The distribution amplitude of the η_c meson

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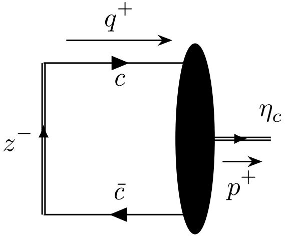
Introduction

The factorization theorem separates cross sections σ into hard H and soft S factors,

$$\sigma = H \otimes S$$

Distribution amplitudes (DAs) appear as soft factors in decays, annihilations and DVMP



On the light-cone metric

$$z^\alpha = (z^+, z^-, z^\perp) \quad z^2 = 0$$

$$\nu \equiv p \cdot z = p^+ z^- \quad x = q^+/p^+$$

We study the DA of the η_c-meson

$$2\pi\phi(x) = \int \mathrm{d}z^- \, e^{\mathrm{i}(x-1/2)\nu} M^+(\nu)$$

where the Ioffe-time DA is

$$M^+(\nu) = \left. \langle \eta_c(p) \big| \bar{c}(-z/2)\gamma^+\gamma_5 W(-z/2, z/2) c(z/2) \big| 0 \rangle \right|_{z^+, z^\perp = 0}$$

Move to Euclidean space

Problem

We can only compute $z_\alpha = (z_\perp, z_3, z_4) = (0, 0, 0)$

Solution [3, 7, 10]

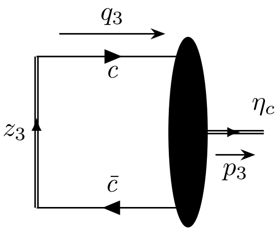
- Generalize $M_\alpha(p, z)$ for $z^2 > 0$ and take $z^2 \to 0$

$$M_\alpha(p, z) = e^{-i\nu/2} \left\langle \eta_c(p) \middle| \bar{c}(0)\gamma_\alpha\gamma_5 W(0, z) c(z) \middle| 0 \right\rangle \Big|_{z_\perp = 0, z_4 = 0}$$

- Remove some higher twist effects

$$M_\alpha(p, z) = 2p_\alpha \mathcal{M} + z_\alpha \mathcal{M}'$$

- Set $p_\alpha = (0_\perp, p_3, E)$, $z_\alpha = (0_\perp, z_3, 0)$
- Choose $\alpha = 4$ to isolate $\mathcal{M}(\nu, z^2)$



Short distance factorization

Form the renormalized quantity [1, 6, 8]

$$\frac{\mathcal{M}(p,z)\mathcal{M}(0,0)}{\mathcal{M}(0,z)\mathcal{M}(p,0)} = \tilde{\phi}(\nu, z^2) + z^2 \times \text{higher twist}$$

Match to the $\overline{\text{MS}}$ light-cone quantity at $\mu = 3\,\text{GeV}$ [10]

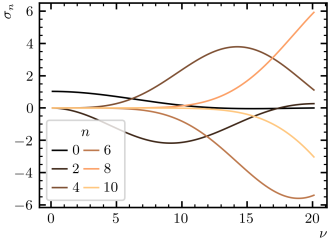
$$\tilde{\phi}(\nu, z^2) = \int_0^1 \mathrm{d}w\, C(w, \nu, z\mu) \int_0^1 \mathrm{d}x \cos\left[w\nu\,(x - 1/2)\right] \phi(x, \mu)$$

Expand the DA in a series of Gegenbauer polynomials [11]

$$\phi(x, \mu) = (1-x)^{\lambda - 1/2} x^{\lambda - 1/2} \sum_{n=0}^{\infty} d_{2n}^{(\lambda)} \tilde{G}_{2n}^{(\lambda)}(x), \quad \text{note} \quad \lambda(\mu)$$

so that

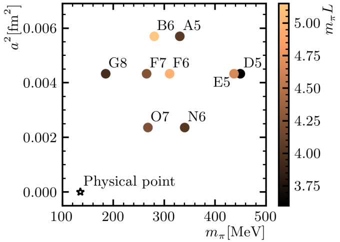
$$\tilde{\phi}(\nu, z^2) = \sum_{n=0}^{\infty} d_{2n}^{(\lambda)} \sigma_{2n}^{(\lambda)}(\nu, z^2)$$



The CLS lattice ensembles

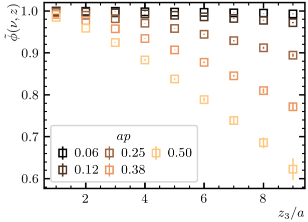
$N_f = 2$ Coordinated Lattice Simulations [2, 4]

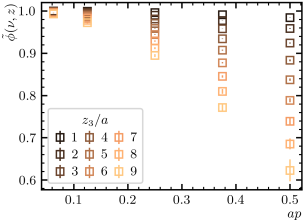
- Wilson gauge action
- $\mathcal{O}(a)$-improved Wilson quarks
- $\kappa_u = \kappa_d := \kappa_\ell$
- No electromagnetism
- No Symanzik program for $M_\alpha(p, z) \to \mathcal{O}(a)$ lattice artifacts
- Between 1000 and 2000 measurements per ensemble



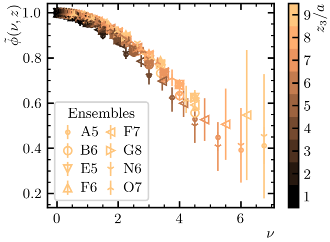
- Interpolate to $\langle \eta_c |$ solving an 4×4 GEVP with 4 smearings levels
- Comparing D5 and E5 yields negligible finite-volume effects

Ensemble G8





Entire dataset



- Entire dataset has $z_3 < 0.68\,\text{fm}$
- The analysis with $z_3 < 0.5\,\text{fm}$ yields fully compatible results

Continuum extrapolation

Make all terms dimensionless with $\Lambda_{\text{QCD}}^{(2)}$ [12]

$$\tilde{\phi}_e(\nu, z^2) = \tilde{\phi}(\nu, z^2) + z^2 C_1(\nu) + aB_1(\nu) + \frac{a}{|z|} A_1(\nu) + \frac{a}{|z|}\Big((m_{\eta_c} - m_{\eta_c,\text{phy}})D_1(\nu) + (m_\pi^2 - m_{\pi,\text{phy}}^2)E_1(\nu)\Big)$$

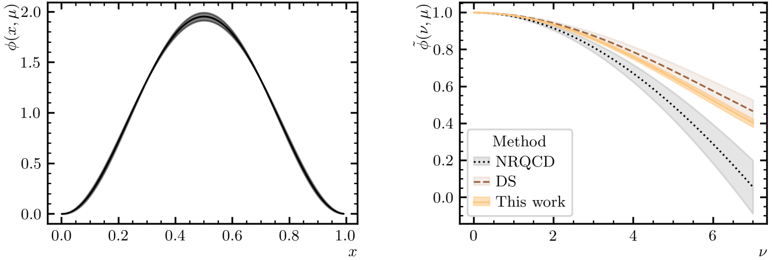
The main ingredients are the

- continuum $\tilde{\phi}(\nu, z^2)$

$$\tilde{\phi}(\nu, z^2) = \frac{4^\lambda \sigma_0^{(\lambda)}(\nu, z^2)}{B\left(\frac{1}{2}, \frac{1}{2} + \lambda\right)}$$

- higher-twist continuum C_1
- z-dependent A_1, and global B_1 lattice artifacts
- mass-dependent corrections D_1 and E_1

We obtain $\lambda = 2.73 \pm 0.12 \pm 0.12 \pm 0.06$



We compare to alternative theory predictions [5, 9]

	This work	Dyson-Schwinger	NRQCD
$\langle \xi^2 \rangle$	0.134(6)	0.118(18)	0.171(23)
$\langle \xi^4 \rangle$	0.043(4)	0.036(9)	0.018 808(19)

where $\xi \equiv -1 + 2x$. There is no DA fitted from experiment

Summary and outlook

We compute the η_c DA with $N_f = 2$ CLS ensembles and obtain

$$\phi(x, \mu) = \frac{4^{\lambda}(1-x)^{\lambda-1/2}x^{\lambda-1/2}}{B\left(1/2, 1/2+\lambda\right)}$$

defined at $\mu = 3\,\mathrm{GeV}$ and

$$\lambda = 2.73 \pm 0.12 \pm 0.12 \pm 0.06$$

In this analysis, we have seen that

- The comparison with Dyson-Schwinger is good
- Analysis choices yield sizable systematic uncertainties
- Finite-size effects are negligible

In the future, we will tackle

- The vector state J/ψ
- Missing sea-quarks with $N_f = 2+1+1$ ensembles

Complete set of CLS ensembles

id	β	a [fm]	L/a	m_π [MeV]	κ_ℓ	κ_c
A5	5.2	0.0755(9)(7)	32	331	0.13594	0.12531
B6			48	281	0.13597	0.12529
D5	5.3	0.0658(7)(7)	24	450	0.13625	0.12724
E5			32	437	0.13625	0.12724
F6			48	311	0.13635	0.12713
F7			48	265	0.13638	0.12713
G8			64	185	0.136417	0.12710
N6	5.5	0.0486(4)(5)	48	340	0.13667	0.13026
O7			64	268	0.13671	0.13022

The matching kernel

Objective: Compute the matching integrals

$$\tilde{\phi}(\nu, z) = \int_0^1 \mathrm{d}w \, C(w, \nu, z\mu) \int_0^1 \mathrm{d}x \cos\left[w\nu \left(x - 1/2\right)\right] \phi(x, \mu)$$

Definitions: The DA matching kernel is [10]

$$C(w, \nu, z\mu) = \delta(w - 1) - \frac{\alpha_s C_F}{2\pi} \left[\log\left(\frac{\mu^2}{\mu_0^2}\right) B(w, \nu) + L(w, \nu)\right]$$

where the scale μ_0 contains the z^2 dependence

$$\frac{1}{\mu_0^2} \equiv \frac{z^2 \; e^{2\gamma_E + 1}}{4}$$

we take $\mu = 3\,\text{GeV}$

The matching kernel

The contribution $B(w,\nu)$ is [10]

$$B(w,\nu) = \left[\frac{2w}{1-w}\right]_+ \cos\left(\frac{(1-w)\nu}{2}\right) + \frac{2}{\nu}\sin\left(\frac{(1-w)\nu}{2}\right) - \frac{1}{2}\delta(w-1)$$

And the contribution $L(w,\nu)$ is [10]

$$\begin{aligned} L(w,\nu) =& 4\left[\frac{\log(1-w)}{1-w}\right]_+ \cos\left(\frac{(1-w)\nu}{2}\right) \\ & - 2\left(\frac{2}{\nu}\sin\left(\frac{(1-w)\nu}{2}\right) - \frac{1}{2}\delta(w-1)\right) \end{aligned}$$

Given two functions $f(x)$ and $g(x)$ defined in a certain domain, **the plus prescription** is

$$\left[\frac{f(x)}{1-x}\right]_+ g(x) = \frac{f(x)}{1-x}(g(x) - g(1))$$

The matching kernel

Method: Rewrite the relation between $\tilde{\phi}(\nu, z)$ and $\phi(x, \mu)$

$$\tilde{\phi}(\nu, z) = \int_0^1 \mathrm{d}x\, K(x, \nu, \mu z)\phi(x, \mu)$$

write the kernel as a series of Gegenbauer polynomials

$$K(x, \nu, \mu z) = \sum_{n=0}^{\infty} \frac{\sigma_{2n}^{(\lambda)}(\nu, z\mu)}{A_{2n}^{(\lambda)}} \tilde{G}_{2n}^{(\lambda)}(x)$$

and every coefficient in the series is given by

$$\sigma_n^{(\lambda)}(\nu, z\mu) = \sum_{k=0}^{\infty} \left(-\frac{\nu^2}{4}\right)^k \frac{c_{2k}(\nu, z\mu)}{\Gamma(2k+1)} I(n, k, \lambda)$$

See [11] for a similar analysis of PDFs

The matching kernel

The λ-dependent function is the Mellin transform of the Gegenbauer polynomials

$$
\begin{aligned}
I(n,k,\lambda) &\equiv \int_{-1}^{+1} \mathrm{d}g\, g^{2k}(1-g^2)^{\lambda-1/2} G_n^{(\lambda)}(g) \\
&= \frac{2\pi}{4^{\lambda+k} n!} \frac{\Gamma(1+2k)\Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma\left(\lambda+\frac{n+2k+2}{2}\right)\Gamma\left(1+k-\frac{n}{2}\right)}
\end{aligned}
$$

The n-th moment of the kernel is given by

$$
\begin{aligned}
c_n(\nu, z\mu) &= \int_0^1 \mathrm{d}w\, C(w,\nu,z\mu) w^n \\
&= 1 - \frac{\alpha_s C_F}{2\pi}\left[\log\left(\frac{\mu^2}{\mu_0^2}\right) b_n(\nu) + l_n(\nu)\right]
\end{aligned}
$$

The matching kernel

$$I(0,k,\lambda) = B(\lambda + \frac{1}{2}, k + \frac{1}{2})$$
$$I(2,k,\lambda) = 2\lambda k B(\lambda + \frac{3}{2}, k + \frac{1}{2})$$
$$I(4,k,\lambda) = \frac{2}{3}(\lambda+1)\lambda k(k-1) B(\lambda + \frac{5}{2}, k + \frac{1}{2})$$
$$I(6,k,\lambda) = \frac{4}{45}(2+\lambda)(1+\lambda)\lambda k(k-1)(k-2) B(\lambda + \frac{7}{2}, k + \frac{1}{2})$$
$$I(8,k,\lambda) = \frac{2}{315}(3+\lambda)(2+\lambda)(1+\lambda)\lambda k(k-1)(k-2)(k-3) \\ B(\lambda + \frac{9}{2}, k + \frac{1}{2})$$

The matching kernel

The moments of $B(w)$ are given by

$$b_n(\nu) = -\sum_{j=0}^{n-1} \frac{2}{j+2} {}_1F_2\left(1, \frac{j+3}{2}, \frac{j+4}{2}, -\frac{\nu^2}{16}\right)$$
$$- \frac{\nu^2}{24} {}_2F_3\left(1, 1, 2, 2, 5/2, -\frac{\nu^2}{16}\right)$$
$$- \frac{1}{2} + \frac{1}{(n+2)(n+1)} {}_1F_2\left(1, \frac{n+3}{2}, \frac{n+4}{2}, -\frac{\nu^2}{16}\right)$$

Note all hypergeometric functions ${}_pF_q$ have $p \leq q \leftrightarrow$ Converge for all ν values [13]

The matching kernel

The moments of $L(w)$ are given by

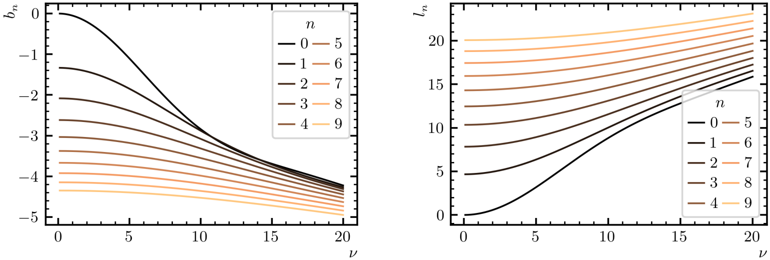
$$I_n(\nu) = 4\sum_{j=0}^{n-1}\binom{n}{j+1}\frac{(-1)^j}{(j+1)^2}\,{}_2F_3\left(\frac{j+1}{2},\frac{j+1}{2},\frac{1}{2},\frac{j+3}{2},\frac{j+3}{2},-\frac{\nu^2}{16}\right)$$

$$+\frac{\nu^2}{8}\,{}_3F_4\left(1,1,1,\frac{3}{2},2,2,2,-\frac{\nu^2}{16}\right)$$

$$+1-\frac{2}{(n+2)(n+1)}\,{}_1F_2\left(1,\frac{n+3}{2},\frac{n+4}{2},-\frac{\nu^2}{16}\right)$$

Note all hypergeometric functions ${}_pF_q$ have $p \leq q \leftrightarrow$ Converge for all ν values [13]

The matching kernel



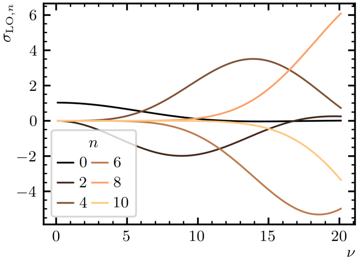
Nuisance functions

The nuisance functions are parametrized just like $\phi(x, \mu)$

$$A_r^{(\lambda)}(x) = (1-x)^{\lambda-1/2} x^{\lambda-1/2} \sum_{s=0}^{S_{a,r}} a_{r,2s}^{(\lambda)} \tilde{G}_{2s}^{(\lambda)}(x)$$

Fourier transform to ν space,

$$A_r^{(\lambda)}(\nu) = \int_0^1 \mathrm{d}x\, A_r^{(\lambda)}(x) \cos(x\nu - \nu/2) = \sum_{s=0}^{S_{A,r}} a_{r,2s}^{(\lambda)} \sigma_{\mathrm{LO},2s}^{(\lambda)}(\nu)$$



Nuisance effects vanish at $\nu = 0$

$$a_{r,0}^{(\lambda)} = 0 \longleftrightarrow \tilde{\phi}(\nu = 0, z) = 1$$

Enough to consider $S_{A_r} = 1$

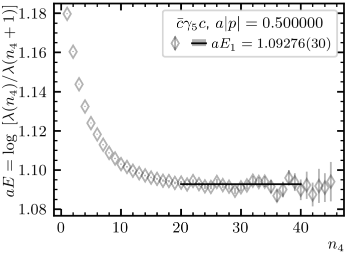
$a_{1,2}$, $b_{1,2}$, $c_{1,2}$, $d_{1,2}$, $e_{1,2}$

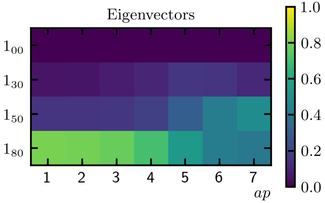
Take the **pseudoscalar interpolator**

$$O = \bar{c}\gamma_5 c \to O_{\text{smeared}} = \bar{c}\gamma_5 (1+c\nabla^2)^i c$$

Use **smearings** $i, j = (0,\ 30,\ 50,\ 80)$

$$C_{ij}(n_4) = \sum_{k=1}^{N} e^{-n_4 E_k} \langle 0|O_i|k\rangle \langle k|O_j^\dagger|0\rangle$$





Solve a $N \times N$ generalized eigenvalue problem

$$C(n_4')u_k = e^{-E_k(n_4'-n_4)} C(n_4) u_k$$

The **optimal interpolator** is

$$\langle \eta_c | = \sum_i \langle O_i | \, (u_1)_i$$

The lattice data

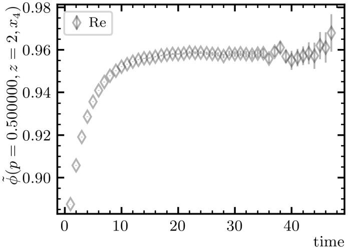
Summarizing, on the lattice we compute

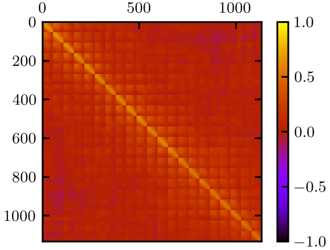
$$M_4(\nu, z^2; e) = e^{-\mathrm{i}\nu/2} \left\langle \eta_c(p) \middle| \bar{c}(0)\gamma_4\gamma_5 W(0,z) c(z) \middle| 0 \right\rangle \Big|_{z_\perp, z_4 = 0}$$

Lattice renormalization [1, 6, 8]

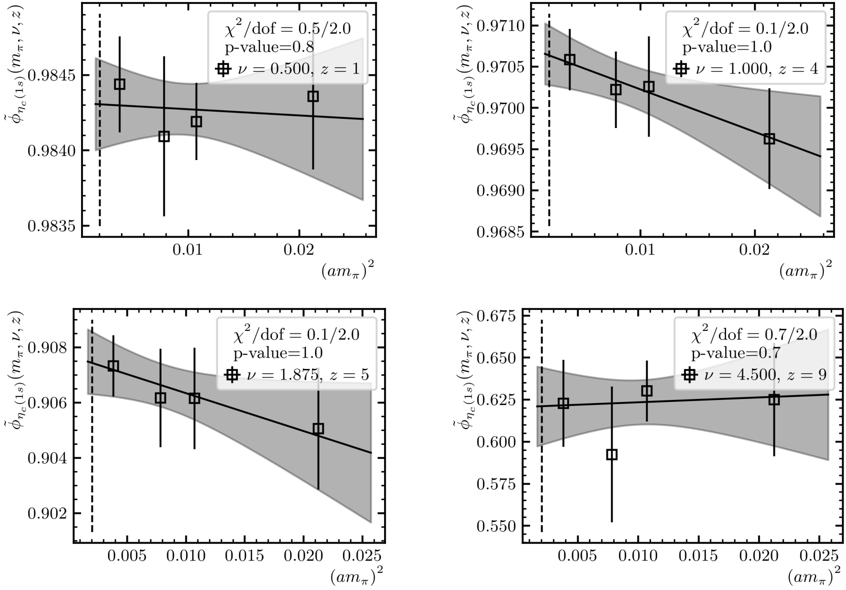
$$\frac{M_4(p,z;e) M_4(0,0;e)}{M_4(0,z;e) M_4(p,0;e)} = \tilde{\phi}(\nu, z^2; e) + z^2 \times \text{higher twist}$$

Remove the regulator and match to the $\overline{\text{MS}}$ scheme





Pion mass dependence



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