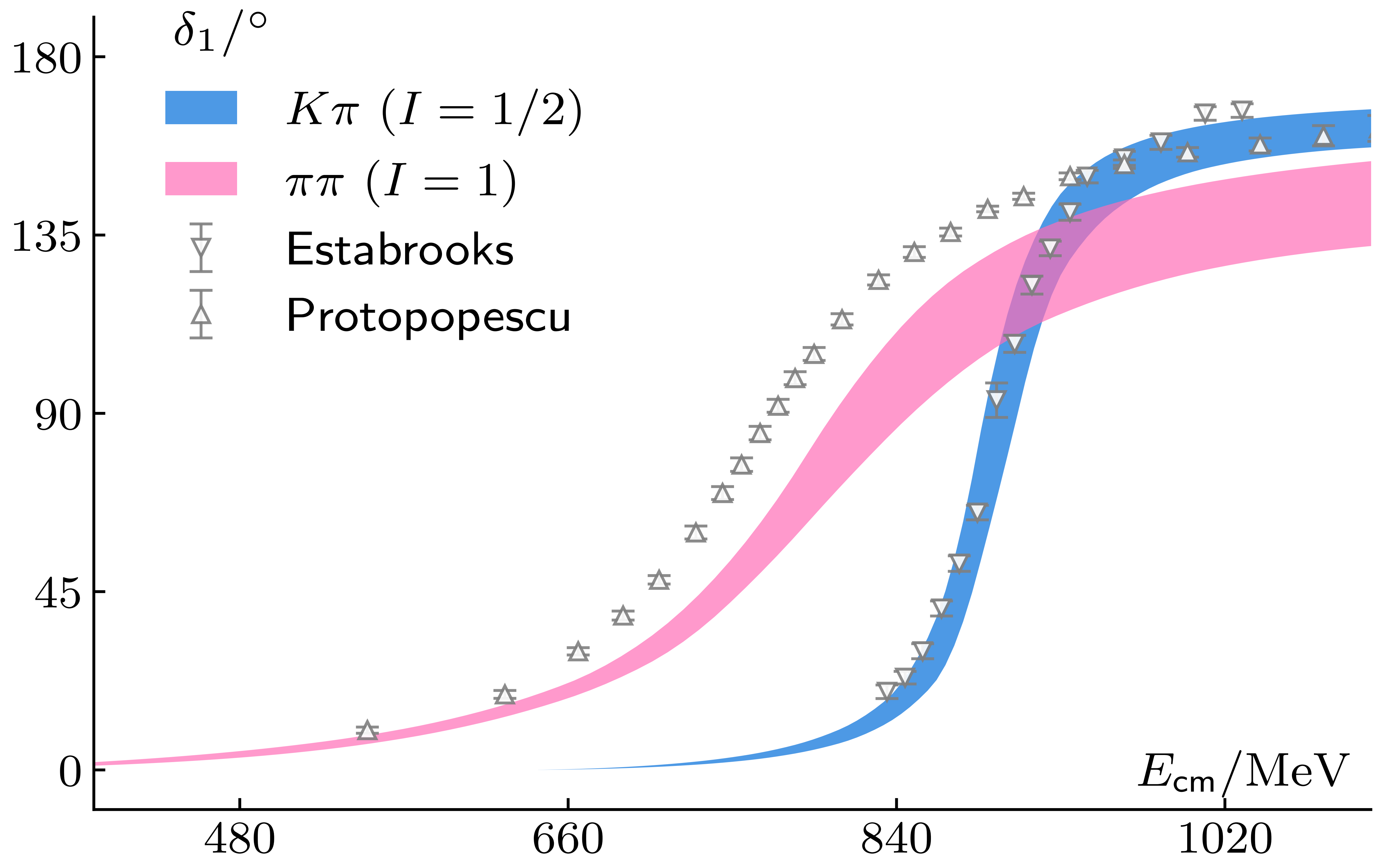
Science and Technology Facilities Council
DiRAC
This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 813942

Extra

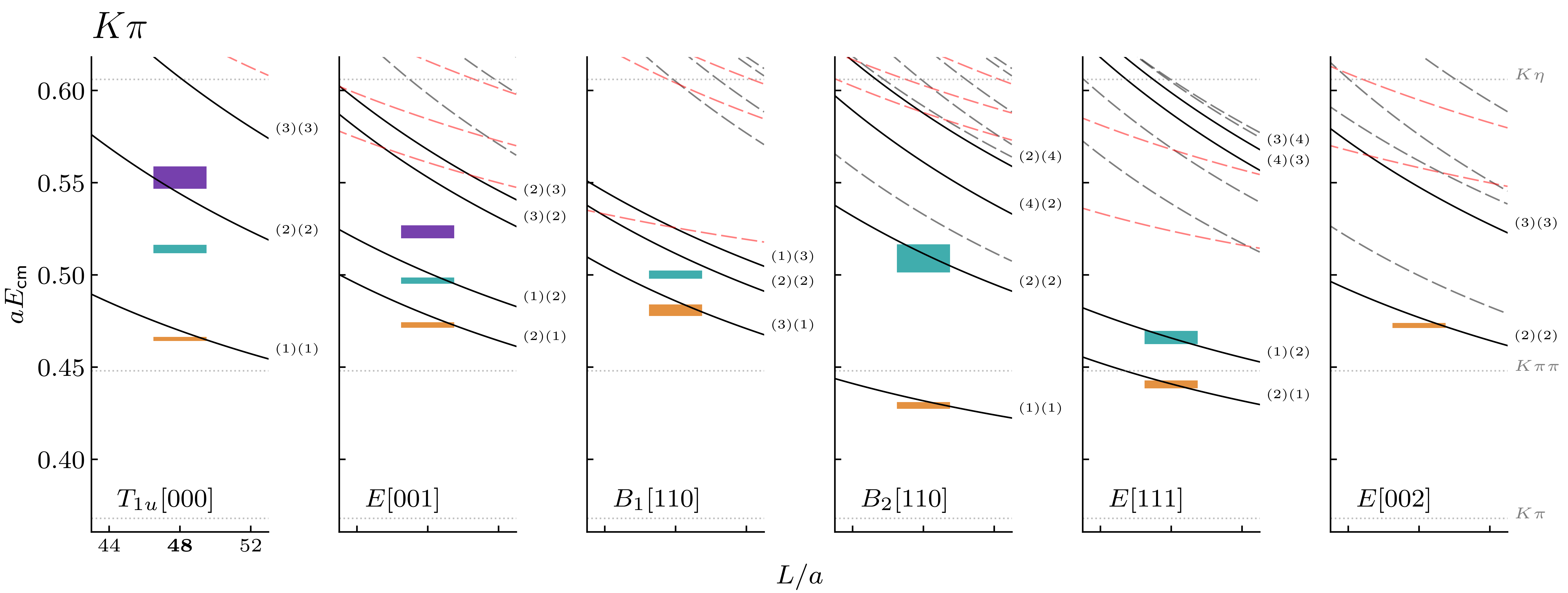


aM
0.51
0.53
−0.02
−0.04
−aΓ
BW run1
BW run2
BW run3
BW run4
BW avg.
ERE run1
ERE run2
ERE run3
ERE run4
ERE avg.
total avg.
weighted frequency
10^1
10^0
10^−1
10^−2
10^−3

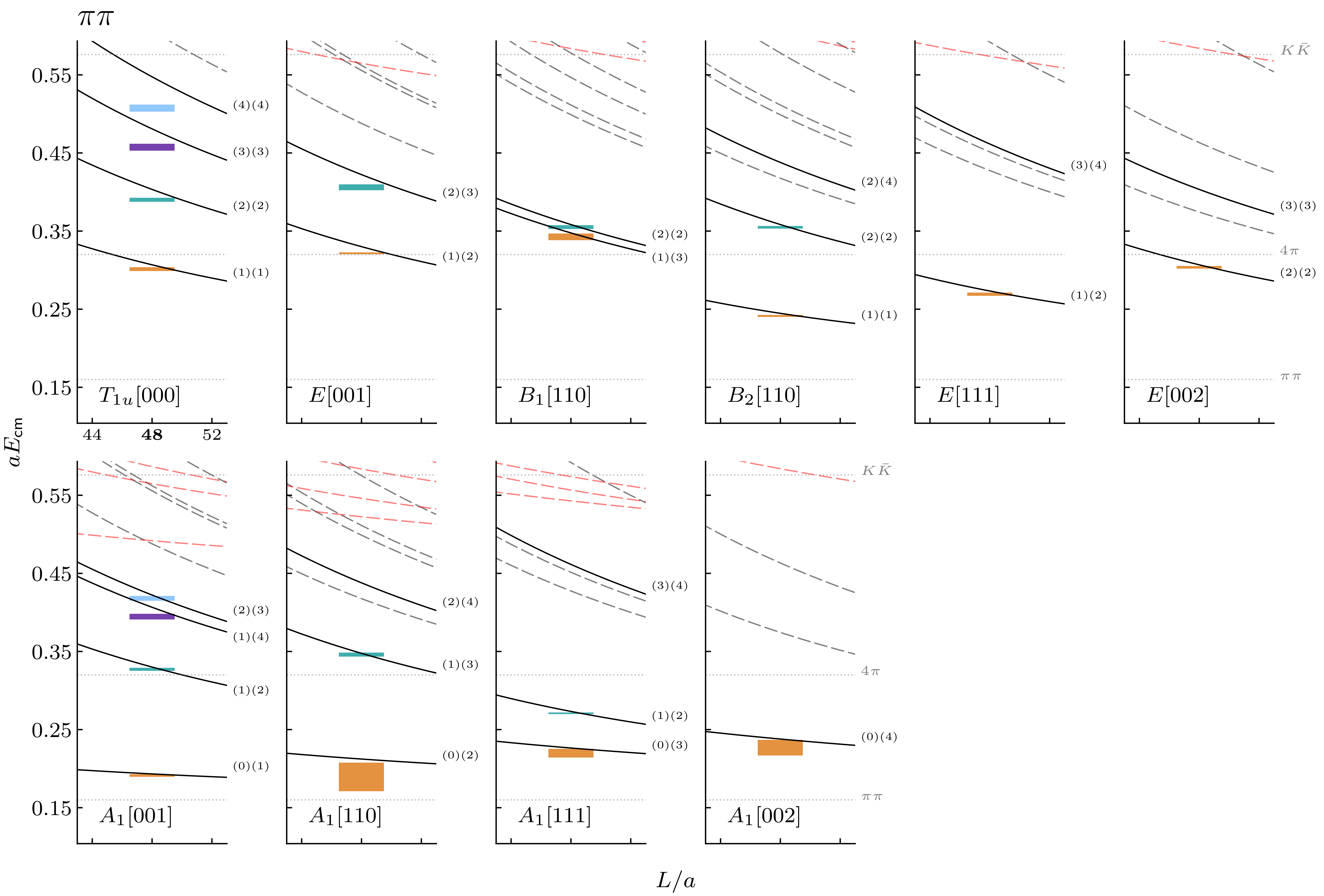

aM
0.44
0.48
−0.07
−0.18
−aΓ
BW
run1
BW
run2
BW
run3
BW
run4
BW avg.
ERE
run1
ERE
run2
ERE
run3
ERE
run4
ERE avg.
total avg.
weighted frequency
10^2
10^−2
10^−6
10^−10



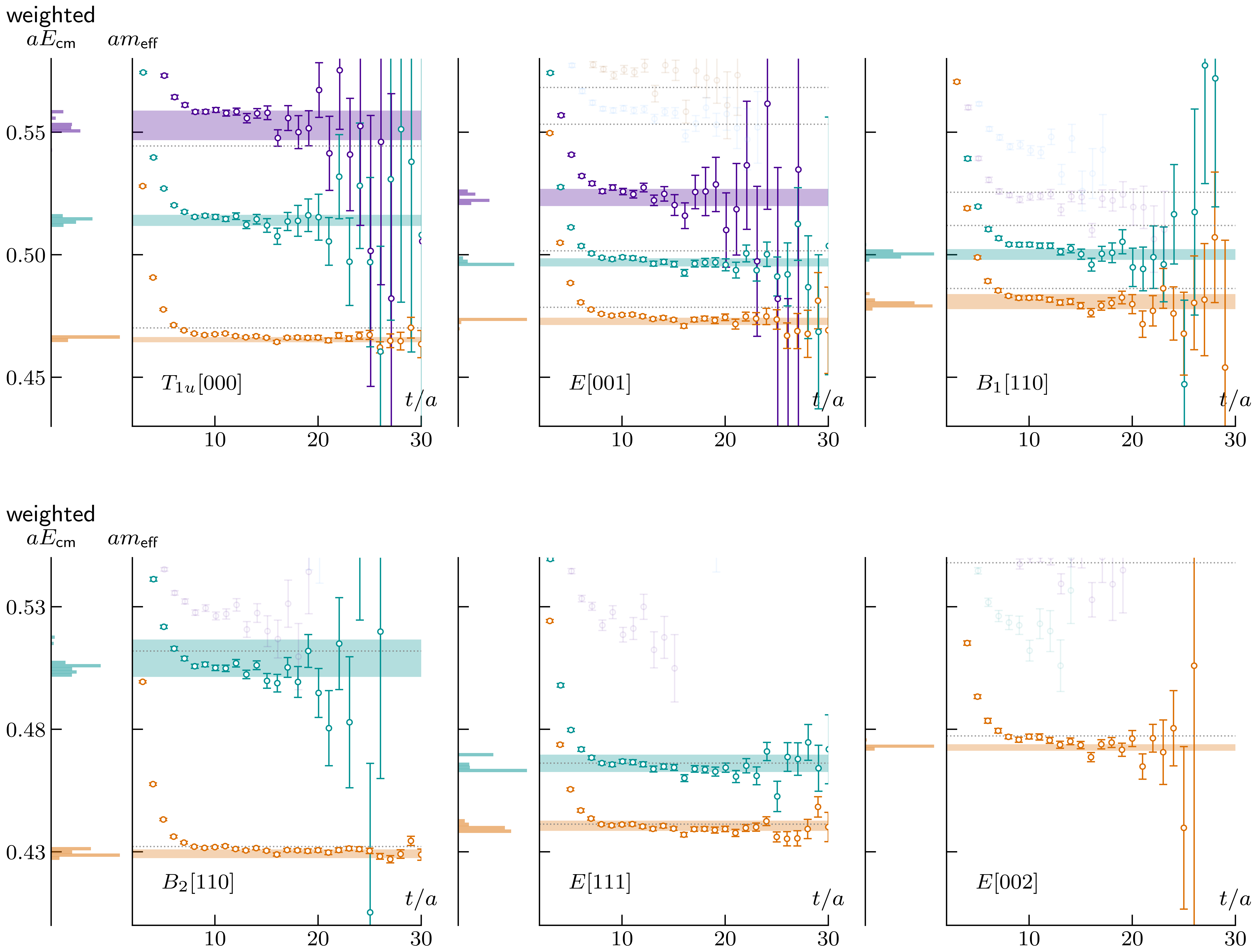
$\delta_1/°$
180
135
90
45
0
$K\pi$ $(I = 1/2)$
$\pi\pi$ $(I = 1)$
Estabrooks
Protopopescu
E_{cm}/MeV
480
660
840
1020



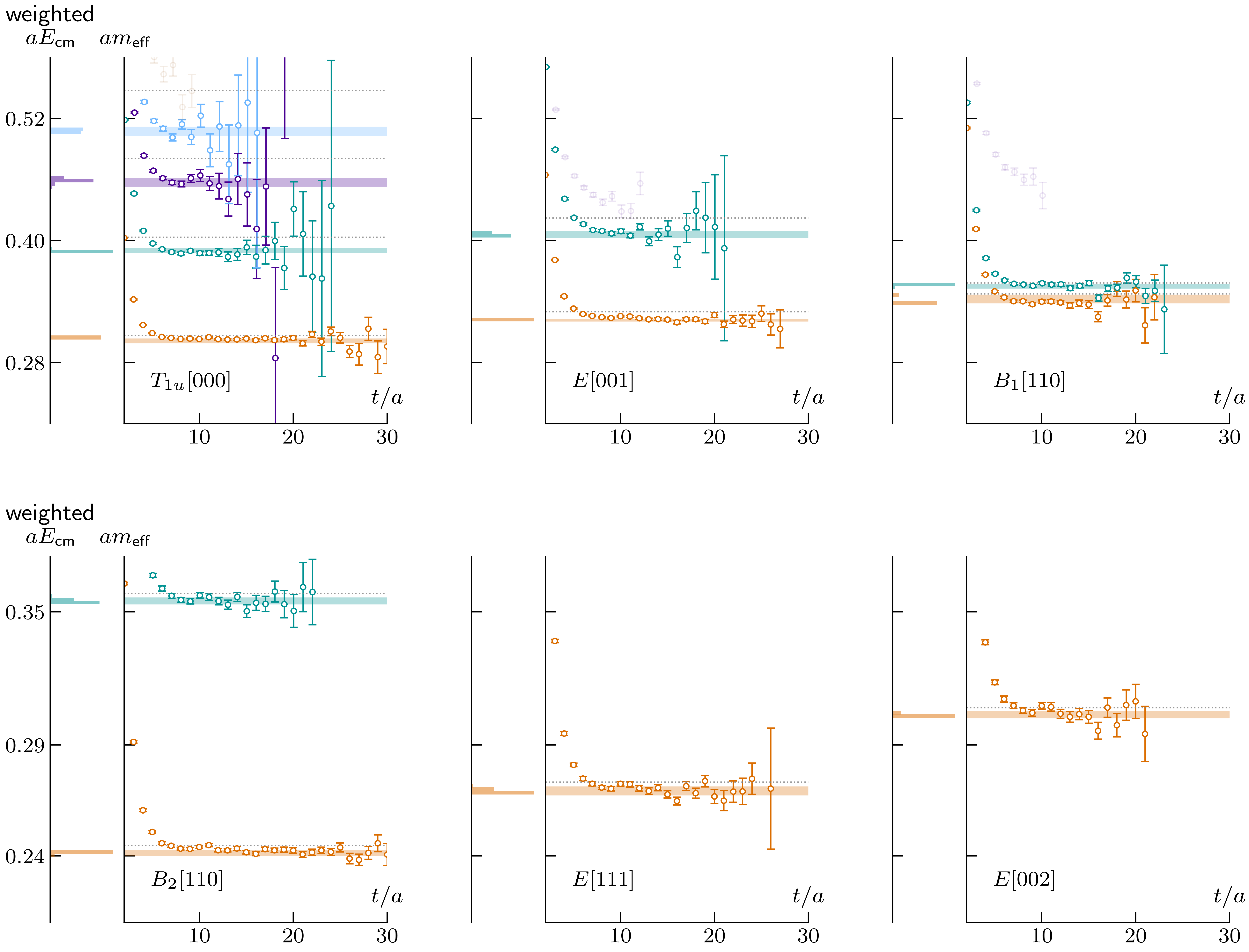
$K\pi$
0.60
0.55
0.50
0.45
0.40
$aE_{\rm cm}$
(3)(3)
(2)(2)
(1)(1)
(2)(3)
(3)(2)
(1)(2)
(2)(1)
(1)(3)
(2)(2)
(3)(1)
(2)(4)
(4)(2)
(2)(2)
(1)(1)
(3)(4)
(4)(3)
(1)(2)
(2)(1)
(3)(3)
(2)(2)
$K\eta$
$K\pi\pi$
$K\pi$
$T_{1u}[000]$
$E[001]$
$B_1[110]$
$B_2[110]$
$E[111]$
$E[002]$
44
48
52
L/a

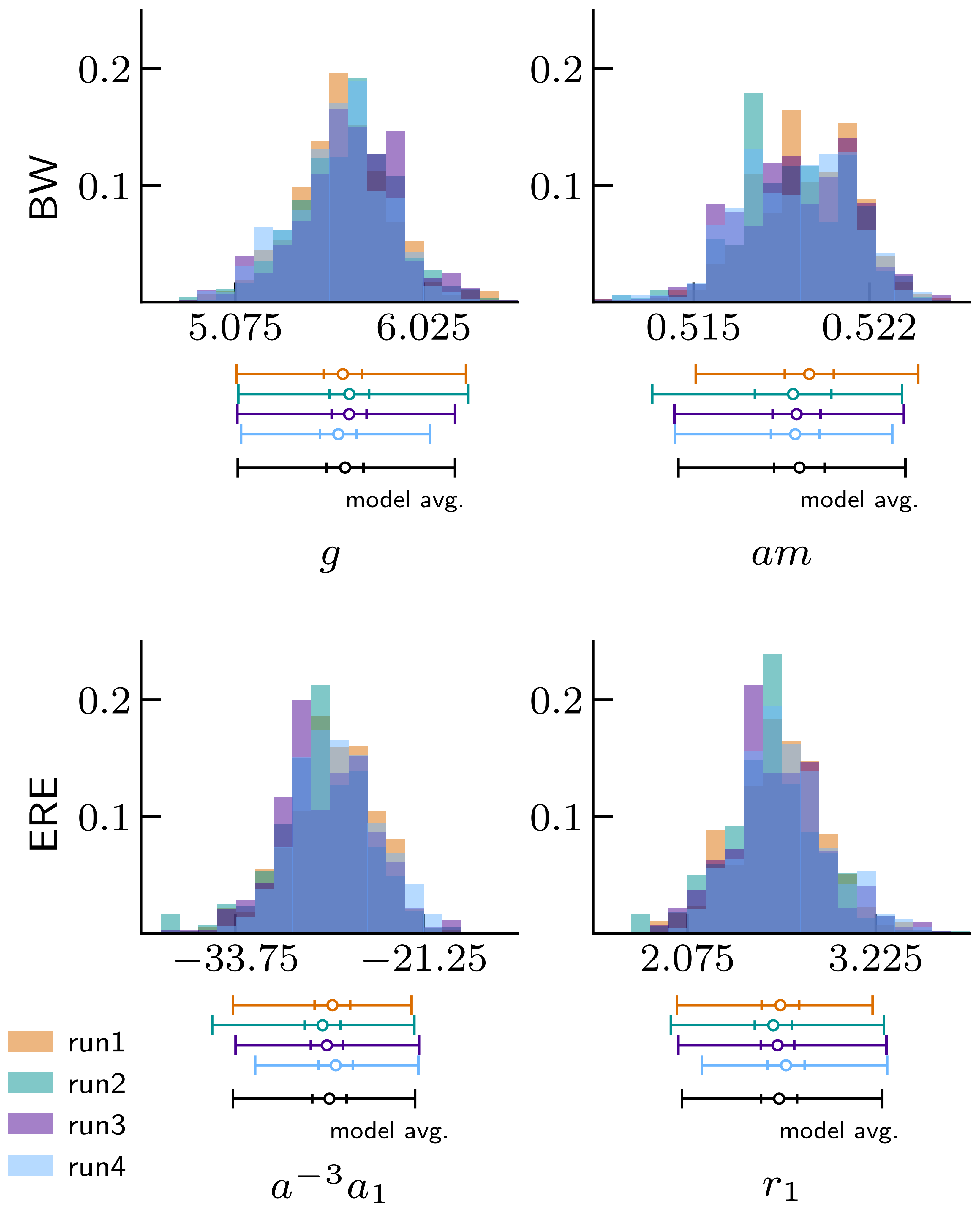


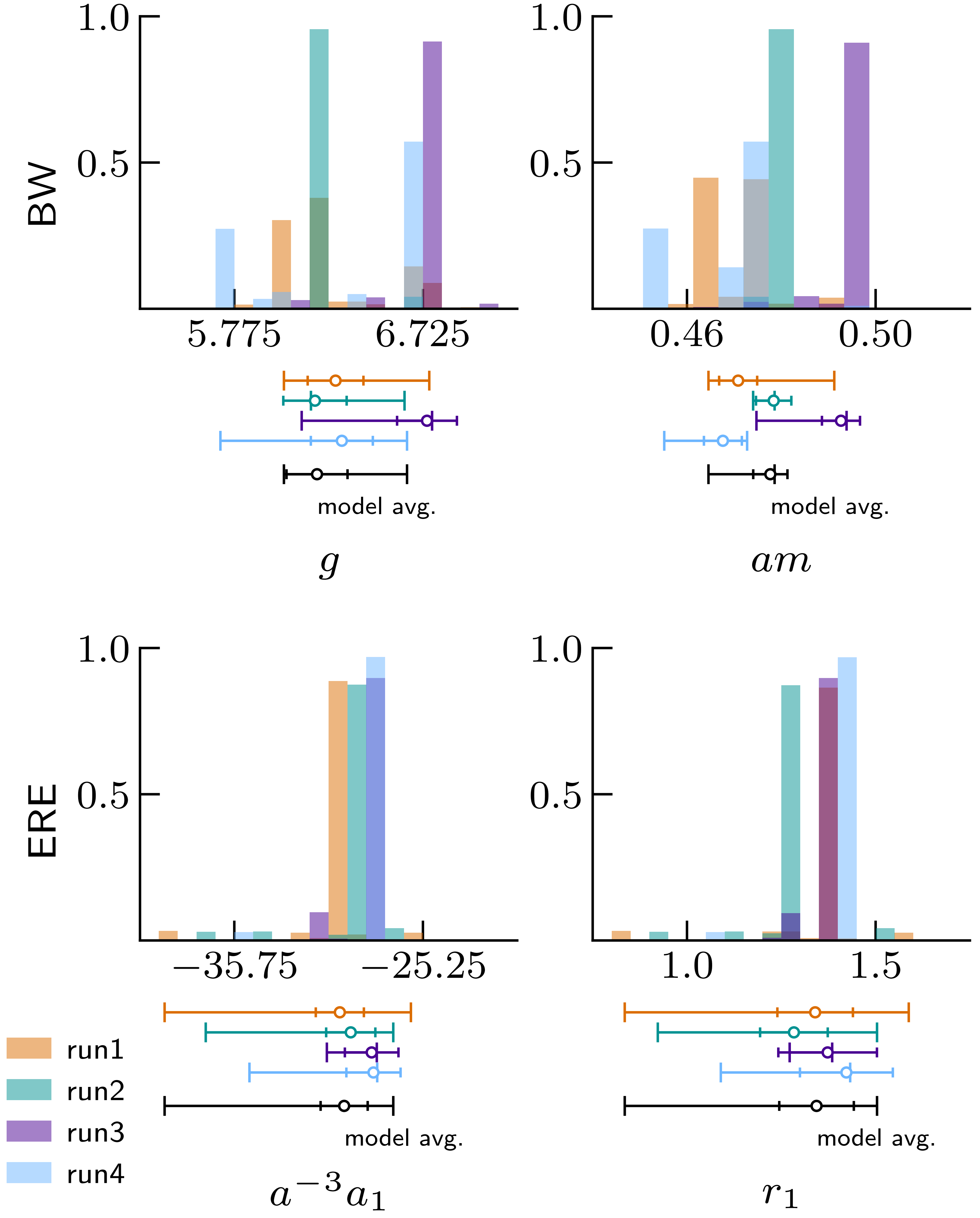
ππ
aE_cm
L/a
0.55
0.45
0.35
0.25
0.15
44
48
52
T1u[000]
E[001]
B1[110]
B2[110]
E[111]
E[002]
A1[001]
A1[110]
A1[111]
A1[002]
(4)(4)
(3)(3)
(2)(2)
(1)(1)
(2)(3)
(1)(2)
(1)(3)
(2)(4)
(3)(4)
(0)(1)
(0)(2)
(0)(3)
(0)(4)
(1)(4)
KK̄
4π
ππ



weighted
aE_{cm}
am_{eff}
0.55
0.50
0.45
$T_{1u}[000]$
$E[001]$
$B_1[110]$
t/a
10
20
30
weighted
aE_{cm}
am_{eff}
0.53
0.48
0.43
$B_2[110]$
$E[111]$
$E[002]$


weighted
aE_{cm}
am_{eff}
0.52
0.40
0.28
$T_{1u}[000]$
$E[001]$
$B_1[110]$
t/a
10
20
30
weighted
aE_{cm}
am_{eff}
0.35
0.29
0.24
$B_2[110]$
$E[111]$
$E[002]$
t/a
10
20
30


BW
ERE
g
am
$a^{-3}a_1$
r_1
model avg.
run1
run2
run3
run4


BW
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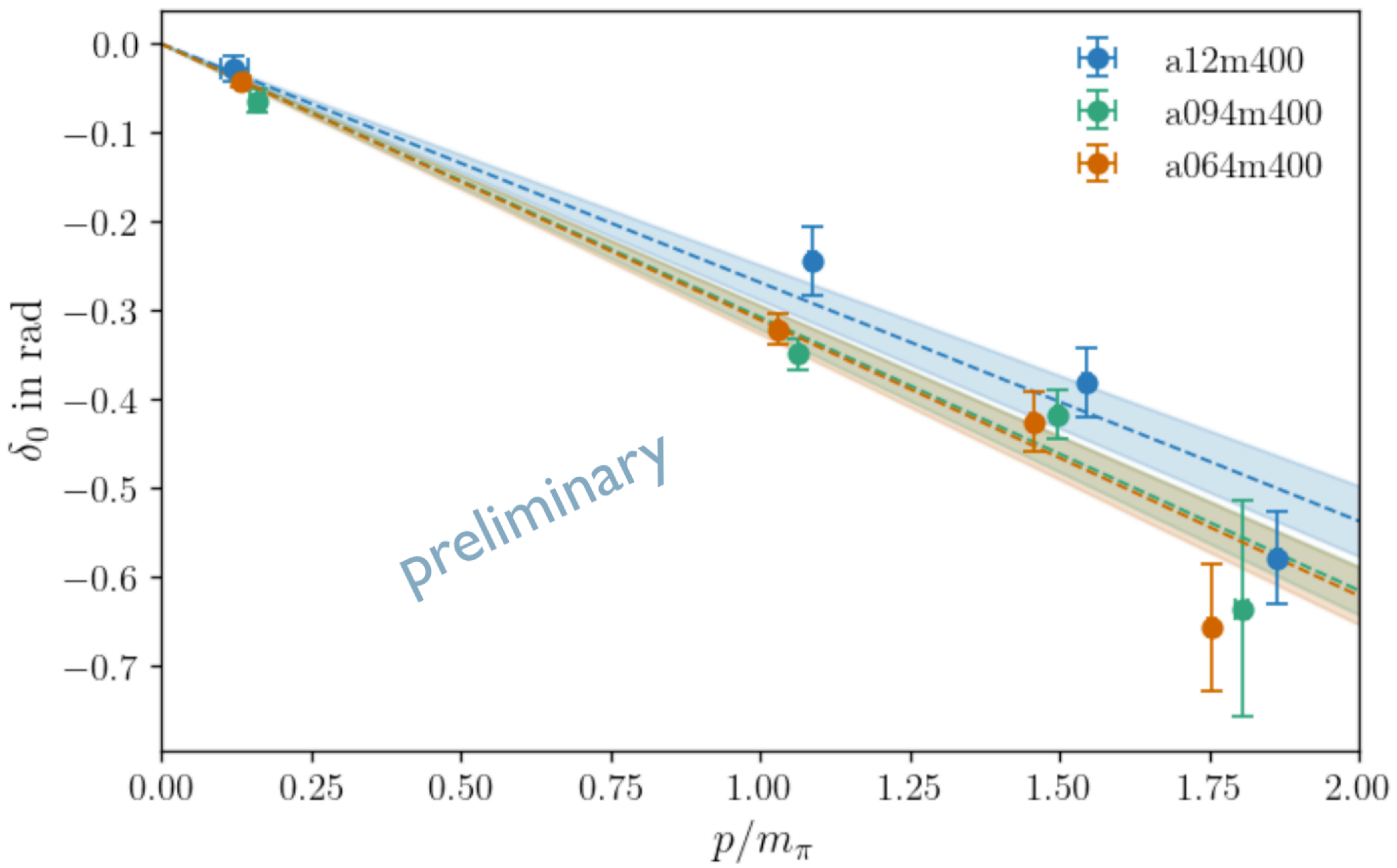
Outlook

[Joswig et al, Lattice2022 & MIT Colloquium]

Hadronic $D \to K\pi$ decays at $SU(3)_f$ point

$$A(D \to h_1 h_2) = \mathcal{C}^{\mathsf{LL}}_{n,L,h_1h_2}\left[\lim_{a\to 0} Z^{\overline{\mathrm{MS}}} \langle n, L|\mathcal{H}_W|D, L\rangle\right]$$

taken from [Hansen, talk at Lattice 2023]



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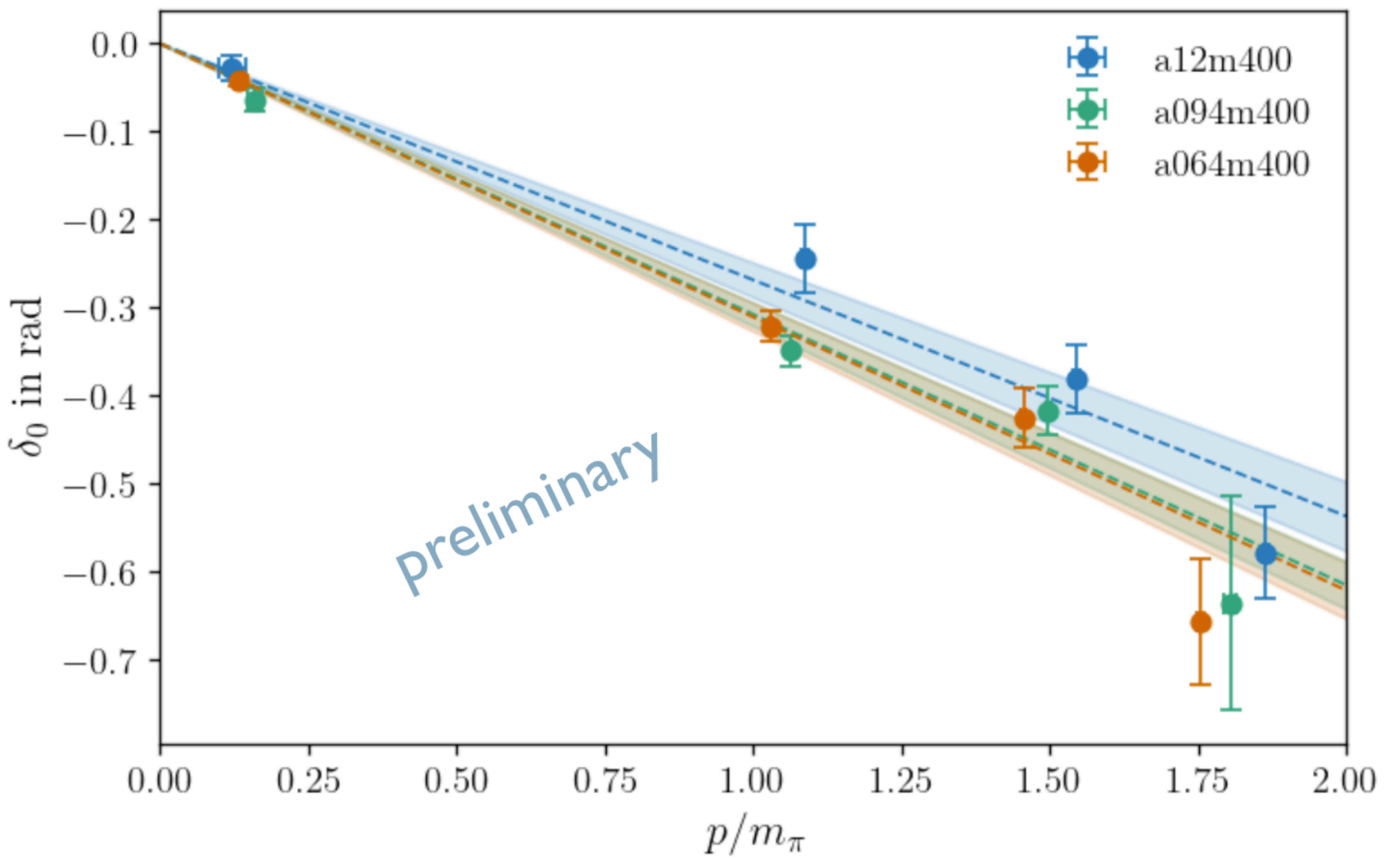


"Heavy-flavour weak decays into resonant scattering states" [Erben et al]

- 232 MeV pion mass, DWF
- Allocated DiRAC project

3pt-functions

$$\langle n, \mathbf{P} \,|\, J^{\mu}(0,\mathbf{q}) \,|\, B, \mathbf{p}_B \rangle$$

e.g. see [Erben, Lattice2024 plenary]
[Leskovec et al, Lattice2022]

$B_{(s)} \to K^* \ell^+ \ell^-$

$B \to \rho \ell \nu$

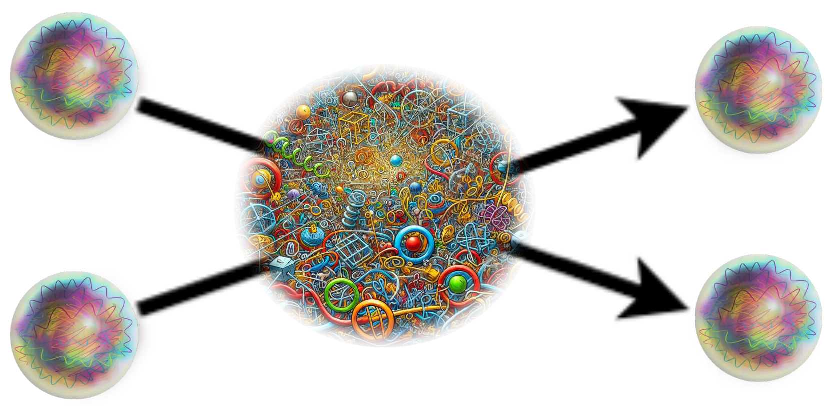
$\vdots$

Phase Shift

Phase Shift

$2 \to 2$ scattering

- S-matrix element



asymptotic states

$$\sim \ {}_{out}\langle \pi(p_1)\pi(p_2) \mid S \mid \pi(p_3)\pi(p_4)\rangle_{in}$$

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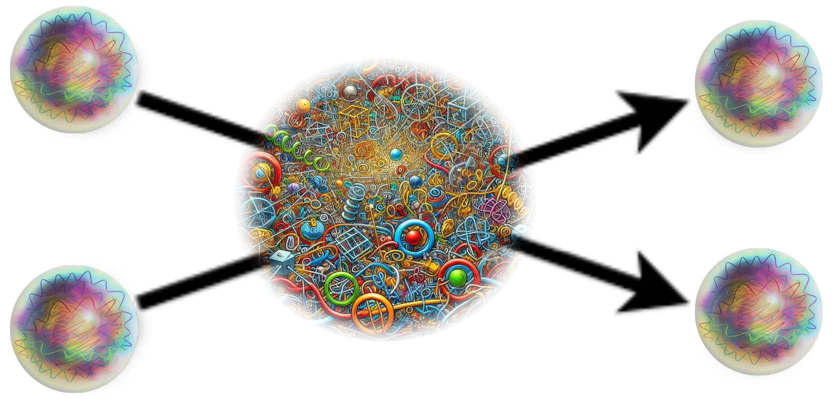

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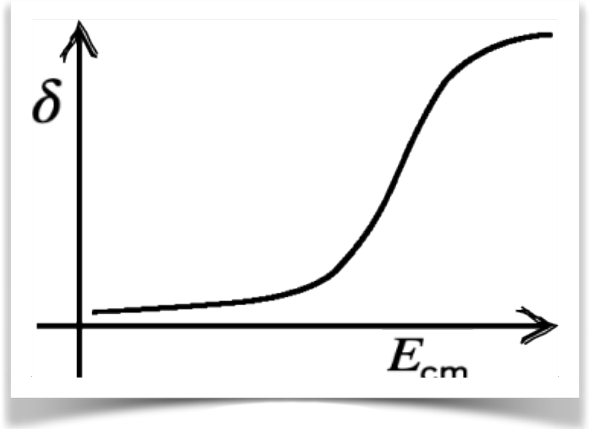
unitarity & symmetry

$$S_\ell(E_{cm}) = e^{2i\delta_\ell(E_{cm})}$$

partial waves



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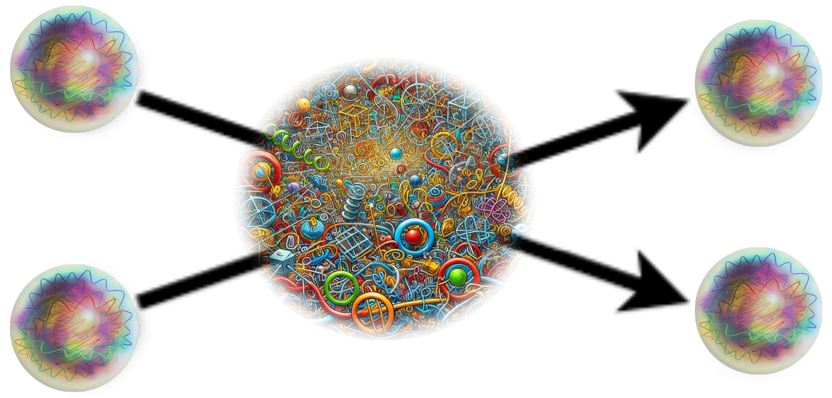

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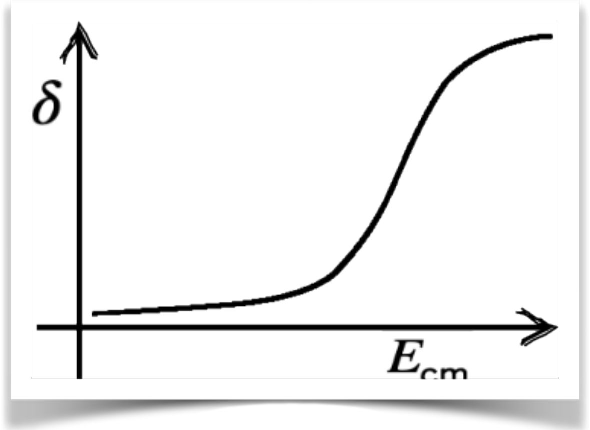
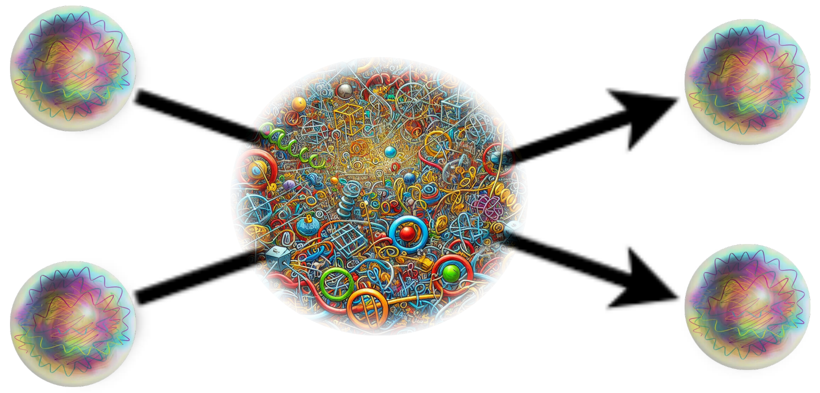
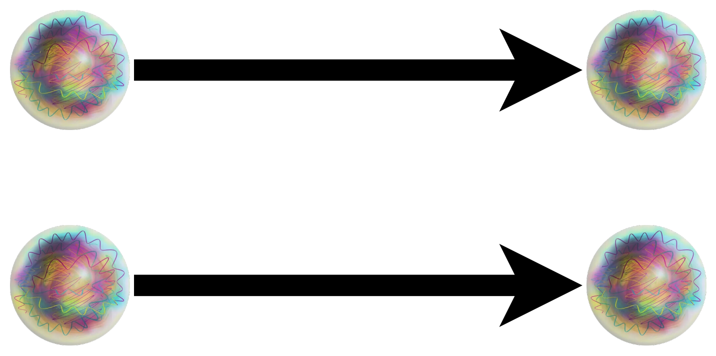


- scattering amplitude



$-$

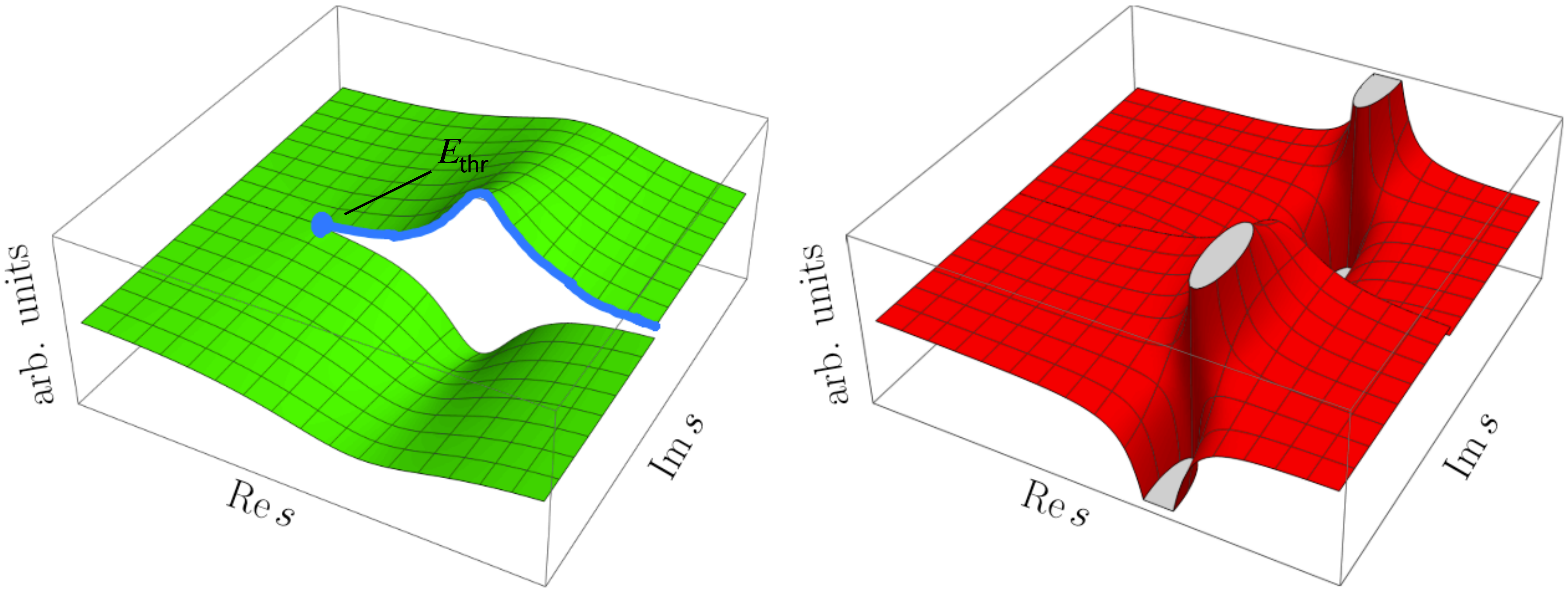


$$\sim \left[S - \mathbf{1}\right]_\ell \equiv$$

$$T_\ell = (\cot\delta_\ell - i)^{-1}$$

Poles

$$T(E_{\text{cm}}) \to T(\sqrt{s}), \qquad \sqrt{s}\ \text{complex}$$



(a) first Riemann sheet (b) second Riemann sheet

[Asner & Hanhart, 50.Resonances, PDG, 2022]

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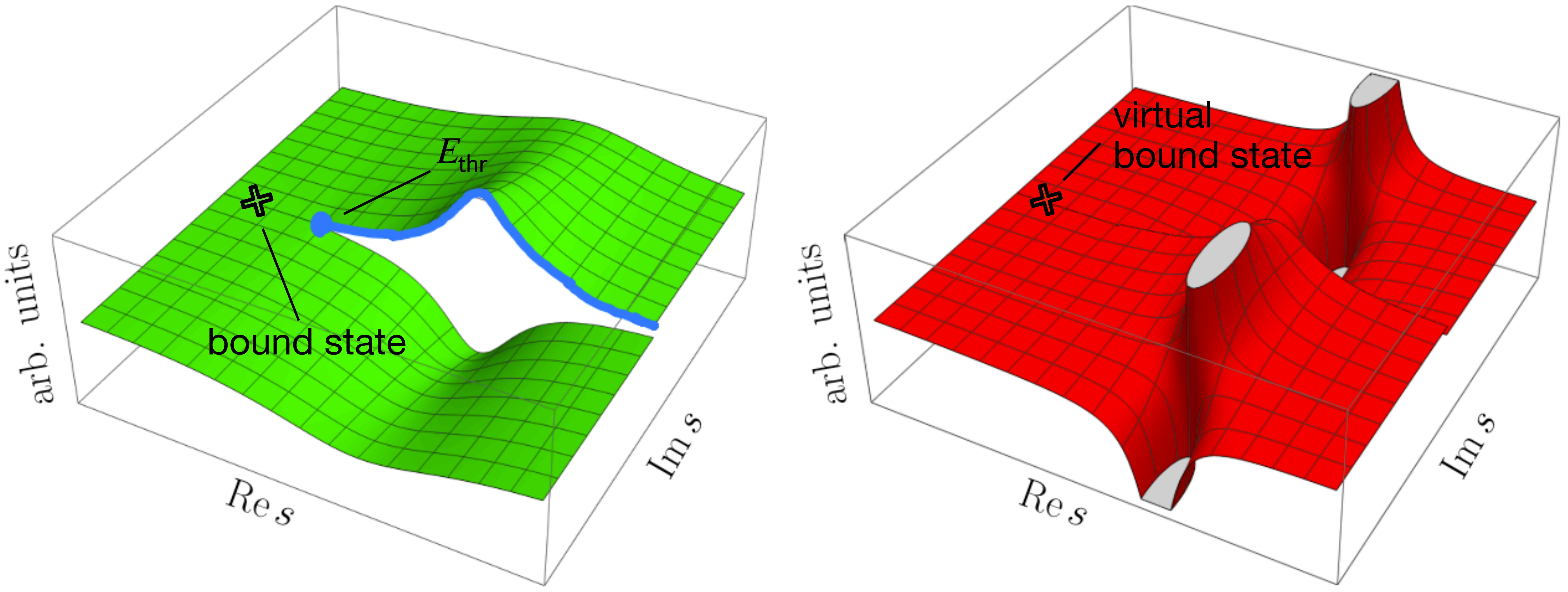
- **bound state**: below E_{thr} on **sheet-I** (or **virtual** on **sheet-II**)



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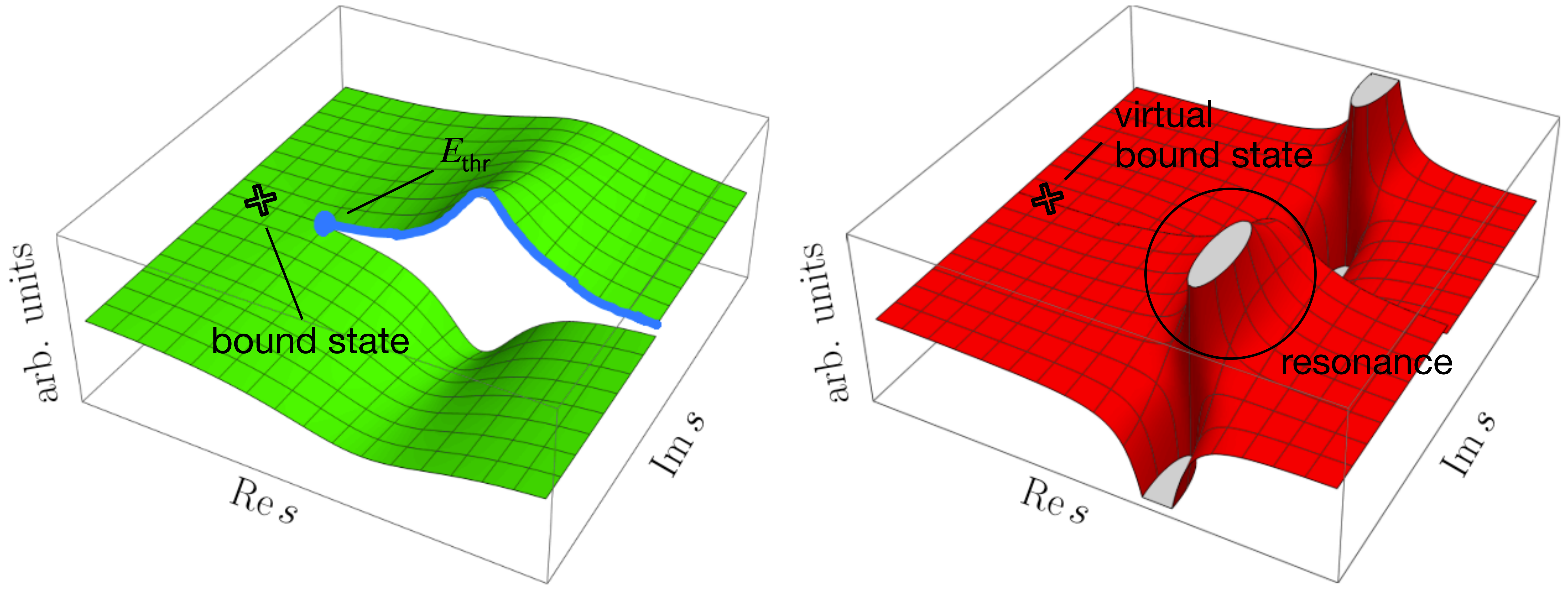

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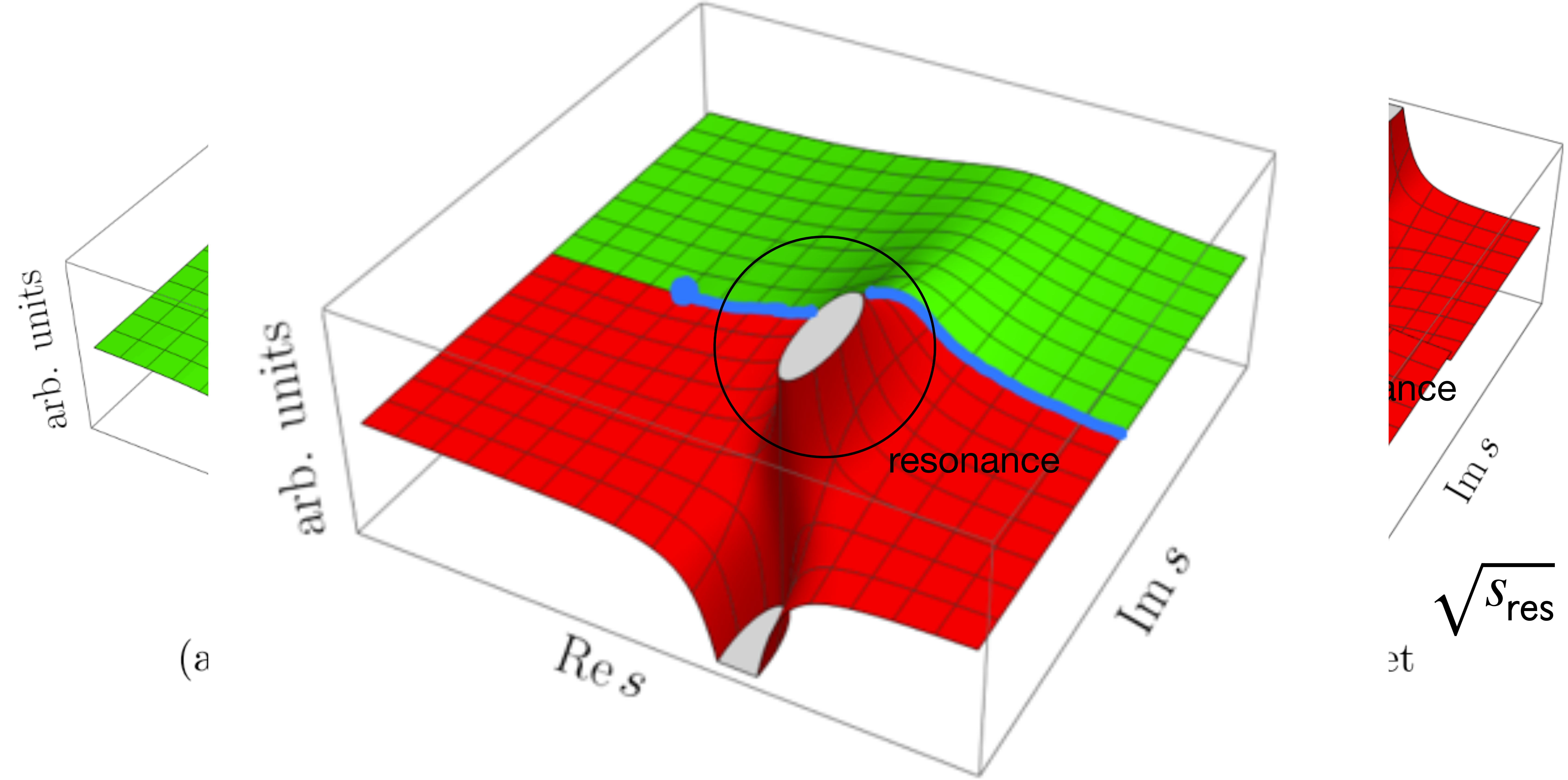

$$\sqrt{s_{\text{res}}} = M - \frac{i}{2}\Gamma$$

What about scattering?

What about scattering?

LSZ: $$\prod_i \int_i e^{-ip_i x_i} (\Box_i + m^2) \langle O_1 O_2 O_3 O_4 \rangle \underset{\text{on-shell}}{\sim} \langle p_1 p_2 | S | p_3 p_4 \rangle$$

can compute nonperturbatively, but…

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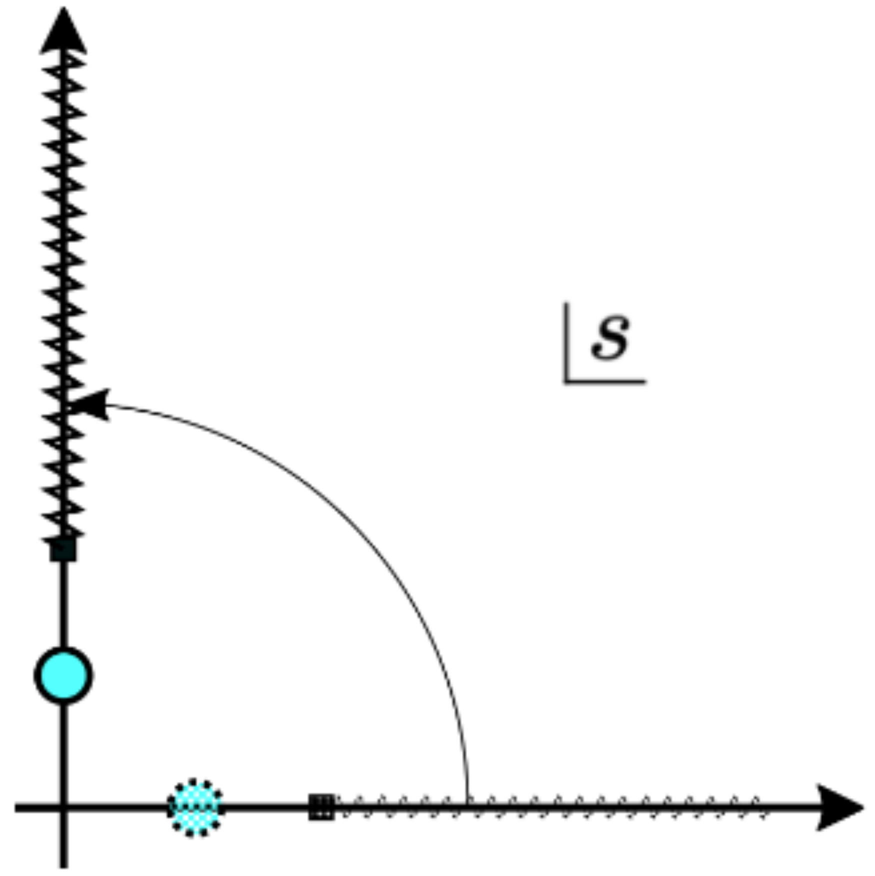

Euclidean

analytical continuation of statistical data

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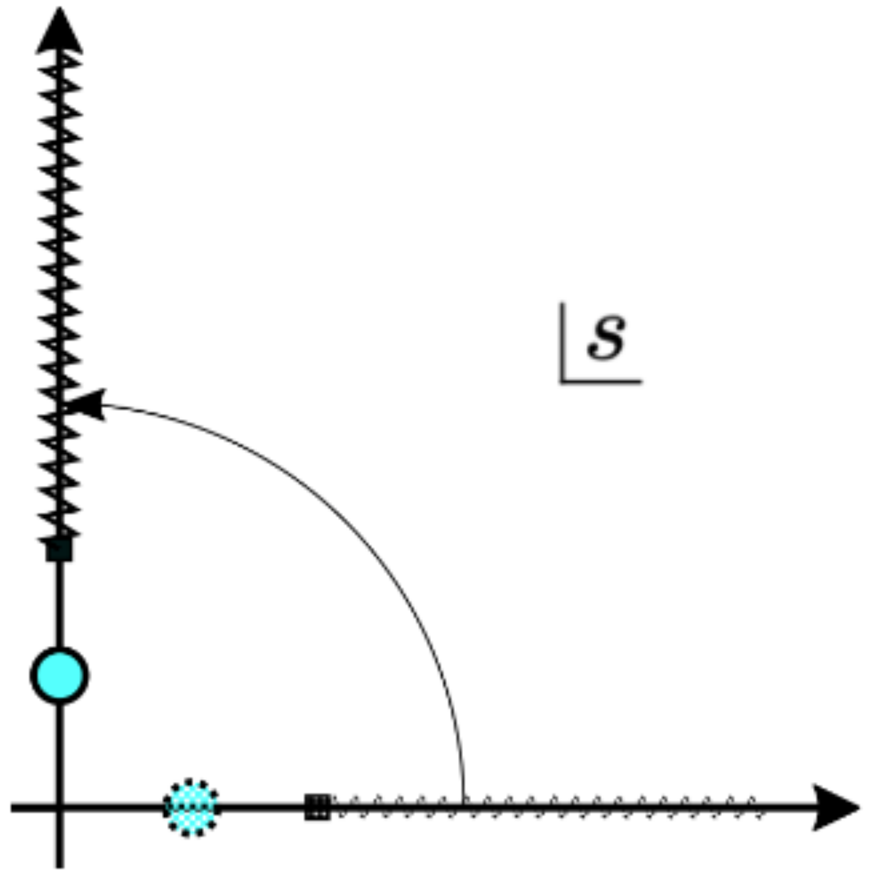

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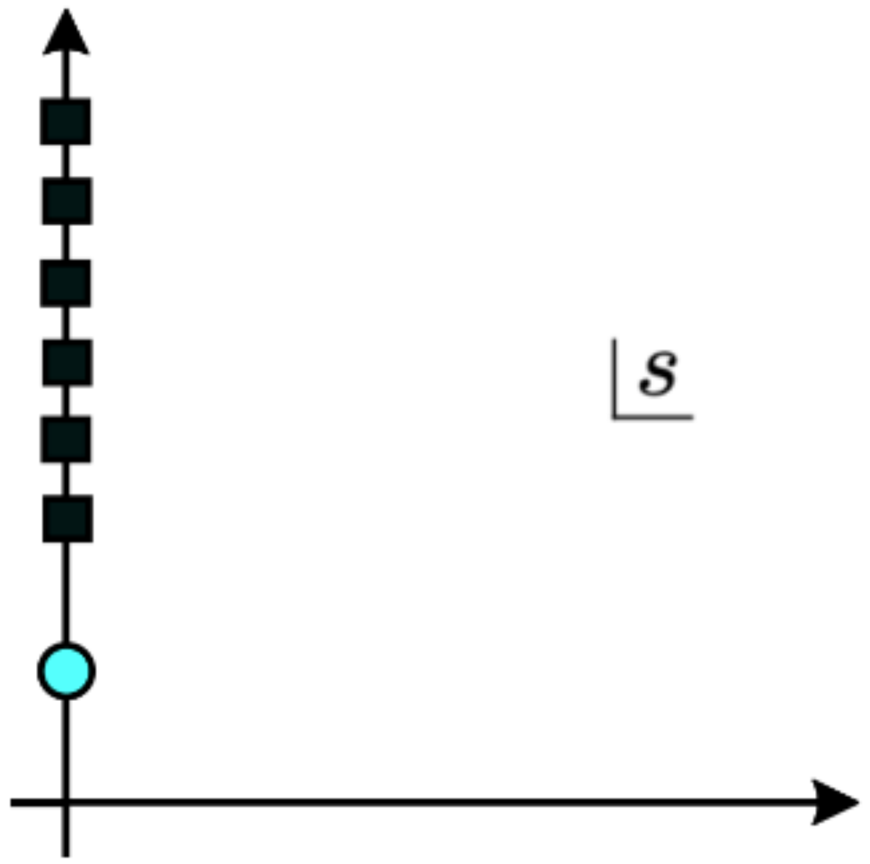

(Periodic) Finite volume

discrete spectrum, L**-dependent**

Two particles in a box

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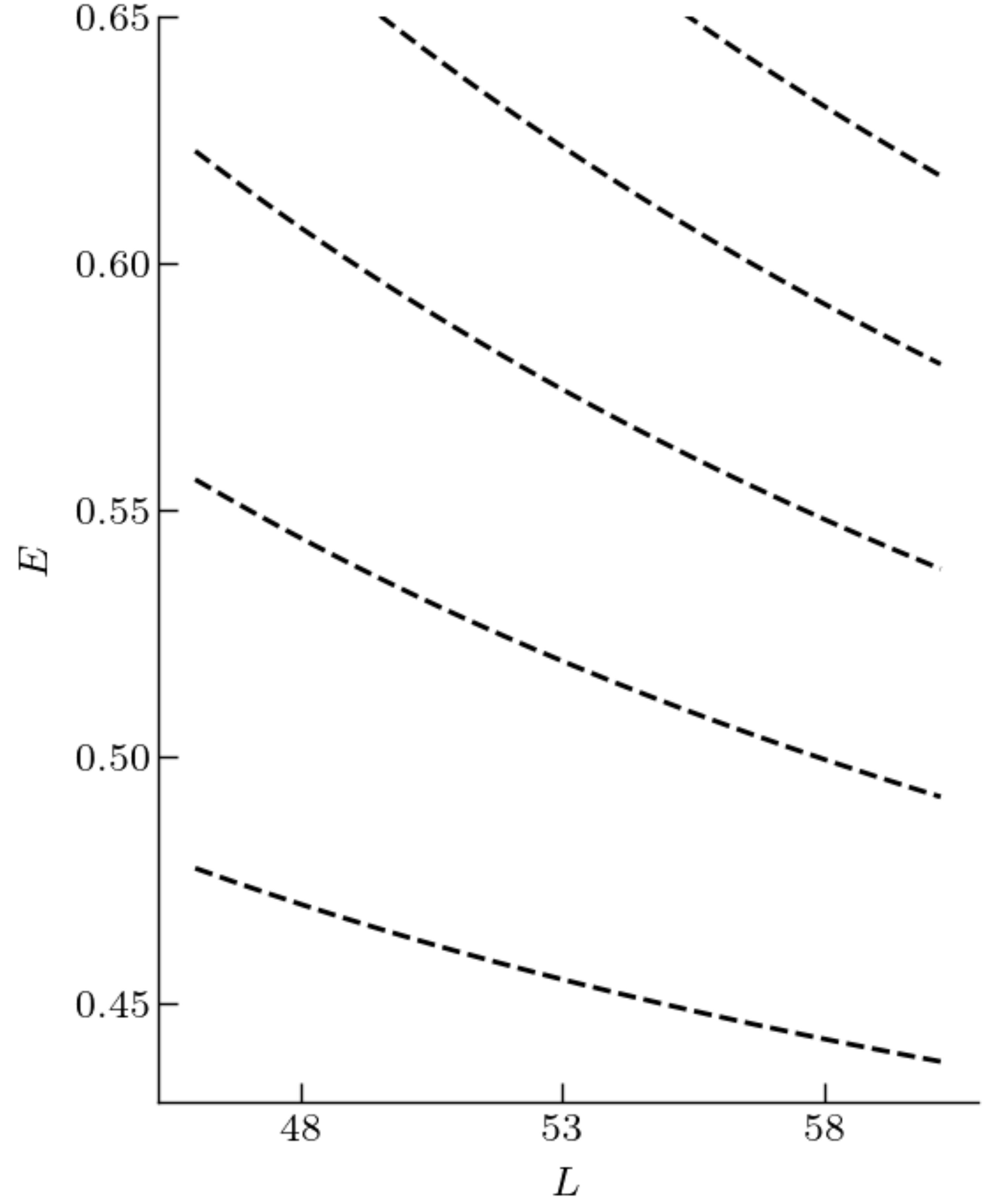
Generic effective dofs: scalar fields, mass m

No interactions: $E(L) = 2\sqrt{m^2 + p^2}$



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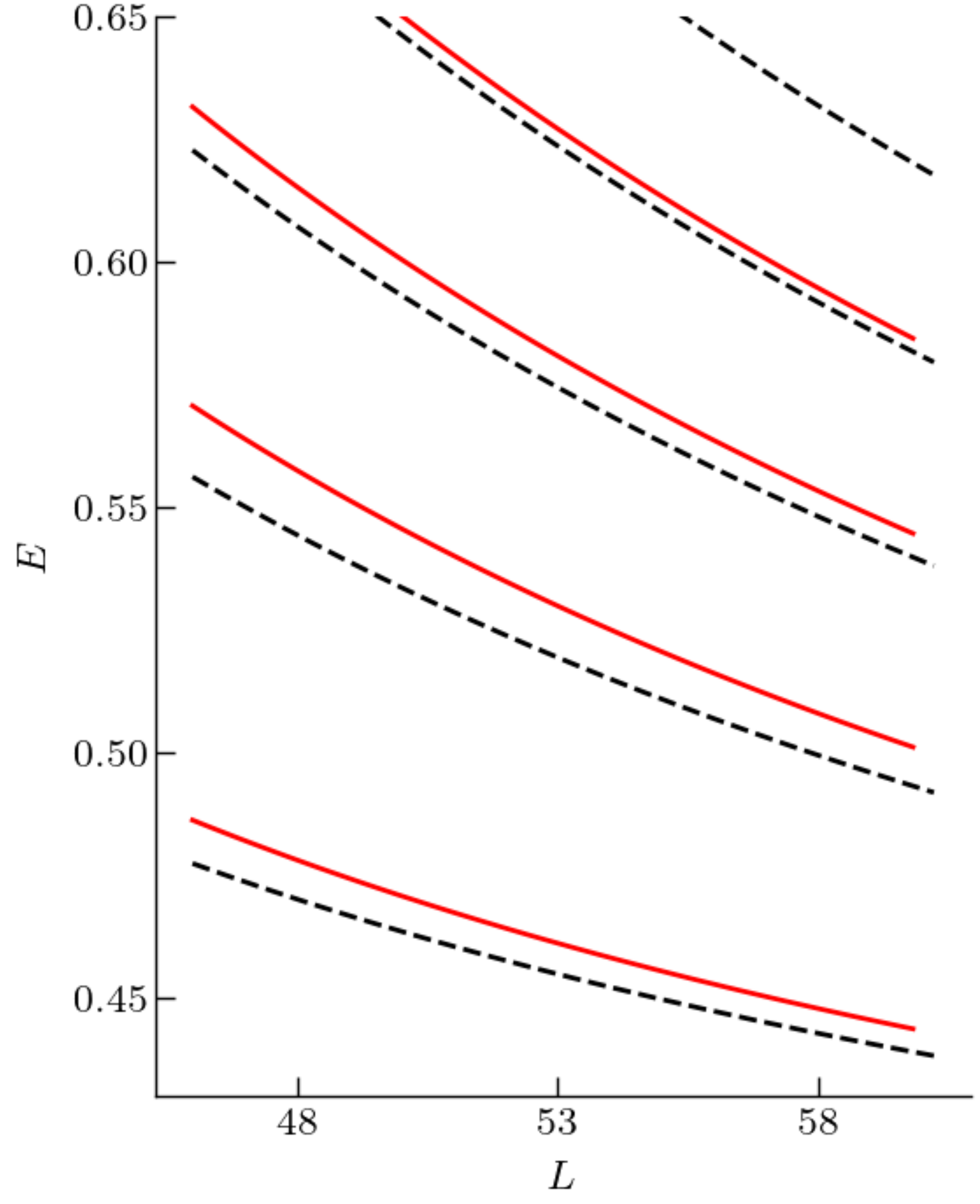

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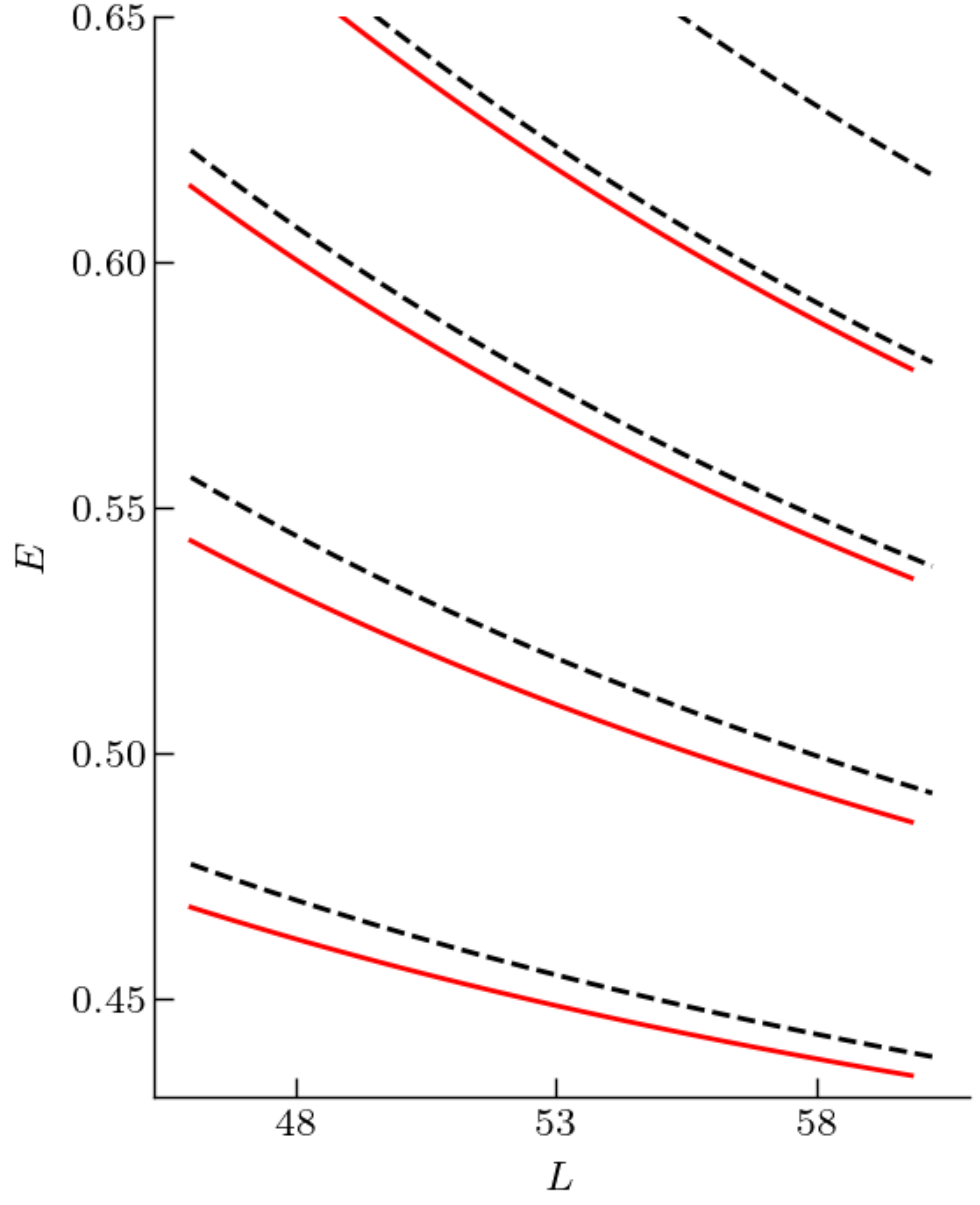

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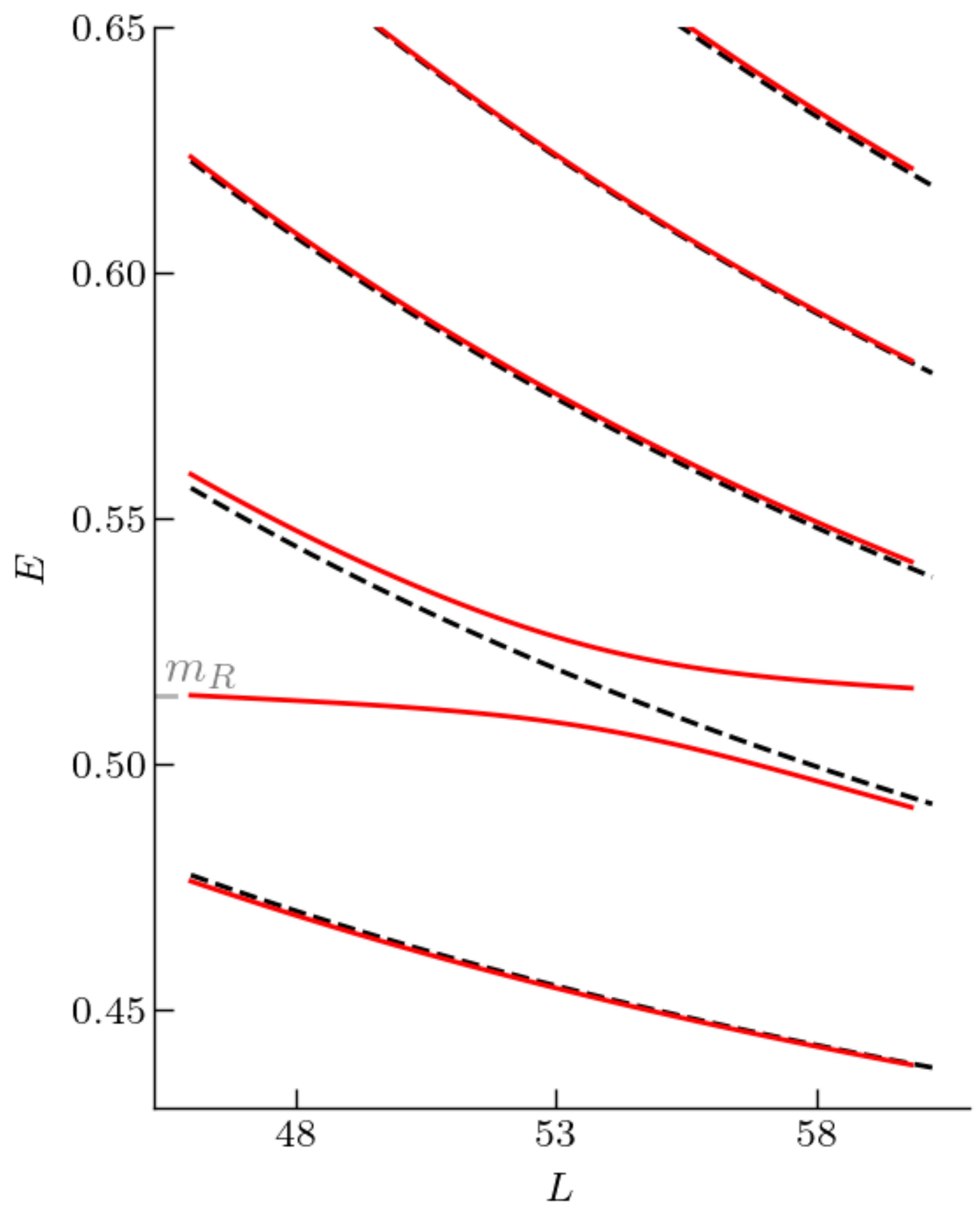

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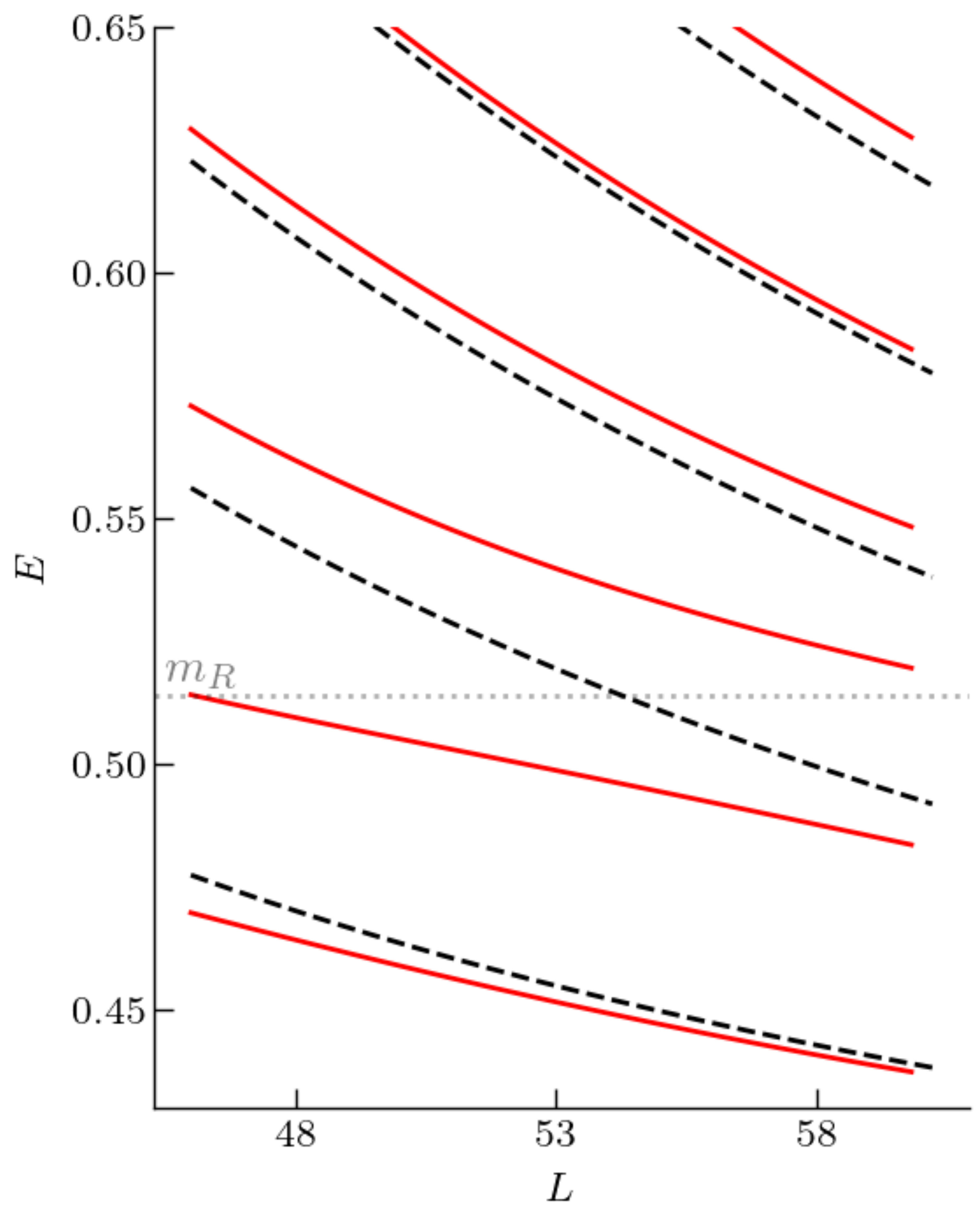

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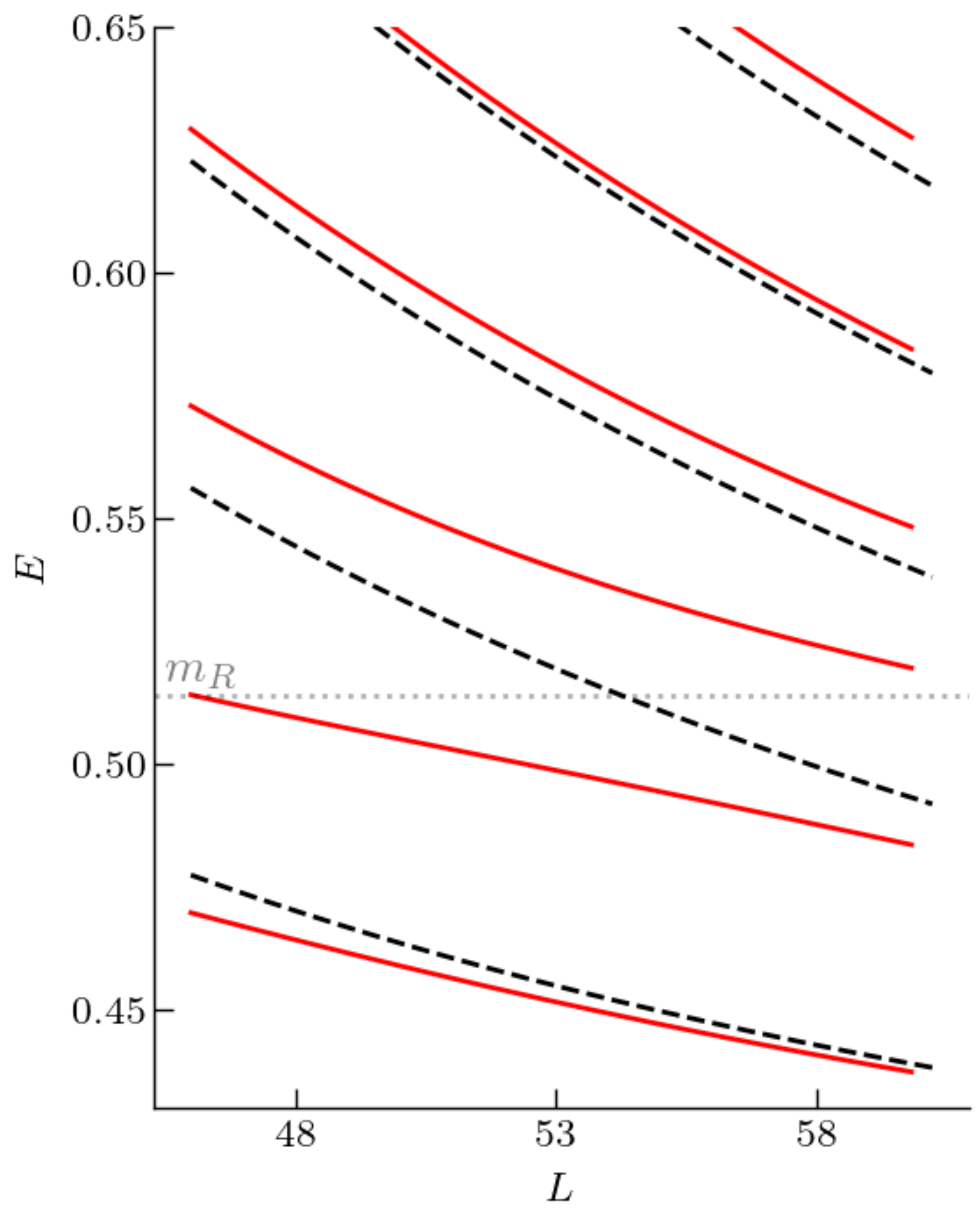

Quantisation Condition (QC)

[Lüscher, 1986]
[Lüscher, 1991]

$$\delta\big(E_{\mathrm{cm}}(L)\big) = n\pi - \phi\big(E_{\mathrm{cm}}(L), L\big), \quad n \in \mathbb{Z}$$

$\rightarrow$driven by $\mathcal{O}(L^{-b})$, neglects $\mathcal{O}(e^{-mL})$

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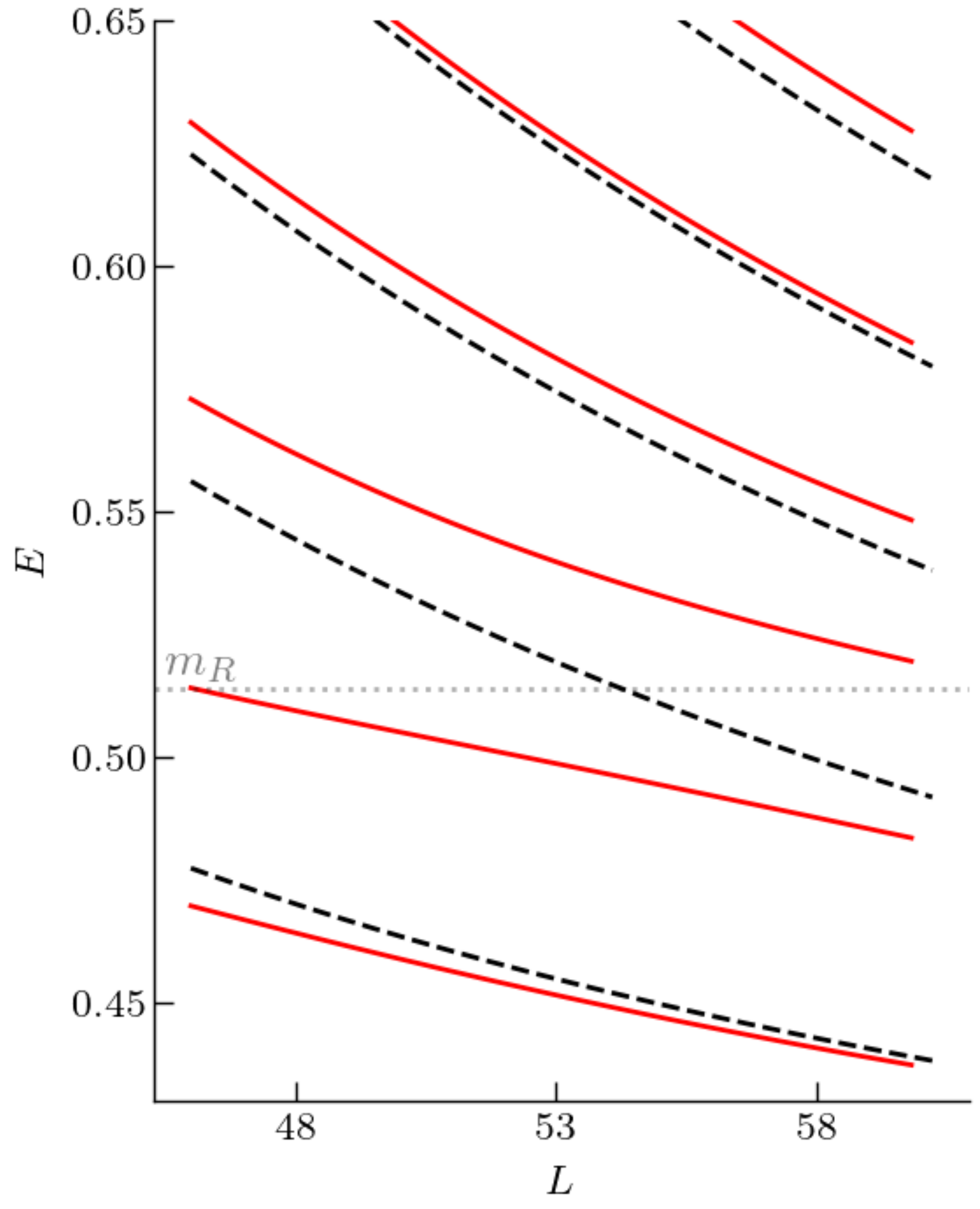

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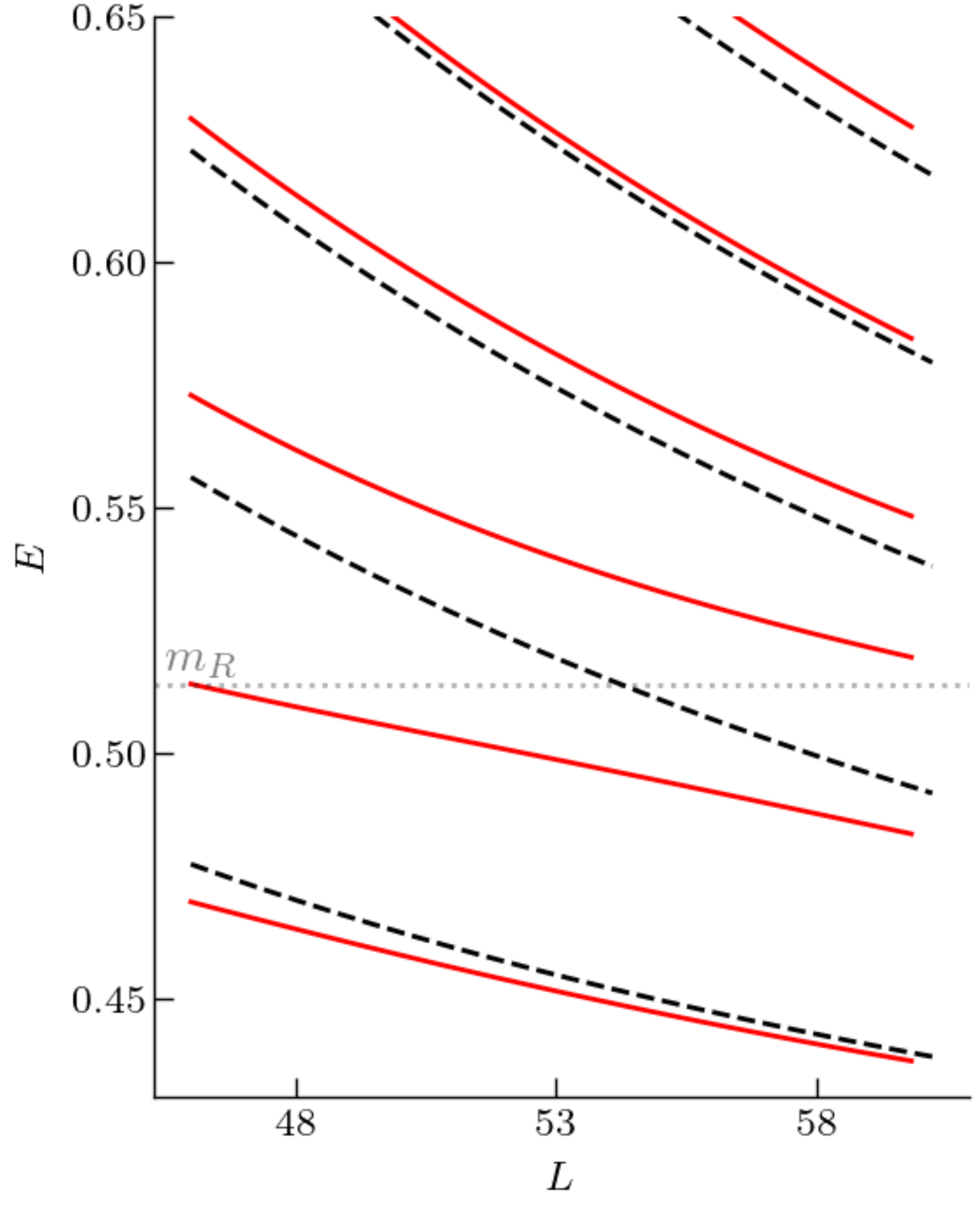

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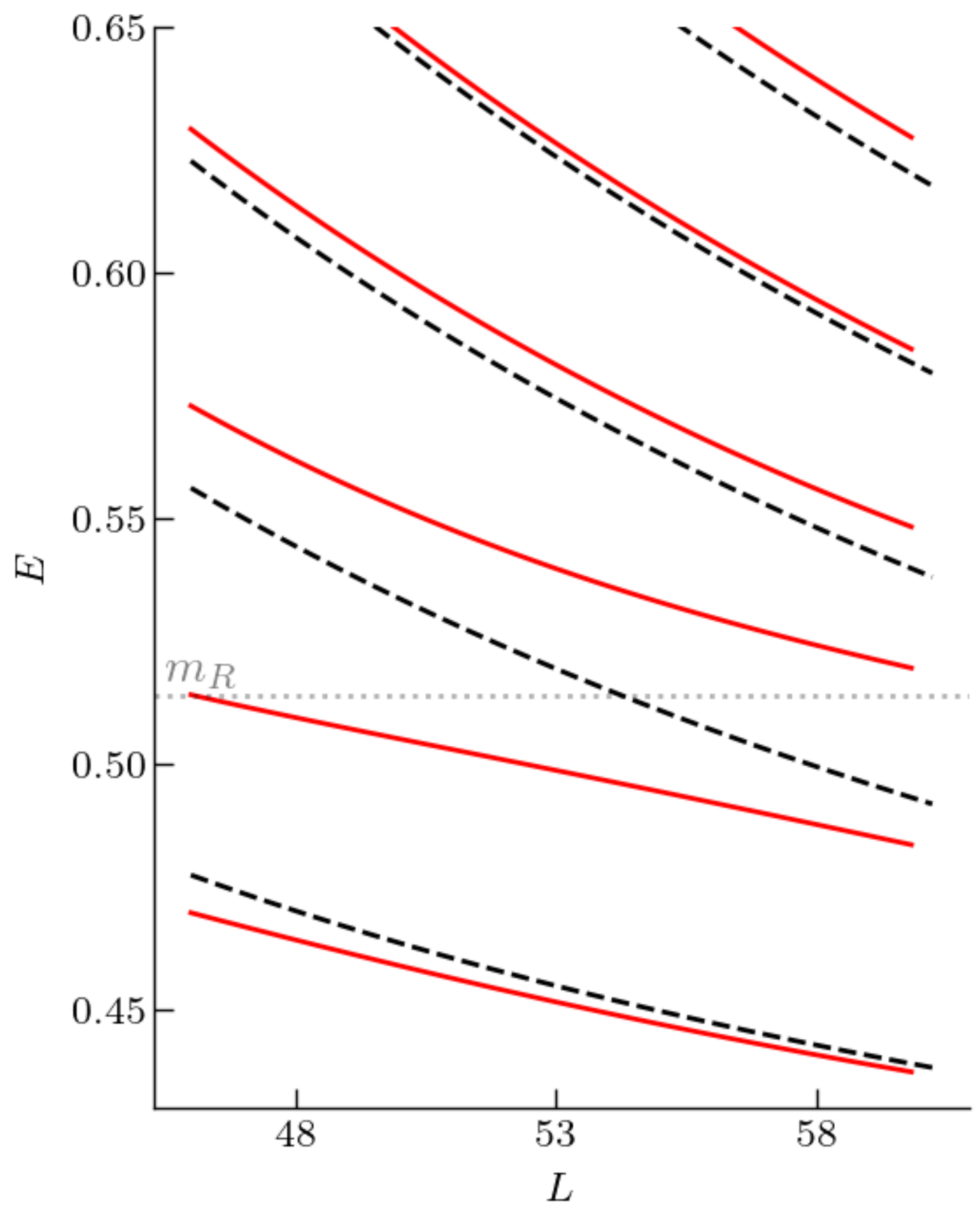

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generalised to multiple channels, spin,…

[Rummukainen & Gottlieb, 1995] [Kim & Sachrajda & Sharpe, 2005] [Hansen & Sharpe, 2012] [Leskovec & Prelovsek, 2012] [Fu, 2012] [Briceno, 2014]…

Two-point functions

$S_{\text{quark}} \propto \bar{\psi} D \psi$: integrate in terms of $D^{-1} = \langle q\bar{q} \rangle$

$$\langle O(x)O(y)^\dagger \rangle = \mathcal{Z}^{-1} \int DU \ \left(O(x)O(y)^\dagger \right)[U] \ e^{-S_{\text{lat}}[U]} \approx \sum_i^{n_{\text{cfg}}} \left(O(x)O(y)^\dagger \right)[U^{(i)}]$$

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Example: bilinear $O_V = \bar{q}\gamma^i q'$

$$\langle \left(\bar{q}\gamma^i q'\right)(x) \; \left(\bar{q}'\gamma^i q\right)(y)^\dagger \rangle_F = \mathrm{tr}\left[\gamma^i D^{-1}_{(q)}(y;x)\gamma^i D^{-1}_{(q')}(x;y)\right]$$

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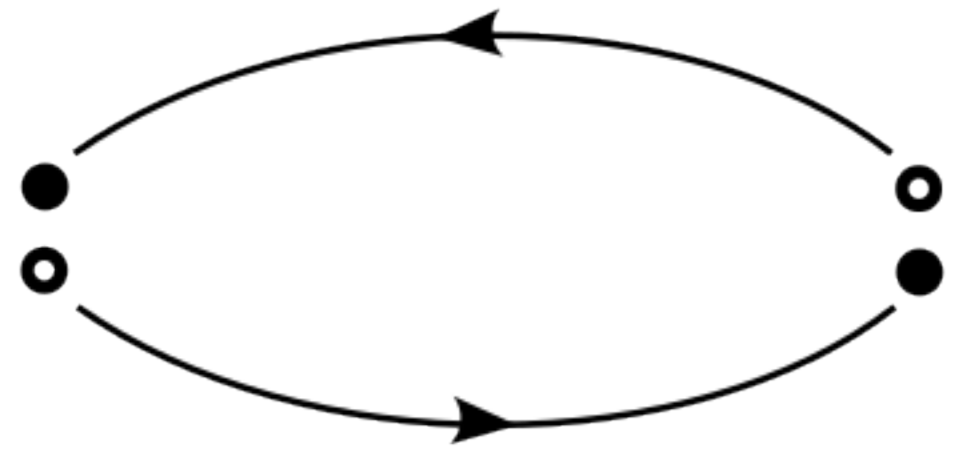
$$\langle \left(\bar{q}\gamma^i q'\right)(\mathbf{P}, t) \; \left(\bar{q}'\gamma^i q\right)(\mathbf{P},0)^\dagger \rangle_F = \sum_{\mathbf{x}} e^{-i\mathbf{x}\cdot\mathbf{P}} \; \mathrm{tr}\left[\gamma^i D^{-1}(0; \mathbf{x}, t) \; \gamma^i D^{-1}(\mathbf{x}, t; 0)\right]$$



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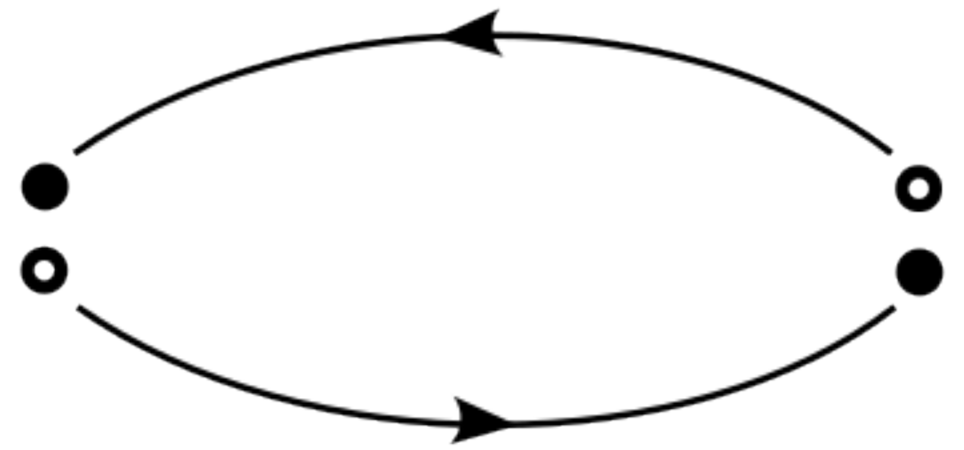

||

$$\sum_n \langle 0 | O_V | n \rangle \langle n | O_V^\dagger | 0 \rangle e^{-tE_n} = \sum_n Z_n e^{-tE_n} \xrightarrow{t \gg 1} Z_0 e^{-tE_0}$$

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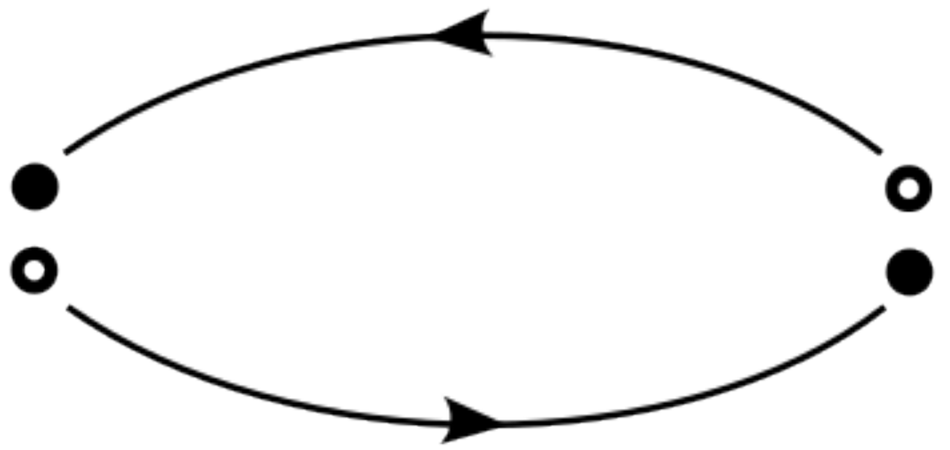

$$||$$

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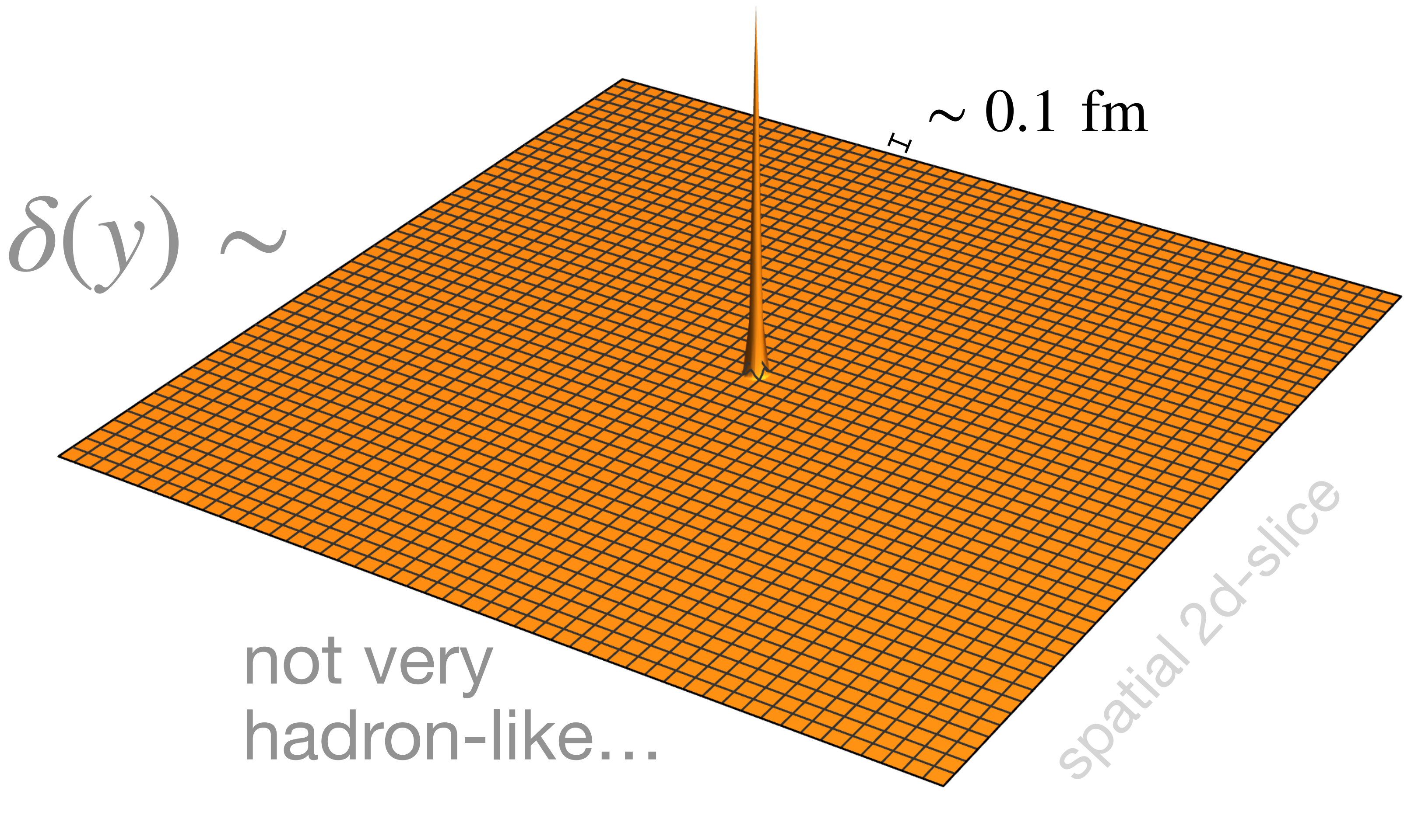
lower m_π, worse

- D^{-1} expensive → compute columns $(D^{-1}\zeta)_b^\beta(x)$
- large freedom to choose source ζ and O

$\zeta \sim \delta(y) \sim$



→ more interpolators? hadronic dimension?

Multi-hadron operators

Multi-hadron operators

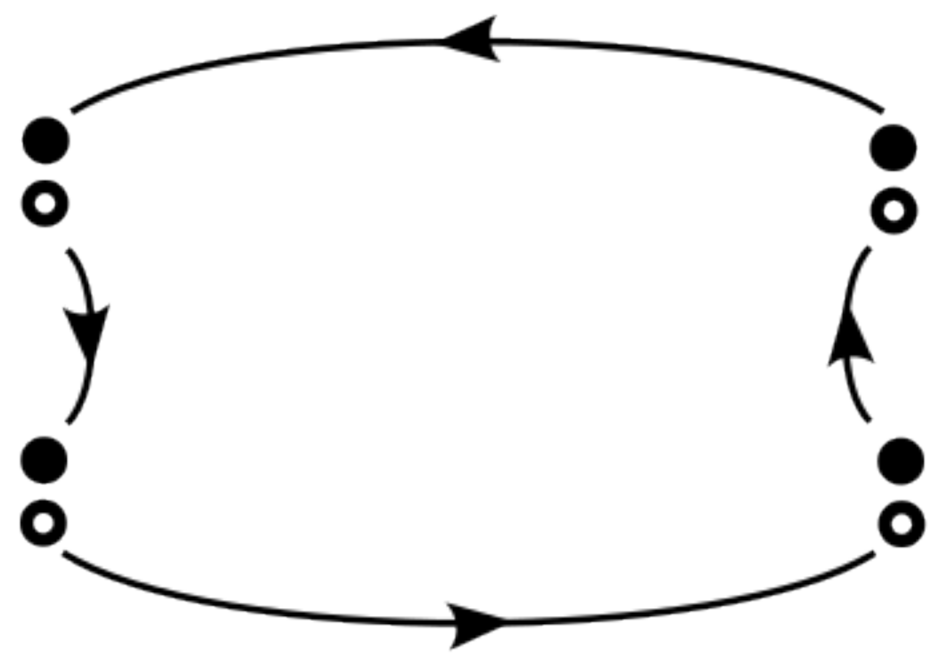
Non-local

$$O_{MM'}(\mathbf{x},\mathbf{y},t) \sim \left(\bar{q}_1\gamma^5 q_1'\right)(\mathbf{x},t)\ \left(\bar{q}_2\gamma^5 q_2'\right)(\mathbf{y},t)$$

$\longrightarrow$

$$\sum_{\mathbf{x},\mathbf{y},\mathbf{z}} e^{-i\mathbf{x}\cdot\mathbf{p}-i\mathbf{y}\cdot\mathbf{q}-i\mathbf{z}\cdot\mathbf{k}} \times \langle O_{MM'}(\mathbf{x},\mathbf{y},t)\, O_{MM'}(\mathbf{z},\mathbf{0},0)^\dagger\rangle_F$$

$\Supset$



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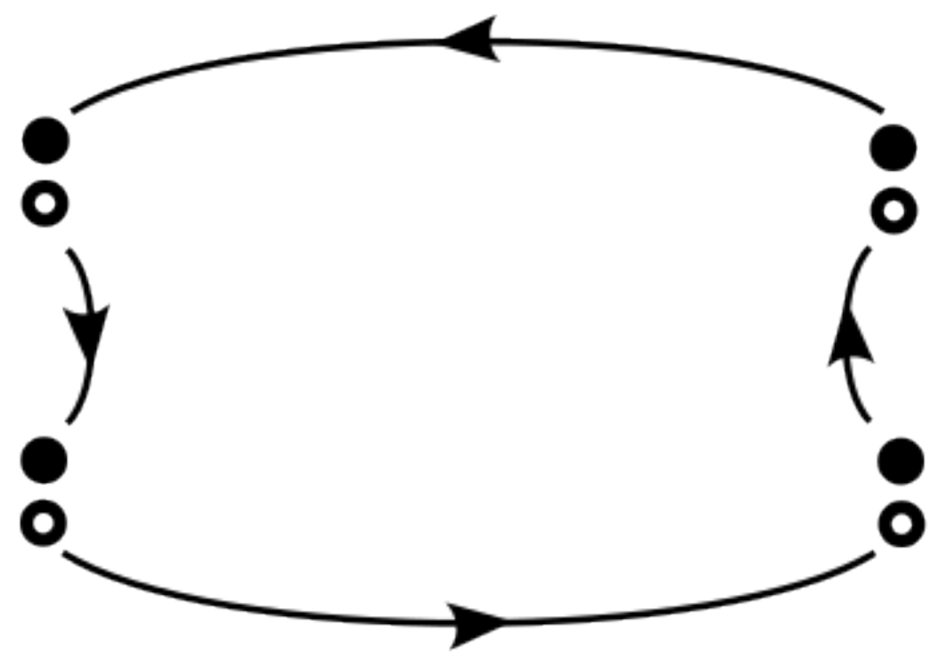
$$O_{MM'}(\mathbf{x},\mathbf{y},t) \sim \left(\bar{q}_1\gamma^5 q_1'\right)(\mathbf{x},t)\ \left(\bar{q}_2\gamma^5 q_2'\right)(\mathbf{y},t) \longrightarrow \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} e^{-i\mathbf{x}\cdot\mathbf{p}-i\mathbf{y}\cdot\mathbf{q}-i\mathbf{z}\cdot\mathbf{k}} \times \langle O_{MM'}(\mathbf{x},\mathbf{y},t)\, O_{MM'}(\mathbf{z},\mathbf{0},0)^\dagger\rangle_F$$

$\supseteq$



Information from $D^{-1}(x;y)^{\alpha\beta}_{ab}$ is needed (all-to-all)

- dimension $4 \times 3\, N_t N^3 \sim \mathcal{O}(10^8)$: unfeasible

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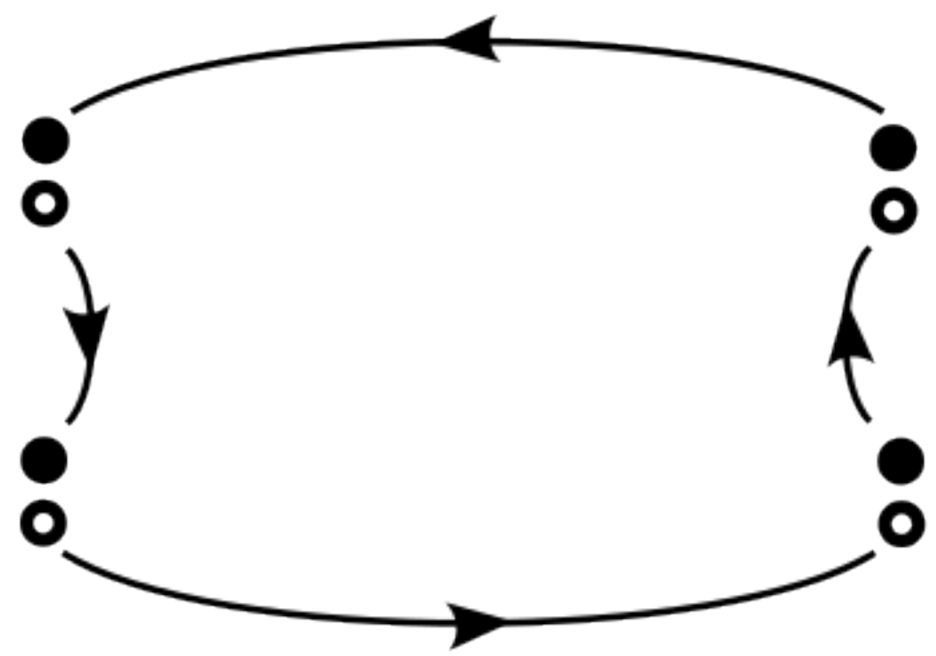
$\ni$



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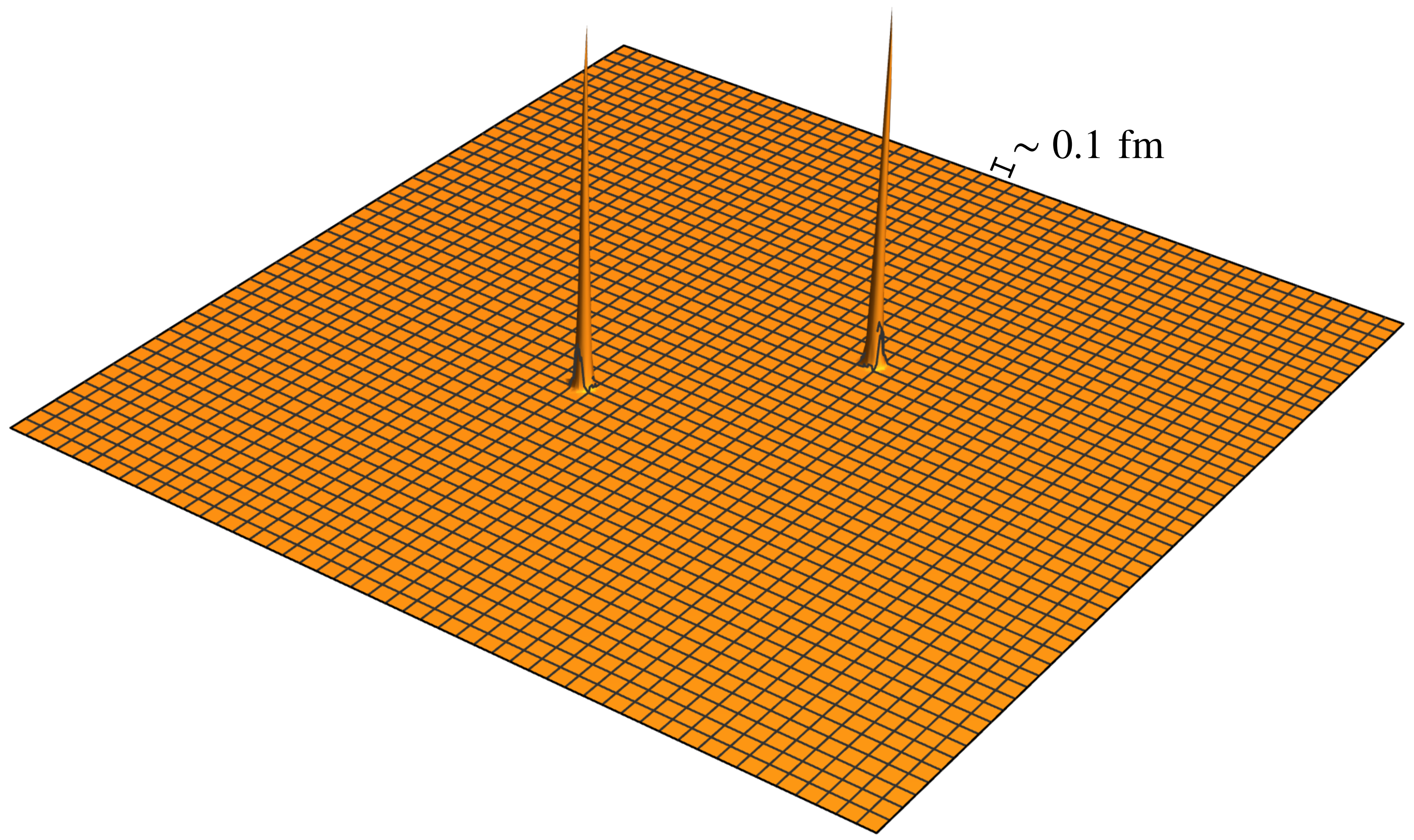

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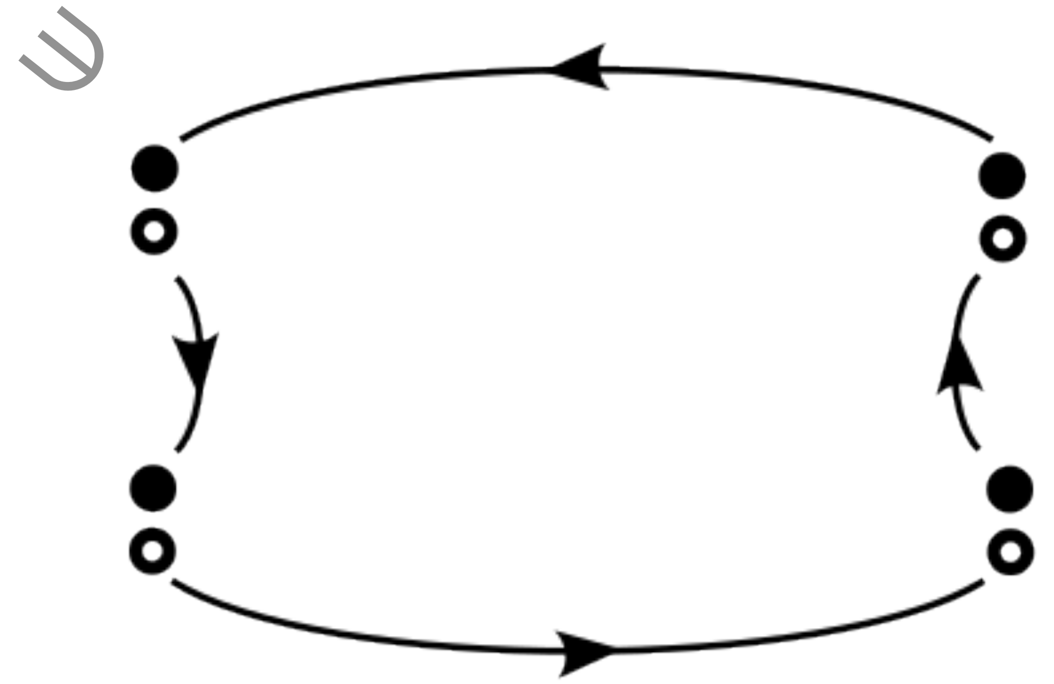

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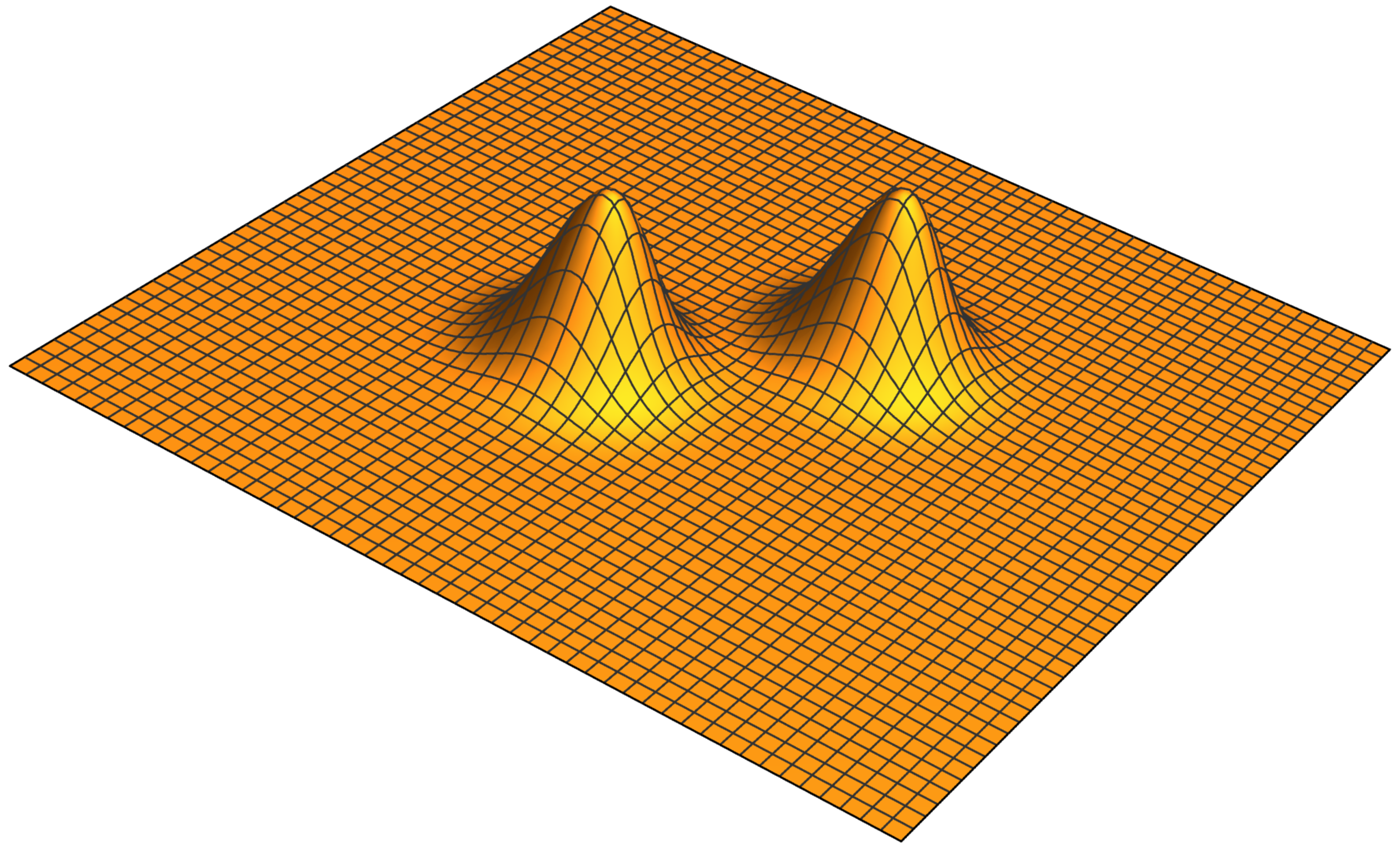

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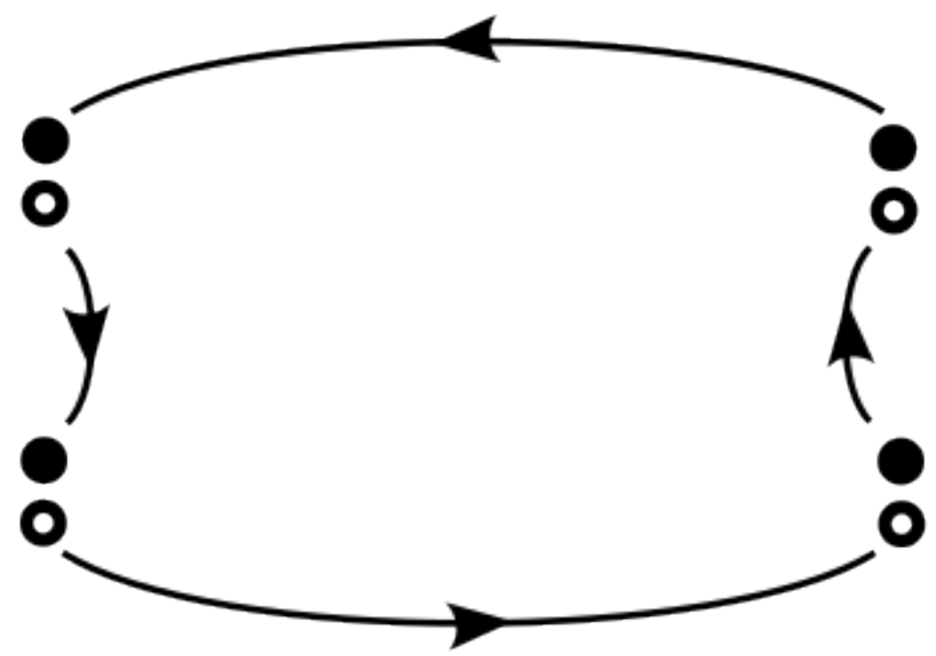

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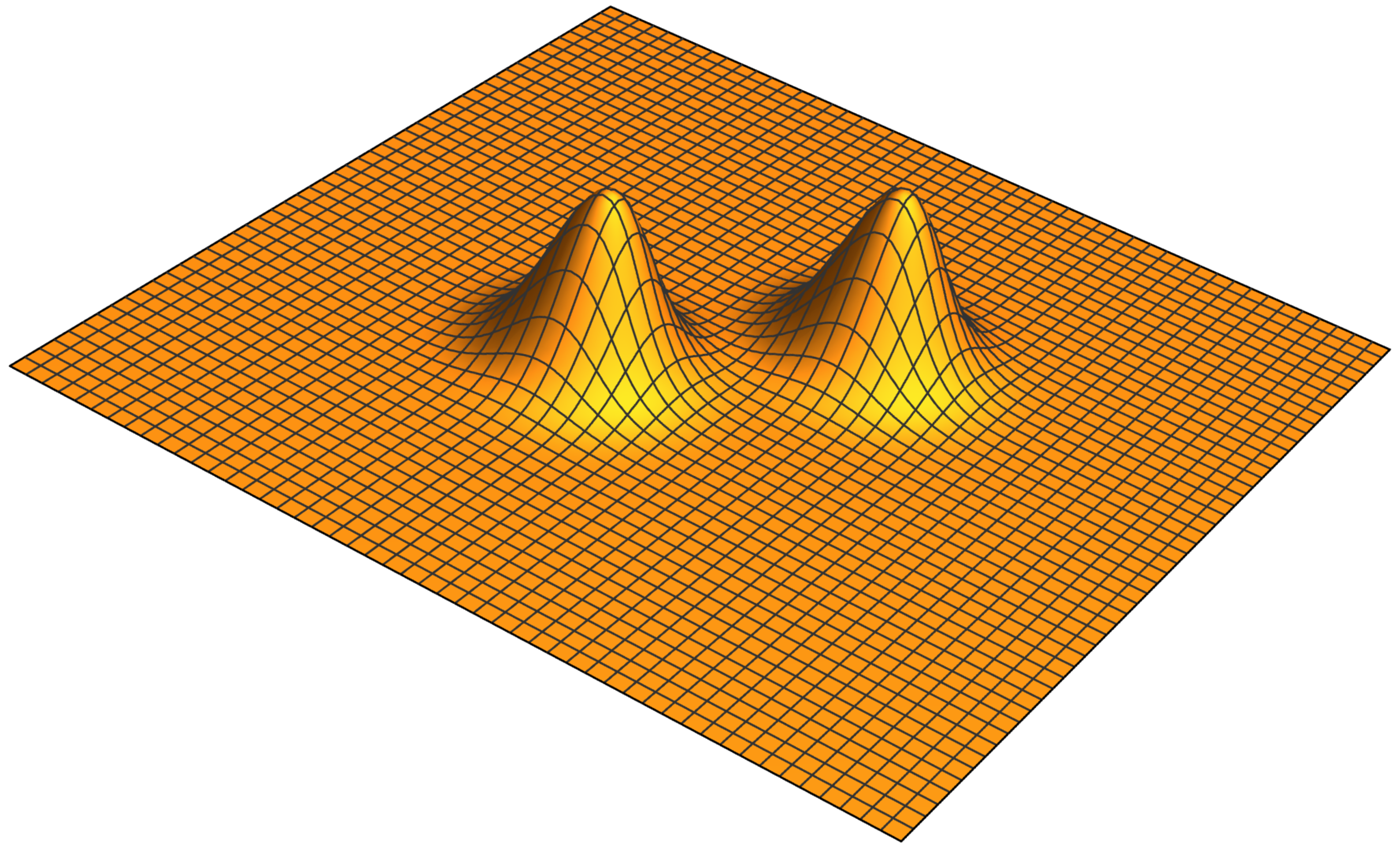
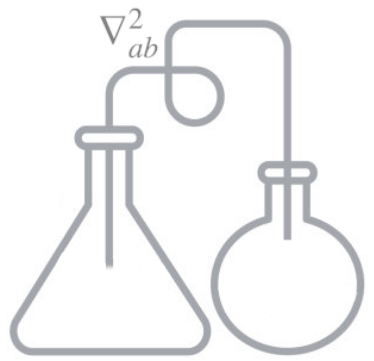
Use freedom to build sources:

- smaller and more efficient <u>basis</u> than entire lattice?



Distillation

[Peardon et al, PRD, 2009] [Morningstar et al, PRD, 2011]



Low-lying N_{vec} eigenvectors of 3D- covariant ***Laplacian*** $-\nabla^2_{ab}(t)$

full propagator

$$D^{-1}(x;y)^{\alpha\beta}_{ab}$$

$$\square(t) = \sum_{k=1}^{N_{vec}} v_k(t) v_k(t)^\dagger$$

projection $q \to \square q$

perambulator

$$\tau(t,t') = v(t)^\dagger D^{-1}(t,t') v(t')$$

space-color encoded in $v_k^a(\mathbf{x}, t)$

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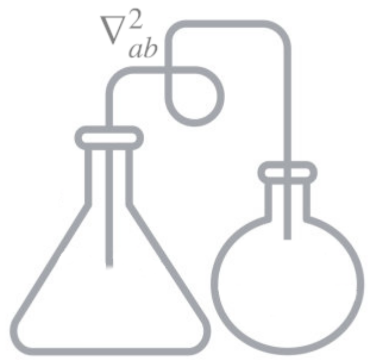

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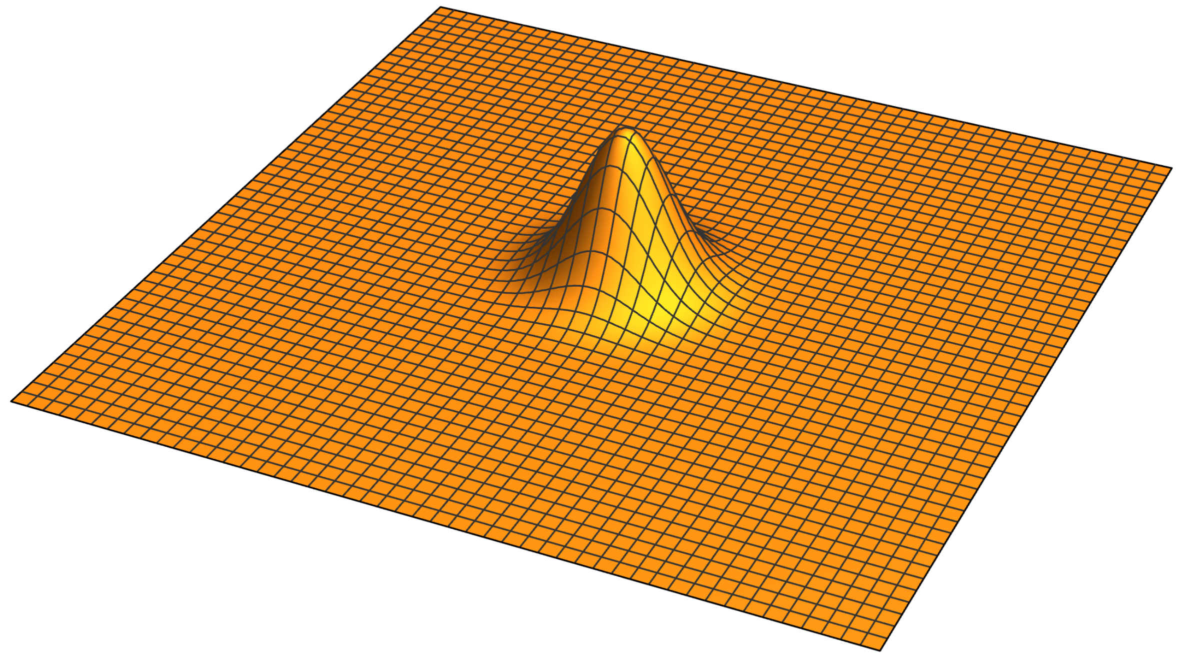
Source is built from N_{vec} inversions $D^{-1}v_k$ for each t'

- their superposition gives source with smeared spatial profile



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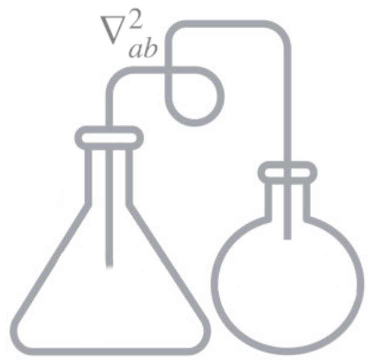

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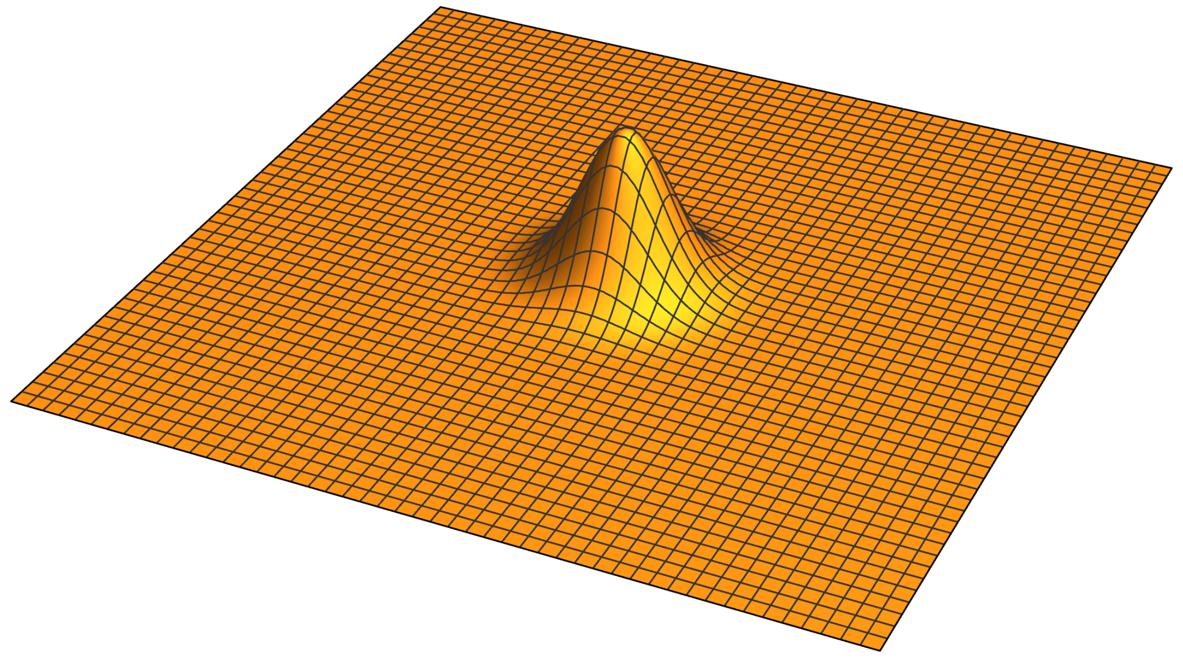

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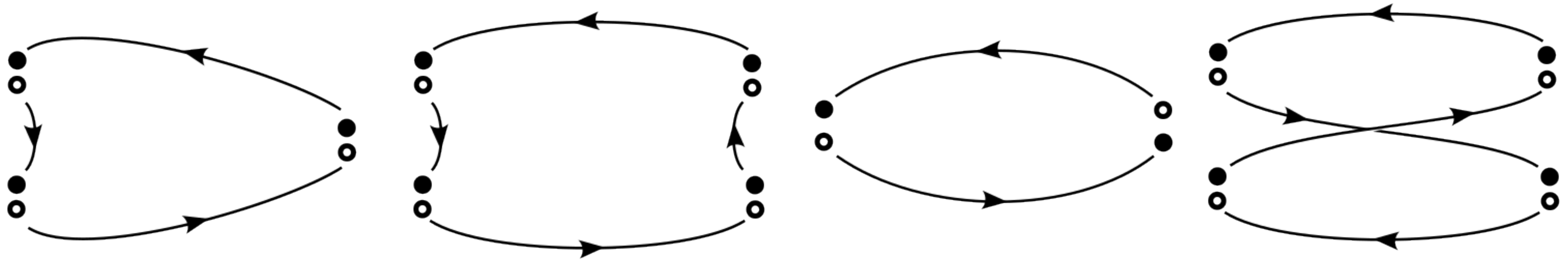
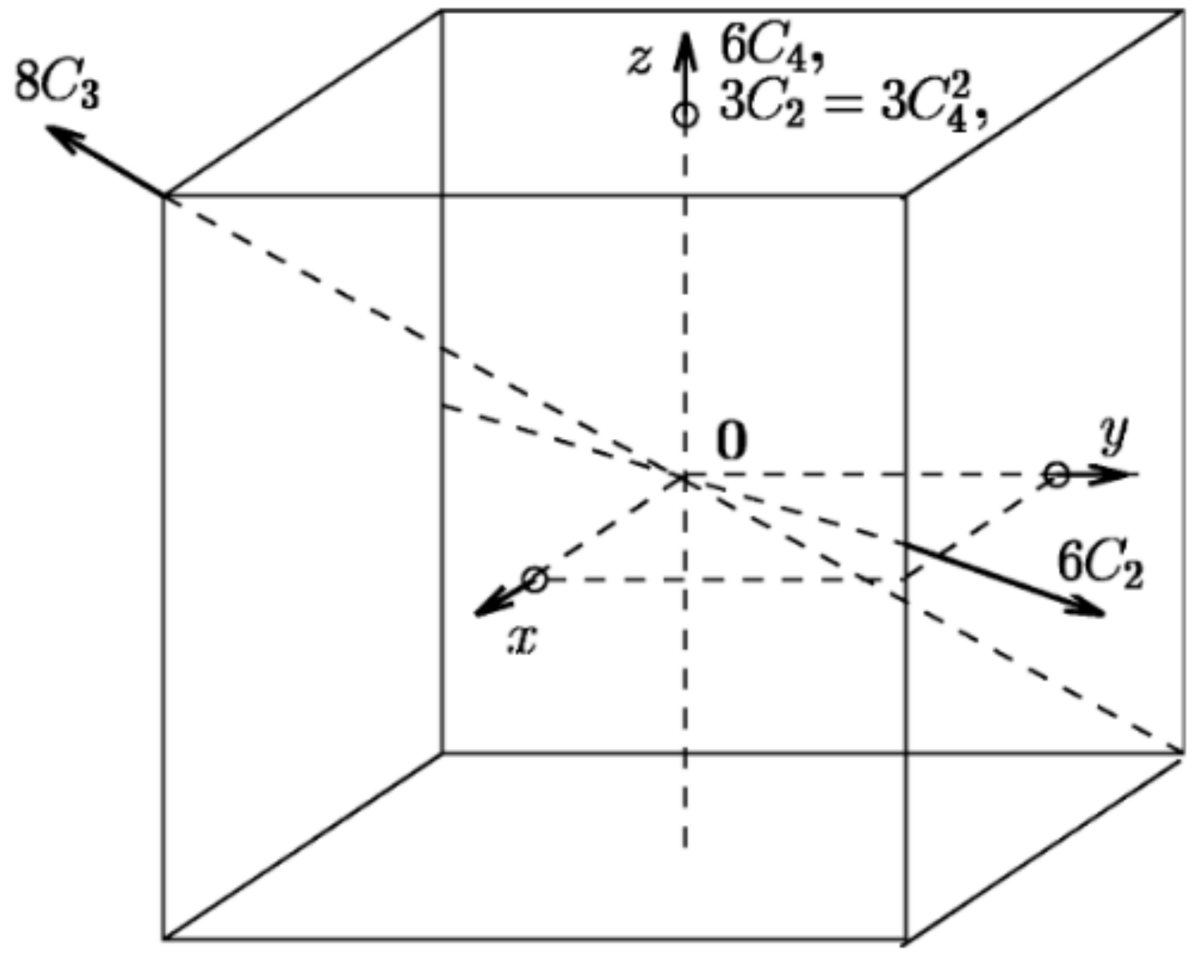
Reusable  , various $\mathbf{p}_i$, γ_μ, …

Hidden until now

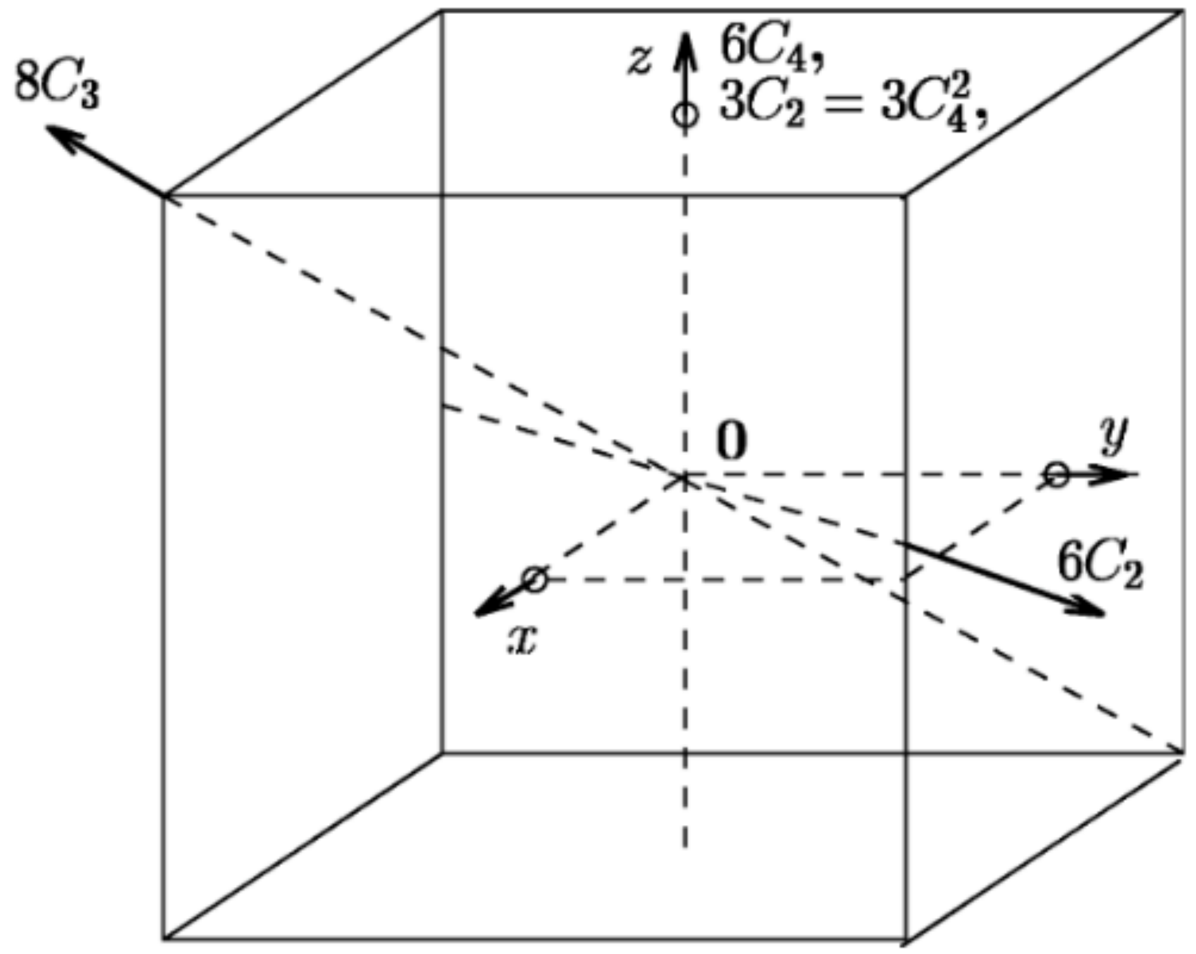
Finite-volume breaks rotational into cubic subgroup

- $SO(3) \rightarrow O$: 24 symmetries of a cube ($\mathbb{Z}$ spin)
- parity $\rightarrow O_h$: 48 elements



[M. S. Dresselhaus, et al. "Group Theory: Application to the Physics of Condensed Matter. 2008"]

Hidden until now



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Rest and moving frames (MF) mapped into O_h subgroups

- all states/operators labelled by irreps $\Lambda[\mathbf{P}]$ of O_h
- parity in MF is not always a good quantum number

"Subduction":

$$J = 1,3,... \rightarrow [000]T_{1u}$$
$$J = 1,2,... \rightarrow [001]E$$
$$\vdots$$

Hidden until now

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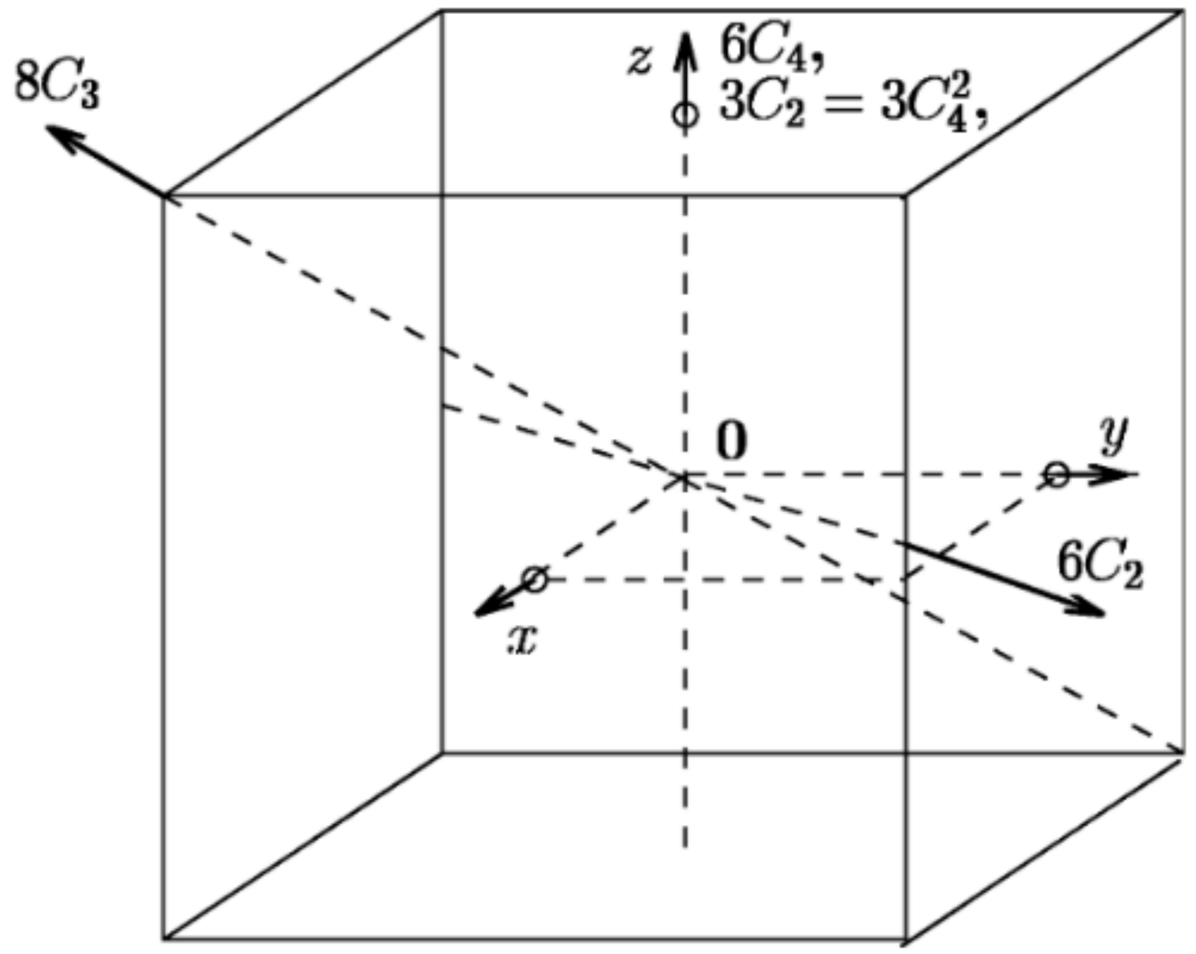

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Rest and moving frames (MF) mapped into O_h subgroups

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- parity in MF is not always a good quantum number

$m_1 \neq m_2$

$J = 0,1,2,... \to [001]A_1$

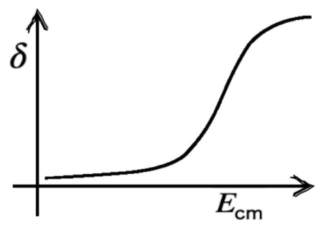
"Subduction":

$J = 1,3,... \to [000]T_{1u}$

$J = 1,2,... \to [001]E$

$\vdots$

Phase-shift model

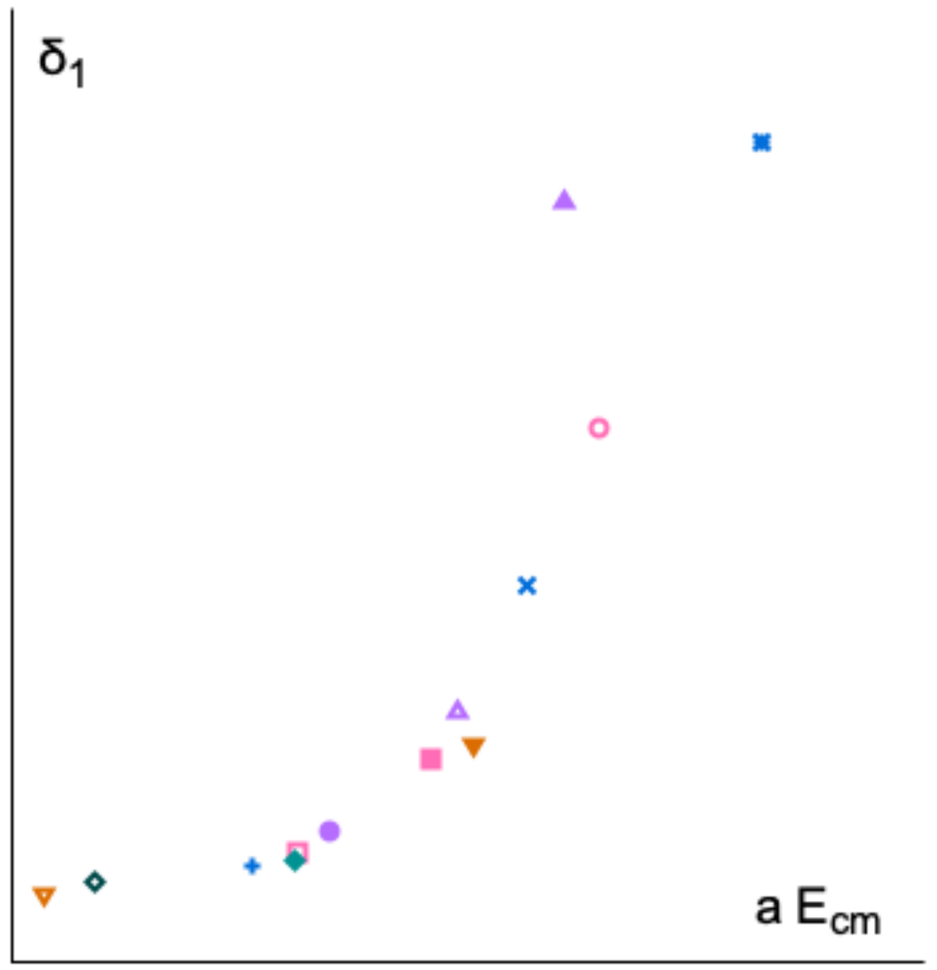


QC reminder:

$$\delta\big(E_{\text{cm}}(L)\big) = n\pi - \phi^{\Lambda}\big(E_{\text{cm}}(L), L\big), \quad n \in \mathbb{Z}$$

$(i) \equiv (n, \Lambda, \text{flavour}),$ lattice

Allows computation of $\delta_1(E_{\text{cm}}^{(i)})$, but poles inaccessible



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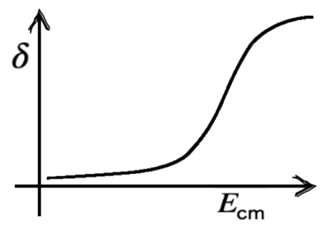

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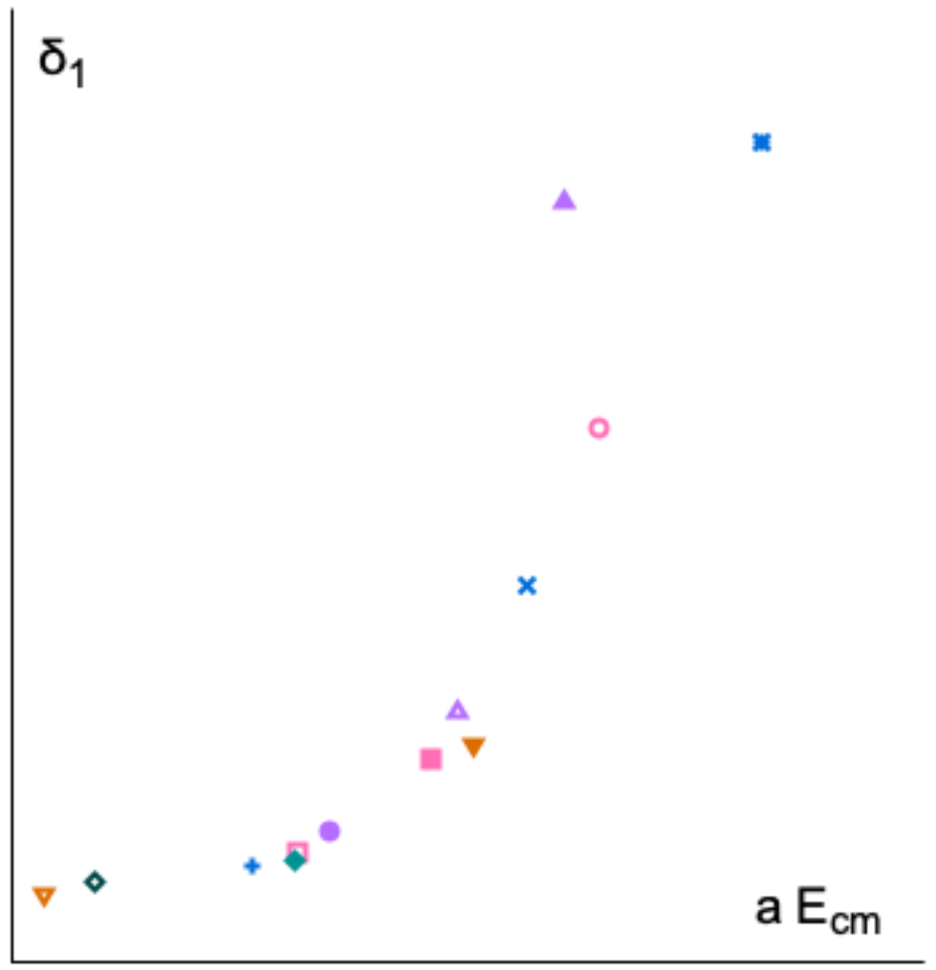
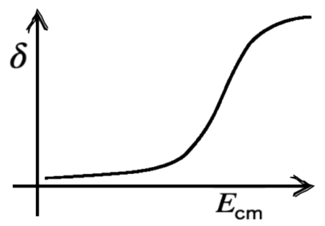
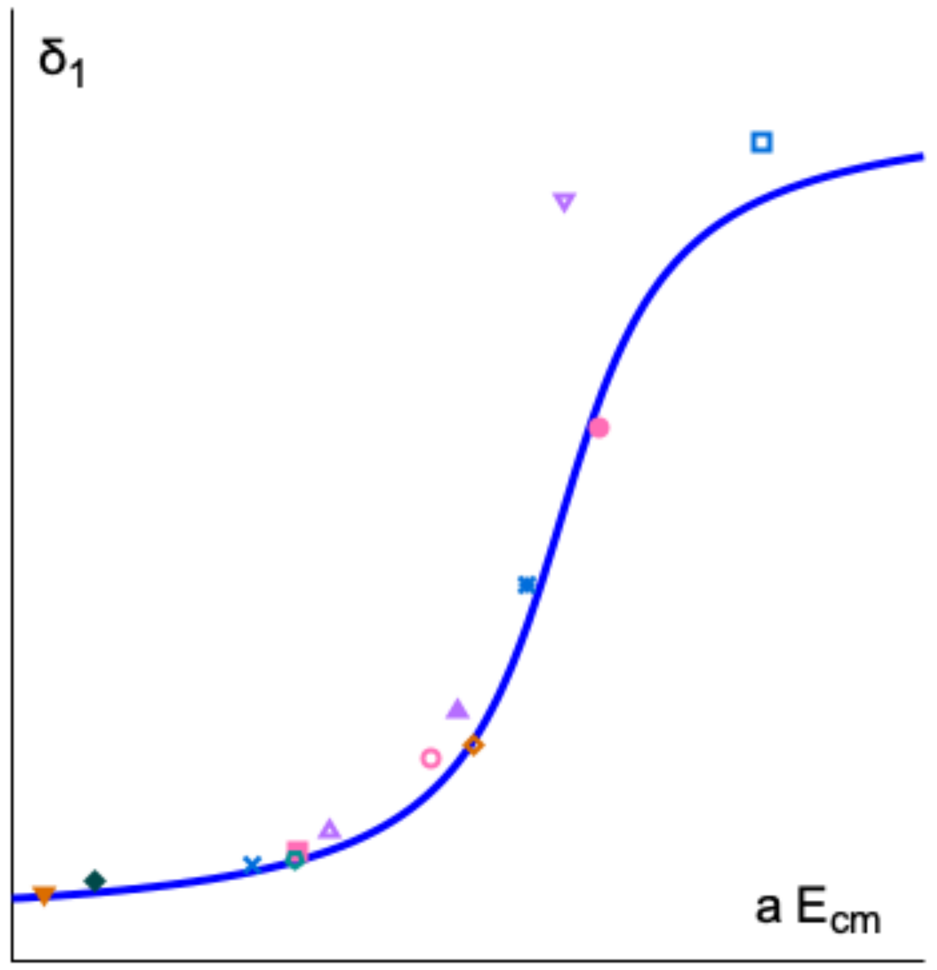


Invert QC: given model δ^{mod} with parameters α^{mod}, find $\mathscr{E}_{\text{cm}}^{(i)}$

Phase-shift model 



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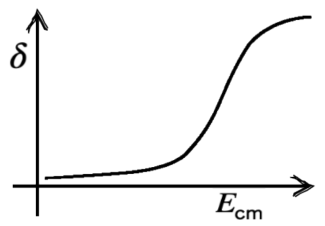

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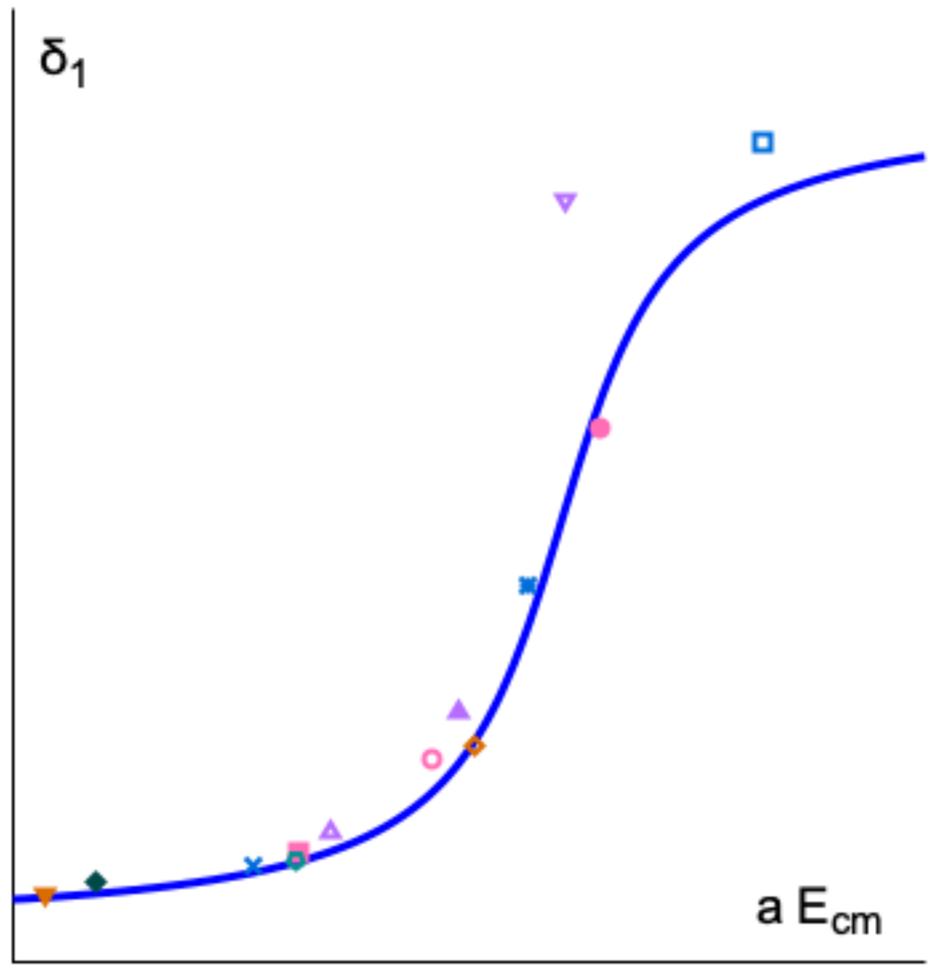
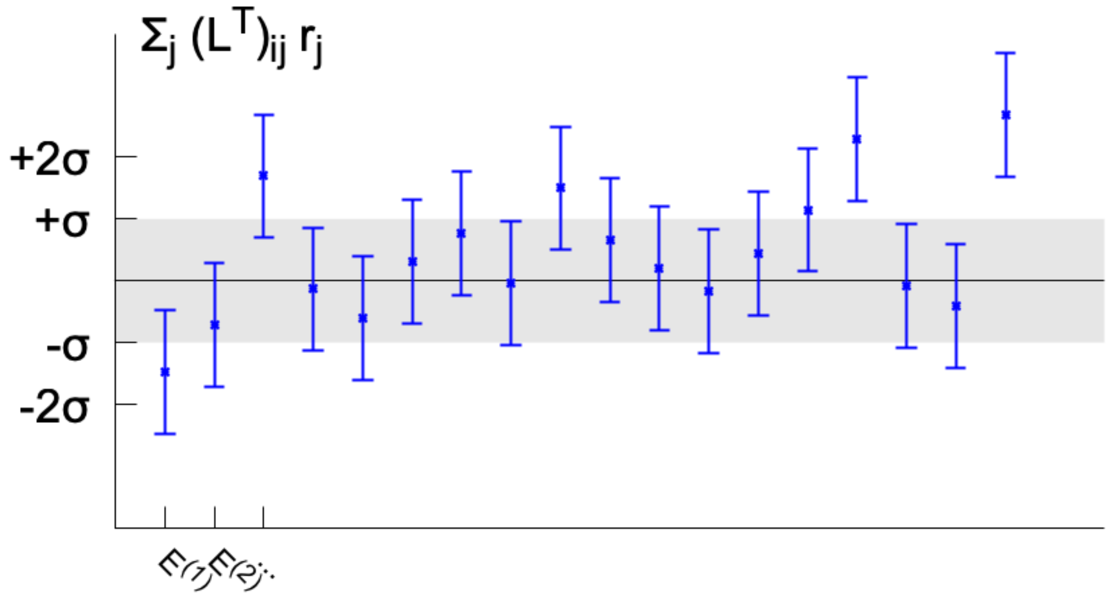


Invert QC: given model δ^{mod} with parameters α^{mod}, find $\mathscr{E}_{\mathrm{cm}}^{(i)}$

Minimise correlated

$$\chi^2_{\mathrm{PS}}(\alpha^{\mathrm{mod}}) = \sum_{i,j} \left[E^i_{\mathrm{cm}} - \mathscr{E}^i_{\mathrm{cm}}(\alpha^{\mathrm{mod}})\right] (\mathrm{Cov}^{-1})_{ij} \left[E^j_{\mathrm{cm}} - \mathscr{E}^j_{\mathrm{cm}}(\alpha^{\mathrm{mod}})\right]$$

to constrain δ^{mod}



Eigenvalue fits

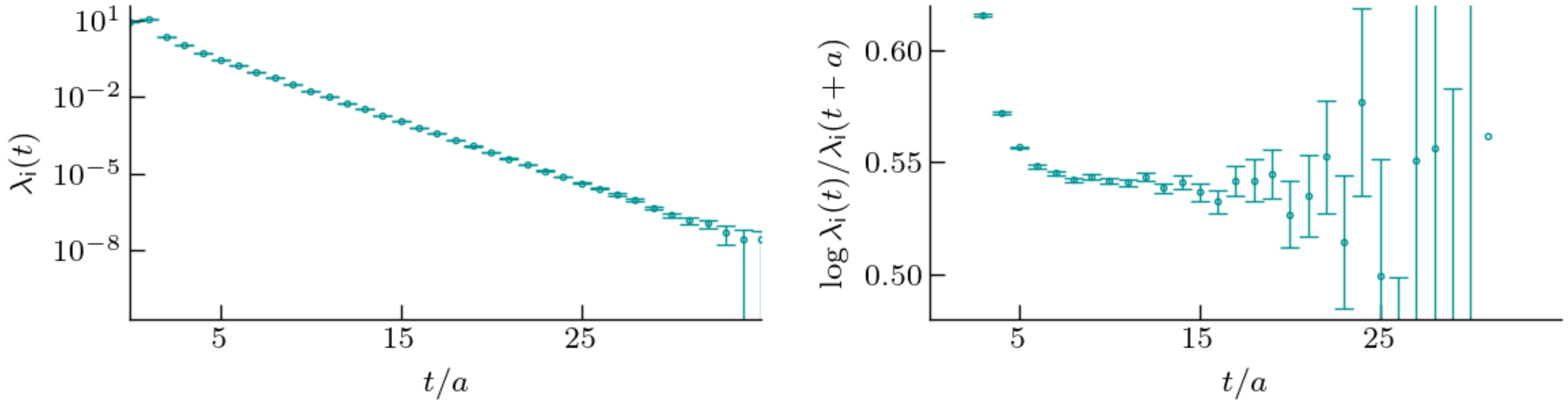
Remember

$$\langle \Omega_n(t)\Omega_n^\dagger(0)\rangle = \sum_n Z_n e^{-tE_n}$$

$$\lambda^n(t) \xrightarrow{t\gg 1} \approx Z_n\, e^{-E^n t}$$

Model: $\lambda^{\text{mod}}(t) = Z_n^{\text{mod}} e^{-tE^n_{\text{mod}}} + \cdots$

Correlated χ^2 to constrain it



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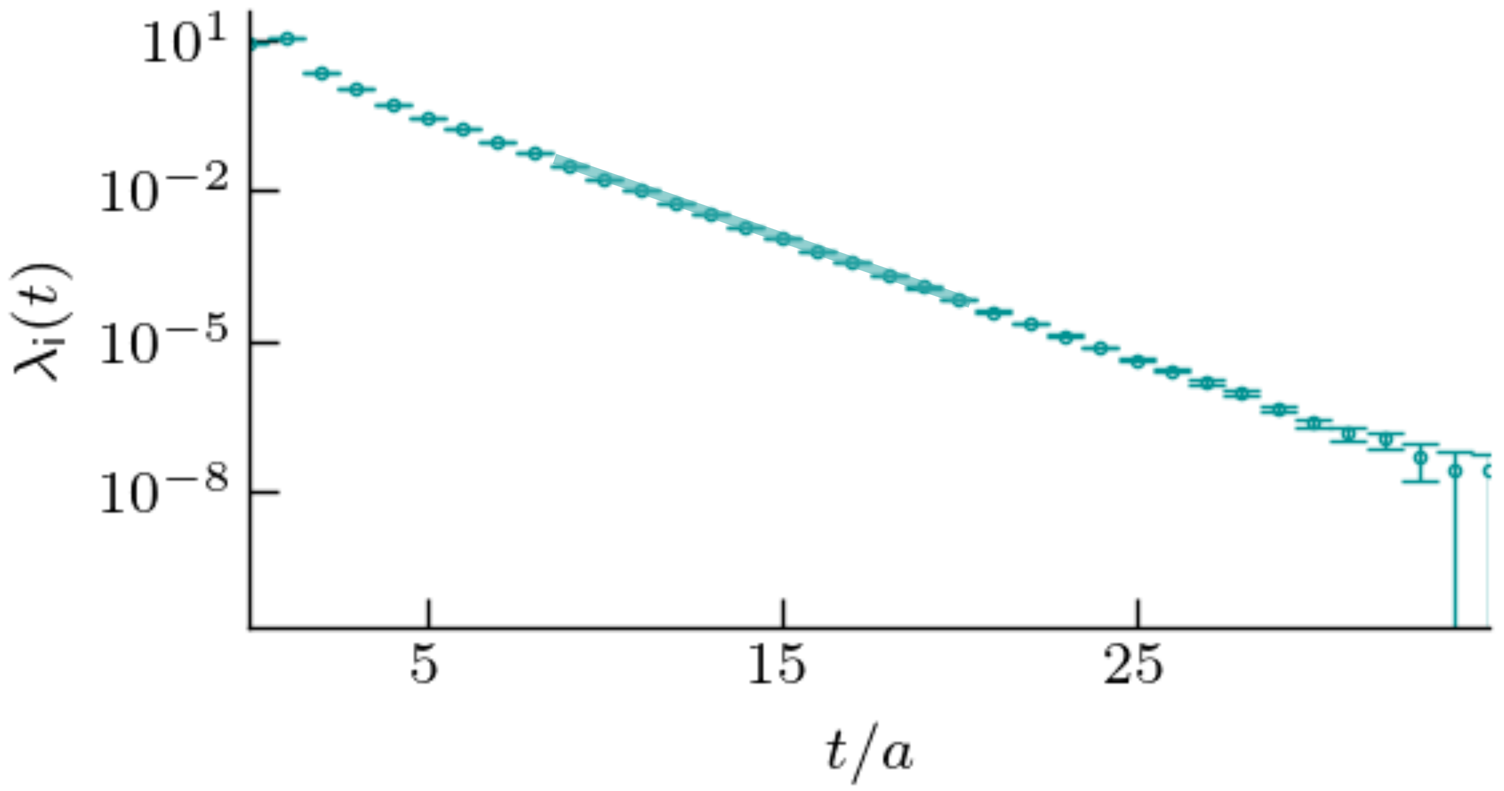

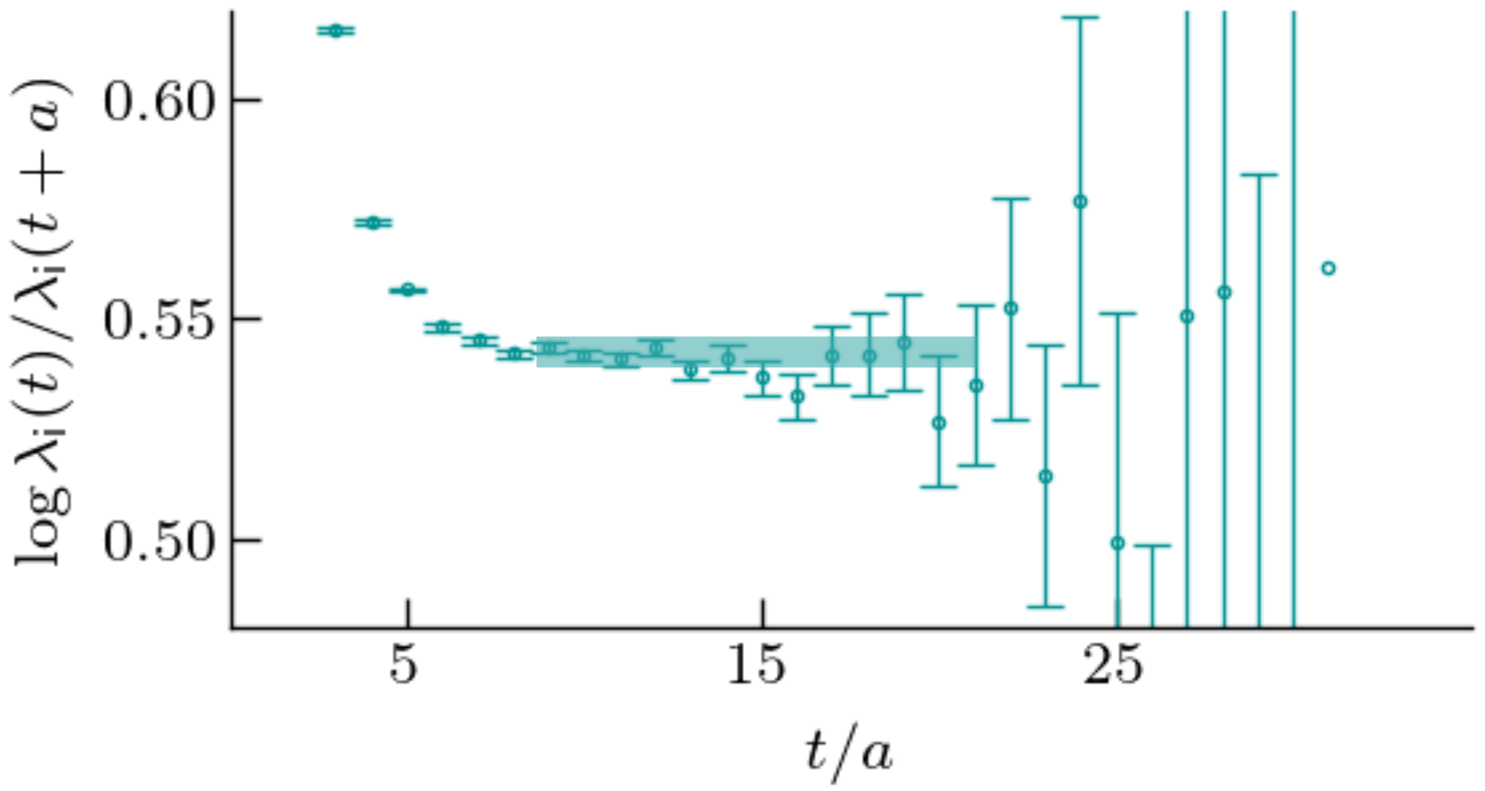

Statistical errors in $\lambda^n(t)$ from MC: few samples : bootstrap N_b replicas

- fit on every replica: $E^n_{\text{mod},b} \quad \rightarrow \quad \sigma^2 \approx \text{Var}\left(E^n_{\text{mod},b}\right)$
- $E^n_{\text{mod},b} \xrightarrow{\text{boost}} E^n_{\text{cm},b}$

Eigenvalue fits

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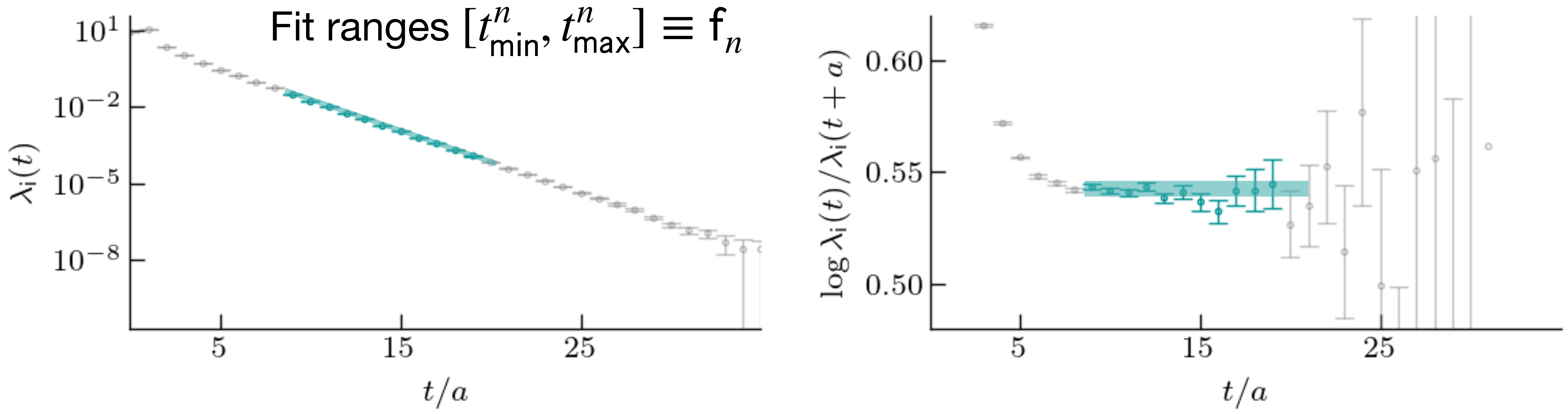

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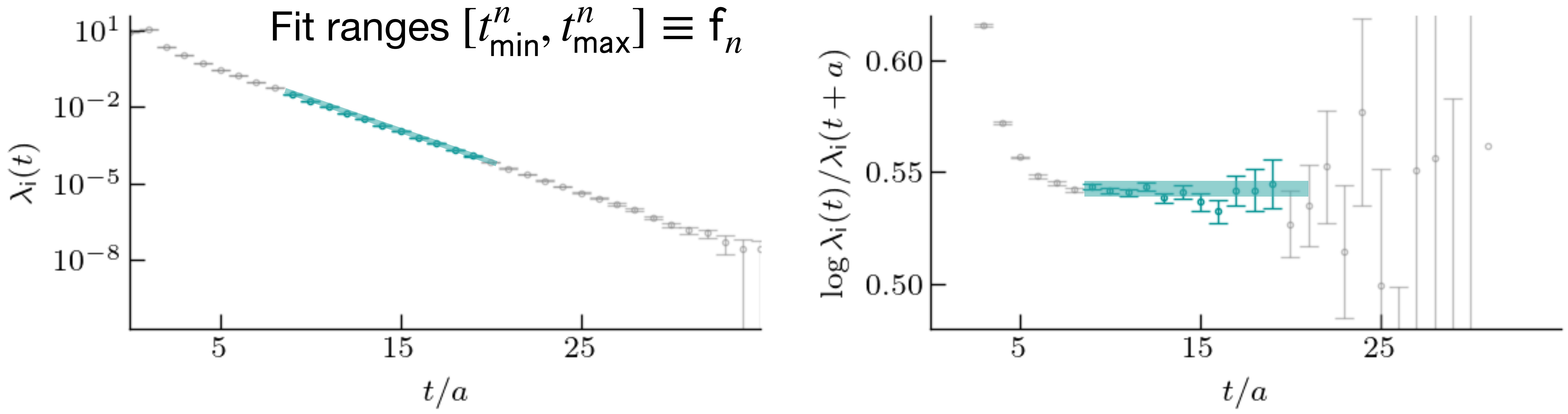

For all levels:

$$\{\mathrm{f}_1, \mathrm{f}_2, \ldots\} = \mathrm{f}$$

$$\downarrow$$

$$\{E_{\text{cm}}\}$$

Statistical errors in $\lambda^n(t)$ from MC: few samples : bootstrap N_b replicas

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Model-averaging

Akaike information criterion (AIC)

- probabilities for different models $w \propto \exp -\frac{1}{2}\left[\overbrace{\chi^2 + 2n^{\text{par}}}^{\text{AIC}}\right]$
- hybrid: “Bayesian” model comparison, but frequentist weights
- uncertainty prescription: spread of final weighted distribution

Model-averaging

Akaike information criterion (AIC)

- probabilities for different models $w \propto \exp -\frac{1}{2}[\overbrace{\chi^2 + 2n^{\text{par}} - n^{\text{data}}}^{\text{AIC}}]$
- hybrid: "Bayesian" model comparison, but frequentist weights
- uncertainty prescription: spread of final weighted distribution

Different $[t_{\text{min}}, t_{\text{max}}] = \mathsf{f} \leftrightarrow$ different models

$\downarrow$

$$w^i_{\text{corr}}(\mathsf{f}_{(i)}) = \exp -\frac{1}{2}\text{AIC}_{\text{corr}}(\mathsf{f}_i)$$

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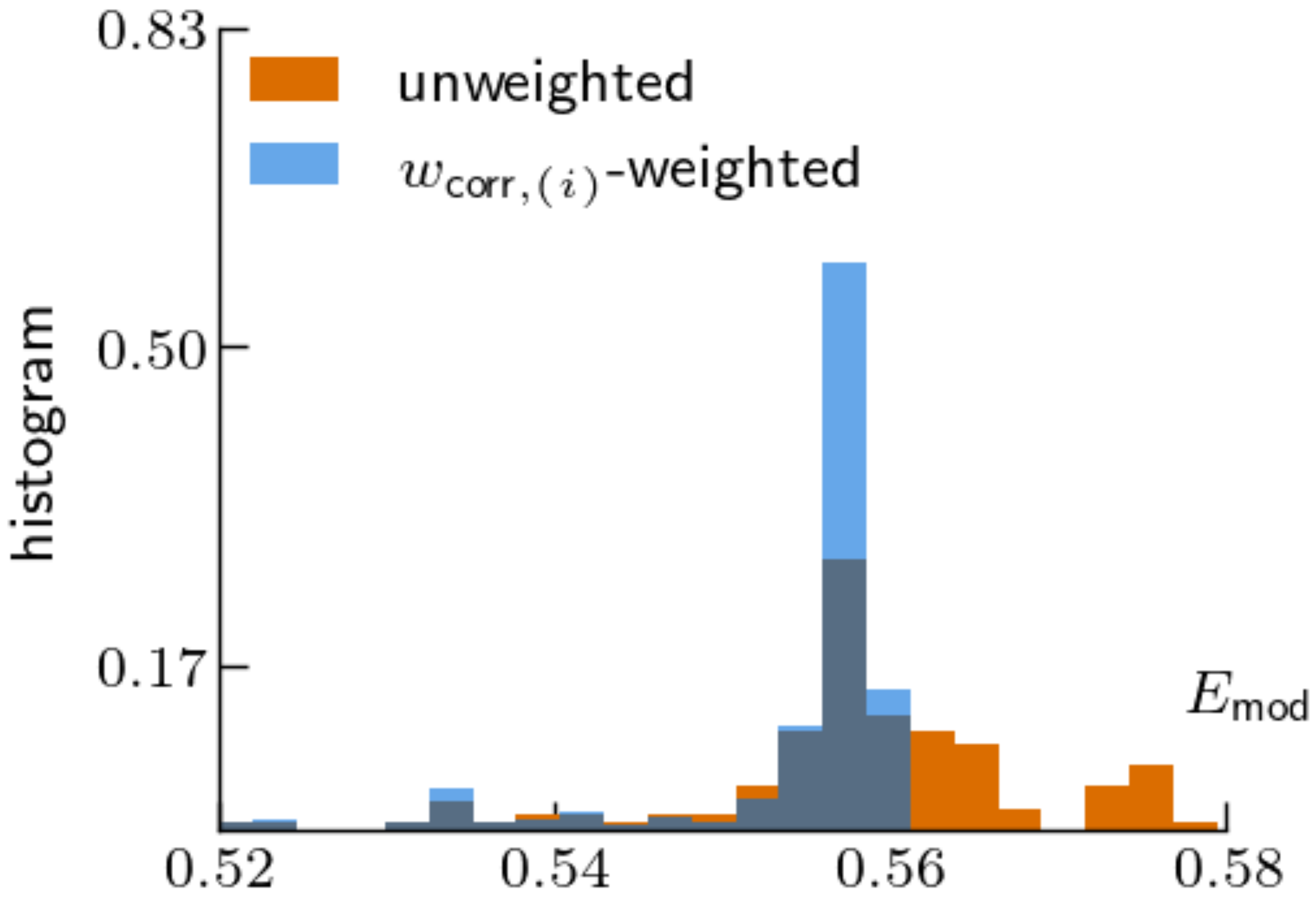
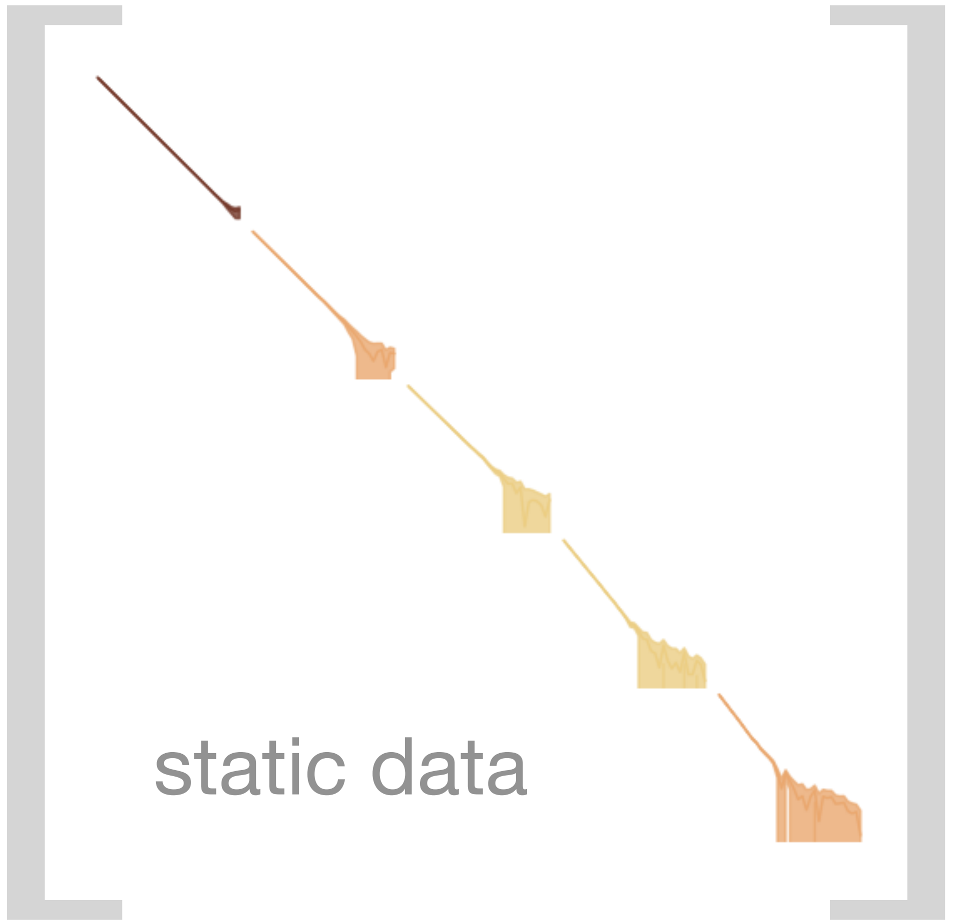
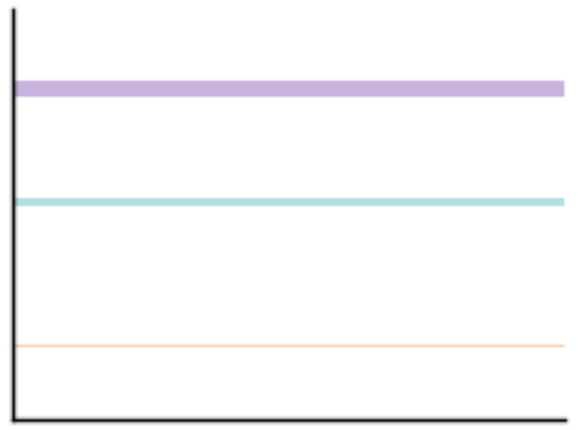




n^{lev} fits

$\lambda_i, \mathrm{f}_i \to E^i_{\text{cm}}$

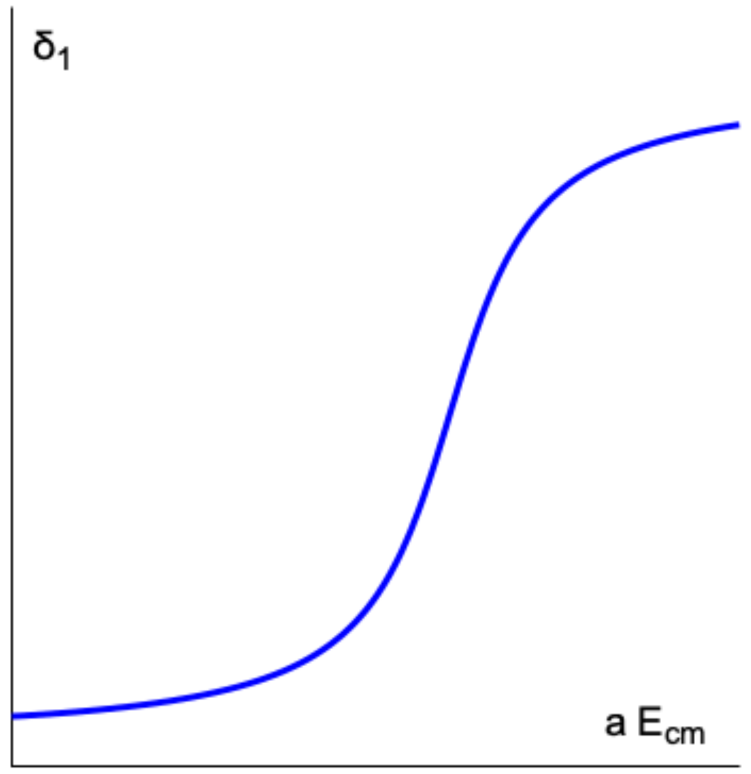
$i \equiv (n, \Lambda, \text{flavour})$

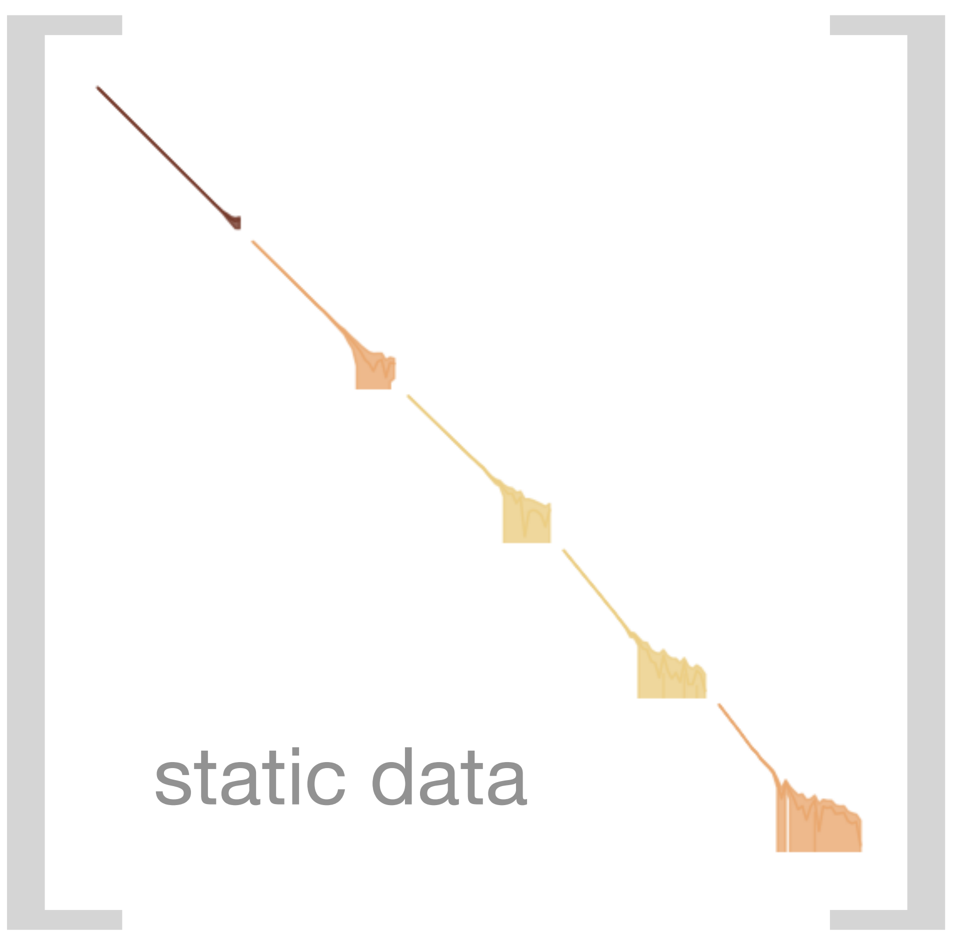


static data

one fit

$\{E_{\text{cm}}\} \to \delta^{\text{mod}}$

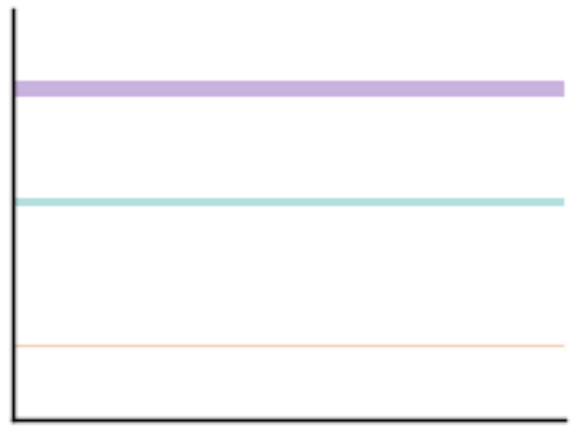




n^{lev} fits

$\lambda_i, \text{f}_i \rightarrow E^i_{\text{cm}}$

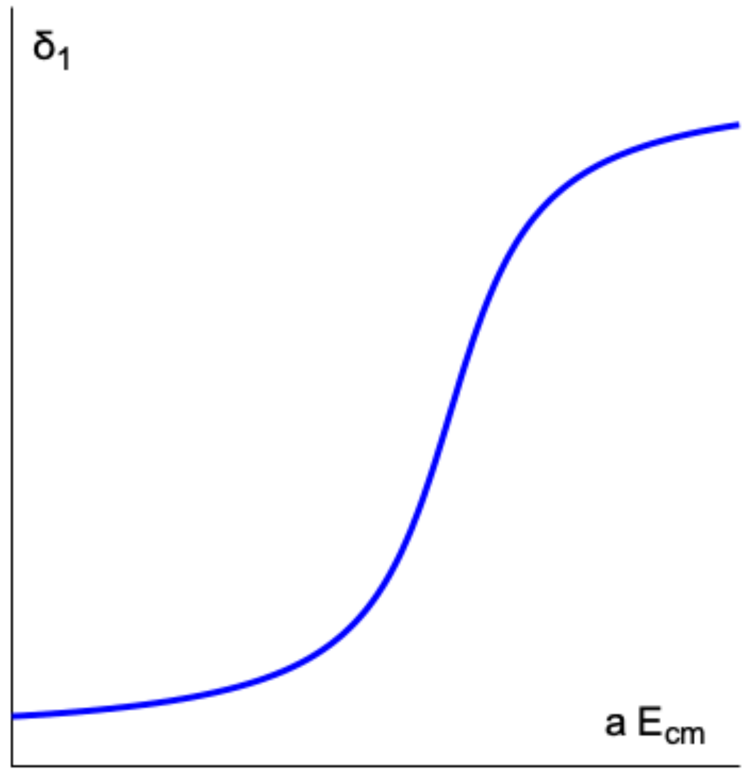
$i \equiv (n, \Lambda, \text{flavour})$



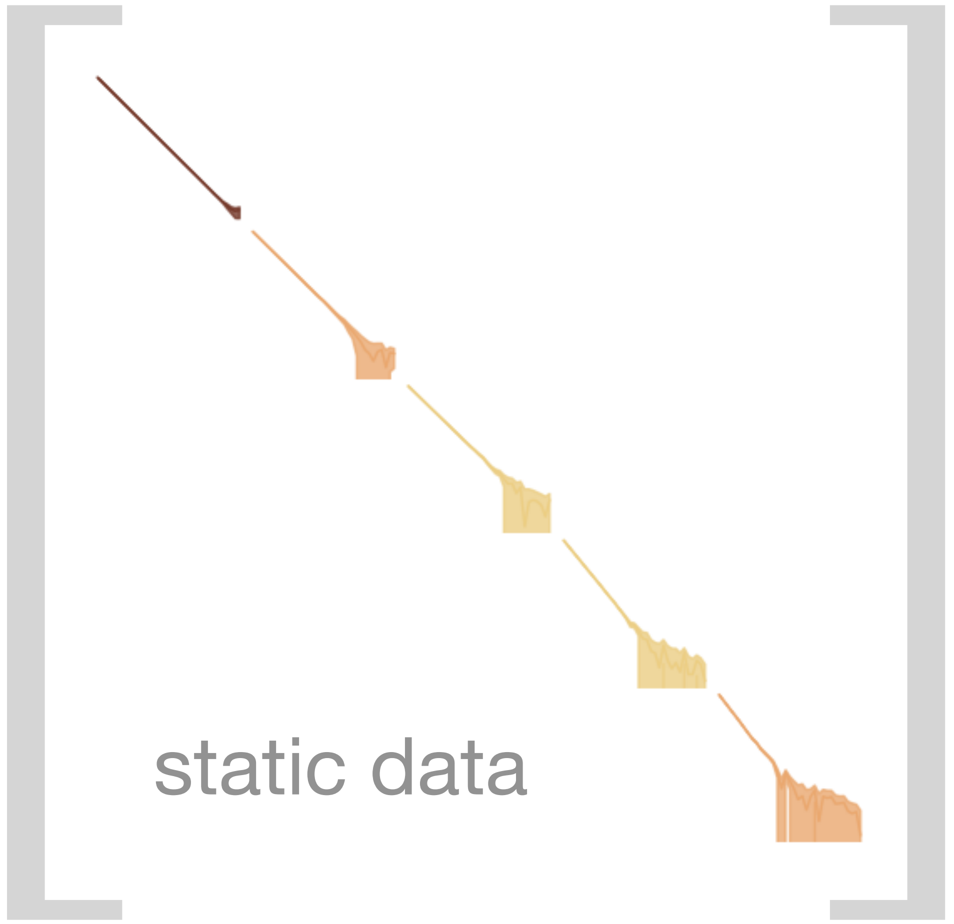
static data

one fit

$\{E_{\text{cm}}\} \rightarrow \delta^{\text{mod}}$



…reliable systematic propagation to scattering?



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$\lambda_i, \mathsf{f}_i \to E^i_{\text{cm}}$

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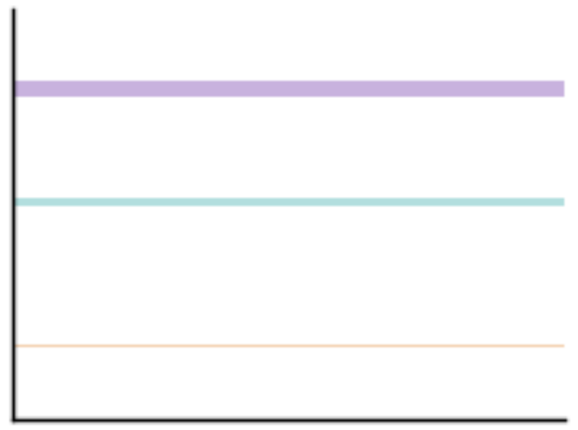
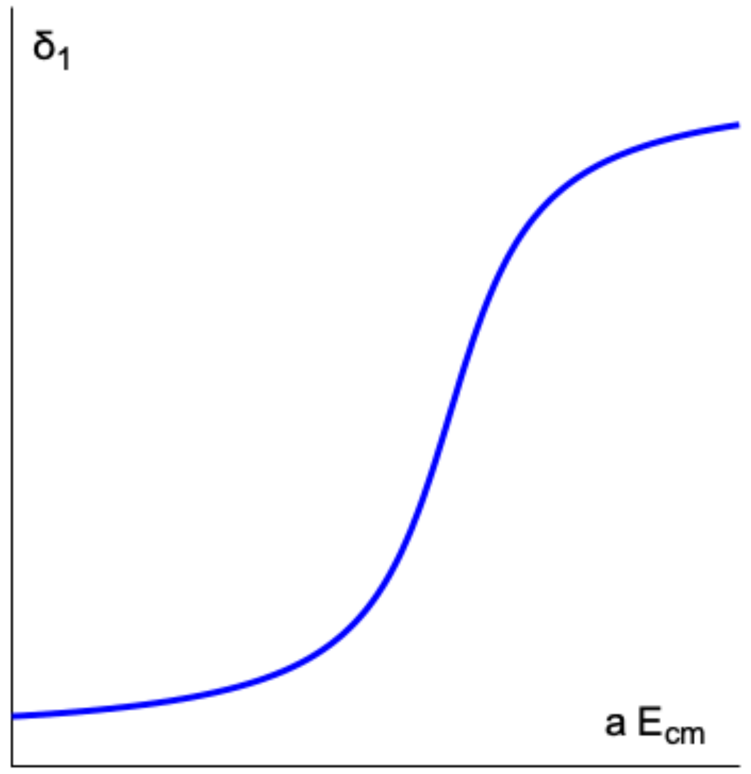


static data

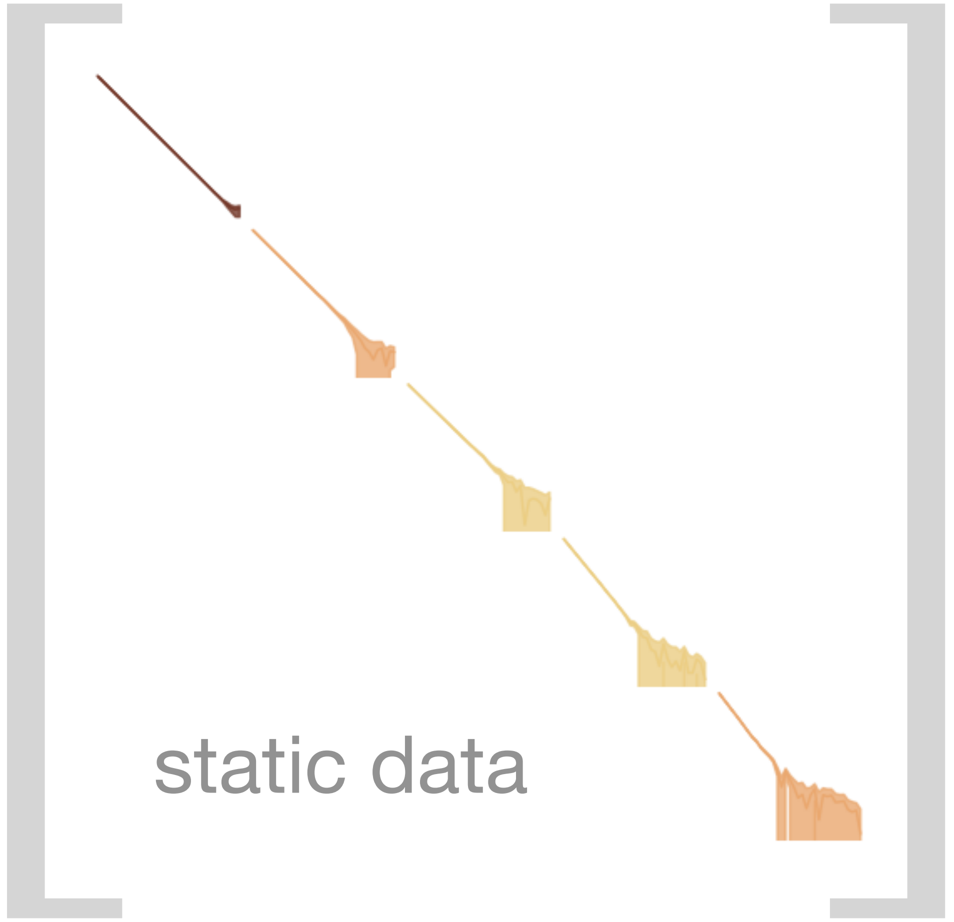
one fit

$\{E_{\text{cm}}\} \to \delta^{\text{mod}}$

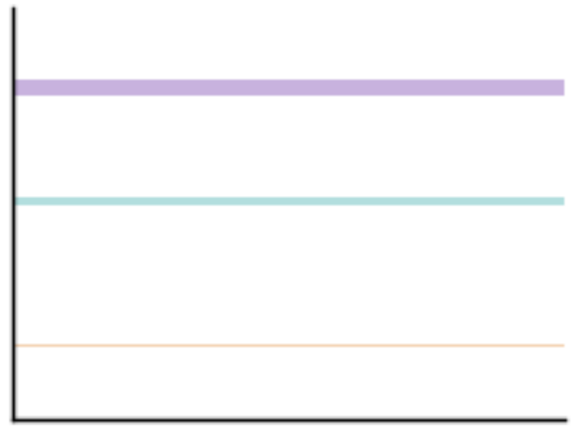


…reliable systematic propagation to scattering?

First, imagine

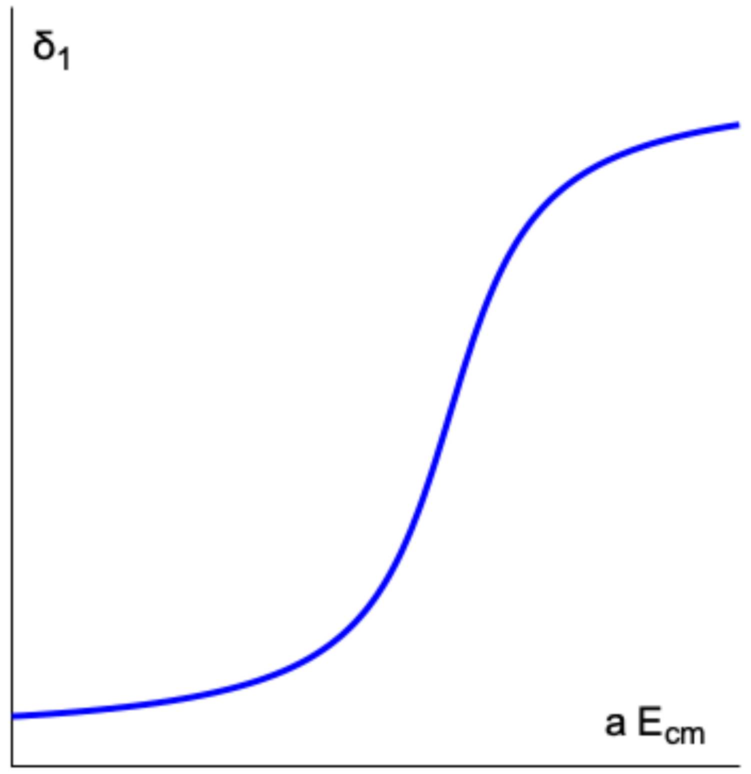


global minimisation unfeasible

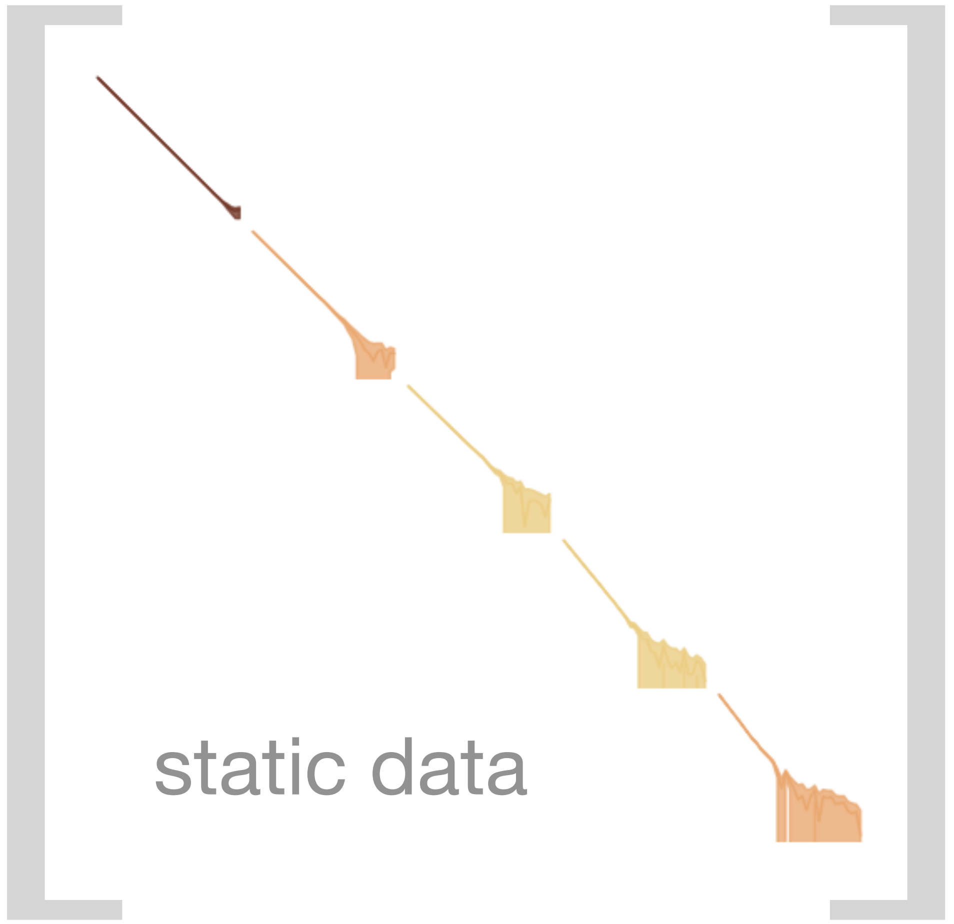


subproduct

one fit $\{\lambda, \mathsf{f}\} \to \delta^{\text{mod}}$

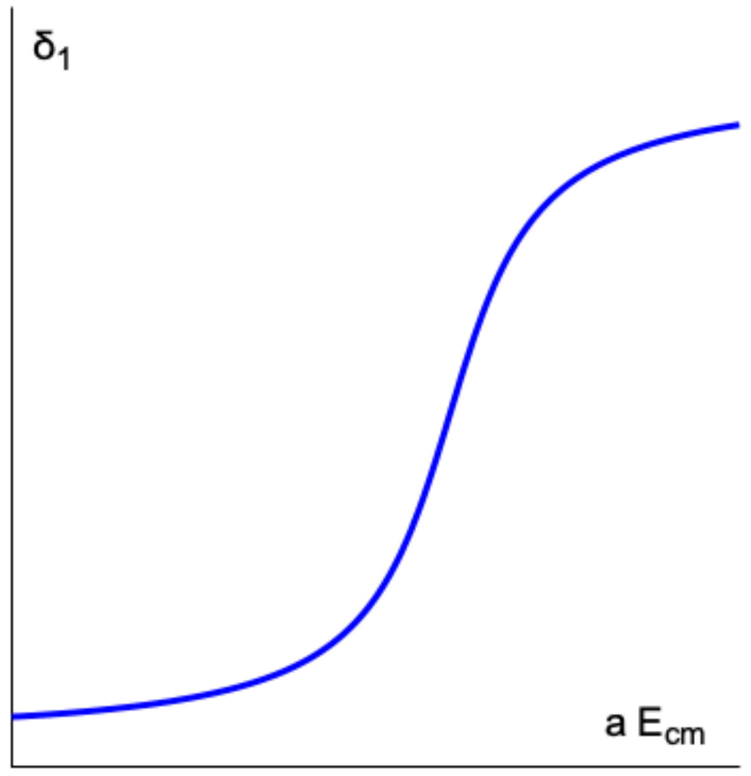


$w_{\text{ideal}}(\mathsf{f}, \delta^{\text{mod}}) \propto e^{-\text{AIC}_{\text{ideal}}(\mathsf{f}, \delta^{\text{mod}})/2}$



n^{lev} fits
$\lambda_i, \mathsf{f}_i \to E^i_{\text{cm}}$

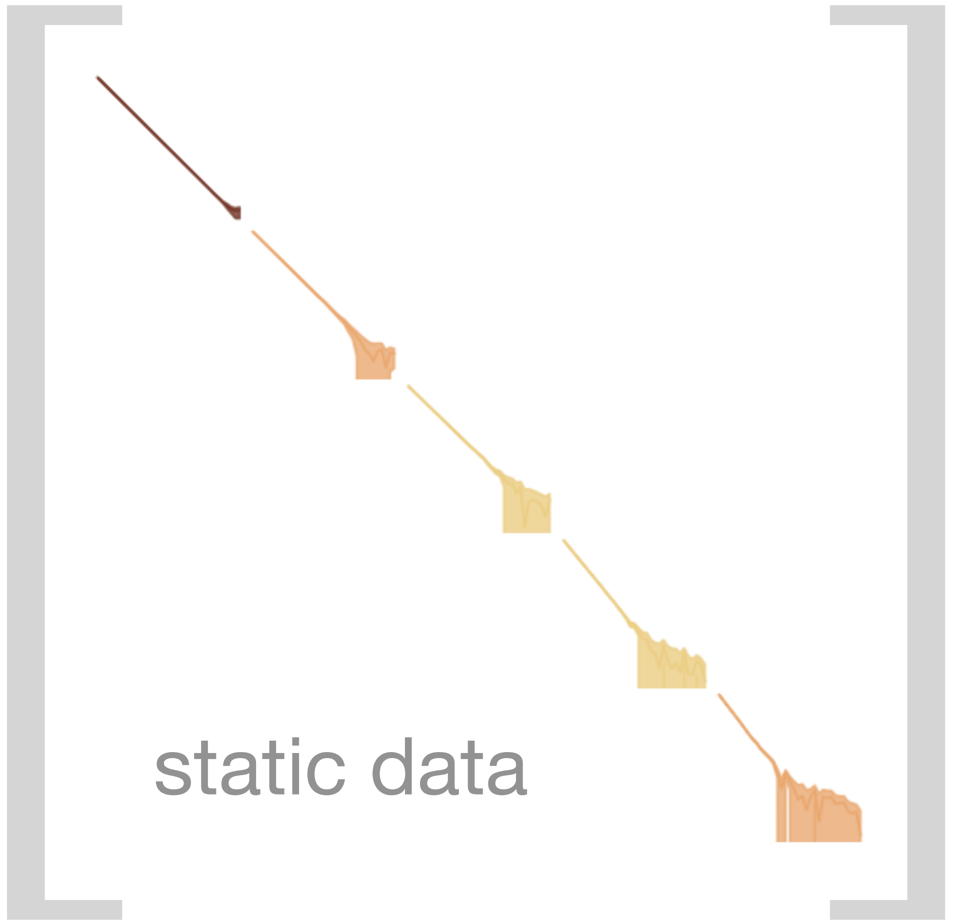
one fit
$\{E_{\text{cm}}\} \to \delta^{\text{mod}}$



back to blocked procedure

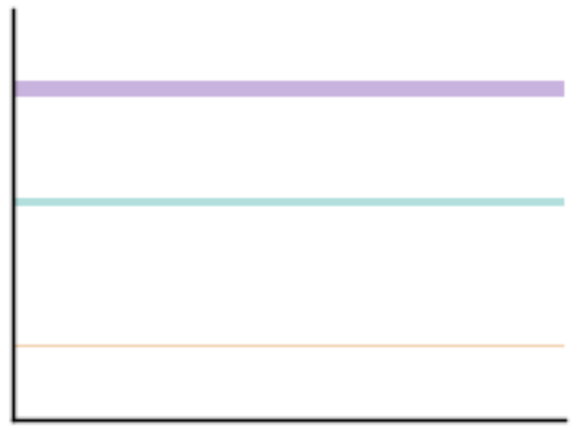
$$w_{\text{ideal}}(\mathsf{f}, \delta^{\text{mod}}) \approx$$

$$w_{\text{t}}(\mathsf{f}, \delta^{\text{mod}}) \propto \underbrace{e^{-\text{AIC}_{\text{PS}}(\mathsf{f}, \delta^{\text{mod}})/2}}_{w_{\text{PS}}} \underbrace{\prod_i e^{-\text{AIC}_{\text{corr}}(\mathsf{f}_{(i)})/2}}_{w_{\text{corr}}}$$



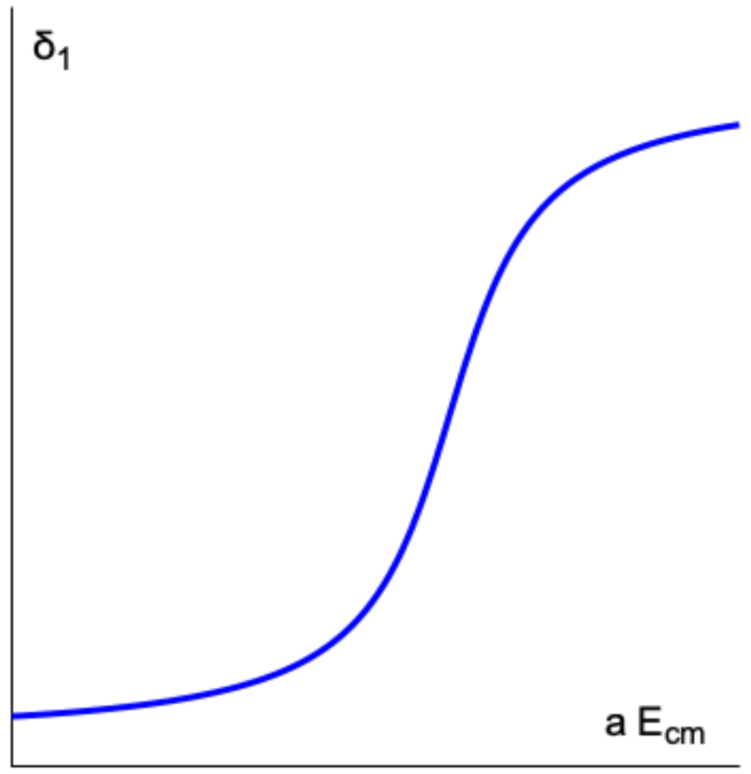
n^{lev} fits

$\lambda_i, \text{f}_i \rightarrow E^i_{\text{cm}}$



one fit

$\{E_{\text{cm}}\} \rightarrow \delta^{\text{mod}}$

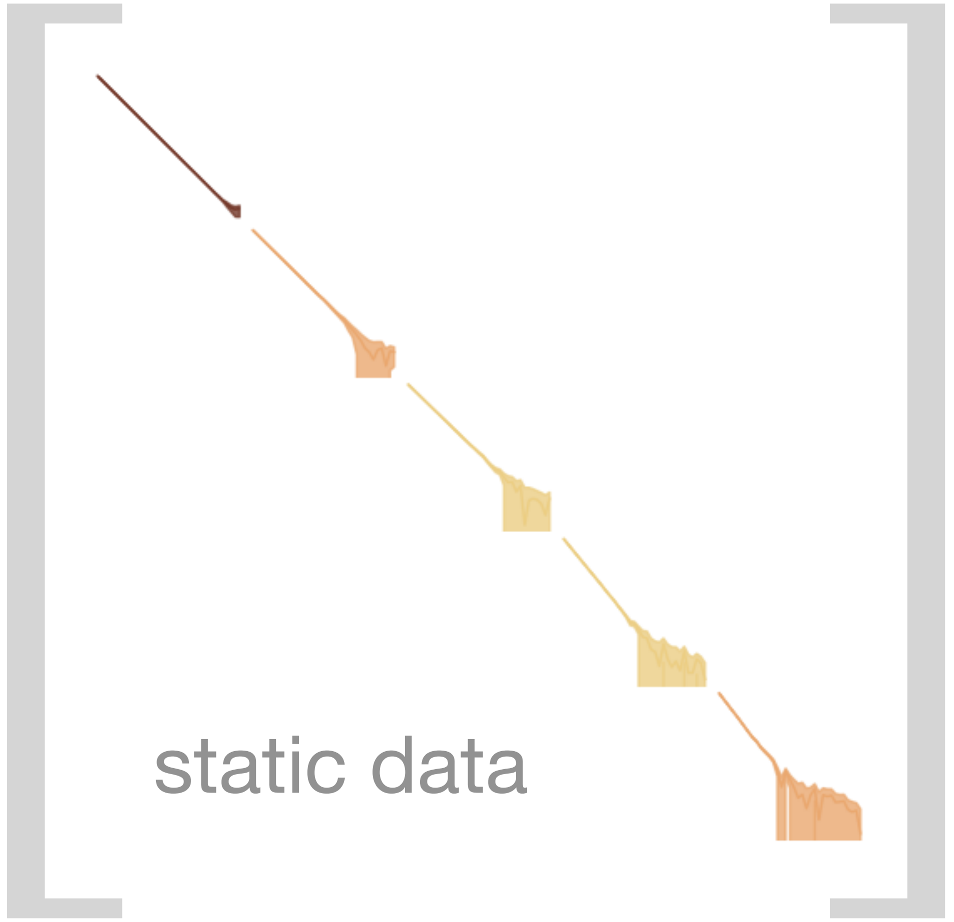


back to blocked procedure

$$w_{\text{ideal}}(\text{f}, \delta^{\text{mod}}) \approx$$

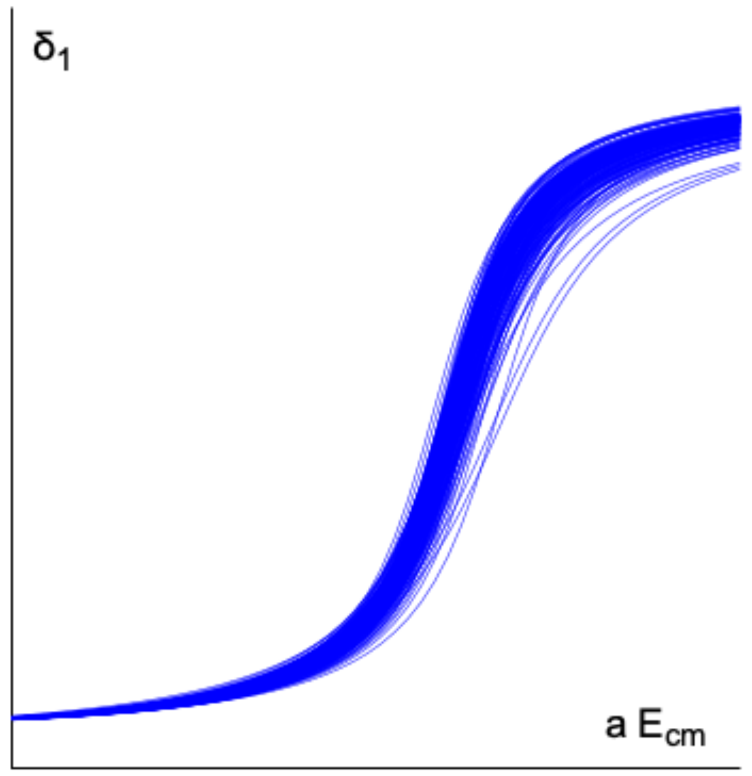
$$w_{\text{t}}(\text{f}, \delta^{\text{mod}}) \propto \underbrace{e^{-\text{AIC}_{\text{PS}}(\text{f},\delta^{\text{mod}})/2}}_{w_{\text{PS}}} \underbrace{\prod_i e^{-\text{AIC}_{\text{corr}}(\text{f}_{(i)})/2}}_{w_{\text{corr}}}$$

Still, too many fit range combinations

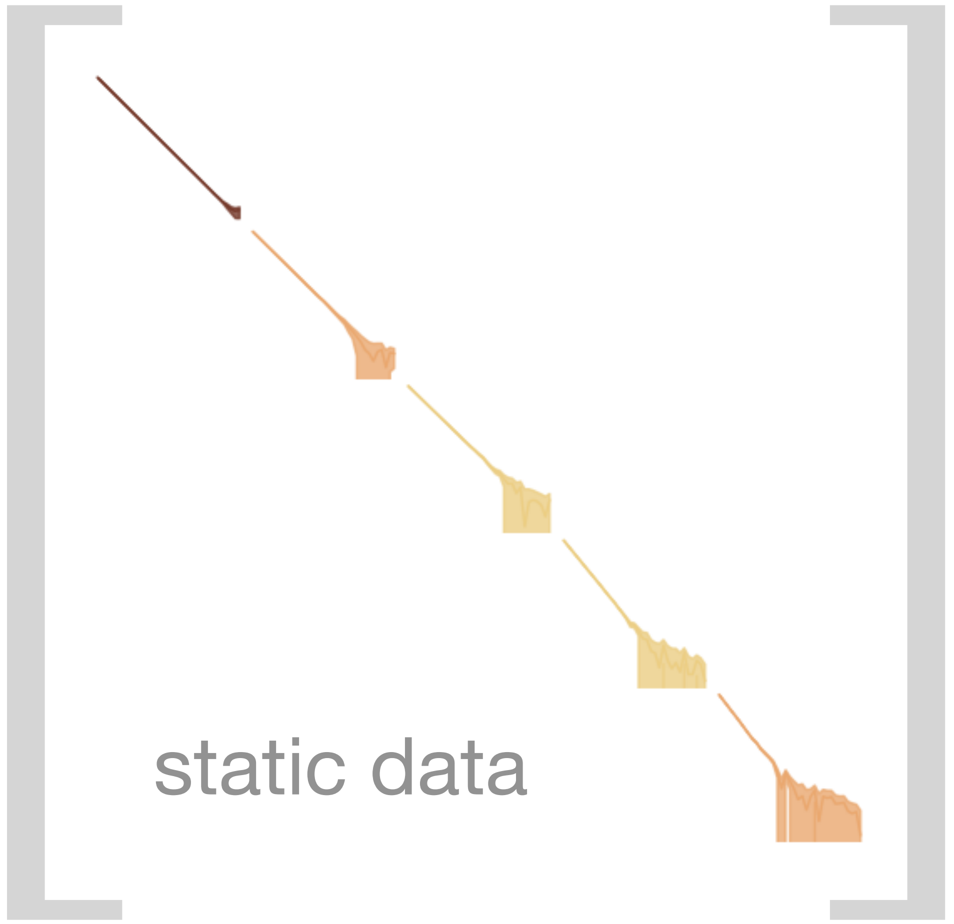


$$\text{dim}\ \{\text{f}\} \sim \mathcal{O}(10^2)^{n^{\text{lev}}} \sim \mathcal{O}(10^{30})$$

⋮

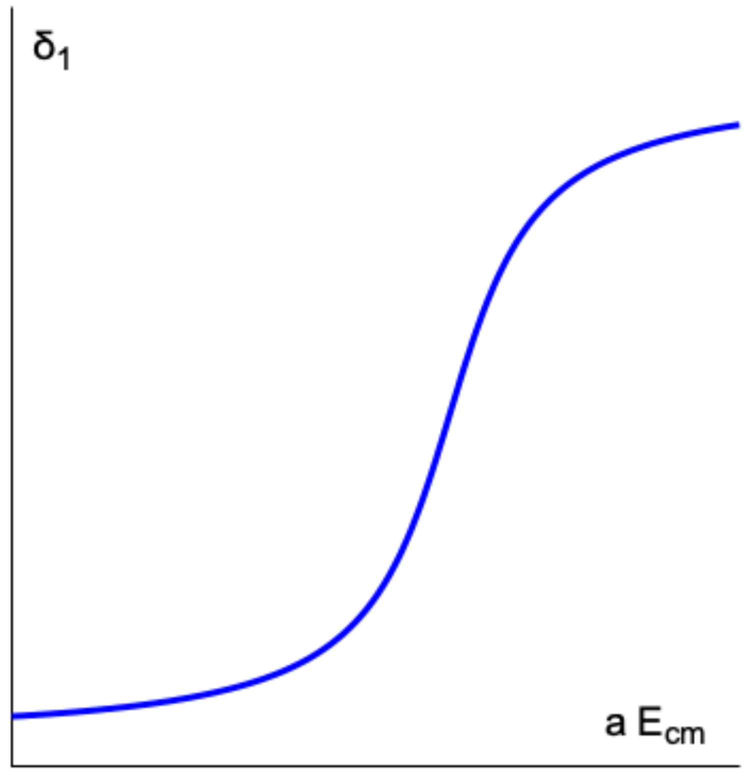


$$\big\{\{\lambda\}, \{\text{f}\}\big\} \rightarrow \{E_{\text{cm}}\} \rightarrow \delta^{\text{mod}}$$



n^{lev} fits
$\lambda_i, \mathsf{f}_i \to E^i_{\text{cm}}$

one fit
$\{E_{\text{cm}}\} \to \delta^{\text{mod}}$

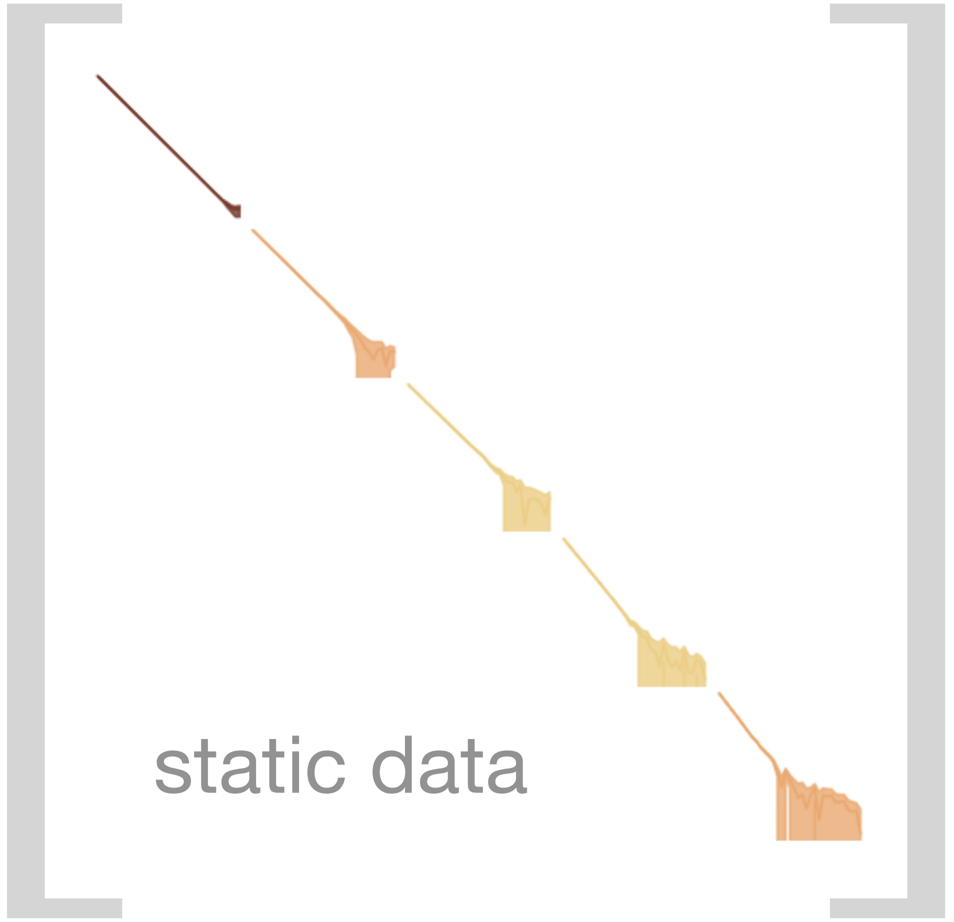


back to blocked procedure

$$w_{\text{ideal}}(\mathsf{f}, \delta^{\text{mod}}) \approx$$

$$w_{\text{t}}(\mathsf{f}, \delta^{\text{mod}}) \propto \underbrace{e^{-\text{AIC}_{\text{PS}}(\mathsf{f},\delta^{\text{mod}})/2}}_{w_{\text{PS}}} \underbrace{\prod_i e^{-\text{AIC}_{\text{corr}}(\mathsf{f}_{(i)})/2}}_{w_{\text{corr}}}$$

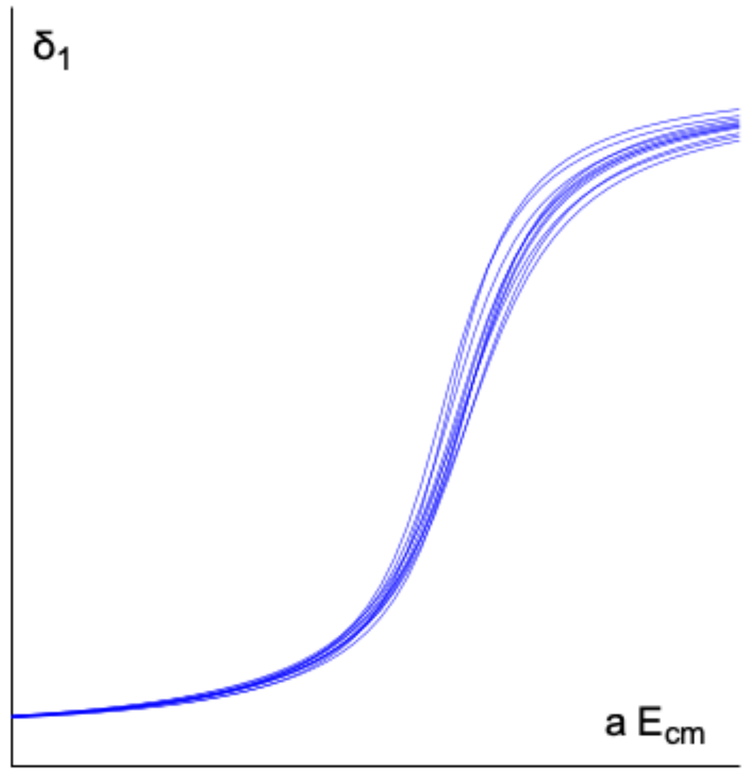
Still, too many fit range combinations



$$\dim \{\mathsf{f}\} \sim \mathcal{O}(10^2)^{n^{\text{lev}}} \sim \mathcal{O}(10^{30})$$

⋮

$$\big\{\{\lambda\}, \{\mathsf{f}\}\big\} \to \{E_{\text{cm}}\} \to \delta^{\text{mod}}$$

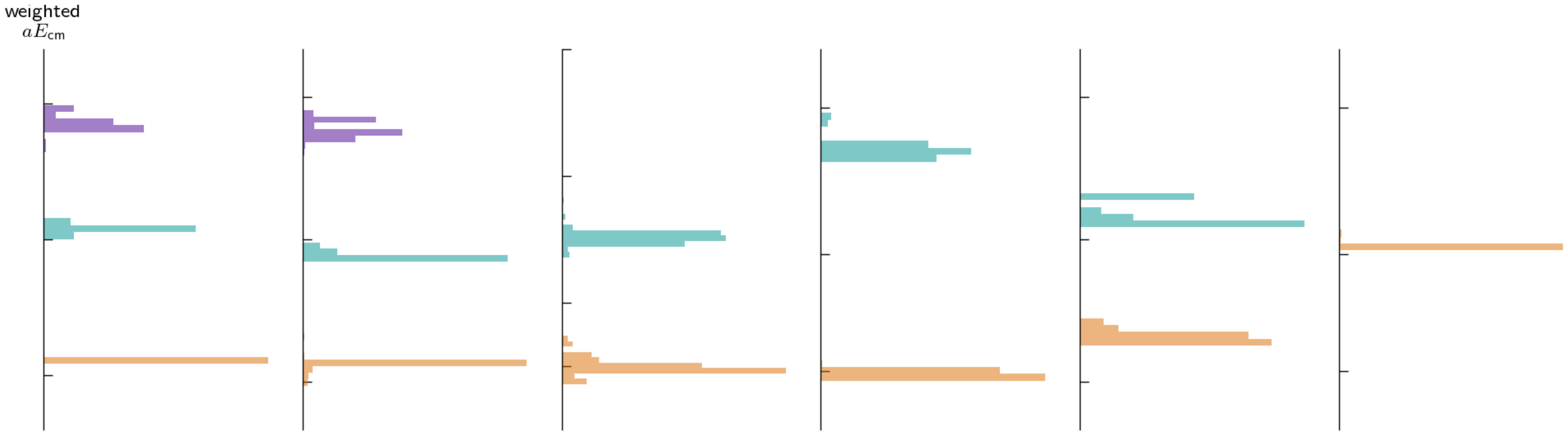


sample?

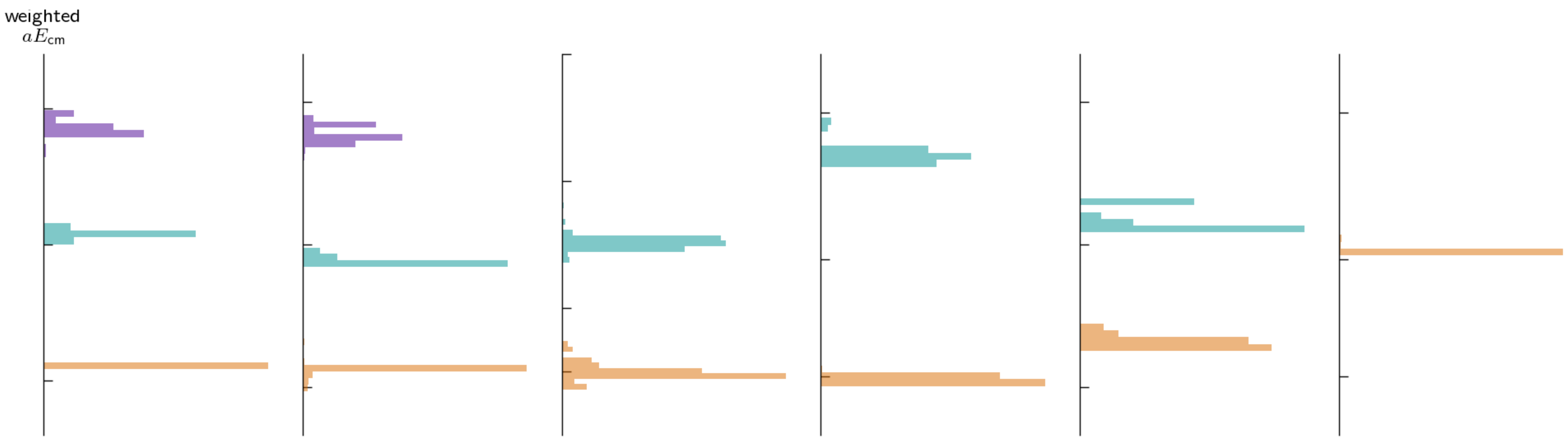
Importance Sample

Proposal

$$w_{\text{corr}}(\mathsf{f}) = \prod_i w_{\text{corr}}^{(i)}(\mathsf{f}^{(i)})$$

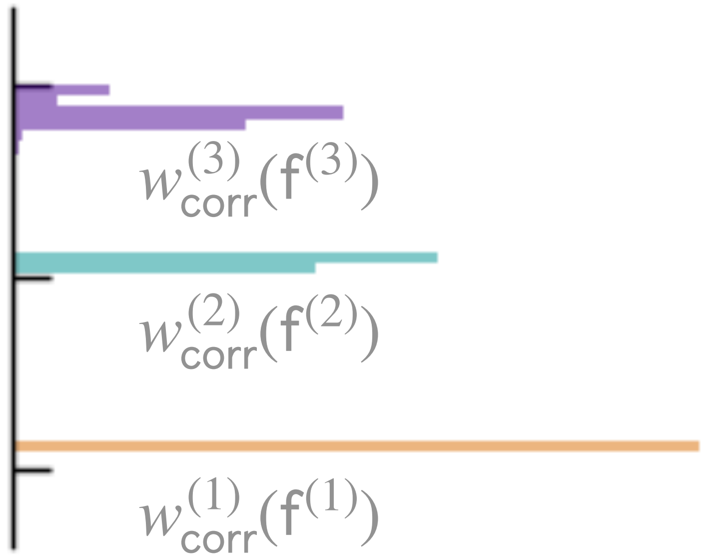


Importance Sample



Proposal

$$w_{\text{corr}}(\mathsf{f}) = \prod_i w^{(i)}_{\text{corr}}(\mathsf{f}^{(i)})$$

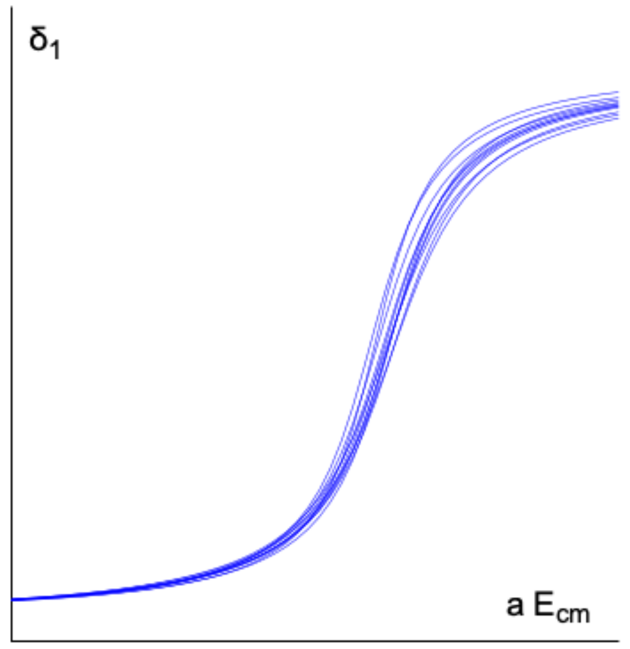


Sample & fit $\lambda^{(i)}$

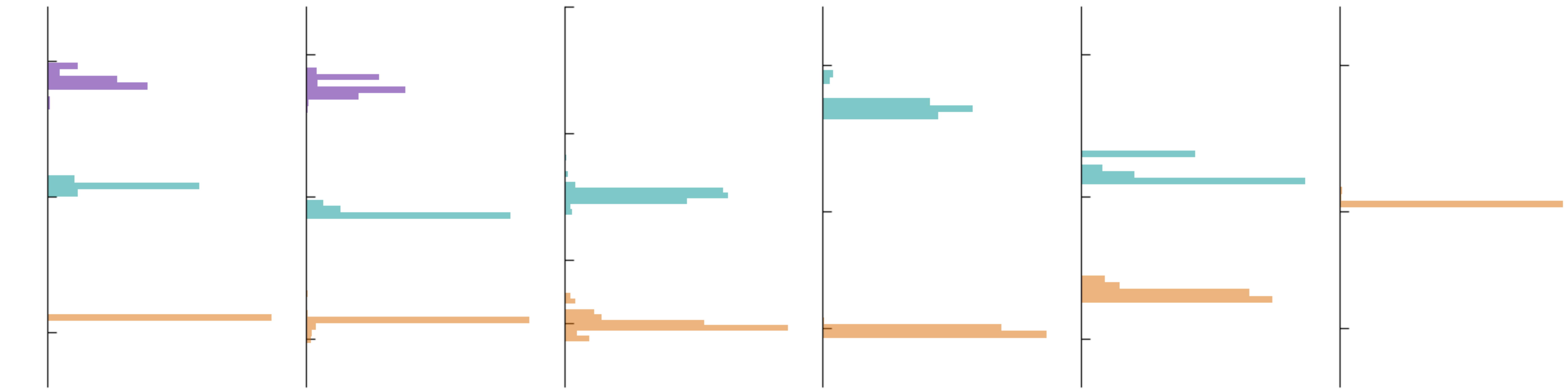
$\{\mathsf{f}_{(1)}, \mathsf{f}_{(2)}, \ldots, \mathsf{f}_{(n^{\text{lev}})}\} \equiv \mathsf{s}$

$$\{E_{\text{cm}}\}^{\mathsf{s}^1}, \{E_{\text{cm}}\}^{\mathsf{s}^2}, \ldots$$

QC fit

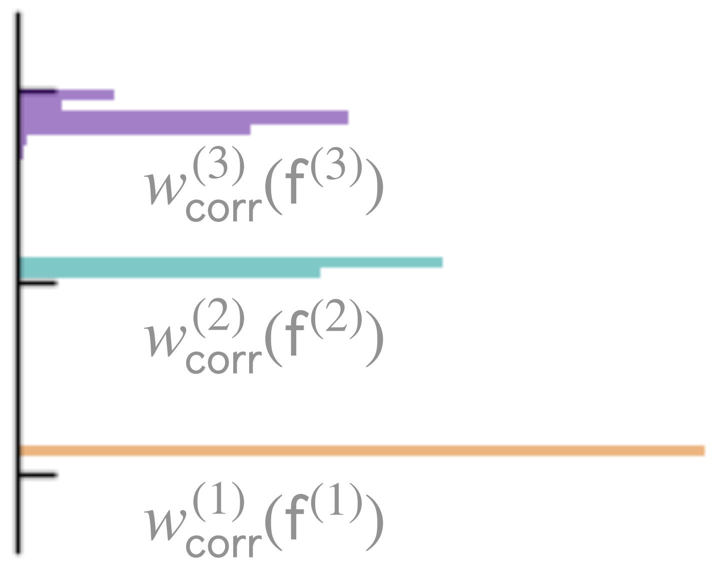


Importance Sample



Proposal

$$w_{\text{corr}}(\mathsf{f}) = \prod_i w^{(i)}_{\text{corr}}(\mathsf{f}^{(i)})$$

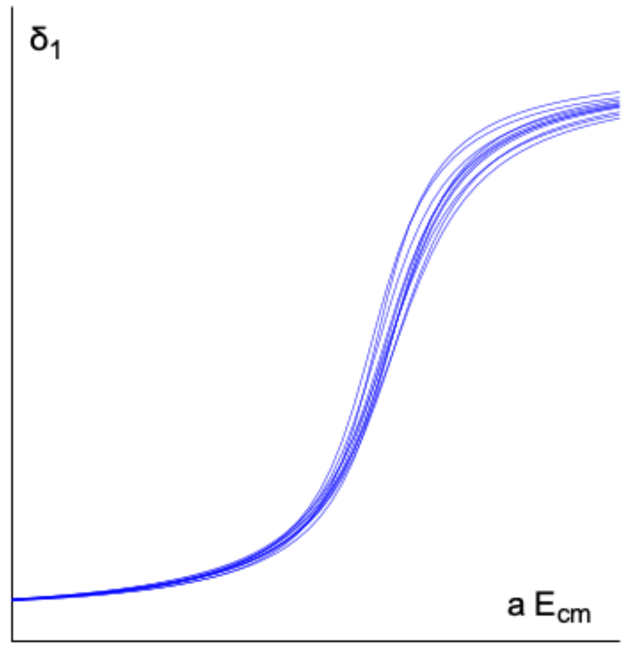


Sample & fit $\lambda^{(i)}$

$\{\mathsf{f}_{(1)}, \mathsf{f}_{(2)}, \ldots, \mathsf{f}_{(n^{\text{lev}})}\} \equiv \mathsf{s}$

$$\{E_{\text{cm}}\}^{\mathsf{s}^1}, \{E_{\text{cm}}\}^{\mathsf{s}^2}, \ldots$$

QC fit

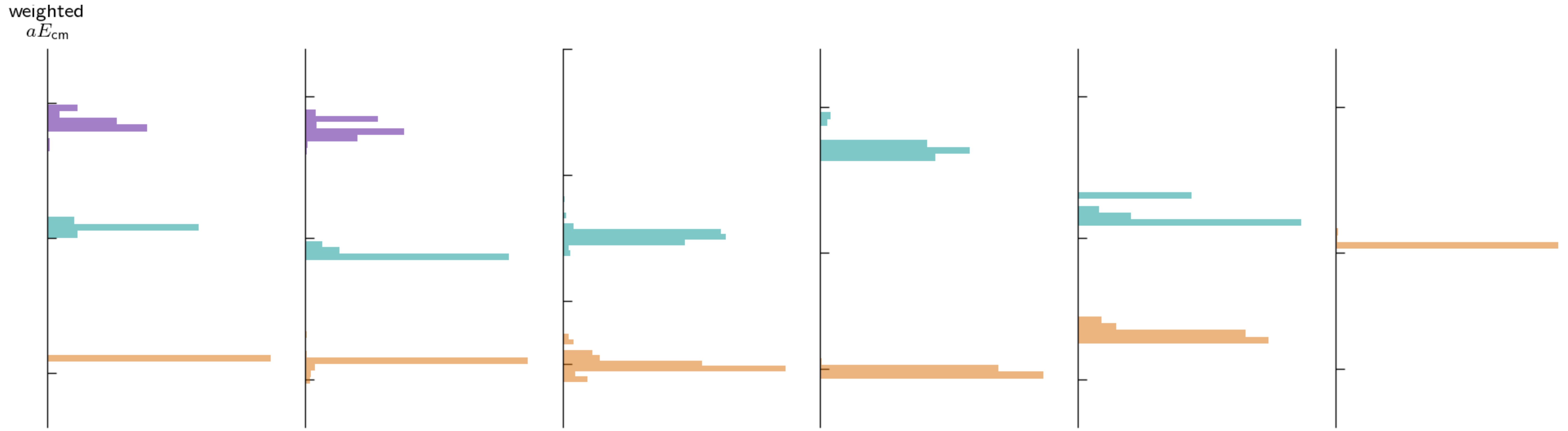


Target $w_t(\mathsf{f}, \delta^{\text{mod}}) = w_{\text{PS}}(\mathsf{f}, \delta^{\text{mod}}) w_{\text{corr}}(\mathsf{f})$ ➡ Reweight $w_{\text{PS}}(\mathsf{f}, \delta^{\text{mod}})$

Model-average estimate

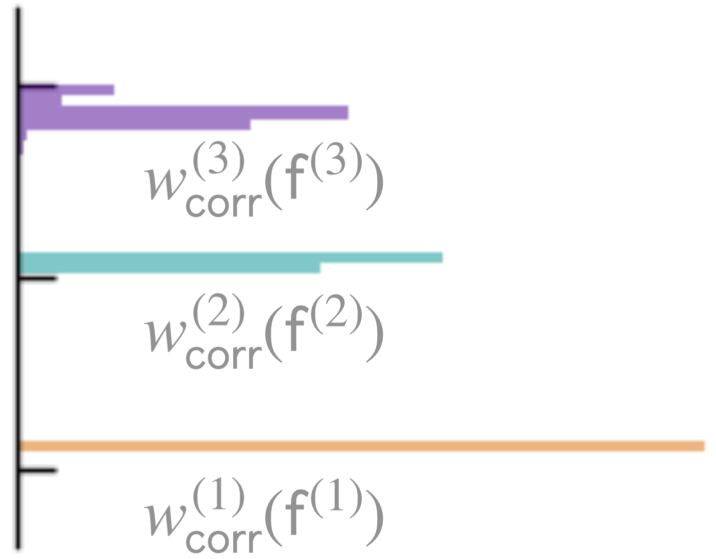
$$\hat{\alpha}^{\text{mod}} = \sum_k \alpha^{\text{mod},\mathsf{s}^k} w^{\text{mod}}_{\text{PS}}(\mathsf{s}^k)$$

Importance Sample



Proposal

$$w_{corr}(\mathrm{f}) = \prod_i w_{corr}^{(i)}(\mathrm{f}^{(i)})$$

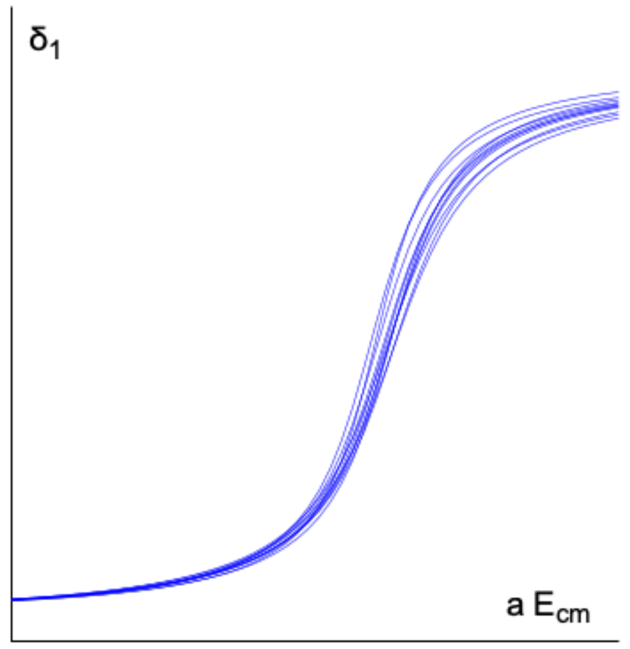


Sample & fit $\lambda^{(i)}$

$\{\mathrm{f}_{(1)}, \mathrm{f}_{(2)}, \ldots, \mathrm{f}_{(n^{lev})}\} \equiv \mathrm{s}$

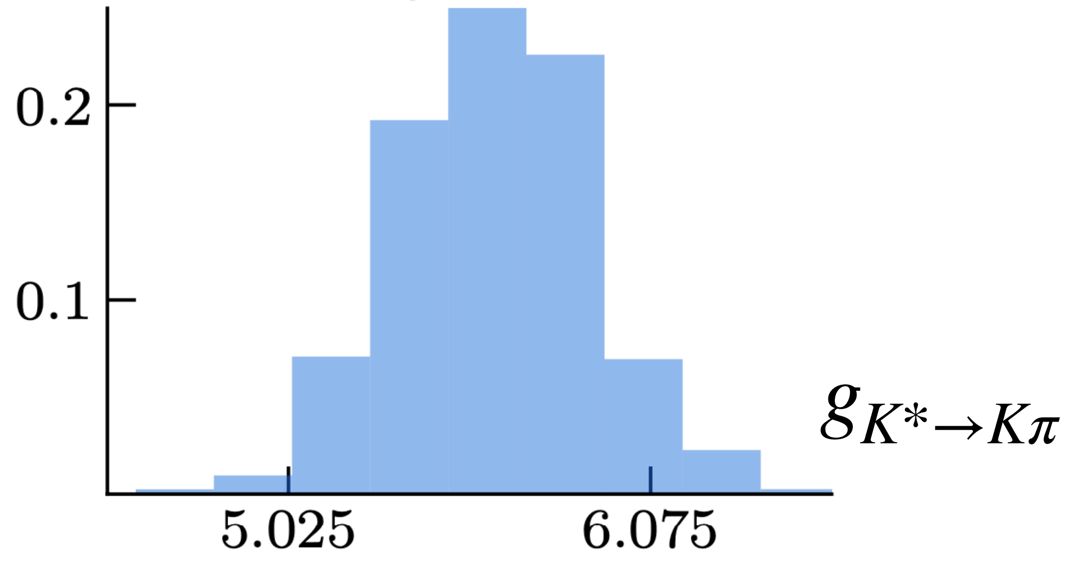
$$\{E_{cm}\}^{\mathrm{s}^1}, \{E_{cm}\}^{\mathrm{s}^2}, \ldots$$

QC fit



Target $w_t(\mathrm{f}, \delta^{mod}) = w_{PS}(\mathrm{f}, \delta^{mod}) w_{corr}(\mathrm{f})$ ➡ **Reweight** $w_{PS}(\mathrm{f}, \delta^{mod})$

central value histogram



Model-average estimate

$$\hat{\alpha}^{mod} = \sum_k \alpha^{mod,\mathrm{s}^k} w_{PS}^{mod}(\mathrm{s}^k)$$

e.g. Breit-Wigner

$$\cot \delta_1^{BW}(\sqrt{s}) = \frac{6\pi(m^2 - s)\sqrt{s}}{p_{cm}^3 g^2}$$

$$\alpha^{BW} = [g, m]$$