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Thermodynamically optimal processes with memory

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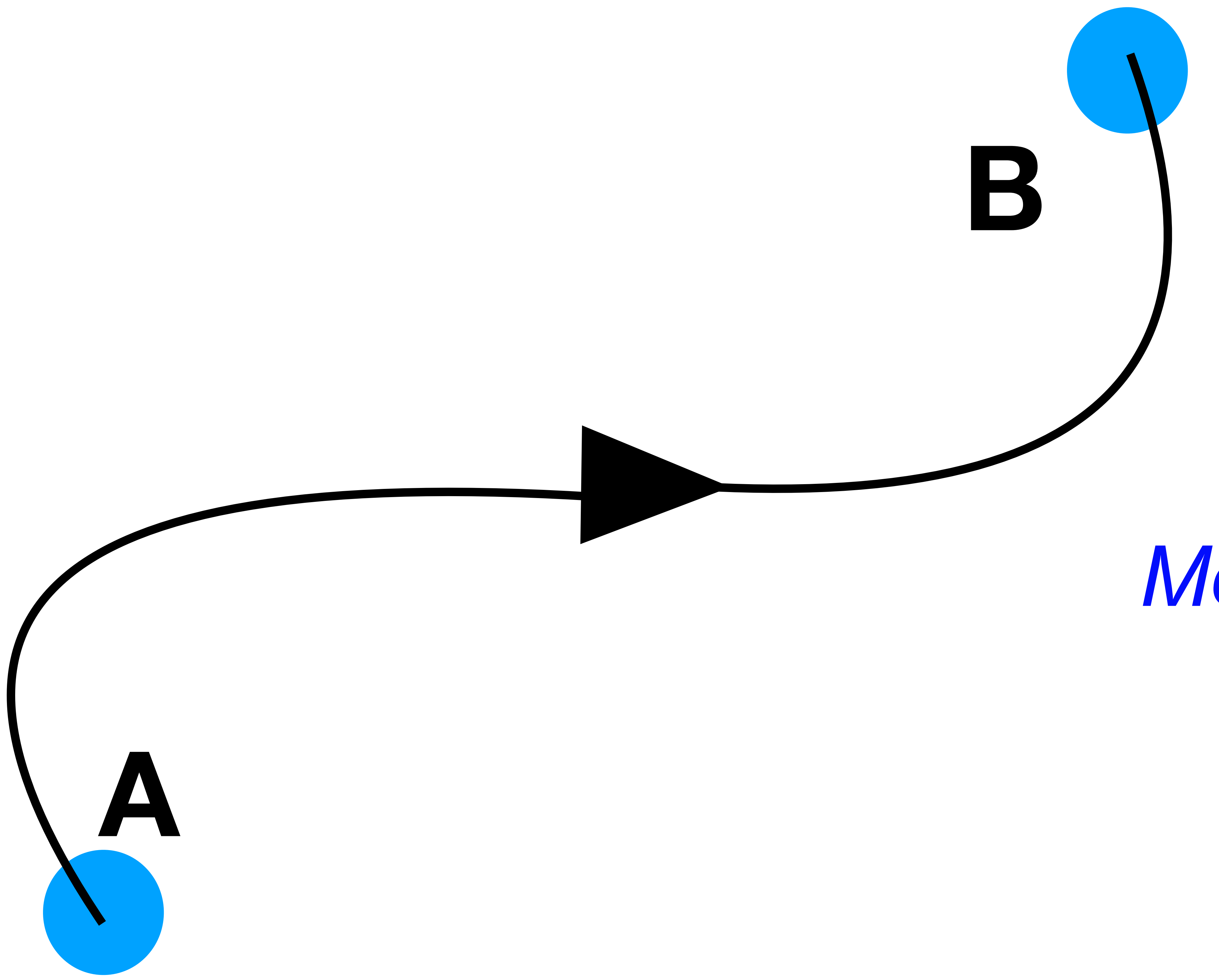
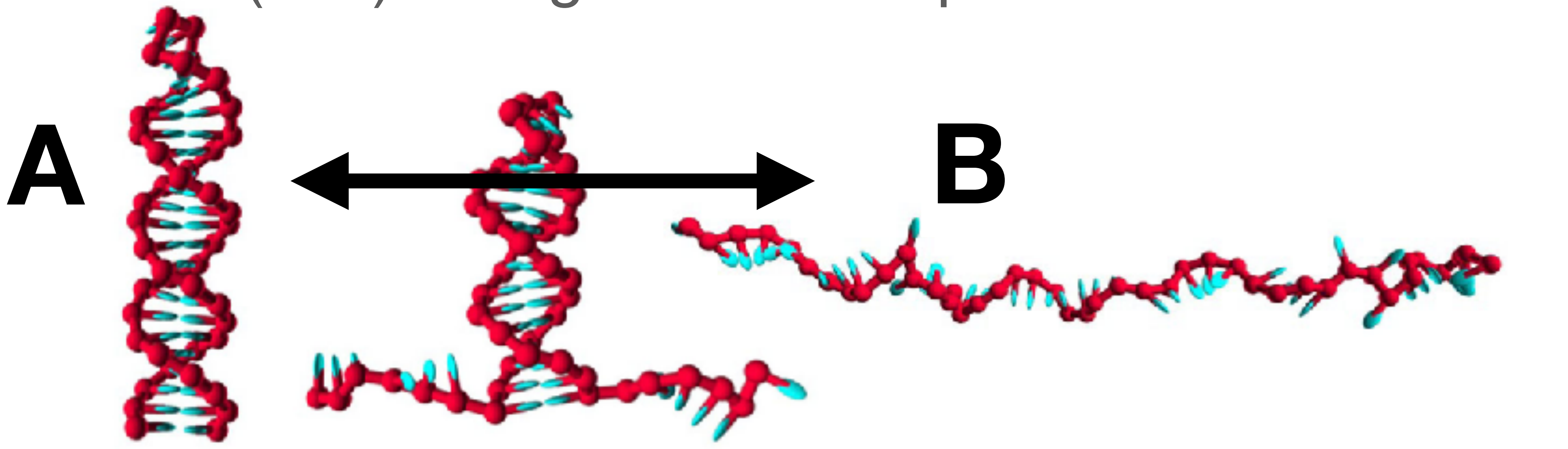
Non-equilibrium thermodynamics:
from chemical reactions to machine learning
Higgs Centre, University of Edinburgh

MSCA Postdoctoral Fellowship (UKRI)



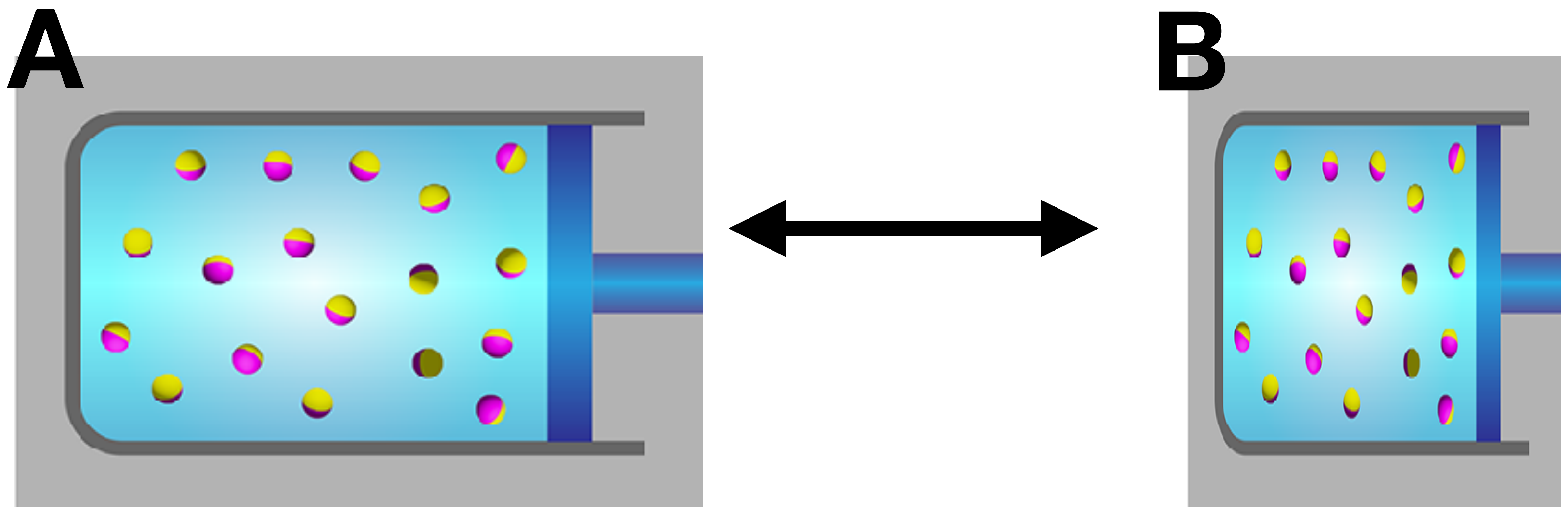
Most thermodynamically efficient way to bring system from A to B in finite time?

(Un-)folding of DNA hairpin



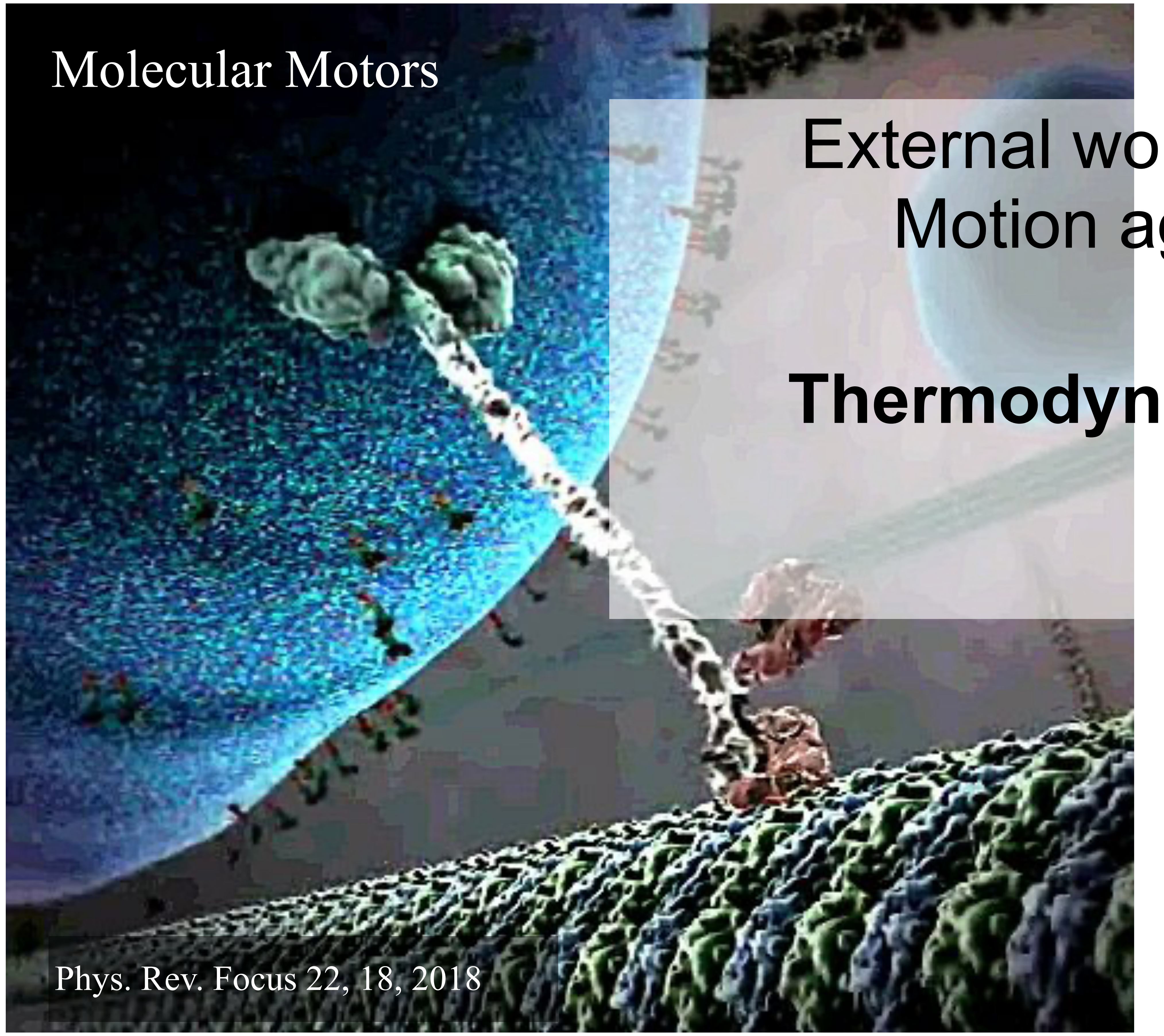
[M.C. Engel, J.A. Smith, M.P. Brenner, PRX (2023)]
[S. Blaber, D. A Sivak, J. Phys. Commun. 7 033001 (2023)]

Isothermal compression of active particles



[Loos, Physics, 17, 20 (2024)]
[L. K. Davis, K. Proesmans, and É. Fodor, PRX (2024)]

Microscale processes as "machines"



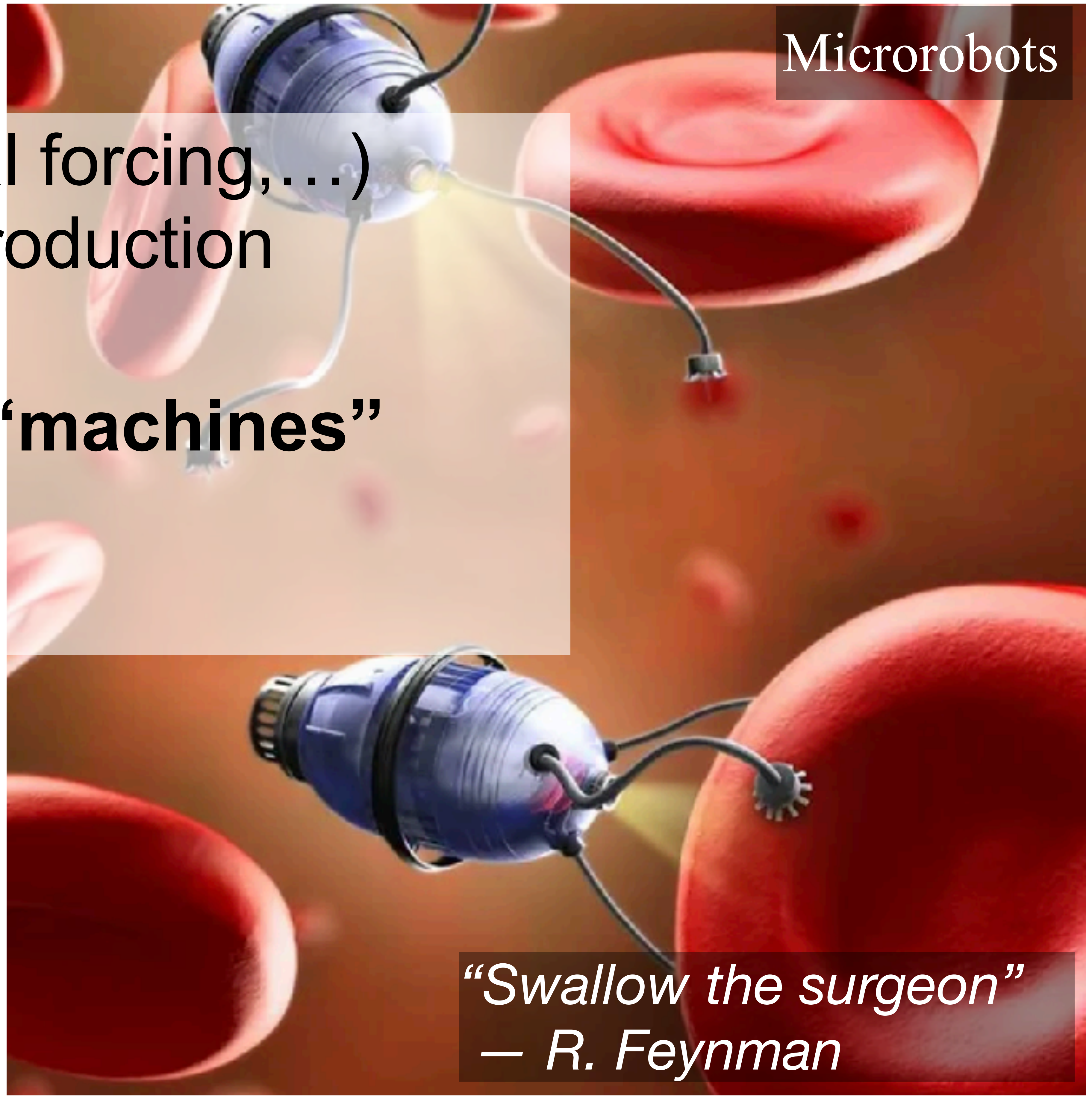
External work input (ATP, external forcing,…)
Motion against friction: heat production

↓

Thermodynamic processes as "machines"

↓

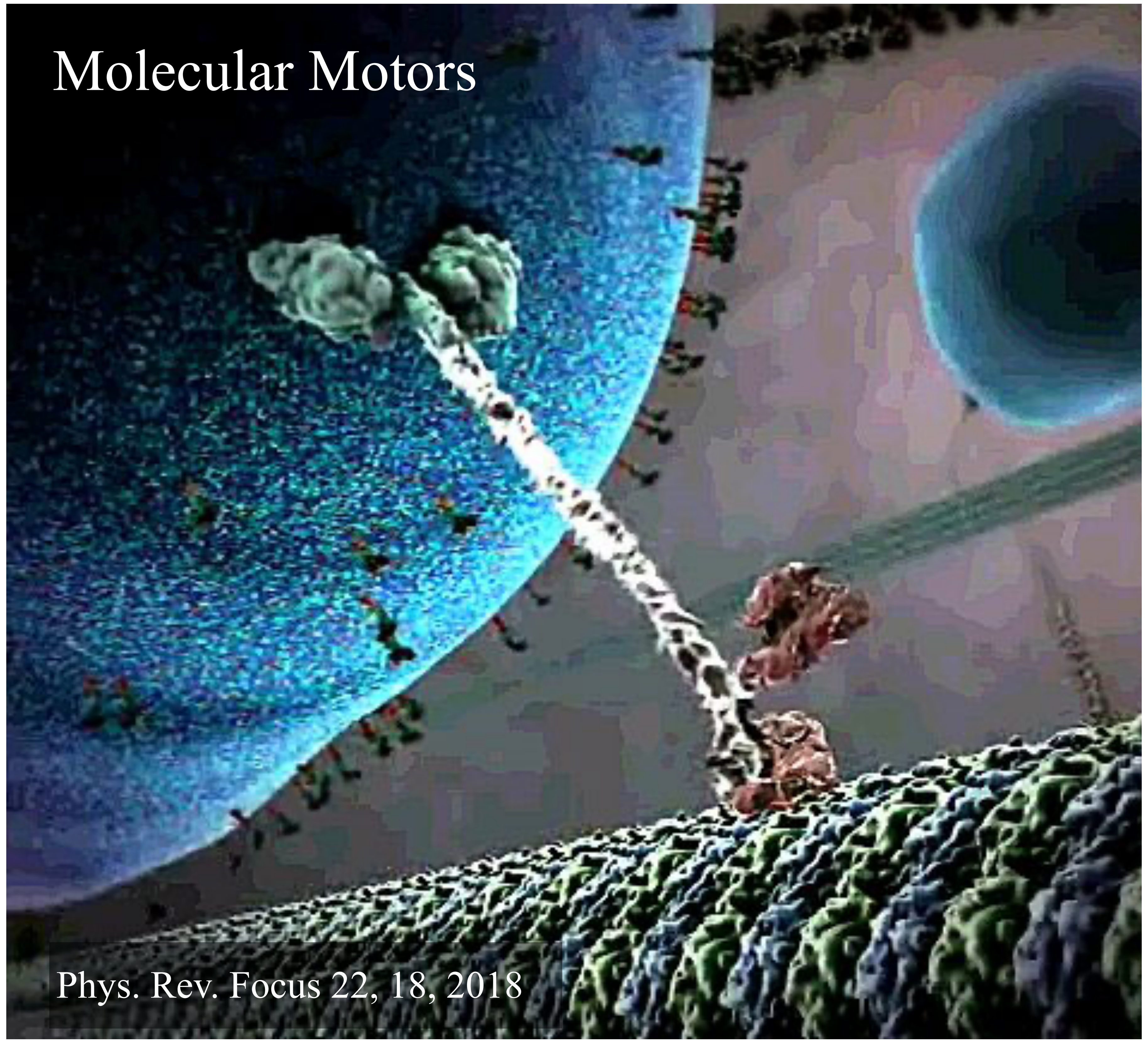
Efficiency?



***Biology**: Has evolution lead to energy-optimisation?*

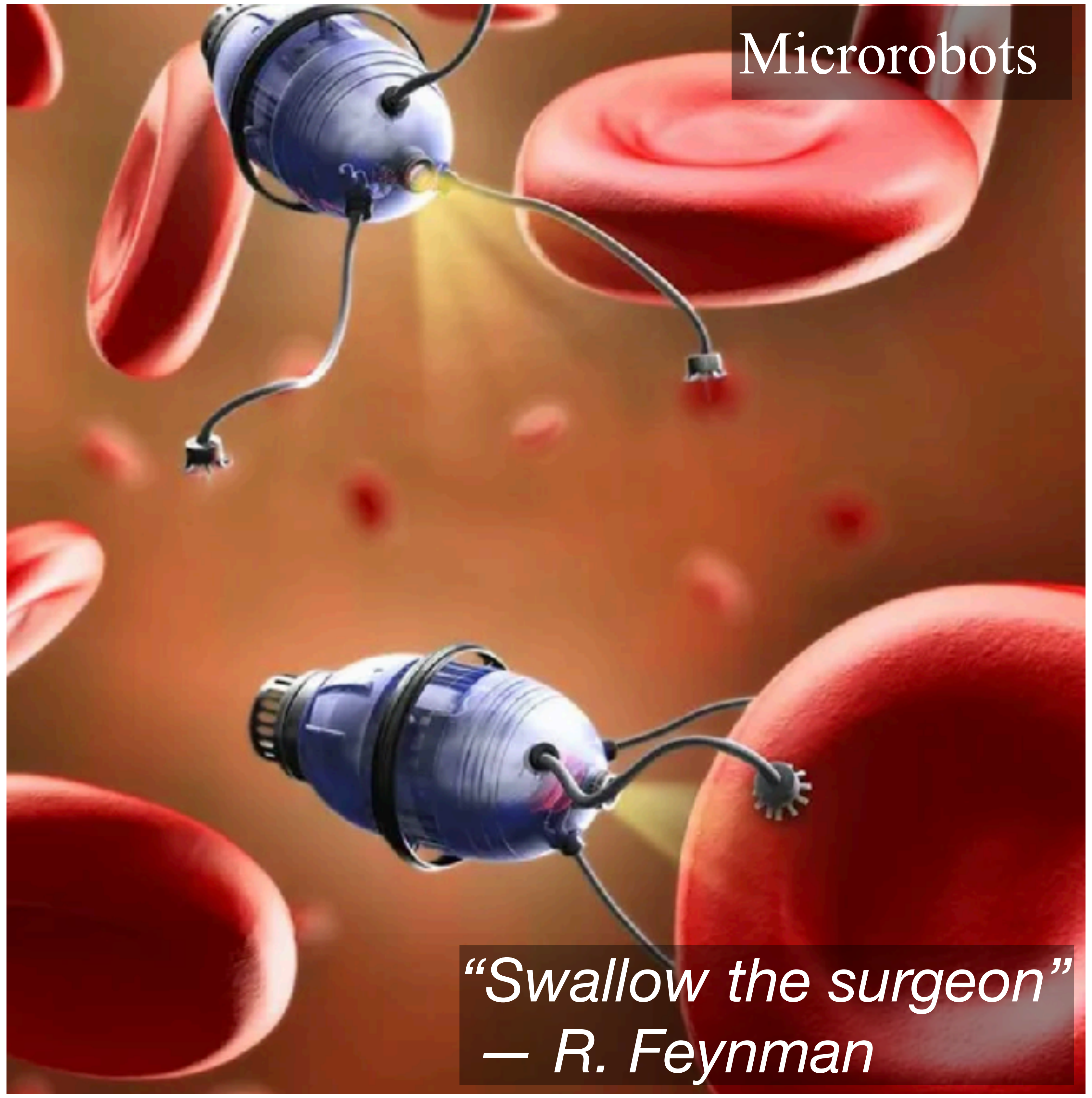
***Engineering**: How to build most efficient robots?*

Microscale processes as "machines"



Challenges at microscale:

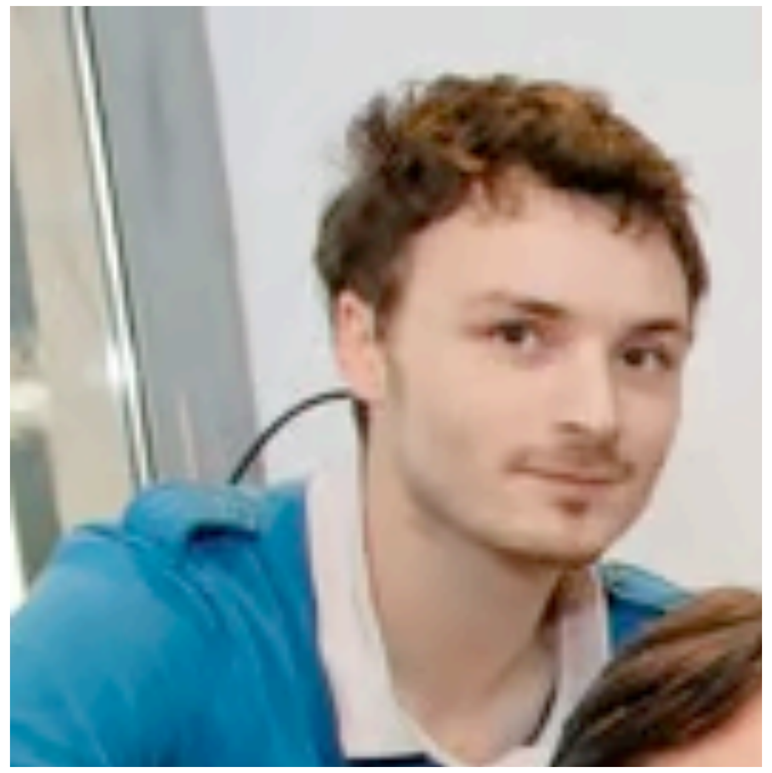
- High friction
- (Non-)thermal noise
- Memory*, e.g., from internal D.O.F. or from viscoelastic environment (cytoskeleton, blood…)



*slow, hidden D.O.F. that participate in dynamics but are not directly controllable or measurable (non-Markovianity)

Thanks to my collaborators

Samuel Monter
(U Konstanz)



Felix Ginot
(U Konstanz)

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Loos, Monter, Ginot, Bechinger,
Physical Review X 14, 021032 (2024)

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Rosalba Garcia-Millan
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Mike Cates
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Garcia-Millan, Schuettler, Cates, Loos,
ArXiv:2407.18542 (2024)

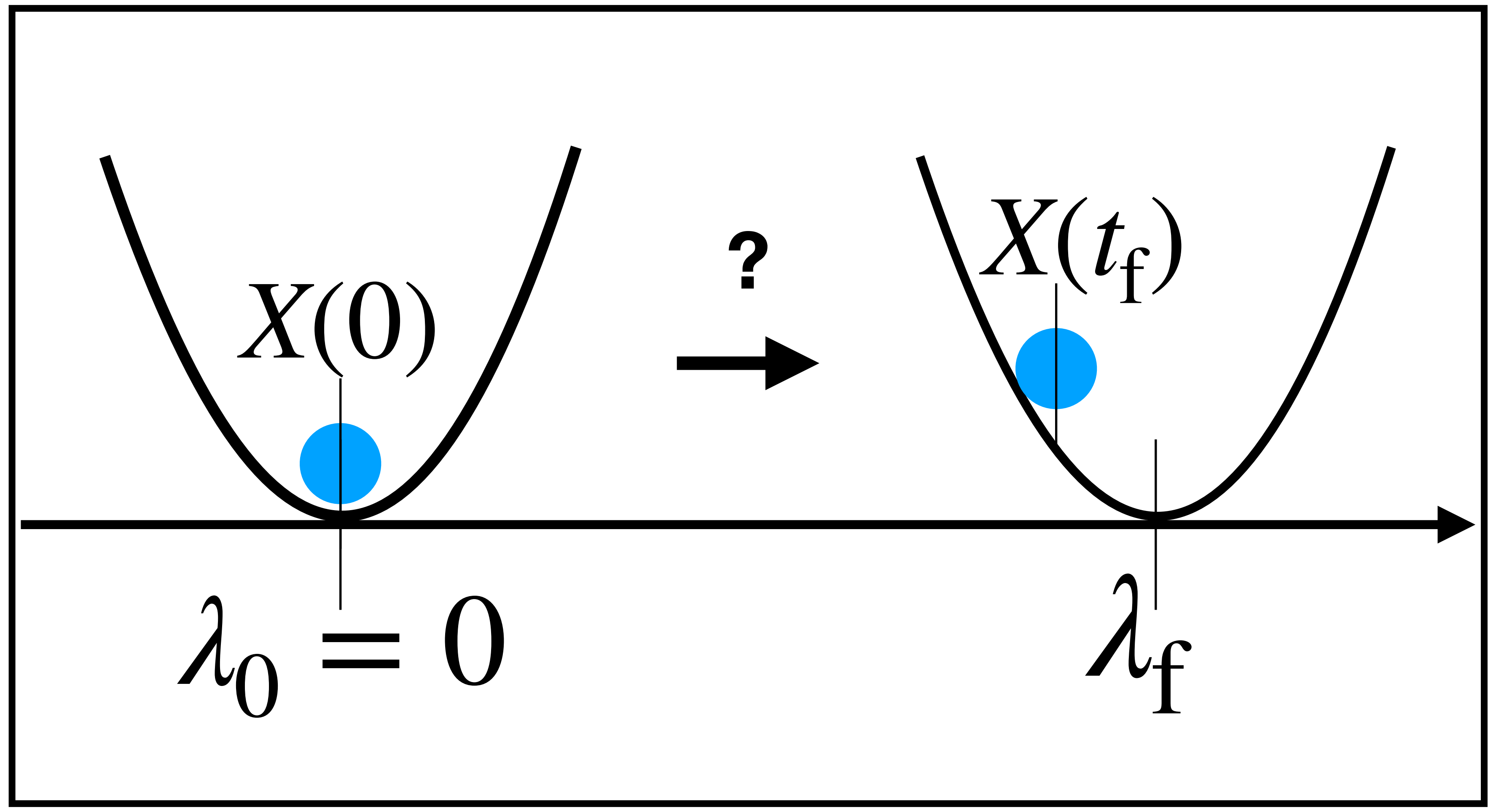
Schuettler, Garcia-Millan, Cates, Loos,
ArXiv:2501.18613 (2025)

Outline

- Background: Stochastic thermodynamics and optimal control

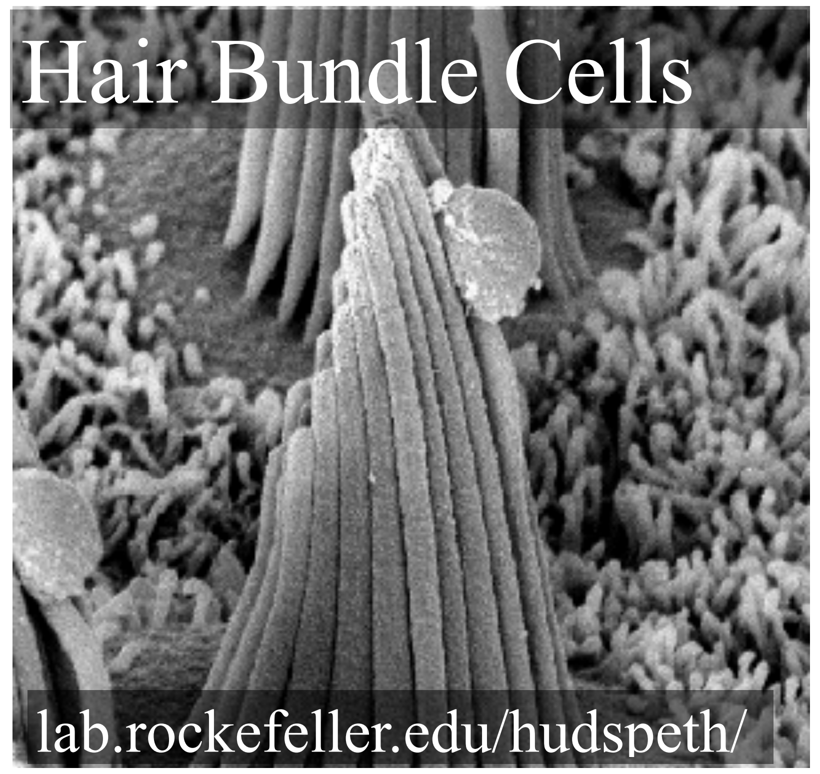
- The optimal dragging problem

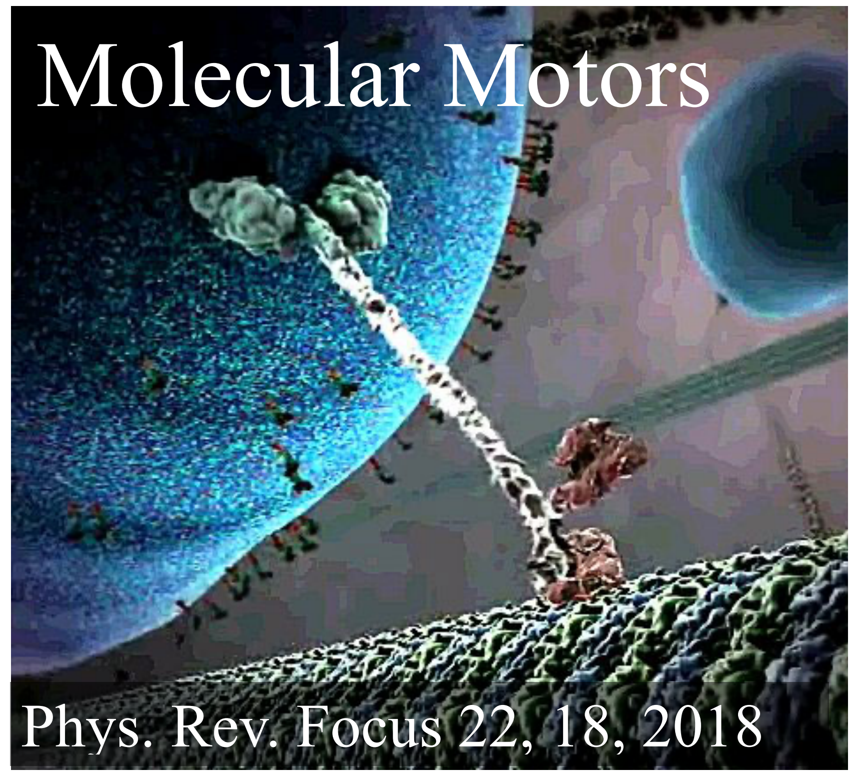
- Impact of **memory** and the role of **symmetry**

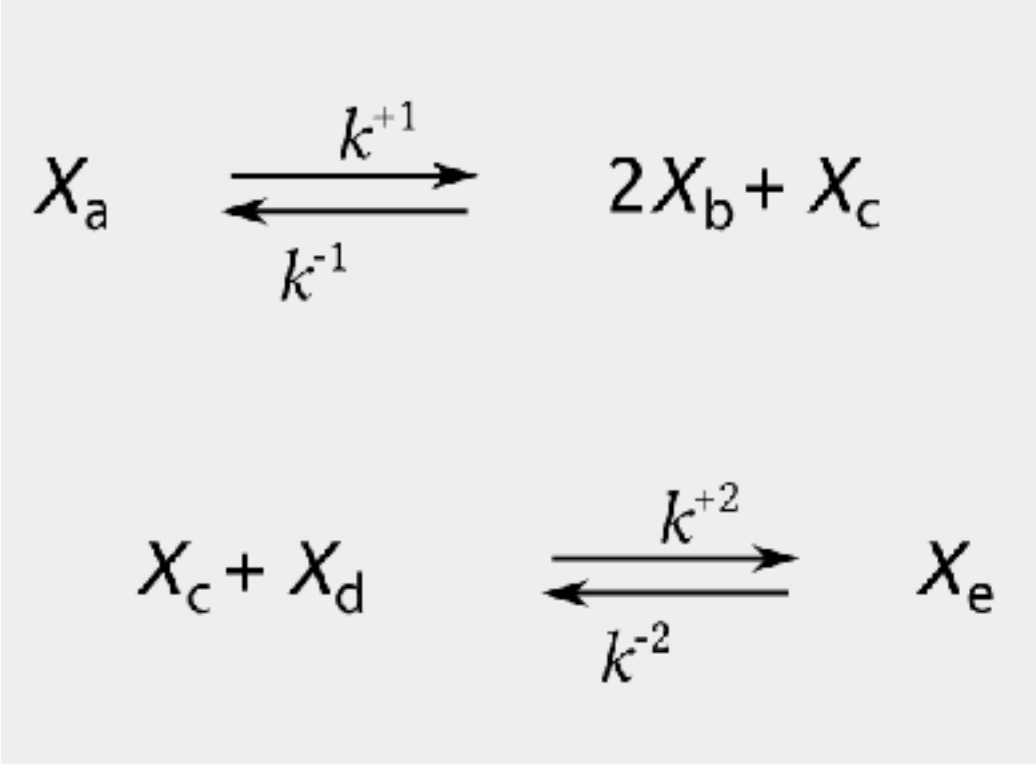


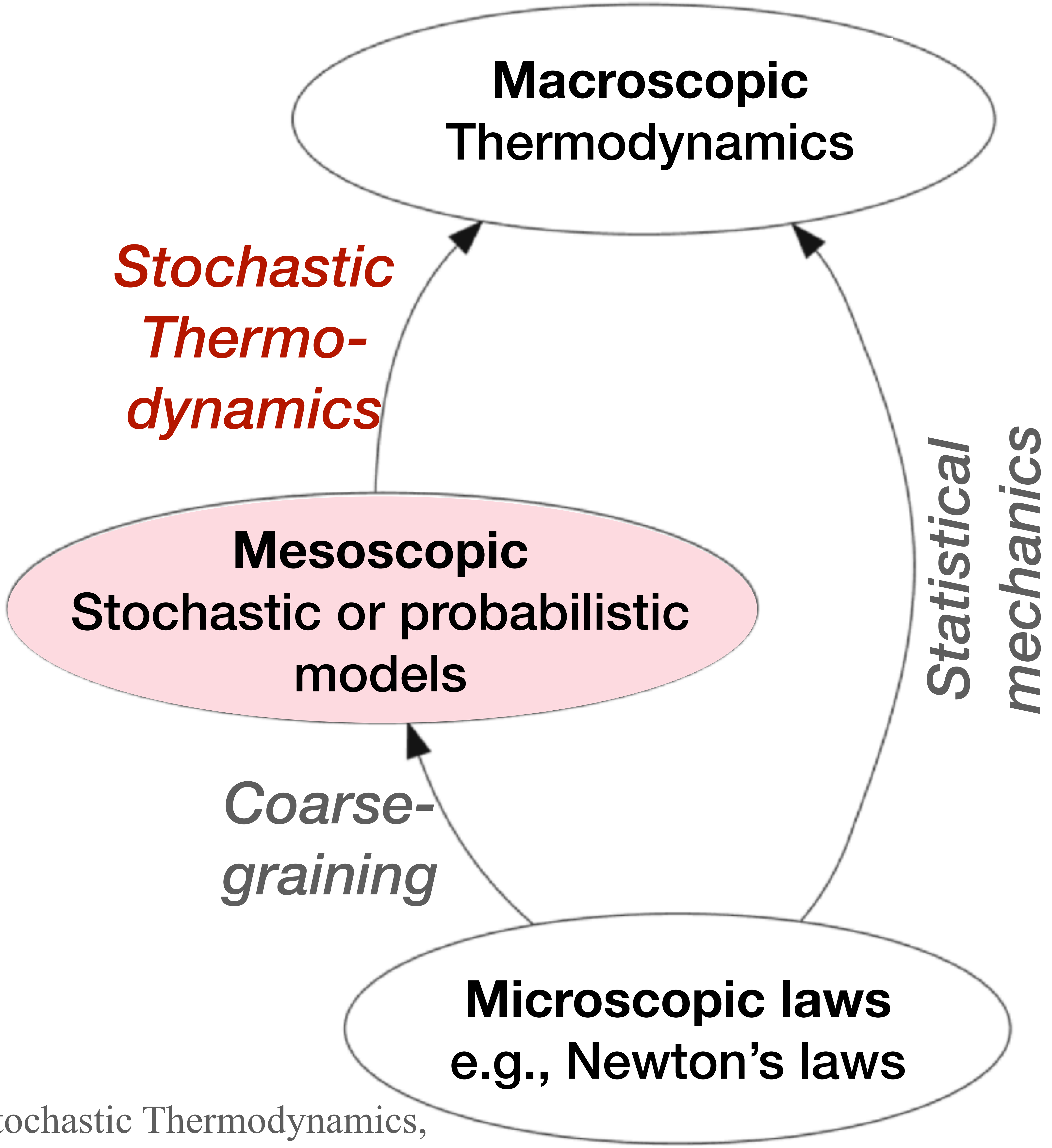
Stochastic thermodynamics "in a nutshell"

- Thermodynamics on mesoscale: Heat, work, entropy as fluctuating quantities $\delta q[X(t)]$
- Bridges between (time-reversible) microscopic and (time-irreversible) macroscopic world
- Universal laws far from equilibrium (Fluctuation theorems, Jarzynski, Crooks, TUR)
- Describe efficiency of mesoscale processes





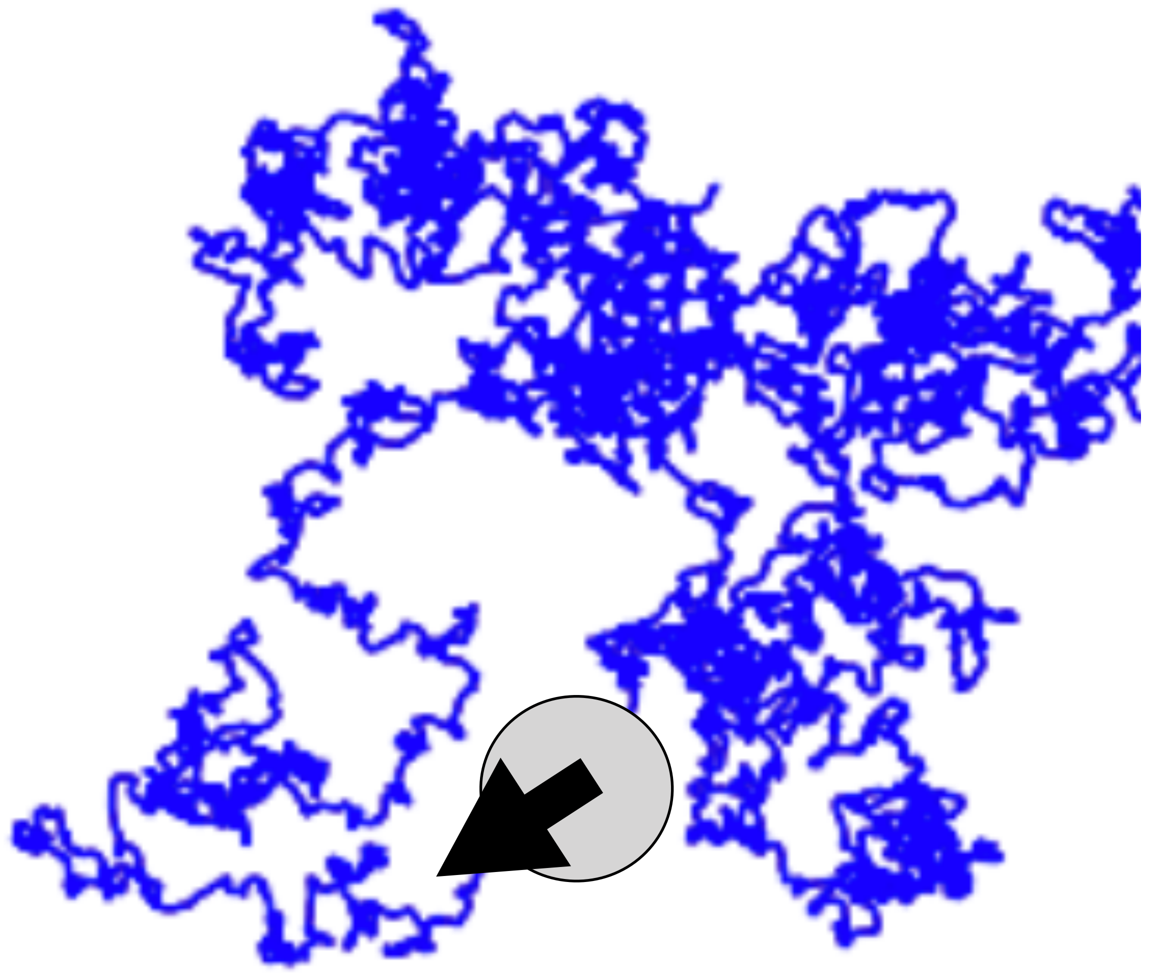


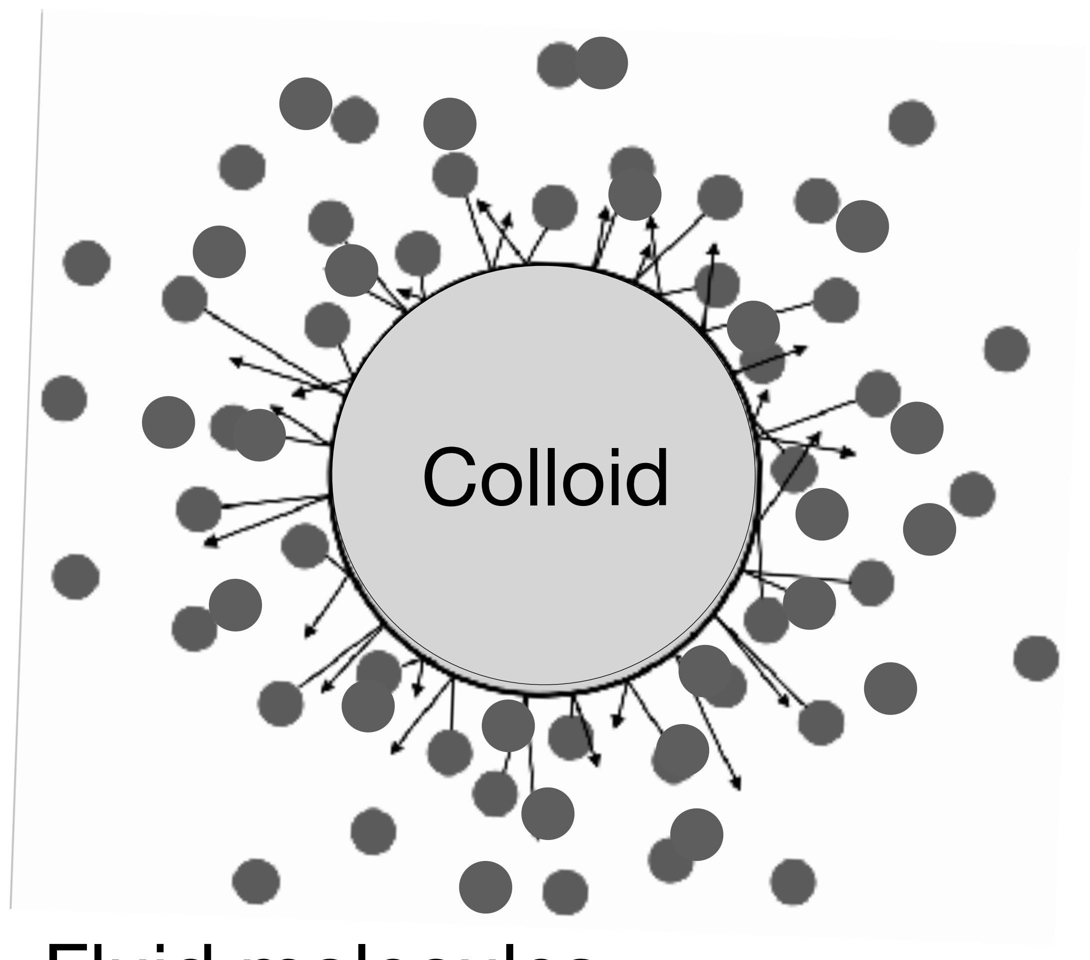


Stochastic Thermodynamics, L. Peliti, S. Pigolotti, 2021

Paradigmatic model: Markovian Langevin equation

- **Brownian motion**





Fluid molecules

Markovian Langevin equation

$$\underbrace{m\ddot{X}(t)}_{\text{inertia}} + \underbrace{\gamma\dot{X}(t)}_{\text{friction}} = \underbrace{F_{\text{ext}}(t)}_{\text{ext. forces}} + \underbrace{\xi(t)}_{\text{noise}}$$

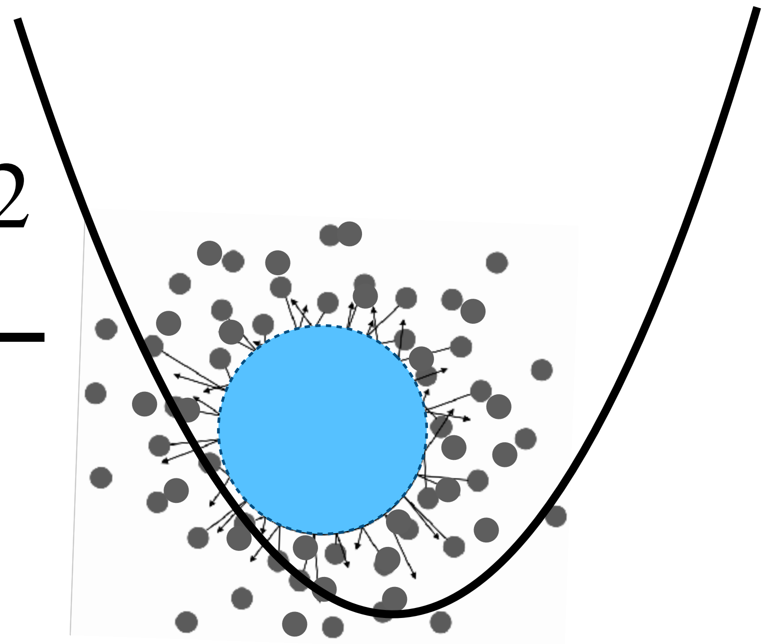
$$\text{white Gaussian noise}: \langle \xi \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = 2k_B T\, \delta(t-t')$$

→ *"Markovian" = no memory, if timescale separation between "system" and "bath"*

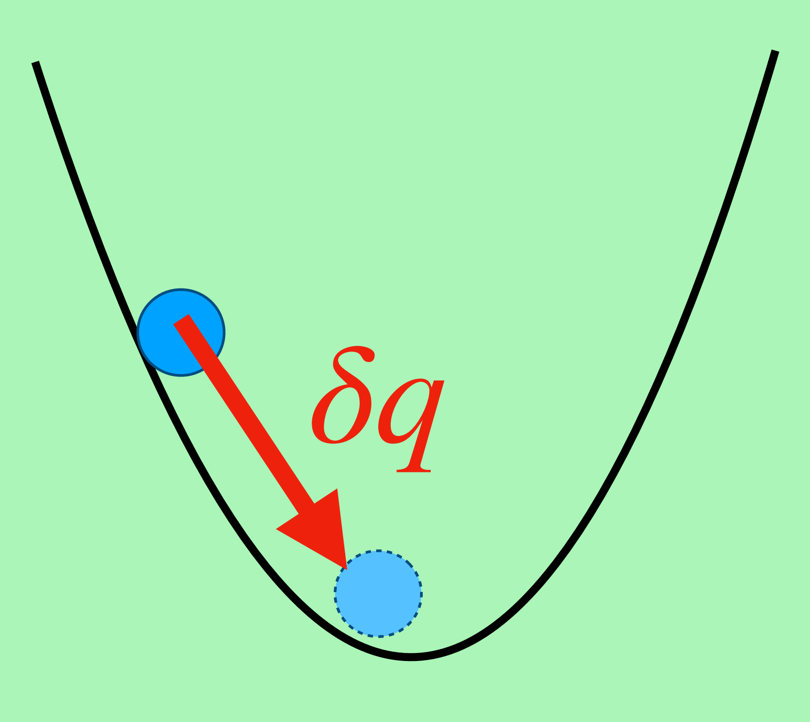
Stochastic Work, Heat, Entropy at small scales

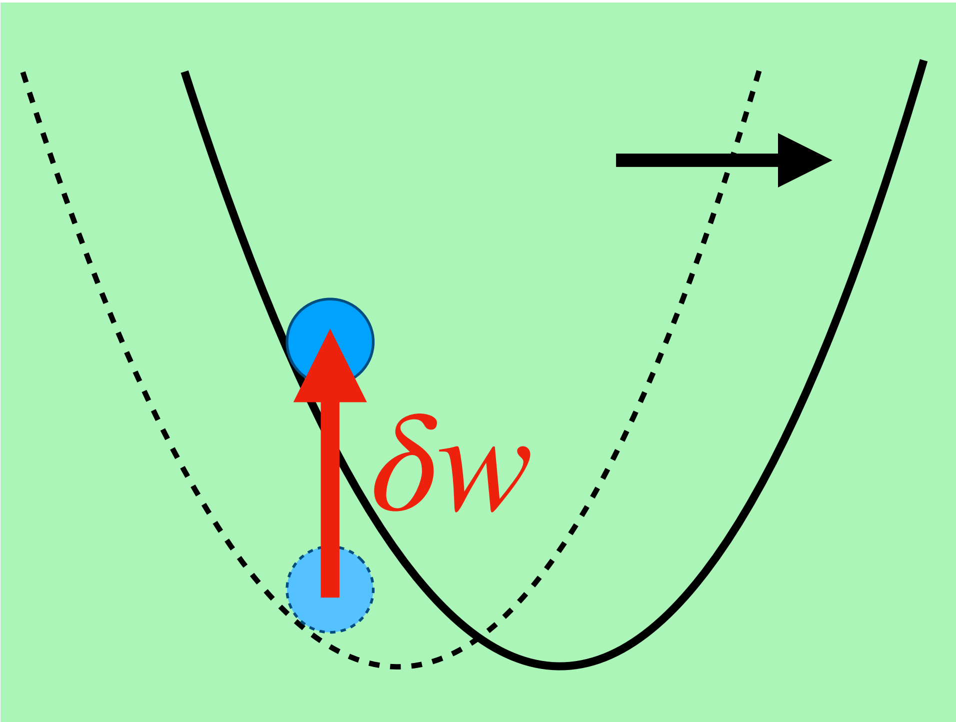
$$\underbrace{\cancel{m\ddot{X}(t)}}_{\text{inertia}} + \underbrace{\gamma\dot{X}(t)}_{\text{friction}} = \underbrace{-\nabla_x V(x,\lambda(t))}_{\text{conservative force}} + \underbrace{\cancel{F_{nc}(t)}}_{\text{non-conservative}} + \underbrace{\xi(t)}_{\text{noise}}$$

$$V = \frac{\kappa[X-\lambda(t)]^2}{2}$$



- Stochastic work and heat:





$$\mathrm{d}V = \underbrace{\frac{\partial V}{\partial X}\circ \mathrm{d}X}_{=\delta q} + \underbrace{\frac{\partial V}{\partial \lambda}\circ \mathrm{d}\lambda}_{=\delta w}$$

$\Rightarrow$1st law :

$\mathrm{d}V = \delta w + \delta q$ ✓

$$\Rightarrow \delta q = [-\gamma\dot{X}(t) + \xi(t)] \circ \mathrm{d}X$$

- Stochastic entropy production:

$$S_{\text{sys}}(x) \propto -\ln \rho(x)$$

$$\Sigma = \ln \frac{P[\{\mathbf{X}(t')\}_0^t]}{\hat{P}[\{\hat{\mathbf{X}}(t')\}_0^t]} = \frac{\int_0^t \delta q}{T} + \Delta S_{\text{sys}}$$

$\Rightarrow$2nd law:

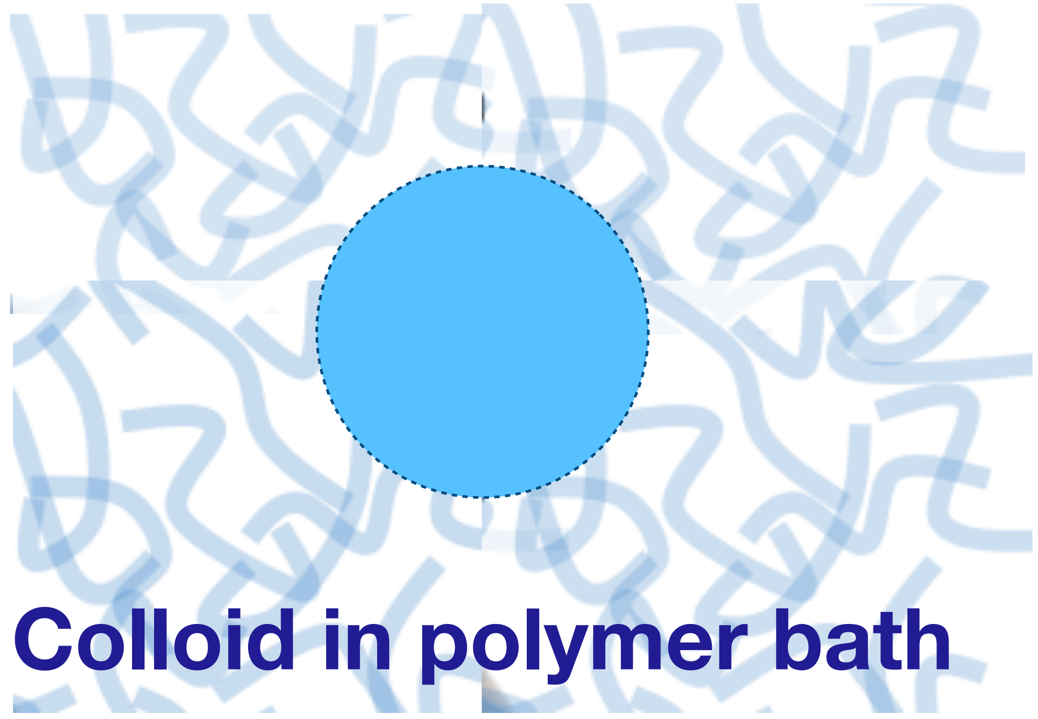
$\langle \Sigma \rangle \geq 0$ ✓

Seifert, PRL 95, 040602 (2005), Sekimoto, *Stochastic Energetics*, Springer 2010

Non-Markovian case: Generalised Langevin equation

(Non-Markovian) Generalized Langevin equation:

$$m\ddot{X}(t) + \underbrace{\int_{-\infty}^{t} \Gamma(t-t')\dot{X}(t')\mathrm{d}t'}_{\text{retarded friction}} = -\nabla_X V(X,\lambda) + \underbrace{\eta(t)}_{\text{coloured noise}}$$



Fluctuation-Dissipation Relation: $\Gamma(|t-t'|) \propto \langle \nu(t)\nu(t')\rangle$ (Equilibrium)

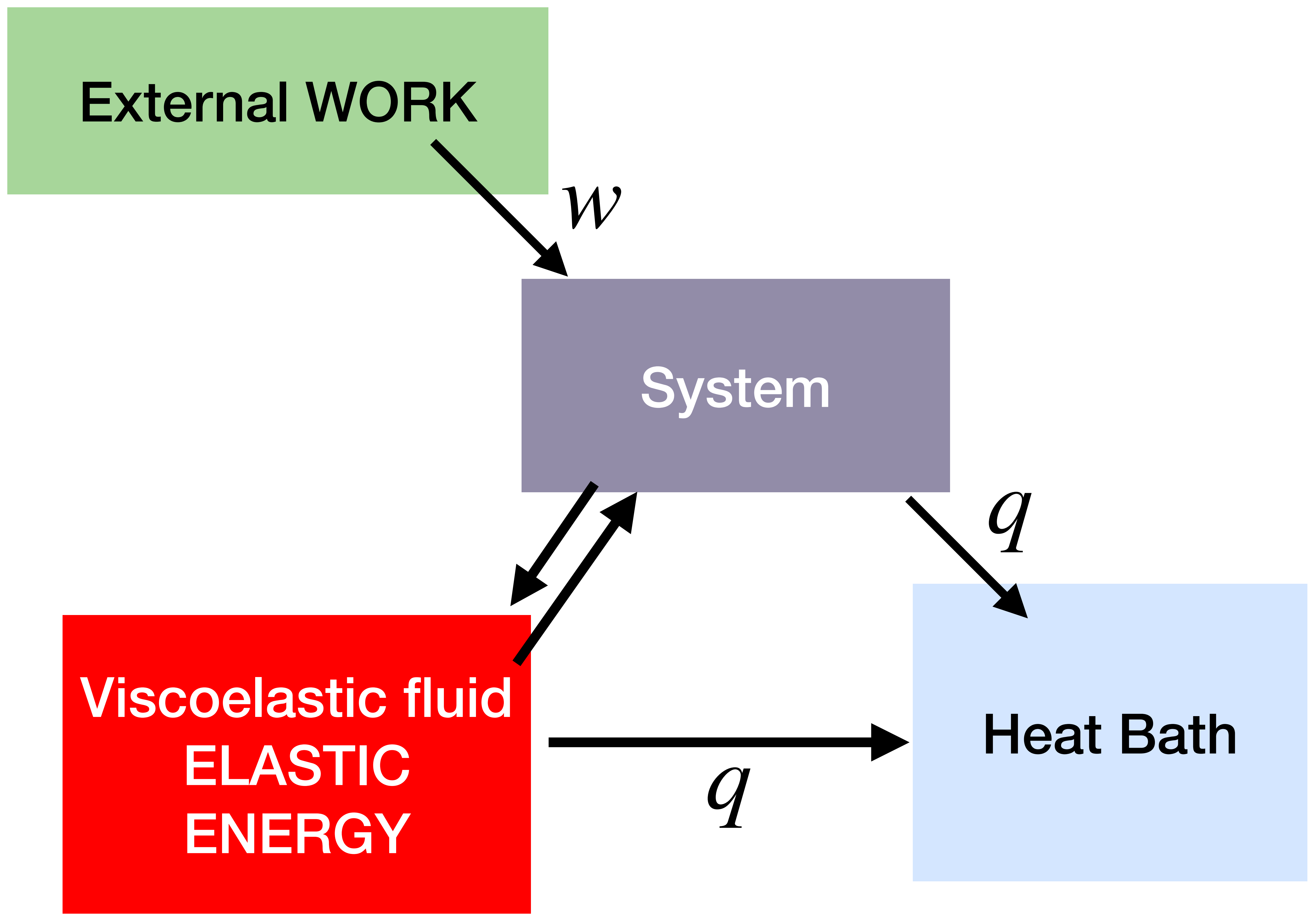
Dimensionality given by spectrum of Γ, in general ∞-dimensional process

- Hidden D.O.F.: "No idea" how much entropy we are missing ($\Sigma^{\text{coarse-gr.}} \leq \Sigma^{\text{full}}$)
  [Esposito. Phys. Rev. E (2012)]

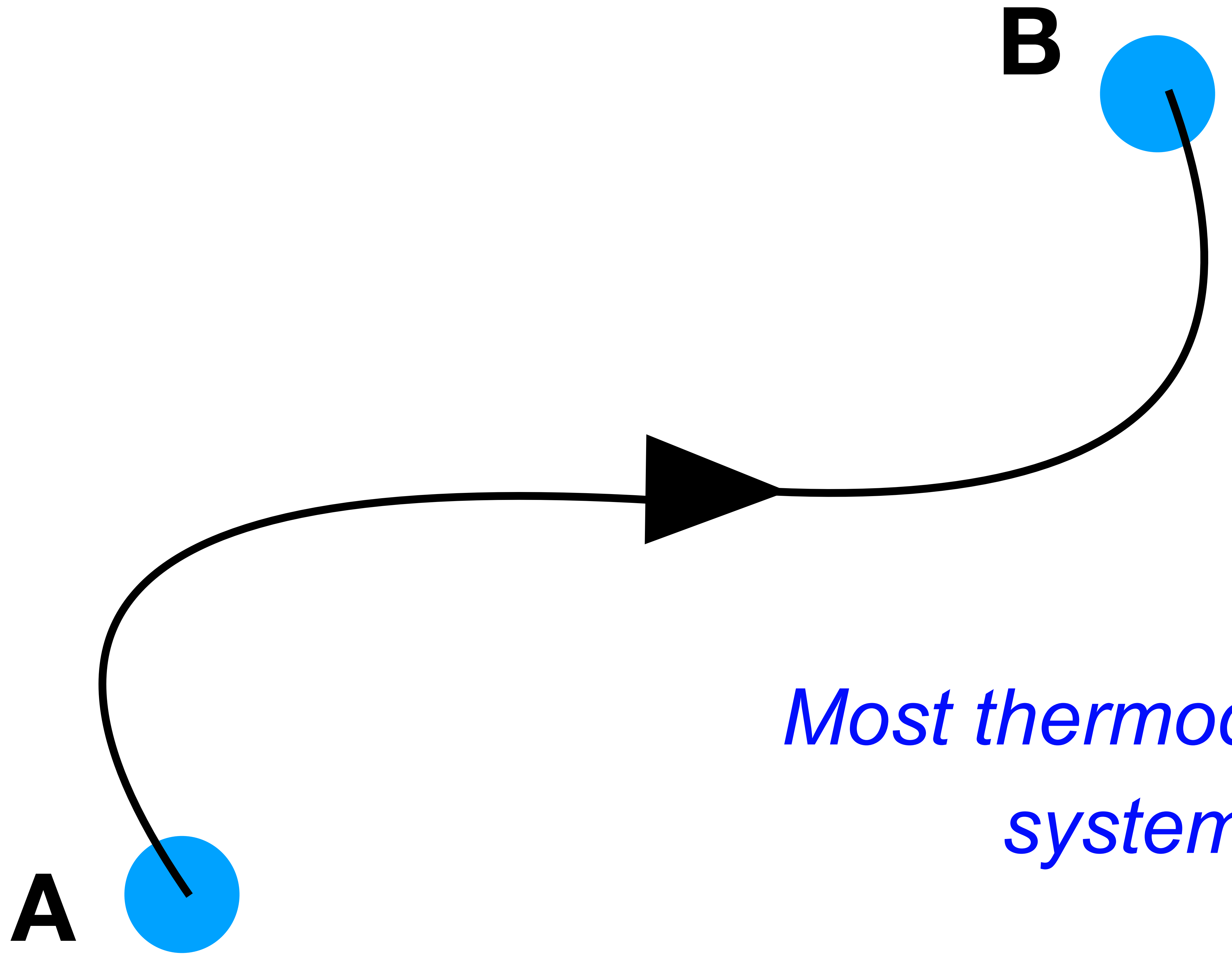
- Acausality of backward process:
  "information-flow between presence and past"
  [Rosinberg, Tarjus, & Munakata, PRE (2017), **Loos** & Klapp, Sci. Rep. (2019), **Loos** & Klapp, NJP (2021), …]

$$\Sigma = \ln \frac{P[\{\mathbf{X}(t')\}_0^t]}{\hat{P}[\{\hat{\mathbf{X}}(t')\}_0^t]} = \frac{\int_0^t \delta q}{T} + \Delta S_{\text{sys}} + \Delta I$$

Thermodynamic process with “additional d.o.f.”



Thermodynamically optimal control



Most thermodynamically efficient way to bring system from A to B in finite time?

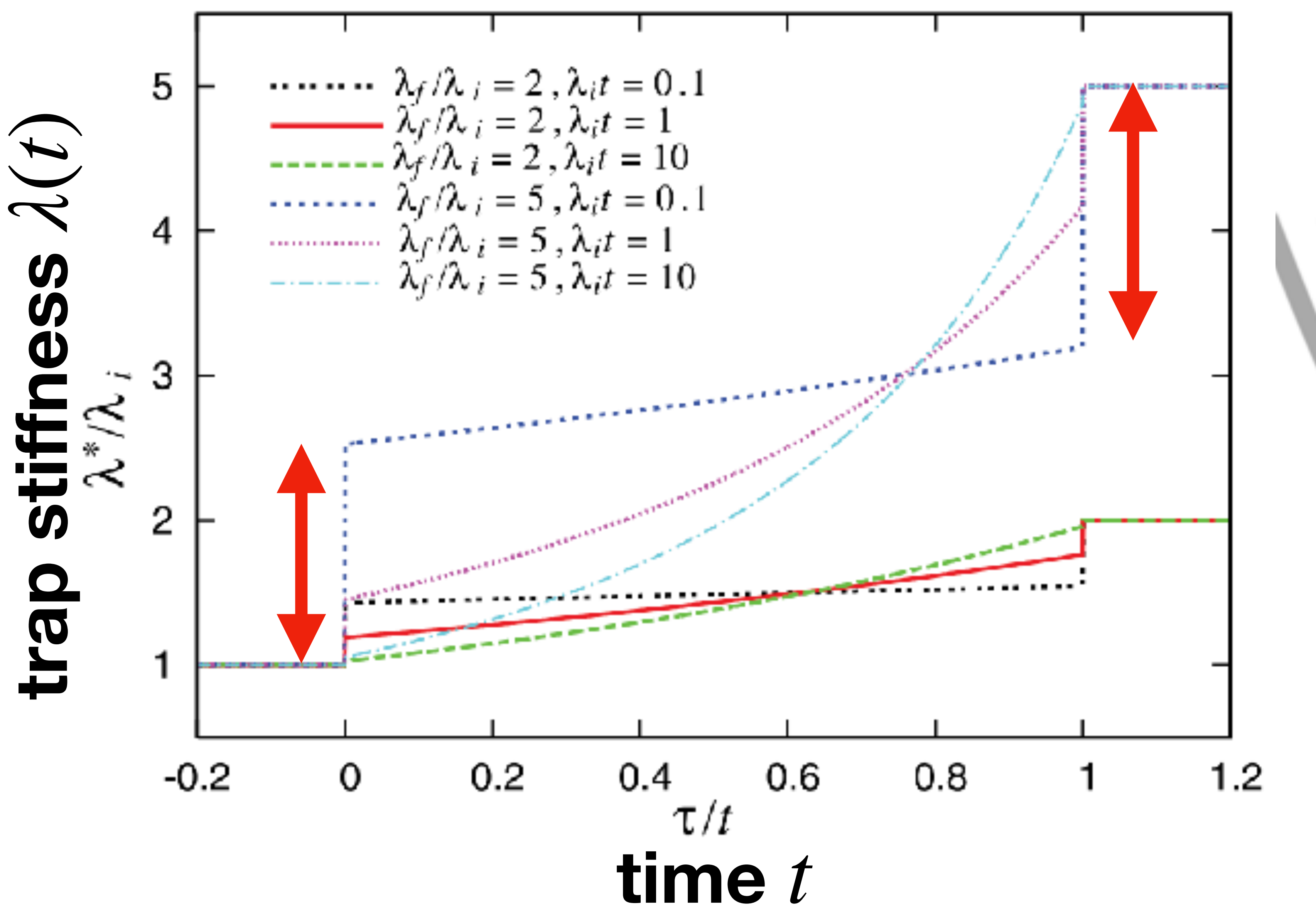
Thermodynamically optimal control — General results so far

- Optimal control theory, variational calculus (Euler-Langrange eq.) + Stochastic calculus

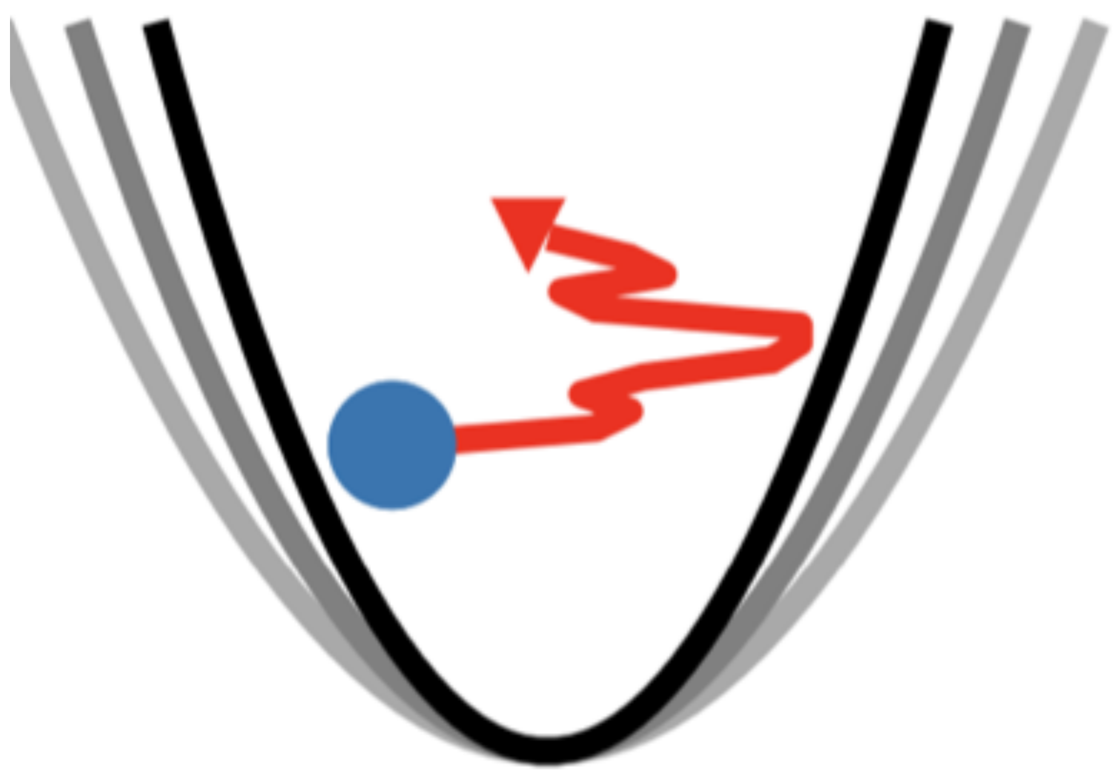
<u>Exact results (Gaussian systems)</u>

Discontinuities of protocol $\lambda(t)$ at beginning and end = "bang-bang solutions"

[Schmiedl & Seifert, PRL 98, 10830 (2007)]



$$U_\lambda(x) = \lambda \frac{x^2}{2}$$



[Zhong PRL (2024)]

- Jumps <u>generic</u> for "fast-protocol" ($t_\mathrm{f} \to 0$) solutions

  [S. Blader, M.D. Louwerse, D. A. Sivak, PRE (2021)]

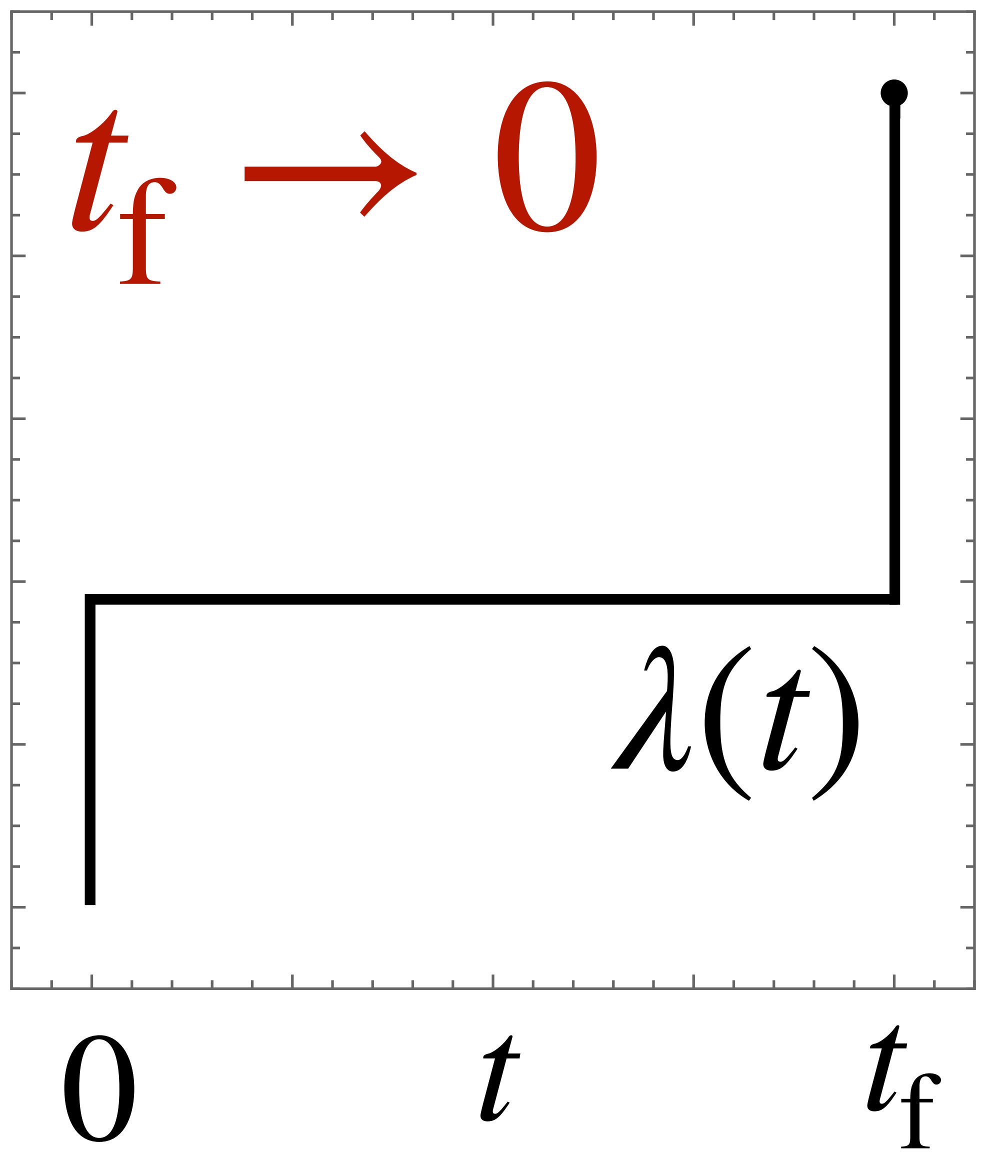
- For nonlinear systems: Approximative schemes

Side note: Thermodynamically optimal control of nonlinear systems

- For nonlinear systems: Geometric approach based on response theory

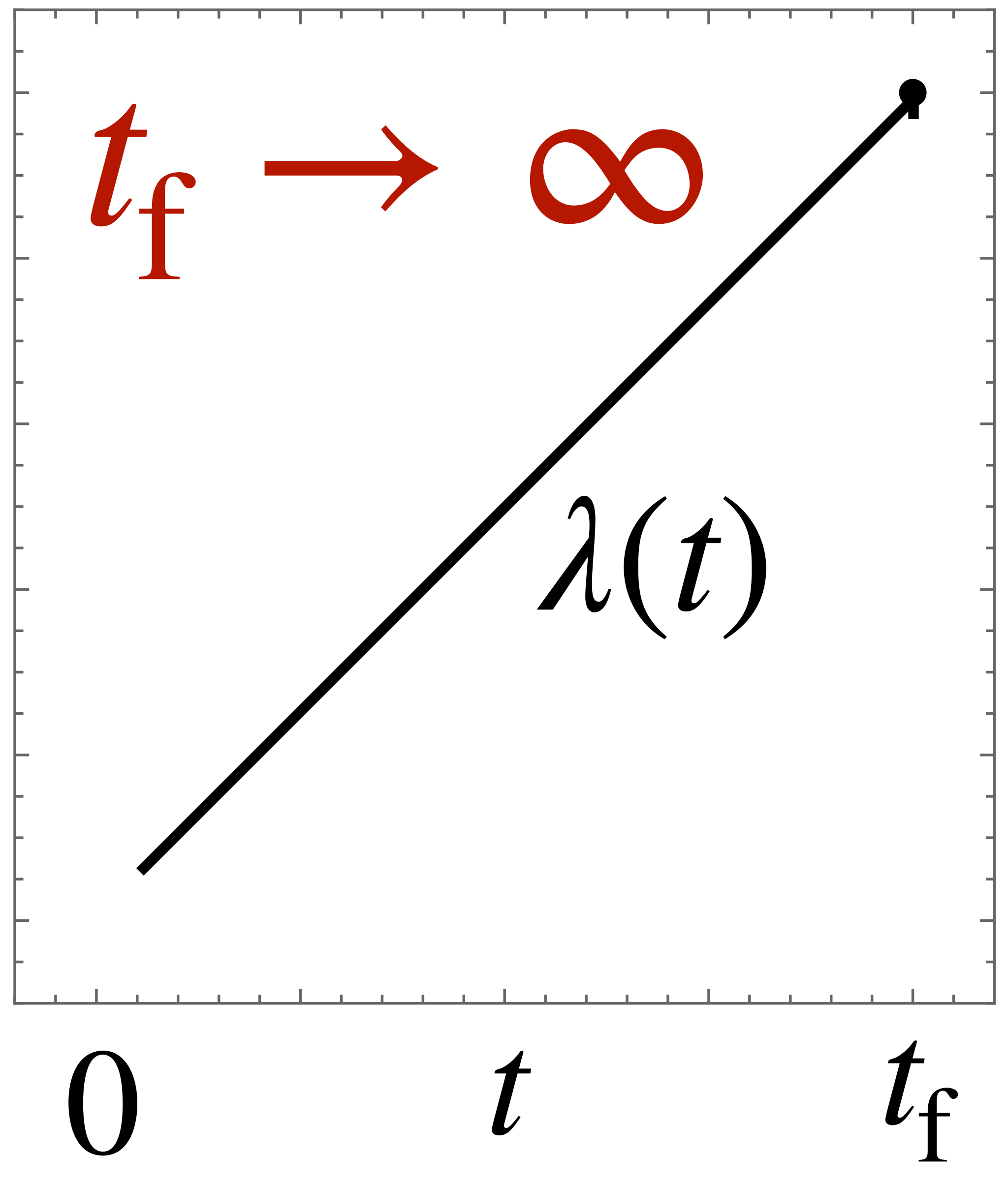
[Crooks, PRL (2007)]
[Sivak, Crooks, PRL (2012)]
[Zhong, DeWeese, PRE (2022)]
[Van Vu and Saito, PRX (2023)]

*Infinitely fast protocols**



→ *finite t_f* ←

Quasistatic limit

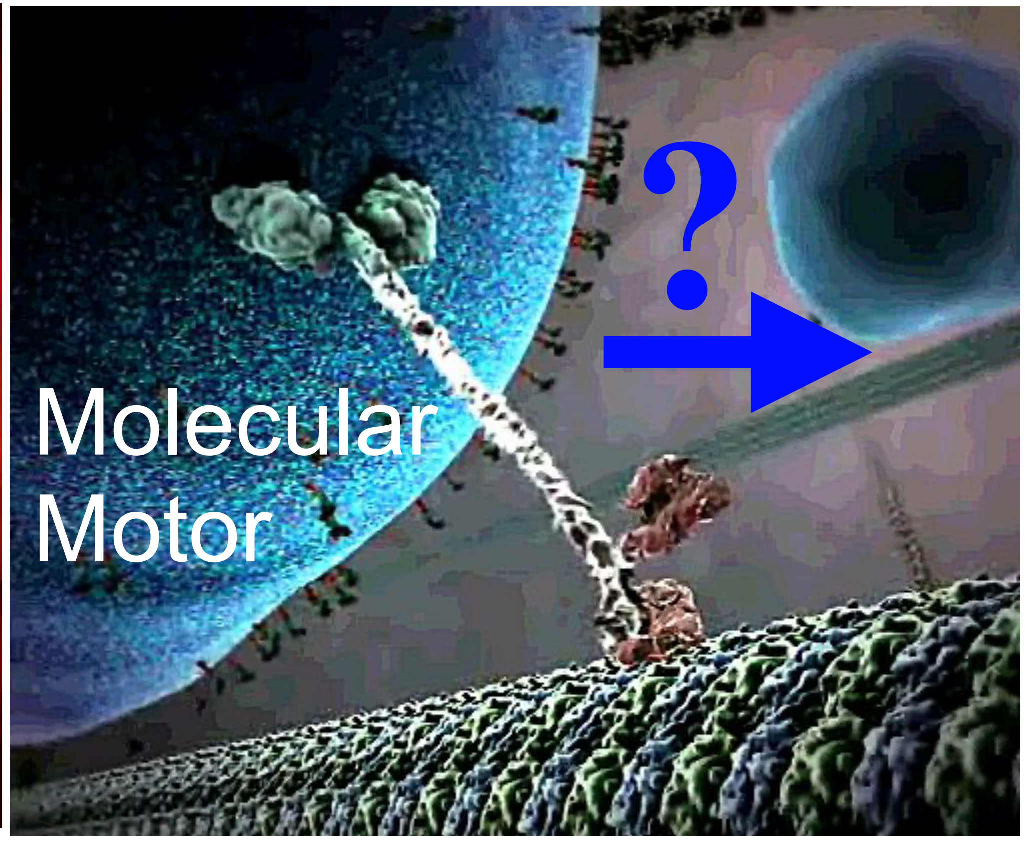


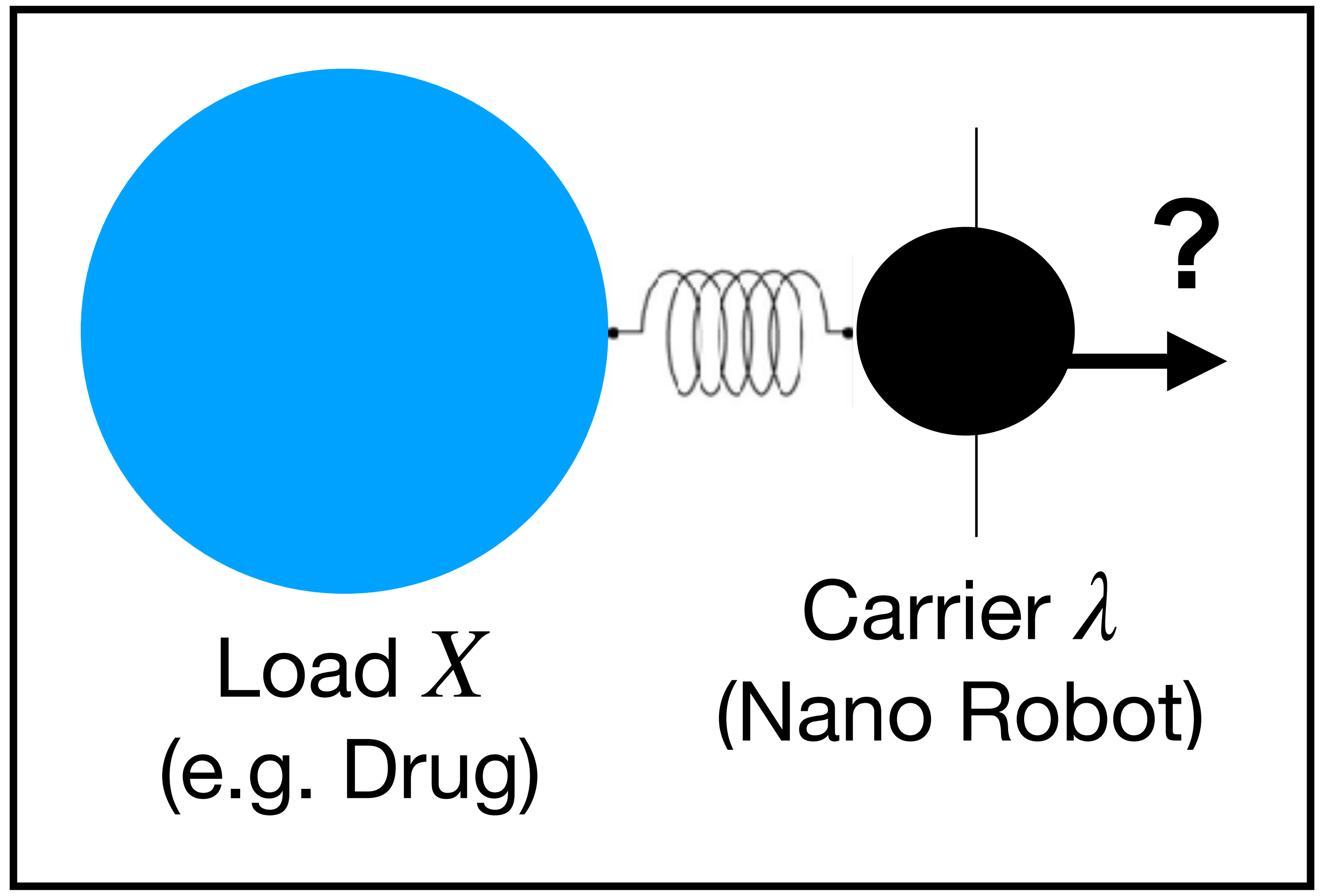
*[S. Blader, M.D. Louwerse, D. A. Sivak, PRE (2021)]

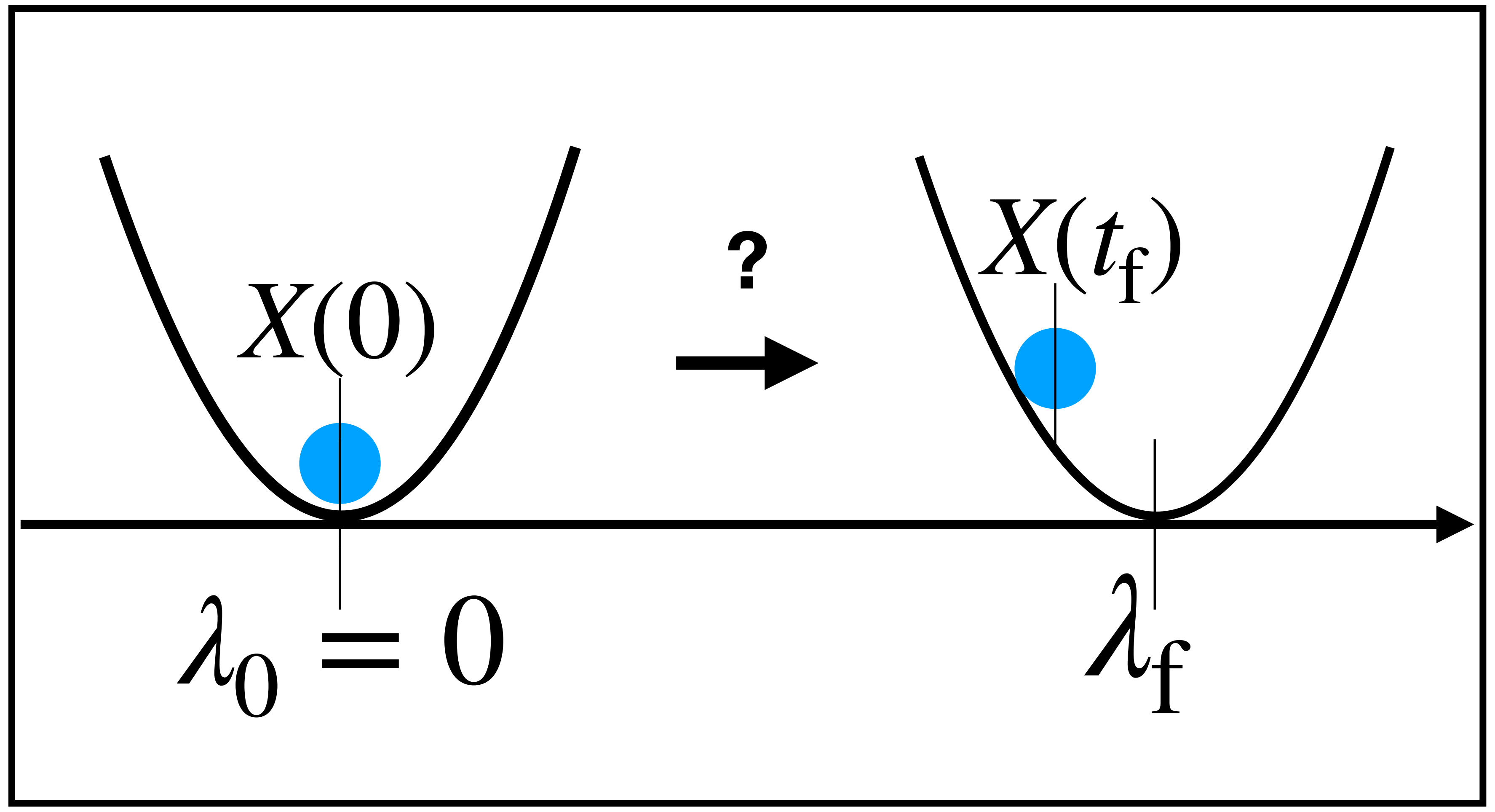
The optimal dragging problem

Abstract/Simplify

Equivalent Problem







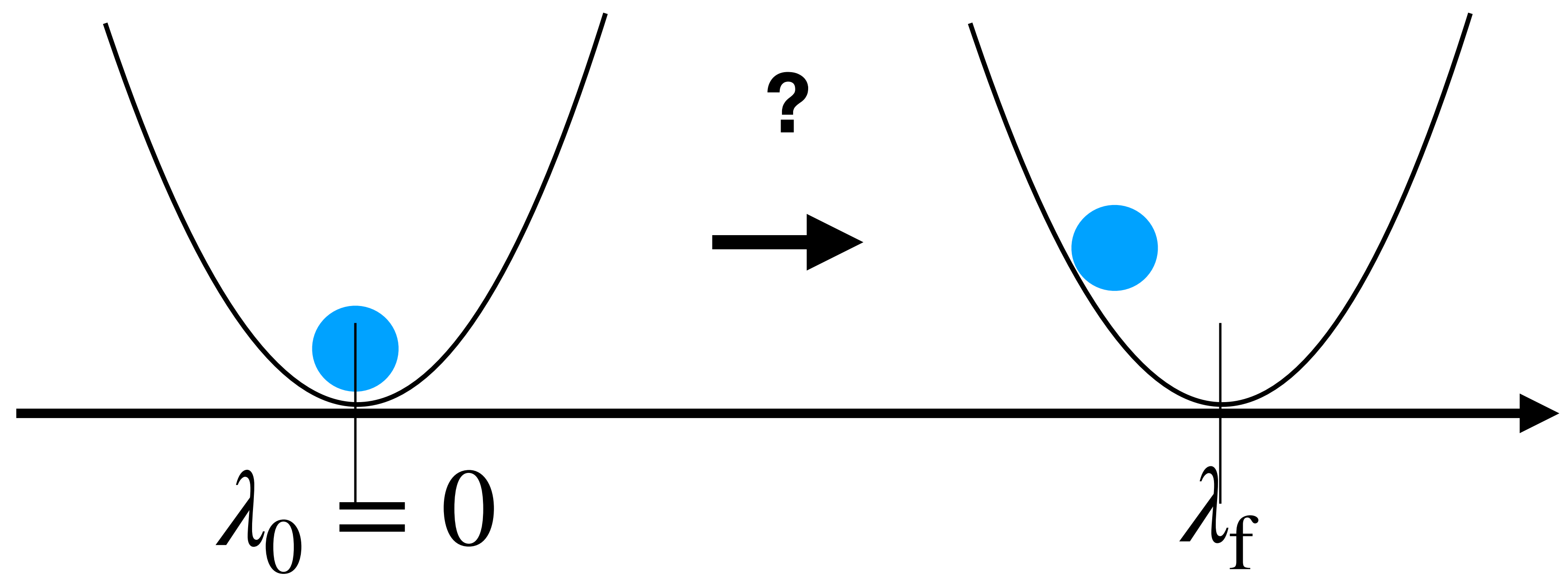
<u>Optimal control problem:</u> Protocol to move λ from $\lambda(0) = \lambda_0$ to $\lambda(t_\mathrm{f}) = \lambda_\mathrm{f}$ in time t_f that minimises average work $\langle W[x, \lambda] \rangle$?

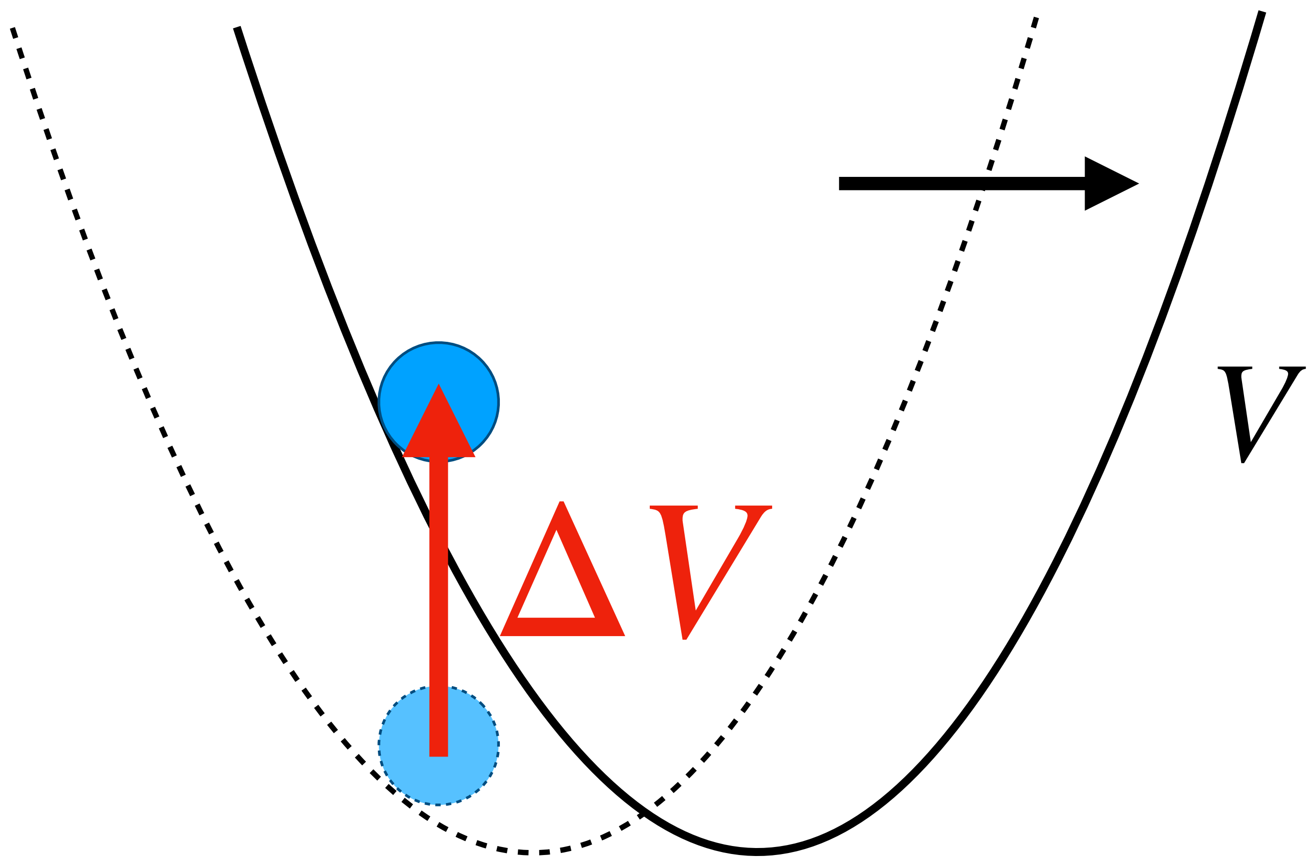
…Simple enough to study analytically!

Minimum work dragging problem

$$\mathrm{d}V = \underbrace{\frac{\partial V}{\partial X}\mathrm{d}X}_{\delta q} + \underbrace{\frac{\partial V}{\partial \lambda}\mathrm{d}\lambda}_{\delta w}$$

- Work spend along dragging:



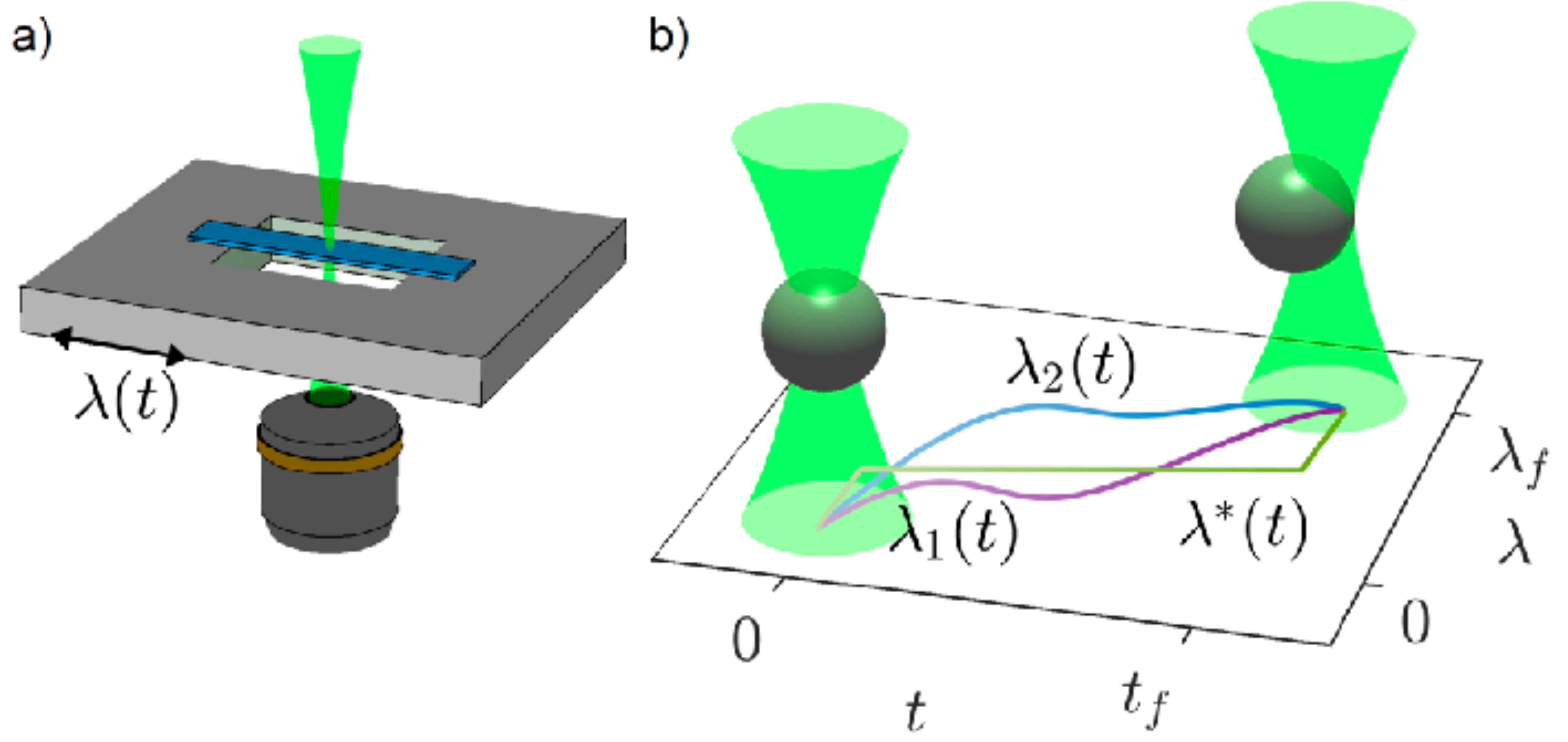


$$V = \frac{\kappa}{2}[X - \lambda(t)]^2$$

λ : Trap center

X : particle position

$$\boxed{W[X(t), \lambda(t)] = \int_0^{\lambda_\mathrm{f}} \frac{\partial V}{\partial \lambda} \circ \mathrm{d}\lambda = \kappa \int_0^{\lambda_\mathrm{f}} \dot{\lambda}(\lambda - X) \circ \mathrm{d}t}$$



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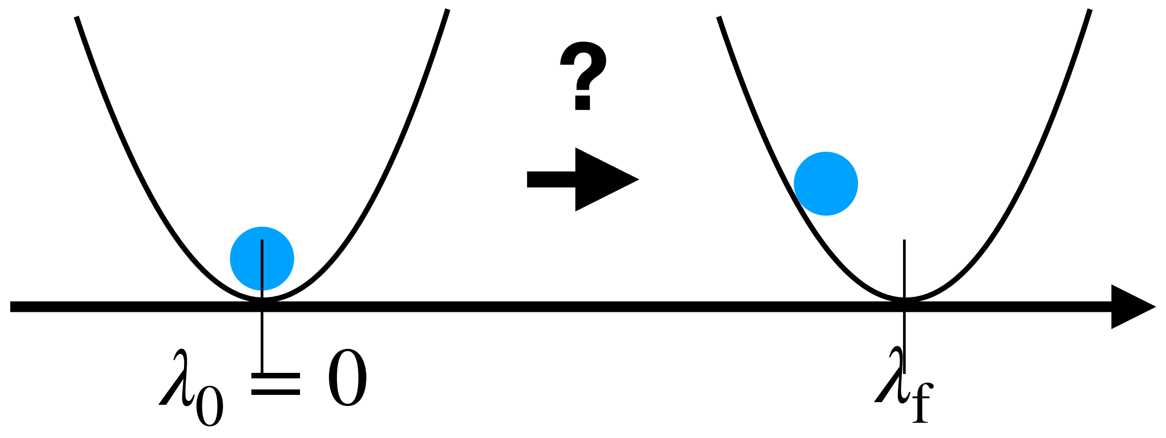
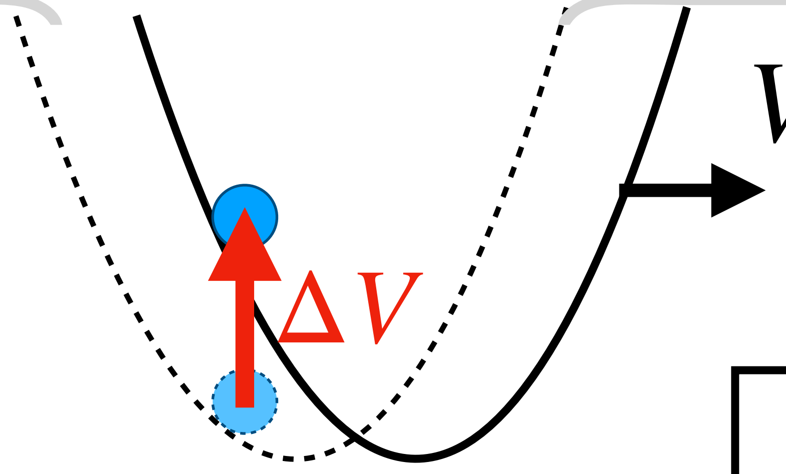
[Schmiedl & Seifert, PRL 98, 10830 (2007)]

Minimum work dragging problem: viscous case

Steady state Steady state Steady state

- Work spend along dragging:





$$V = \frac{\kappa}{2}[X - \lambda(t)]^2$$

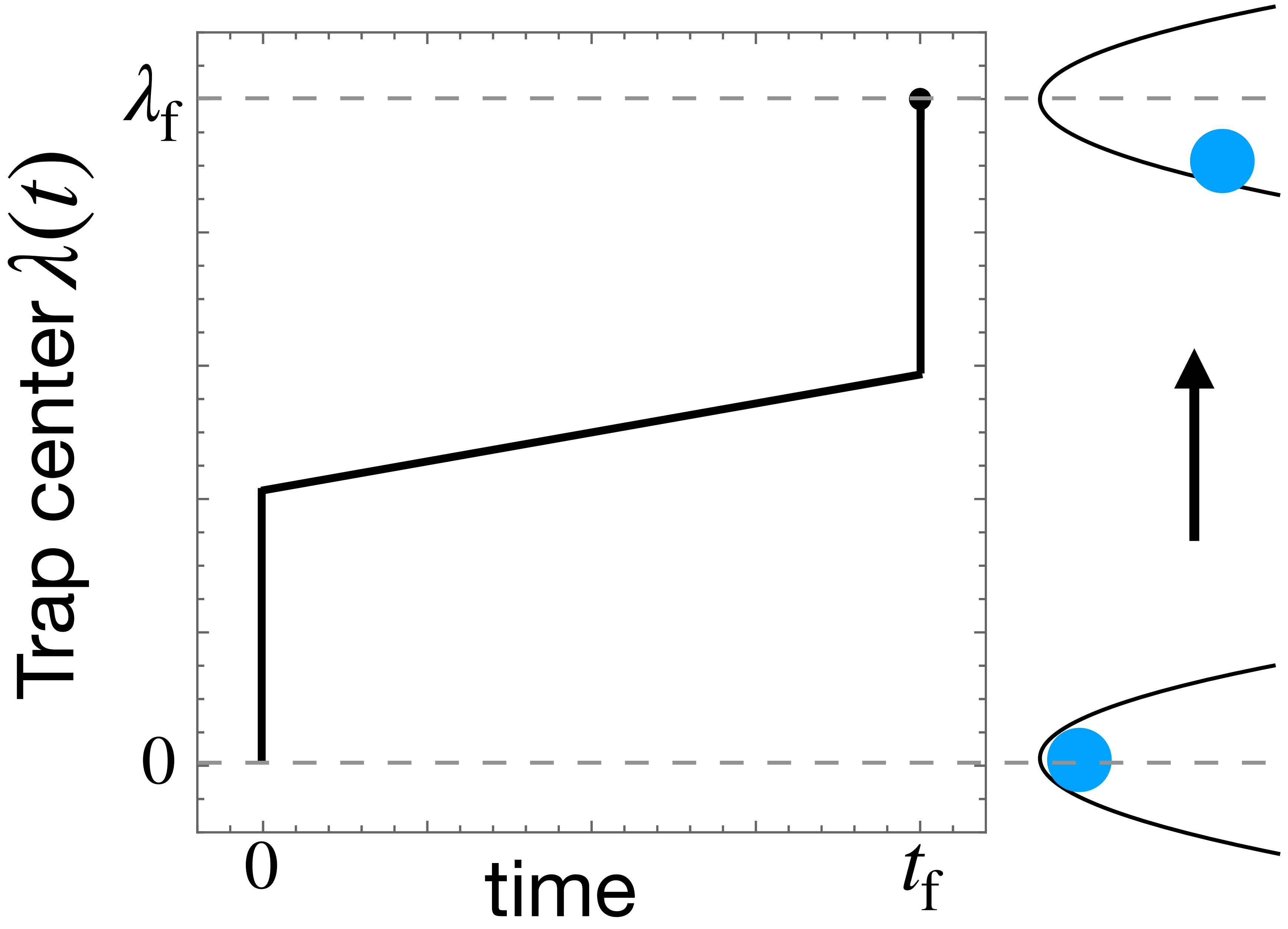
$$\boxed{W[X(t), \lambda(t)] = \int_0^{\lambda_\mathrm{f}} \frac{\partial V}{\partial \lambda} \circ \mathrm{d}\lambda = \kappa \int_0^{\lambda_\mathrm{f}} \dot{\lambda}(\lambda - X) \circ \mathrm{d}t}$$

- <u>Particle in viscous fluid:</u>

Langevin equation: $\tau_0 \dot{X} = -[X - \lambda(t)] + \xi$ $\quad \tau_0 = \gamma_0/\kappa$

$\langle \xi(t)\xi(t') \rangle \propto \delta(t - t')$

Euler-Langrange Eq.



Optimal protocol:

$$\lambda^*(t) = \begin{cases} 0, & t = 0 \\ \Delta\lambda^*(1 + t/\tau_0), & 0 < t < t_\mathrm{f}\,. \\ \lambda_\mathrm{f}, & t = t_\mathrm{f} \end{cases}$$

$$\Delta\lambda^* = \tau_0/(t_\mathrm{f} + 2\tau_0)\lambda_\mathrm{f}$$

[Schmiedl & Seifert, PRL 98, 10830 (2007)]

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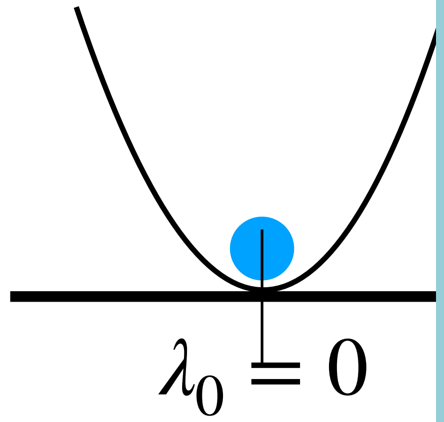
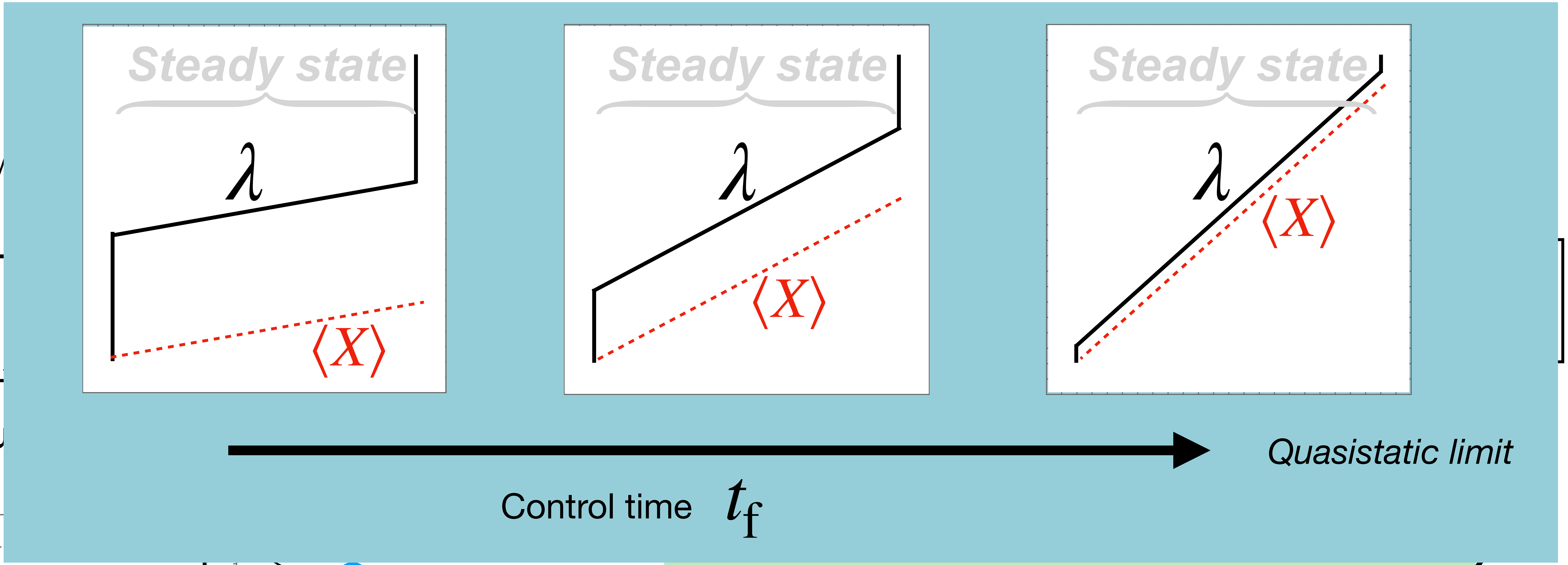
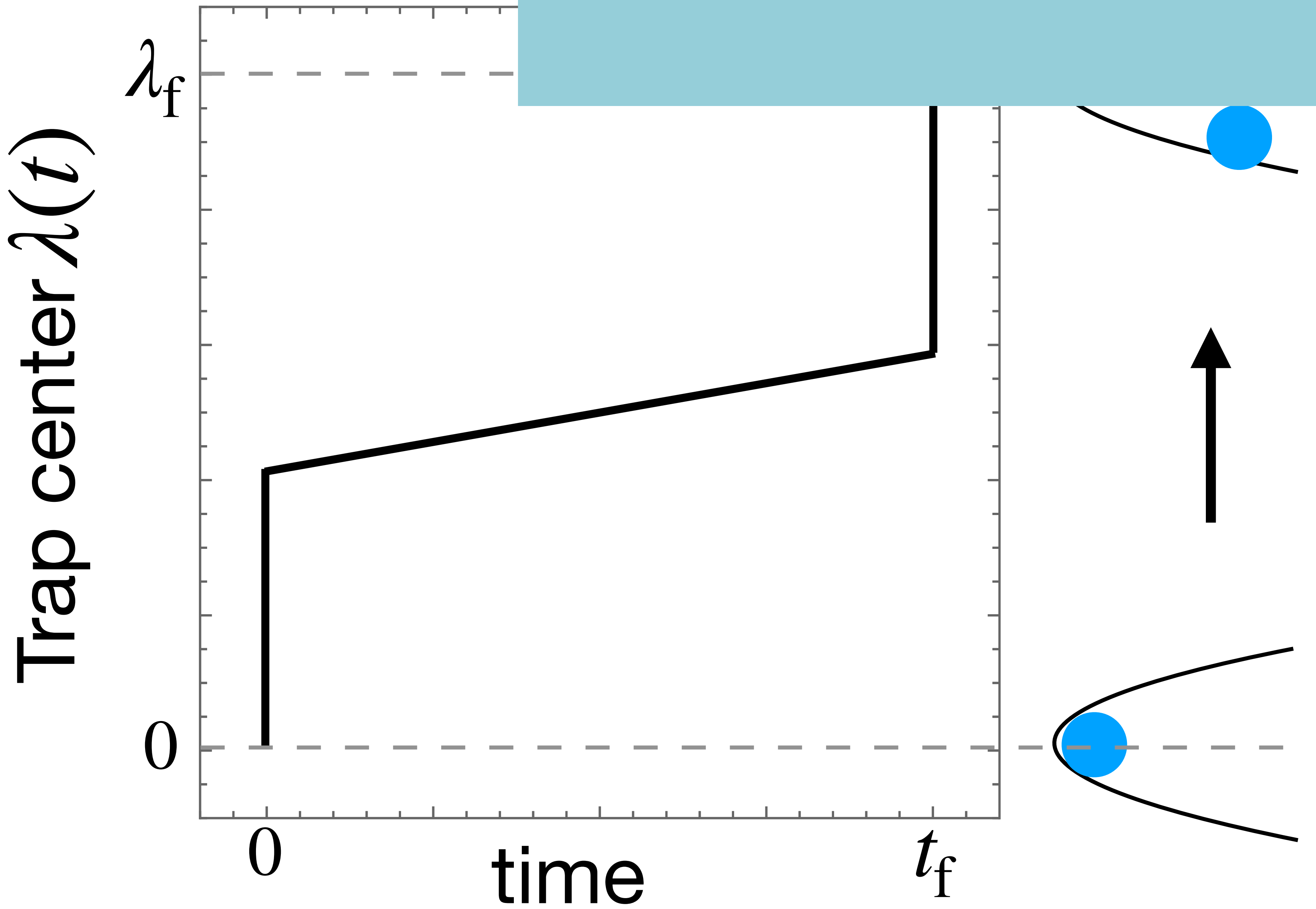
- Work spen



- Particle in vi

Langevin equ





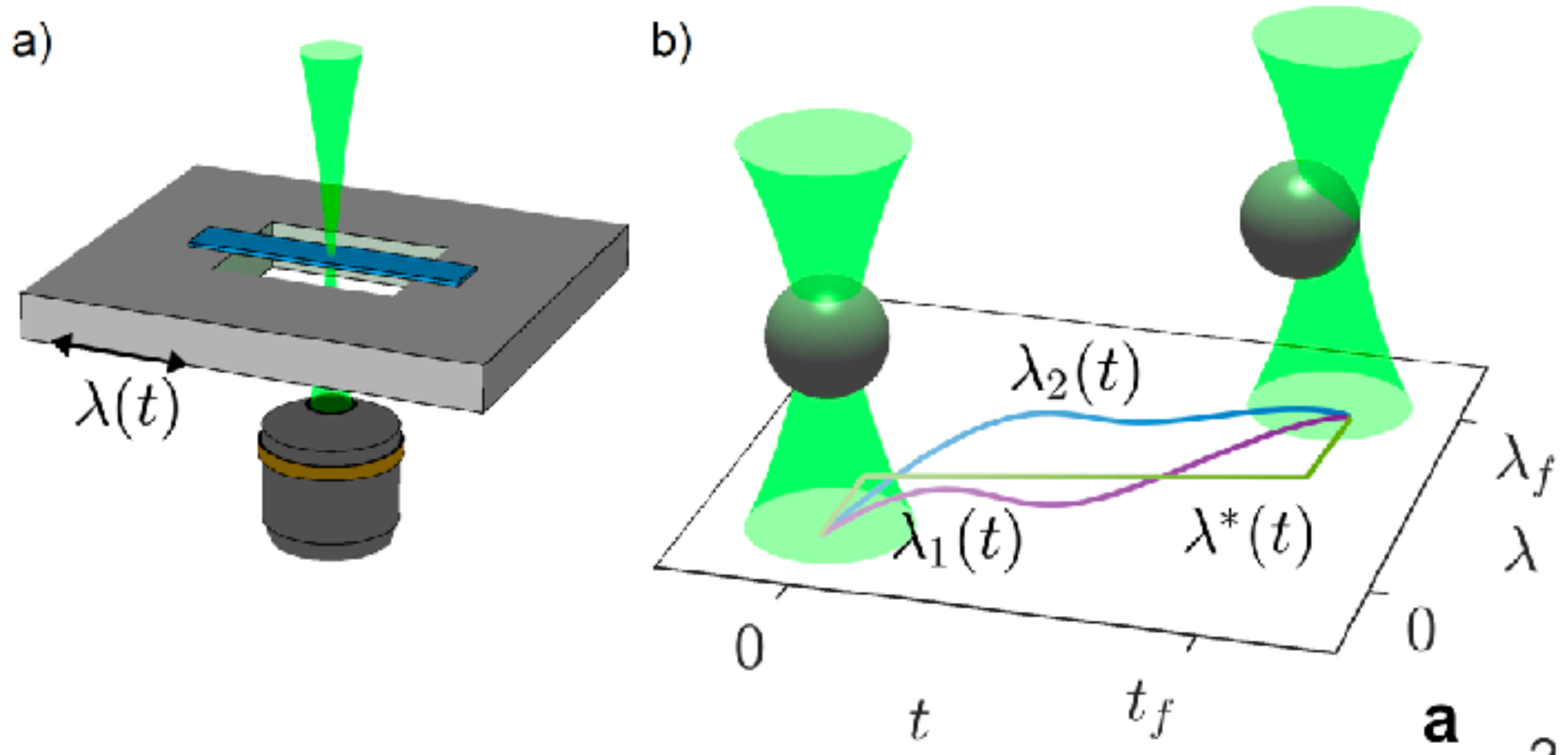
Optimal protocol:

$$\lambda^*(t) = \begin{cases} 0, & t = 0 \\ \Delta\lambda^*(1 + t/\tau_0), & 0 < t < t_f \, . \\ \lambda_f, & t = t_f \end{cases}$$

$$\Delta\lambda^* = \tau_0/(t_f + 2\tau_0)\lambda_f$$

[Schmiedl & Seifert, PRL 98, 10830 (2007)]

Minimum work dragging problem: viscous case

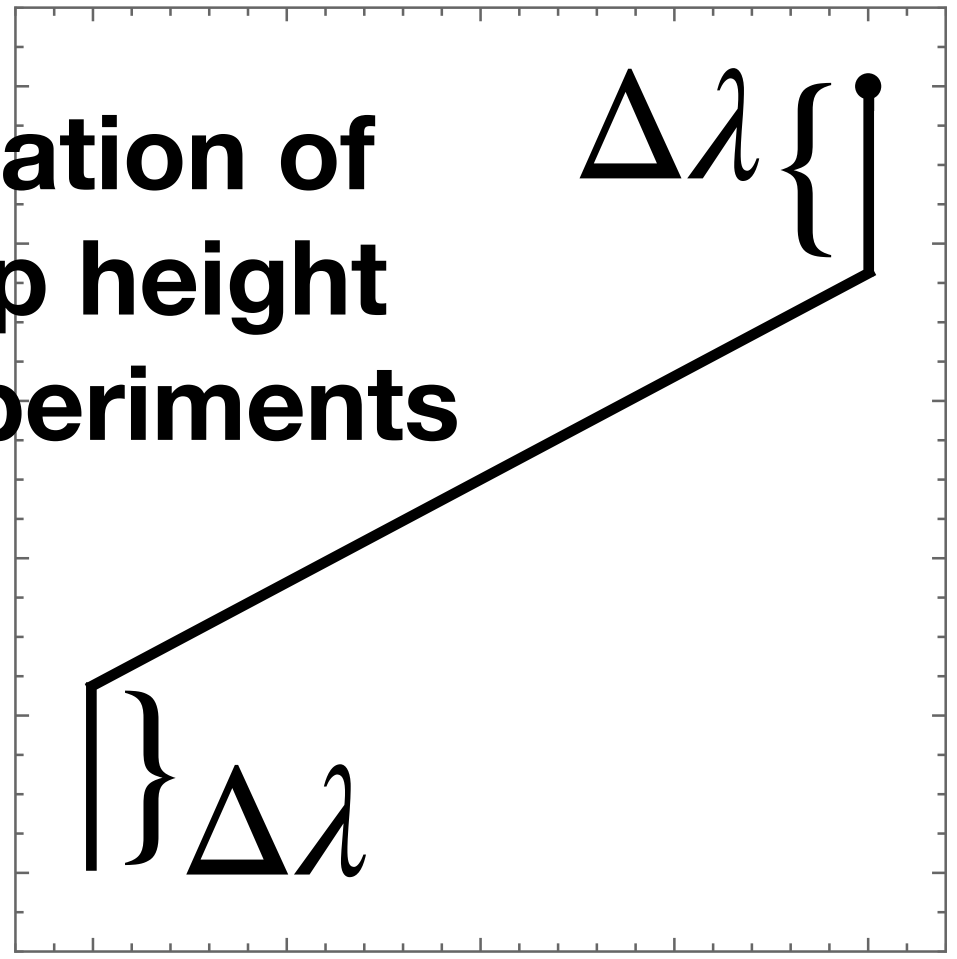


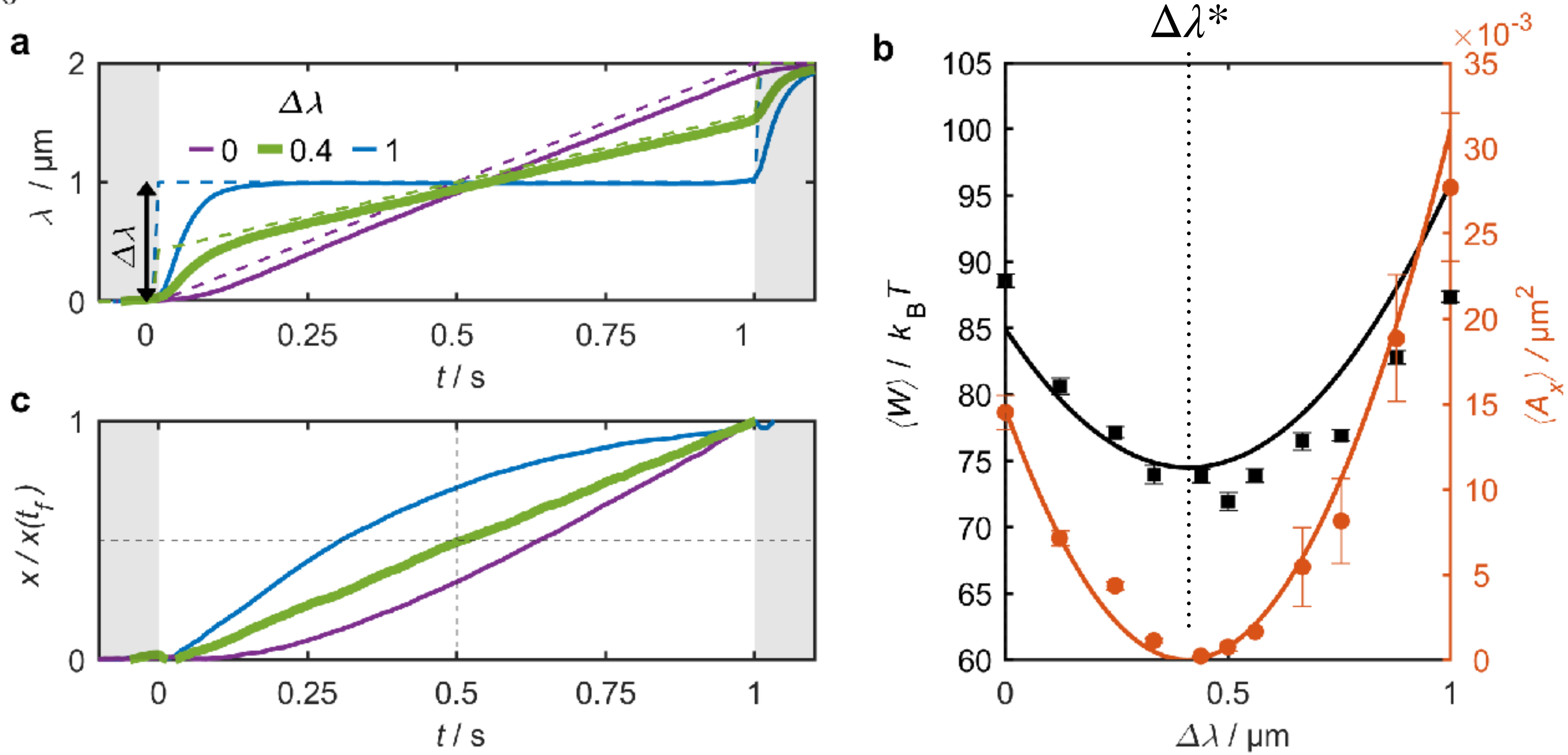
$r = 2.7\,\mu$m

$t_\mathrm{f} = 1$ s, $\lambda_\mathrm{f} = 2\,\mu$m

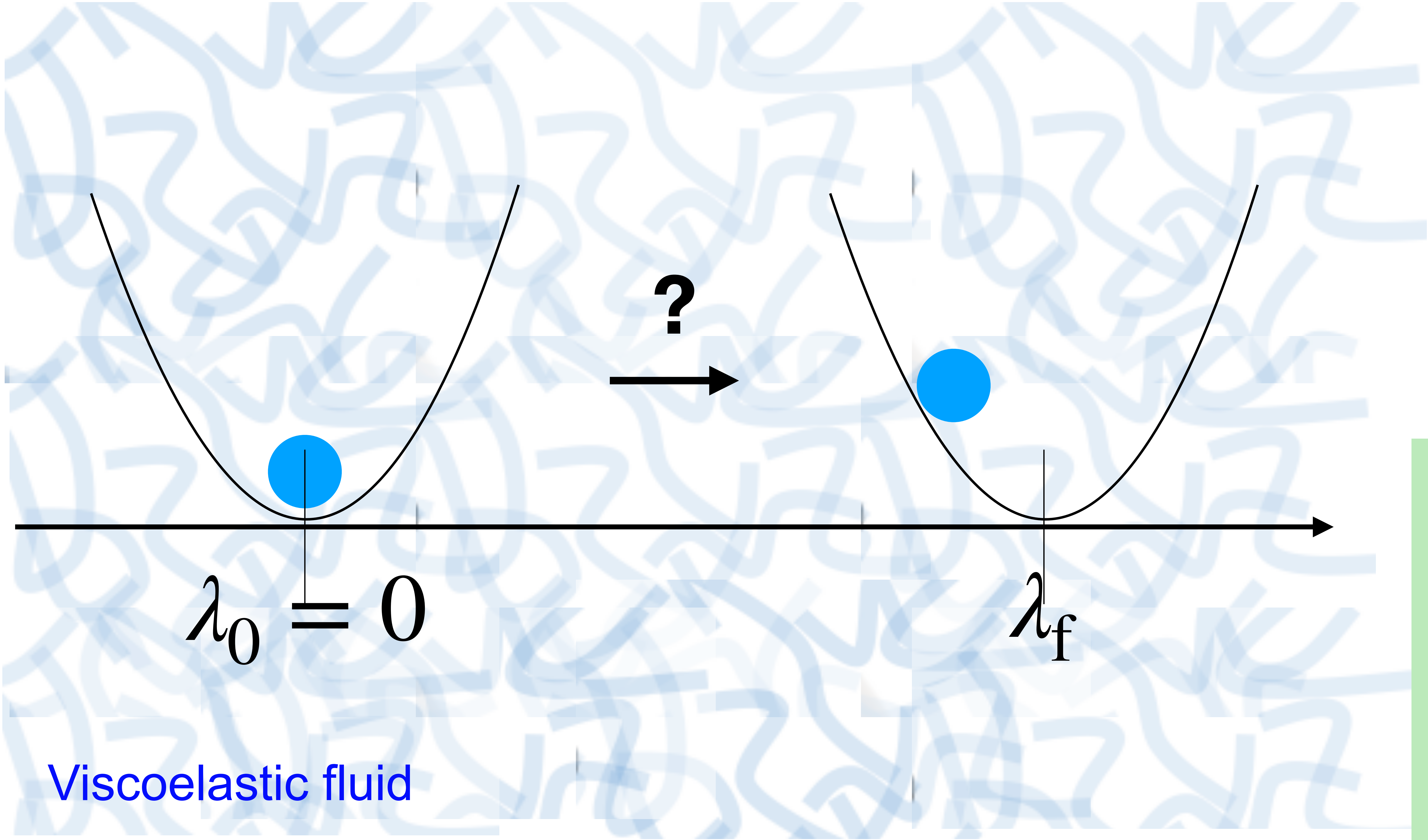
$\tau_0 = 0.33\,s$

Variation of jump height in experiments



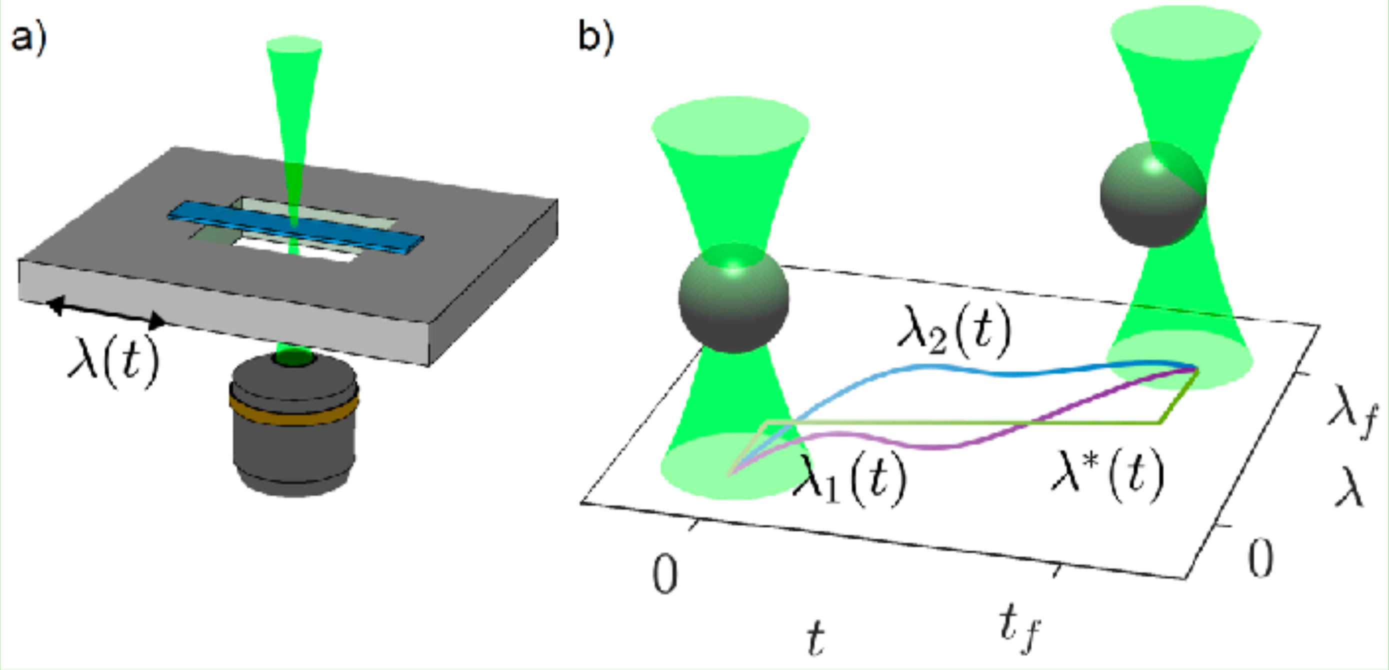


Minimum work dragging problem, non-Markovian case



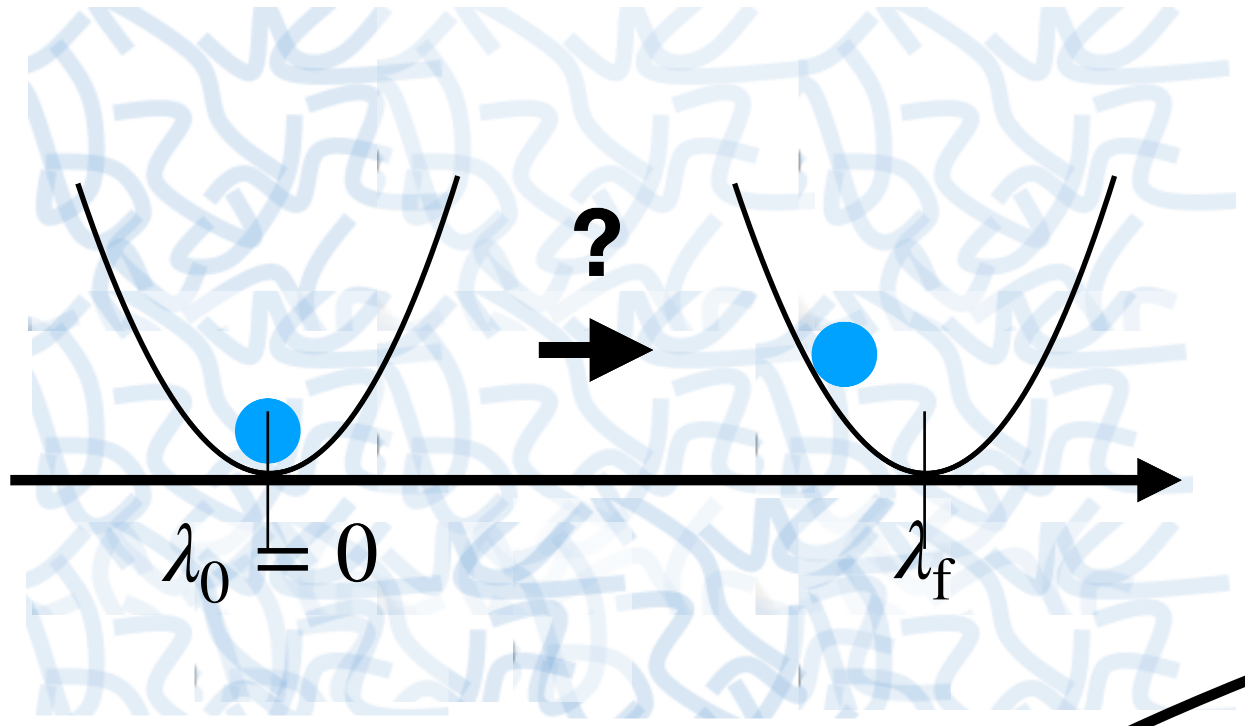
Wormlike micelles/
polymer solution

8mM mixture of CPyCl and NaSal



$t_\mathrm{f} = 10\,\mathrm{s}, \;\; r = 2.7\,\mu\mathrm{m}$
$\lambda_\mathrm{f} = 3\,\mu\mathrm{m}$ $\tau_\mathrm{b} \approx 17\,s$

Minimum work dragging problem, non-Markovian case



(non-Markovian) Generalized Langevin equation:

$$\tau_0 \dot{X}(t) + \underbrace{\int_{-\infty}^{t} \Gamma(t-t')\dot{X}(t')\mathrm{d}t'}_{\text{retarded friction}} = -V' + \xi(t) + \underbrace{\eta(t)}_{\text{coloured noise}}$$

$$\langle \eta(t) \rangle = 0$$

"Maxwell model":

$$\Gamma(t-t') \propto \langle \eta(t)\eta(t') \rangle \propto \sum_i e^{-|t-t'|/\tau_{\mathrm{b,i}}}$$

Memory time: τ_{b}

N coupled Markovian eqs.

$$\tau_p \dot{X} = -V' - \sum_i (X - Y_i) + \xi_X$$

$$\tau_{b,1}\dot{Y}_1 = -(Y_1 - X) + \xi_{Y,1}$$

$$\tau_{b,2}\dot{Y}_2 = -(Y_2 - X) + \xi_{Y,2}$$

Minimum work dragging problem, non-Markovian case

Generalized Langevin eq:

$$\tau_0 \int_{-\infty}^{t} \Gamma(t-t')\dot{X}(t')\mathrm{d}t' = -\left[X-\lambda(t)\right] + \sqrt{2k_\mathrm{B}T/\kappa^2}\,\eta(t)\,,$$

$\downarrow$ + one fictitious variable per timescale of medium

N coupled Markovian Langevin eq.

$\downarrow$

$$\langle W\rangle = \kappa \int_0^{t_\mathrm{f}} \dot{\lambda}(\lambda - \langle X\rangle)\,\mathrm{d}t$$

$$= \ldots = \int_0^{t_\mathrm{f}} \mathscr{L}[\langle X\rangle, \langle \dot{X}\rangle, \langle X_\mathrm{b}\rangle, \langle \dot{X}_\mathrm{b}\rangle]\,\mathrm{d}t$$

Euler-Langrange Eq.

$$\dot{\boldsymbol{z}} = \mathbb{A}\boldsymbol{z}, \quad \text{with} \quad \boldsymbol{z} = \begin{pmatrix} x & v & x_b & \Lambda \end{pmatrix}^T,$$

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2\tau_b\tau_p} & 0 & -\frac{1}{2\tau_b\tau_p} & -\frac{1}{2\tau_p} \\ \frac{1}{\tau_b} & 0 & -\frac{1}{\tau_b} & 0 \\ 0 & -\frac{1}{\tau_b} & 0 & \frac{1}{\tau_b} \end{pmatrix}.$$

$\downarrow$ "Project back" onto physical variable x

$$x^*(t) = \frac{1}{2(\tau_b+\tau_p)}\left[2\tau_p\mathcal{C}_1 t + \tau_b(\tau_b+t)\mathcal{C}_2 + \tau_b^2\left\{-\mathcal{C}_2\cosh\frac{\sqrt{\tau_b+\tau_p}}{\tau_b\sqrt{\tau_p}}t + \frac{\sqrt{\tau_p}(2\mathcal{C}_1-\mathcal{C}_2)}{\sqrt{\tau_b+\tau_p}}\sinh\frac{\sqrt{\tau_b+\tau_p}}{\tau_b\sqrt{\tau_p}}t\right\}\right]$$

$$x_b^*(t) = \frac{1}{2(\tau_b+\tau_p)}\left[2\tau_p\mathcal{C}_1 + \tau_b\mathcal{C}_2 + \tau_b(2\mathcal{C}_1-\mathcal{C}_2)\cosh\frac{\sqrt{\tau_b+\tau_p}}{\tau_b\sqrt{\tau_p}}t - \tau_b\frac{\sqrt{\tau_b+\tau_p}}{\sqrt{\tau_p}}\sinh\frac{\sqrt{\tau_b+\tau_p}}{\tau_b\sqrt{\tau_p}}t\right]$$

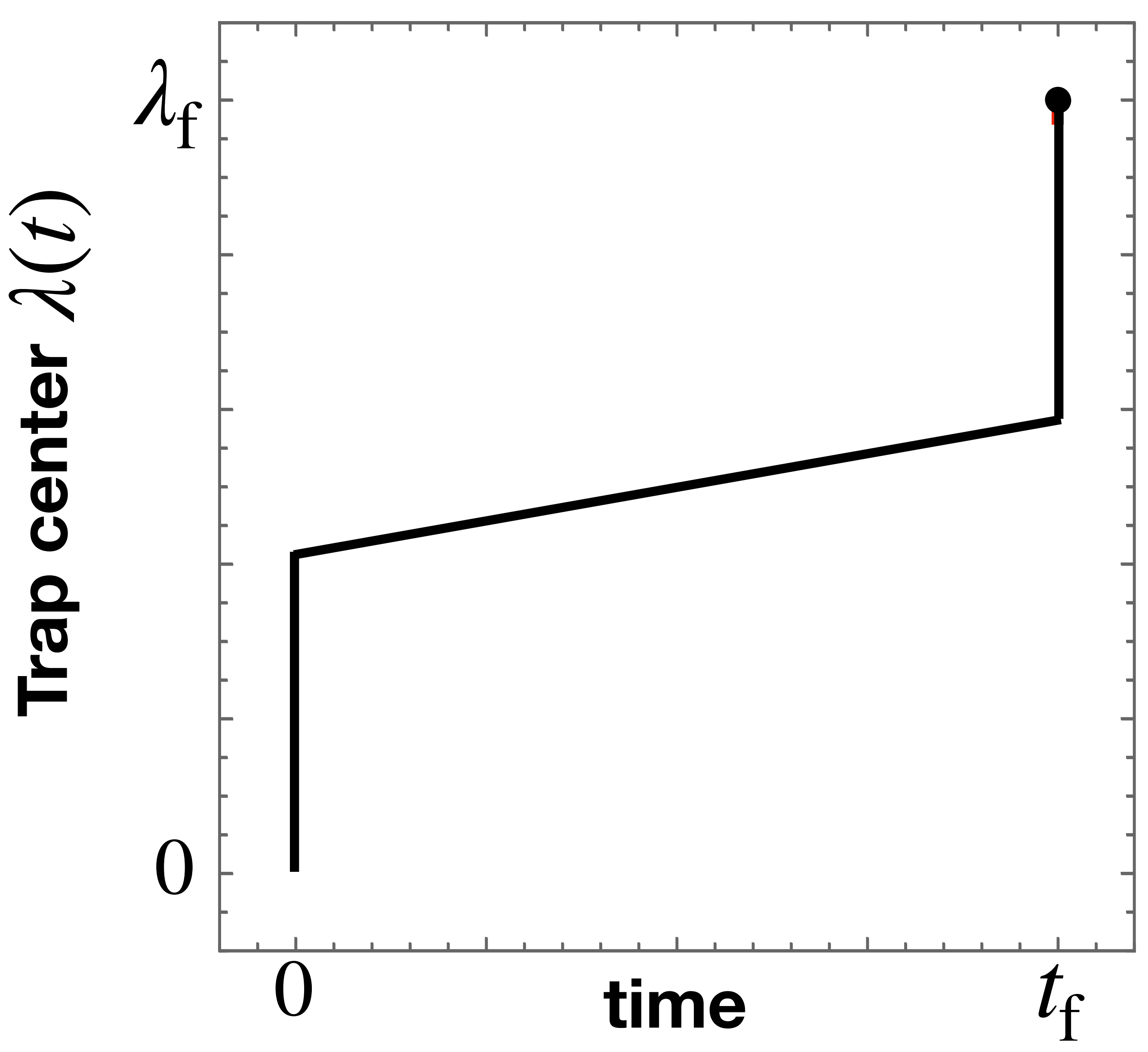
$$\lambda^*(t) = \frac{1}{2(\tau_b+\tau_p)k}\Big[2\tau_p(\tau_b+\tau_p+kt)\mathcal{C}_1 + \tau_b(\tau_b+\tau_p+\tau_b k+kt)\mathcal{C}_2$$
$$-\tau_b(\tau_b+\tau_p+\tau_b k)\mathcal{C}_2\cosh\frac{\sqrt{\tau_b+\tau_p}}{\tau_b\sqrt{\tau_p}}t + \frac{\tau_b\sqrt{\tau_p}(\tau_b+\tau_p+\tau_b k)(2\mathcal{C}_1-\mathcal{C}_2)}{\sqrt{\tau_b+\tau_p}}\sinh\frac{\sqrt{\tau_b+\tau_p}}{\tau_b\sqrt{\tau_p}}t\Big].$$

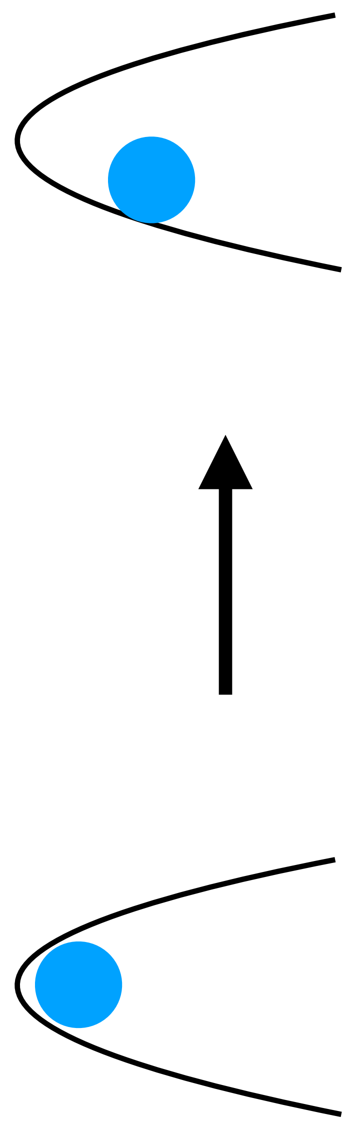
Minimum work dragging problem, non-Markovian case

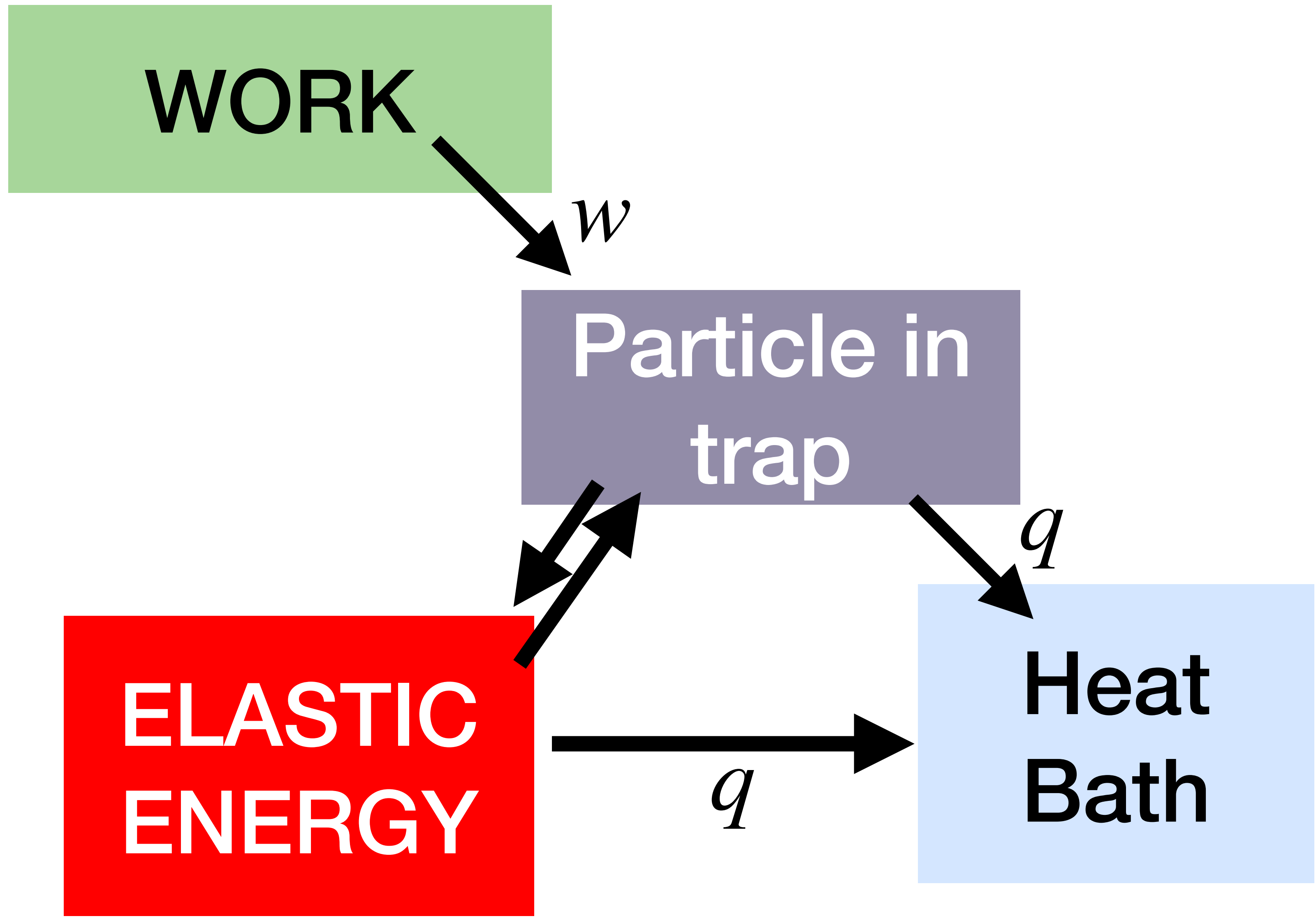
- Viscous Markovian case (black line)
- **Viscoelastic non-Markovian case (red line)**

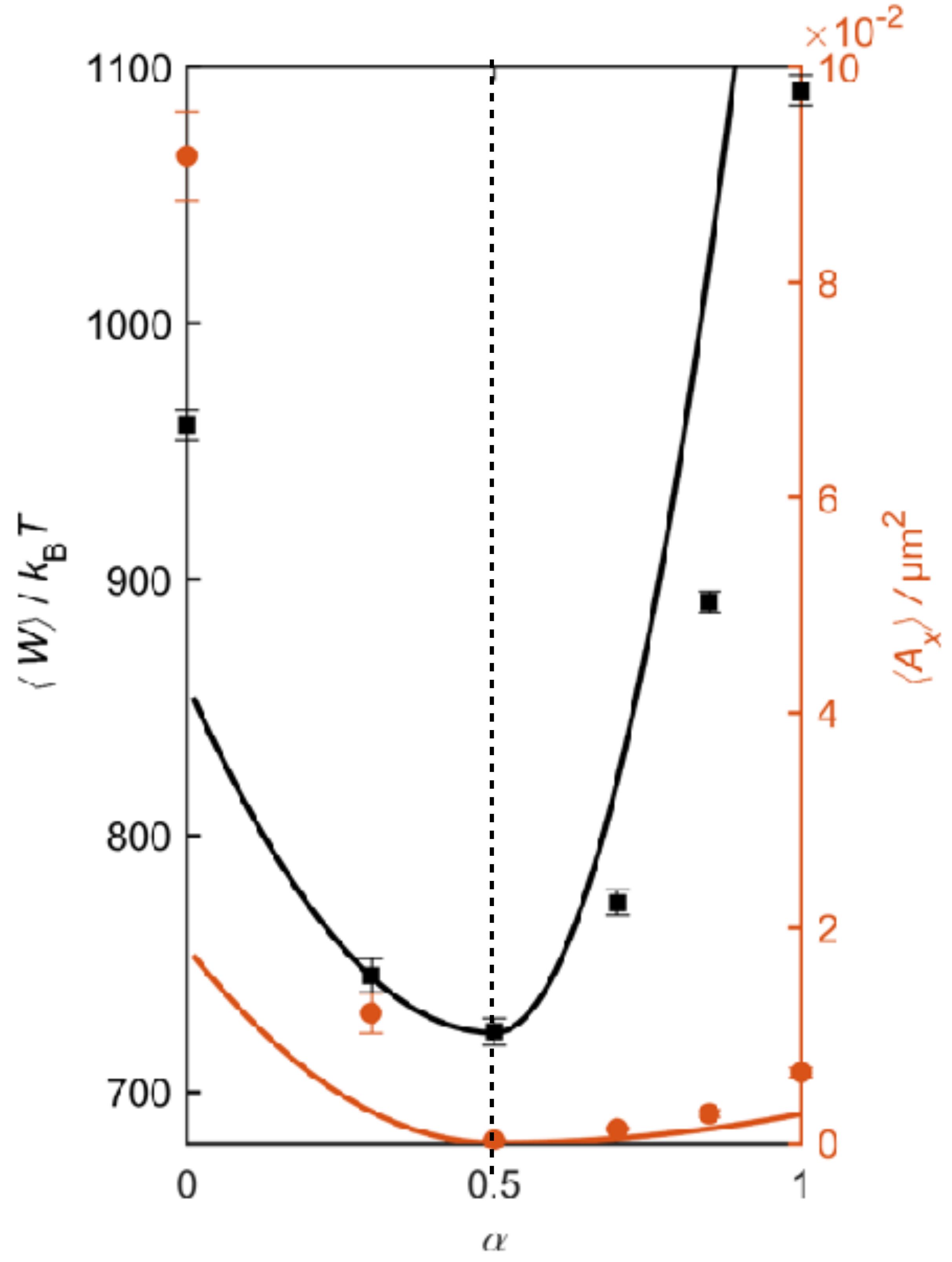
→ Still jumps at beginning and end

→ No constant-power “NESS regime”









Time-reversal symmetry — criterion for optimality

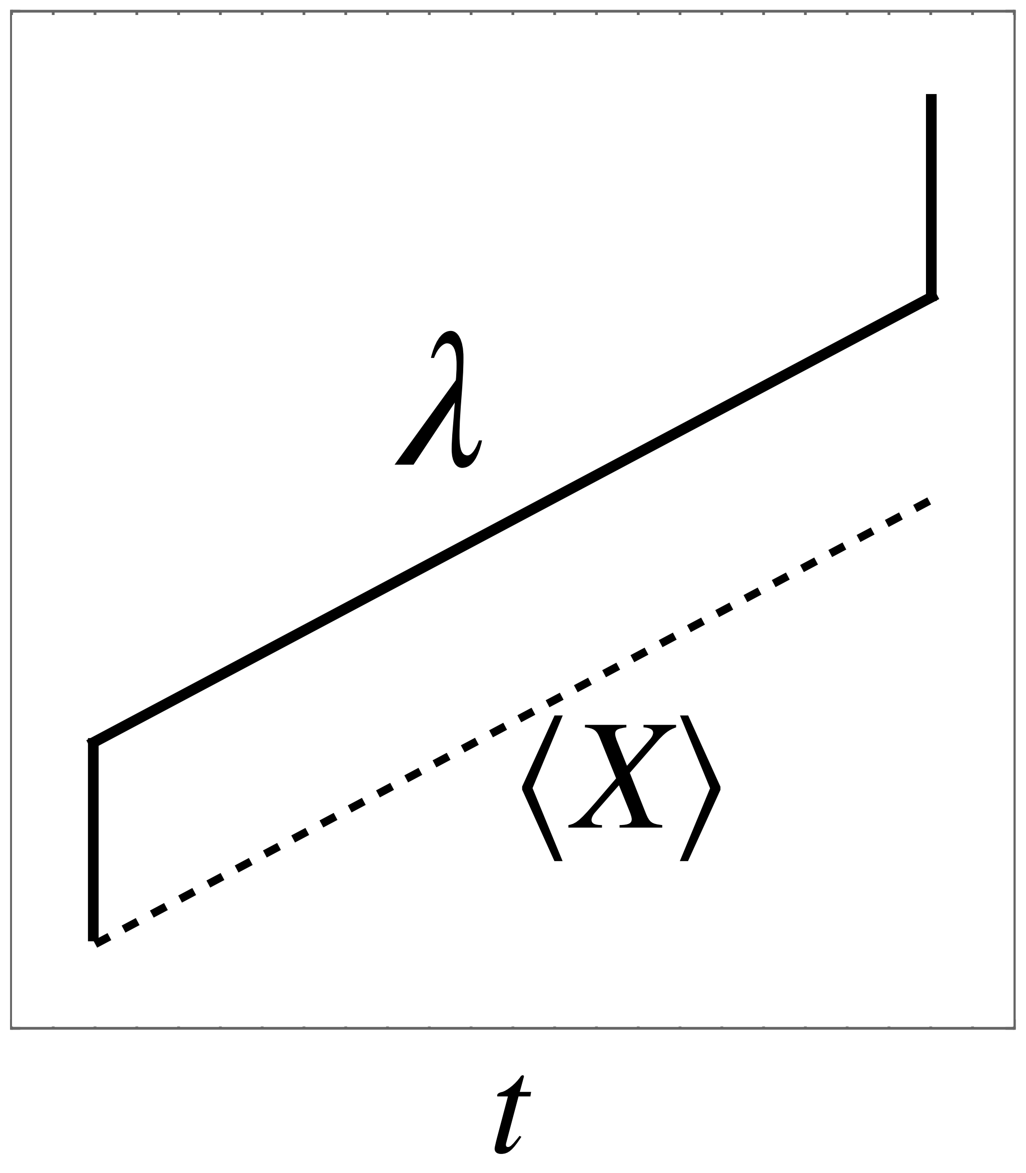
$f = \{\langle X \rangle, \lambda\}$

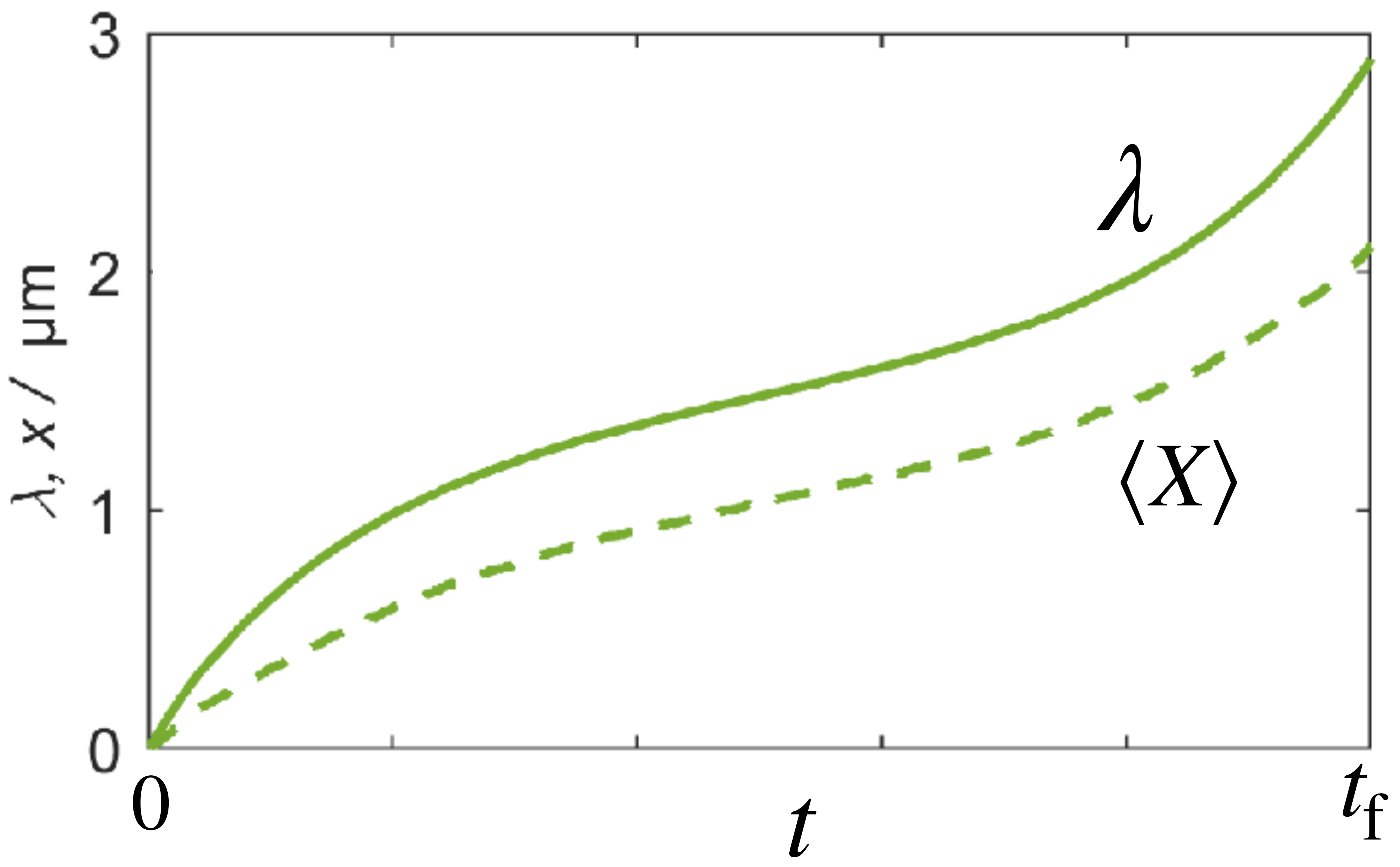
Time-reversal symmetry

$$f(t) = f(t_\mathrm{f}) - f(t_\mathrm{f} - t)$$

$$\dot{f}(t) = -\dot{f}(t_\mathrm{f})$$

Optimal solutions: $\langle X \rangle$ and λ are **time-reversal symmetric** $\rightarrow$ *Why??*



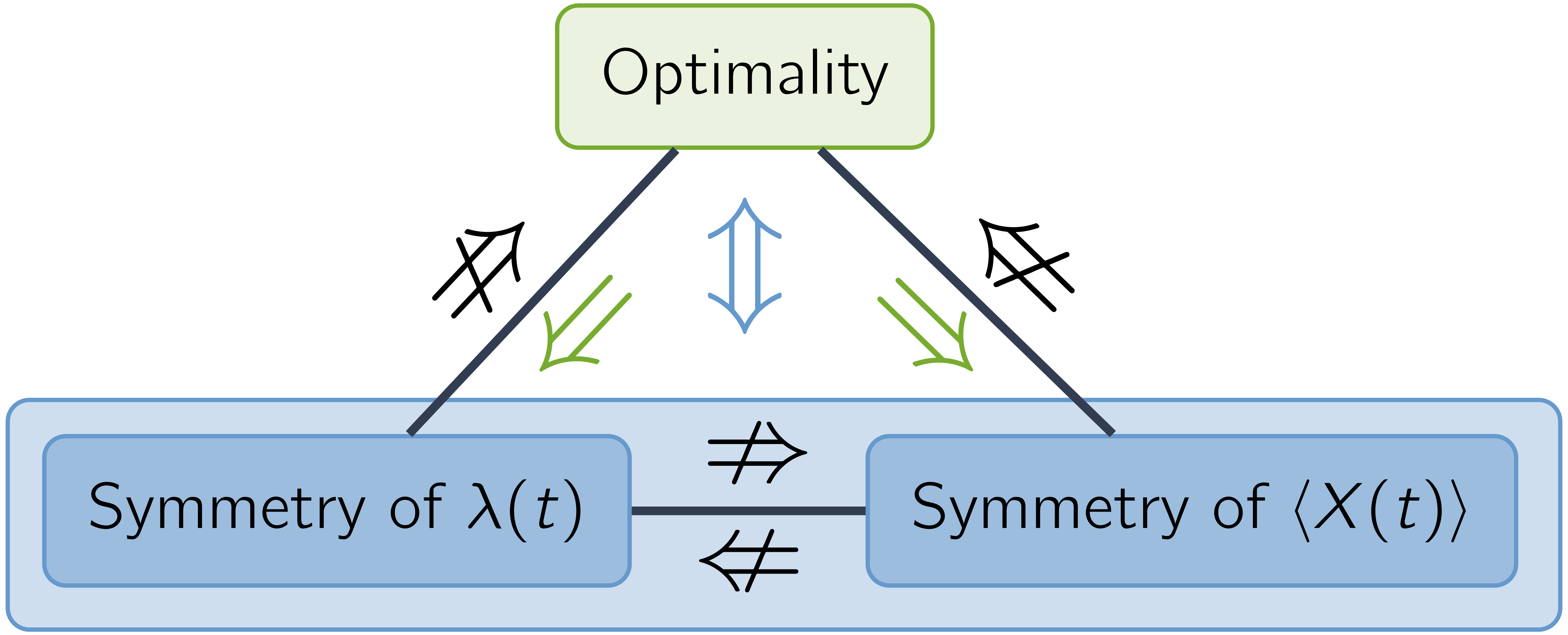


Time-reversal symmetry — criterion for optimality

Time-reversal symmetry

$$f(t) = f(t_\mathrm{f}) - f(t_\mathrm{f} - t)$$

$f = \{\langle X \rangle, \lambda\}$



Proof of Symmetry: General Generalized Langevin E (GGLE)

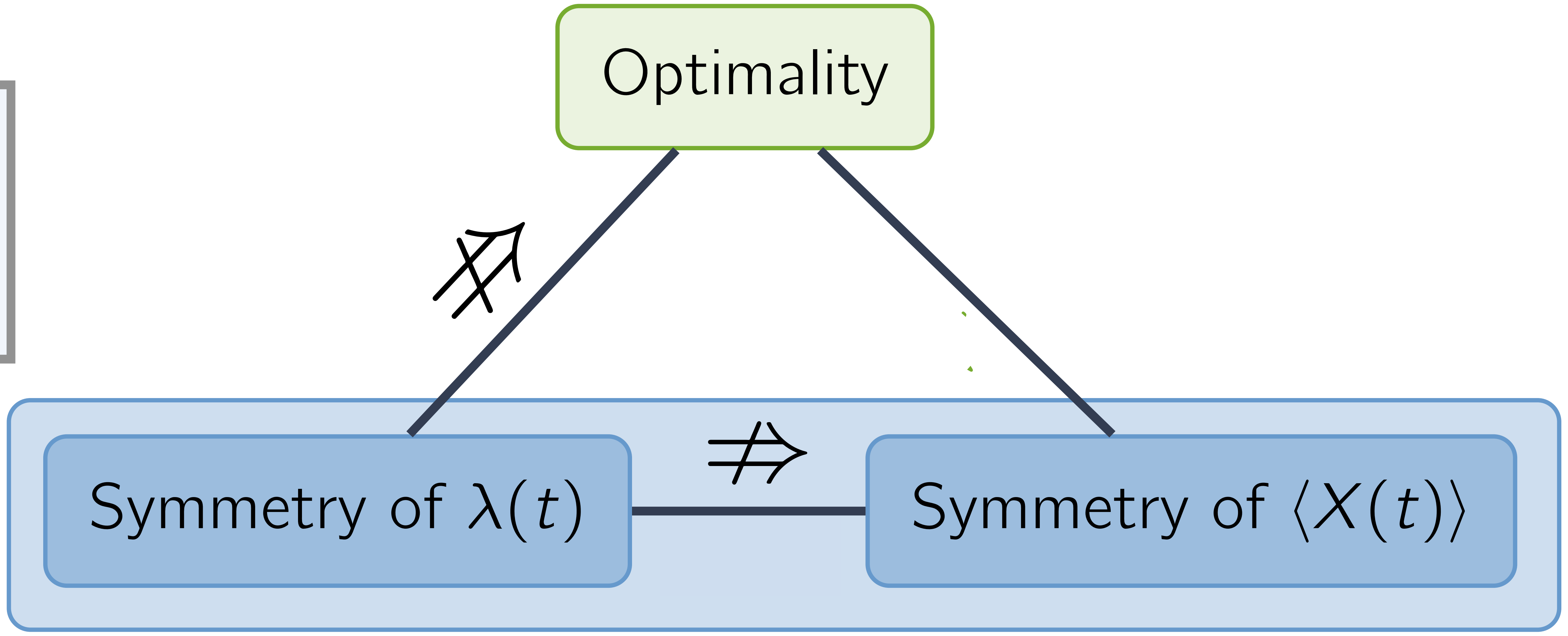
$$m\ddot{X} + \int_{-\infty}^{t} \Gamma(t - t')\dot{X}(t')\mathrm{d}t' = -\kappa[X - \lambda(t)] + \eta(t)$$

Proof using linearity of GGLE, causality, and time-translation invariance of response function and memory kernel Γ *…*

Proof of symmetry property

General Generalized Langevin Equation (GGLE)

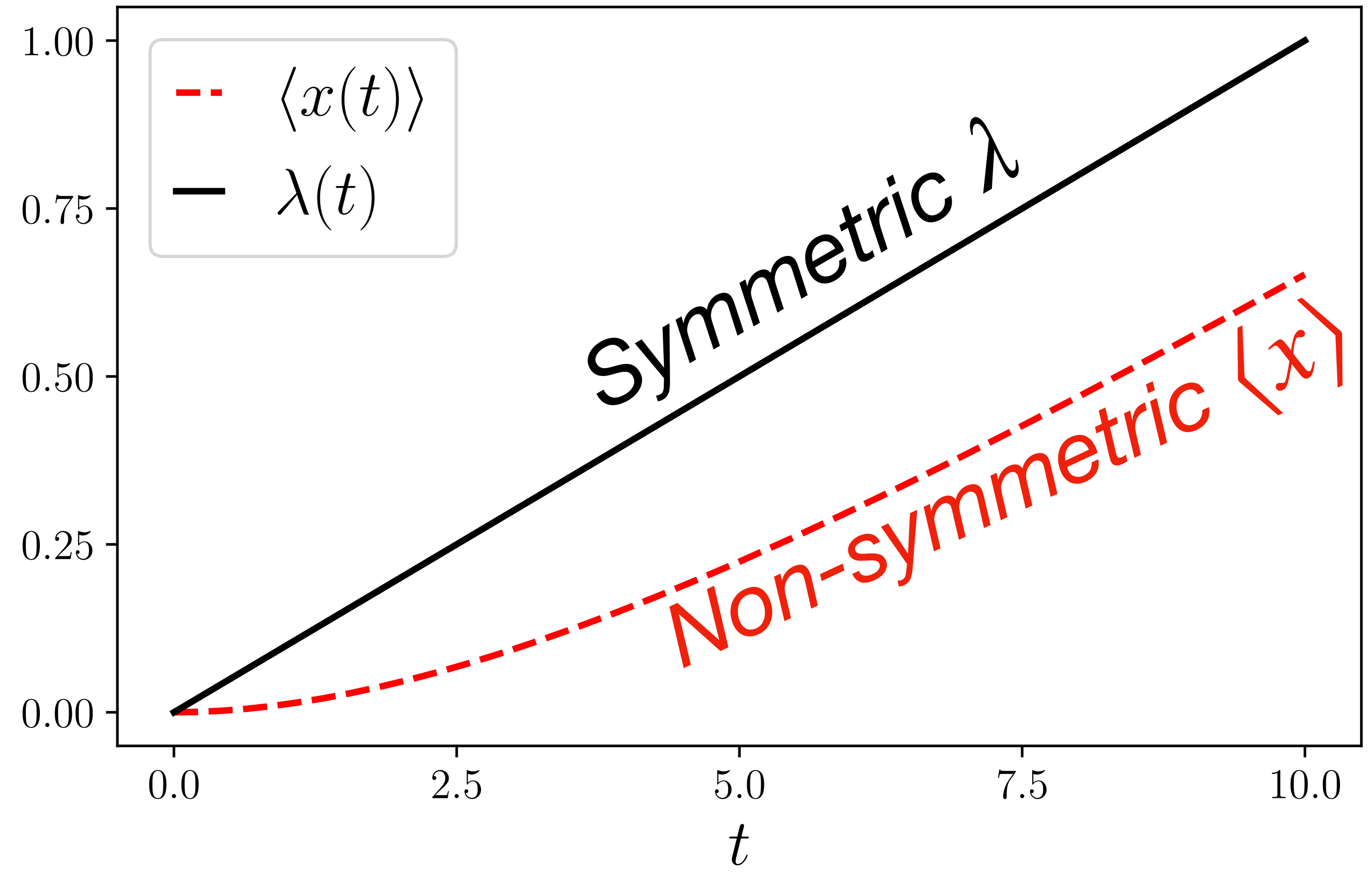
$$m\ddot{X} + \int_{-\infty}^{t} \Gamma(t-t')\dot{X}(t')\mathrm{d}t' = -\kappa[X-\lambda(t)] + \eta(t)$$



Counterexample I: Linear protocol

$$\lambda(t) = (\lambda_\mathrm{f}/t_\mathrm{f})t$$

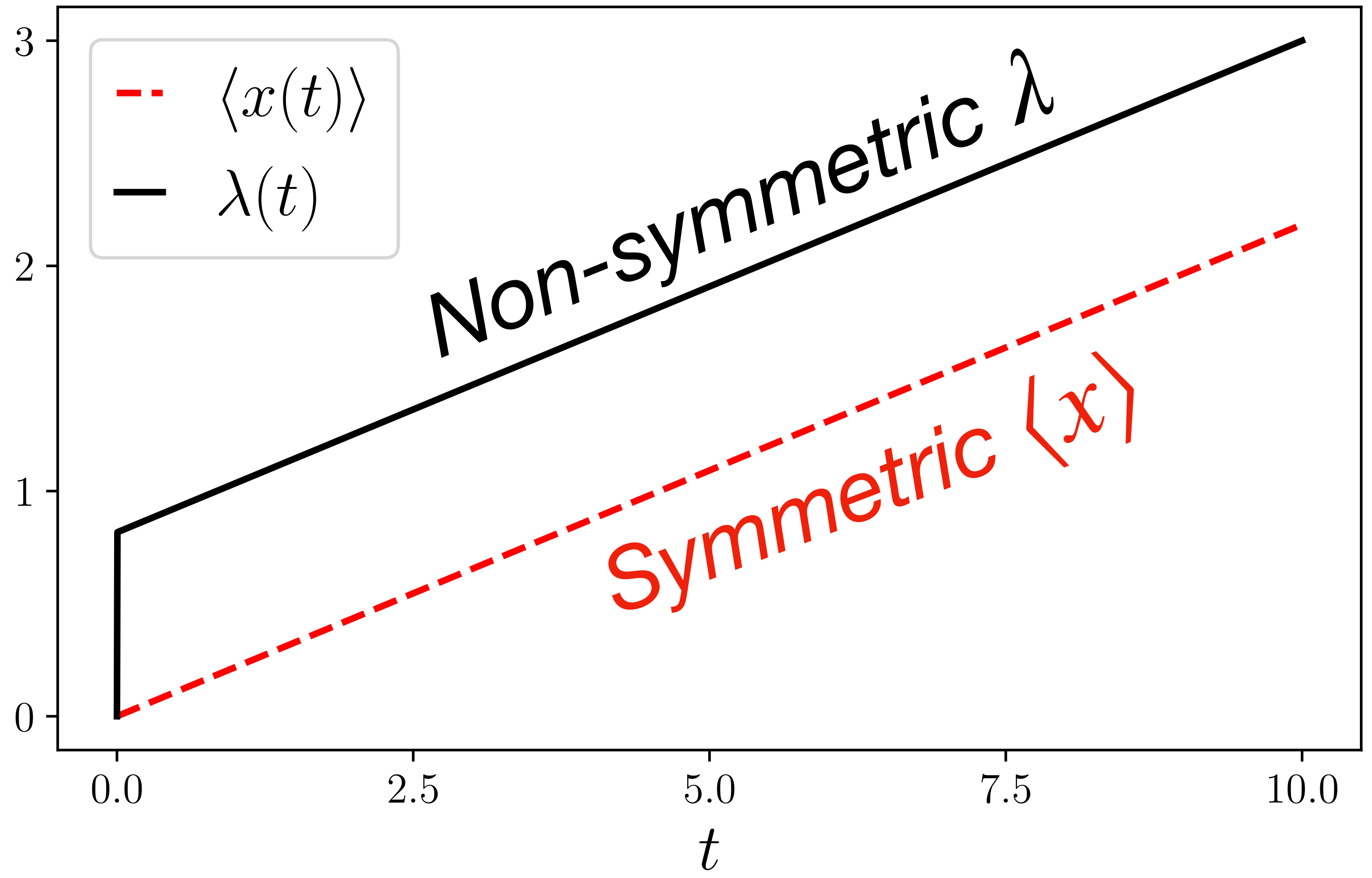
$$\Rightarrow \langle X(t)\rangle = (\lambda_\mathrm{f}/t_\mathrm{f})[t - \tau_0(1-e^{-t/\tau_0})]$$



Counterexample II: Single jump protocol

$$\lambda(t>0) = \frac{\lambda_\mathrm{f}}{(t_\mathrm{f}+\tau_0)}(\tau_0 + t)$$

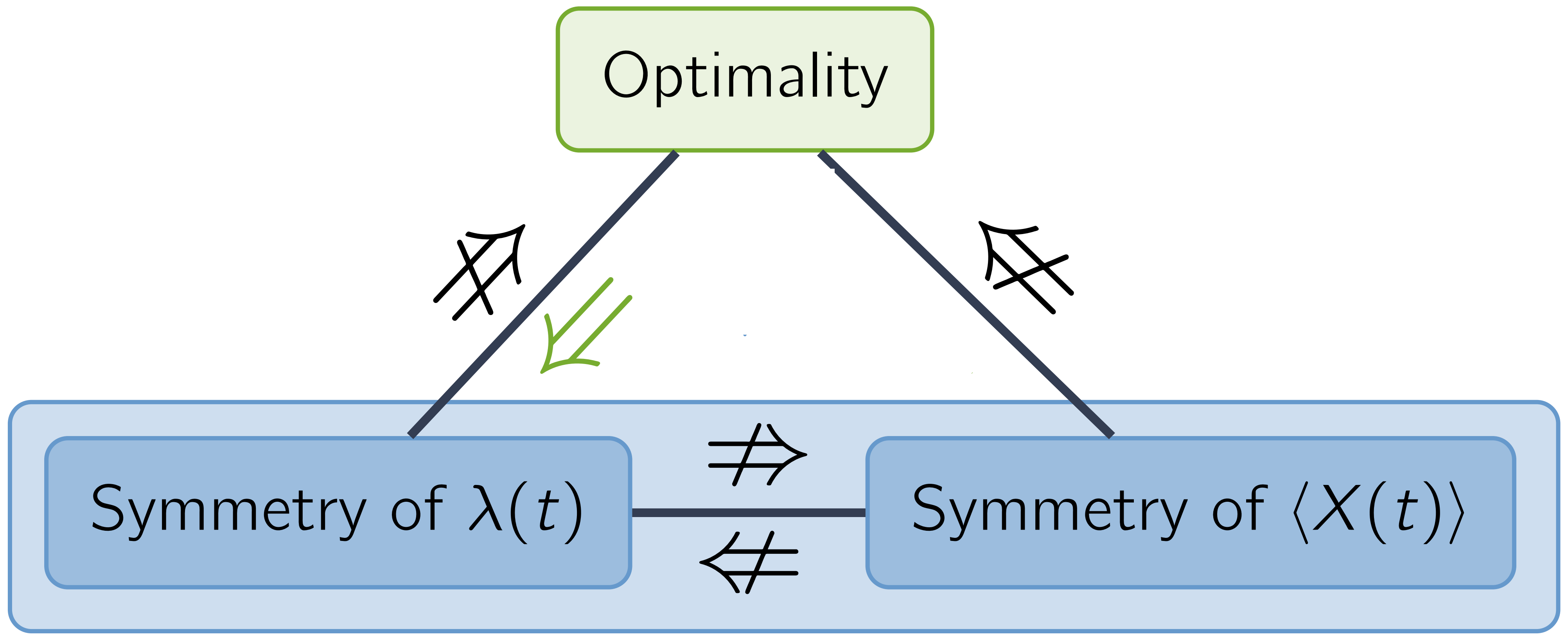
$$\Rightarrow \langle X(t)\rangle = \frac{\lambda_\mathrm{f}}{(t_\mathrm{f}+\tau_0)}t$$



Proof of symmetry property

Notation : $\langle X \rangle = x$

Key Idea: rewrite mean work as functional of $\dot{\lambda}$ (or $\dot{x}$) only!



$$\boxed{m\ddot{X} + \int_{-\infty}^{t} \Gamma(t-t')\dot{X}(t')\mathrm{d}t' = -\kappa[X - \lambda(t)] + \eta(t)}$$

Laplace Domain

$\langle X \rangle = x$

$$\langle W[X(t), \lambda(t)] \rangle = \kappa \int_0^{t_\mathrm{f}} \mathrm{d}t\, \dot{\lambda}(t)\, [\lambda(t) - x(t)] = \frac{\kappa \lambda_\mathrm{f}^2}{2} + \kappa \int_0^{t_\mathrm{f}} \mathrm{d}t\, \dot{\lambda}(t) x(t)$$

$$x(t) = \int_0^t \Phi(t-t')\dot{\lambda}(t')dt'.$$

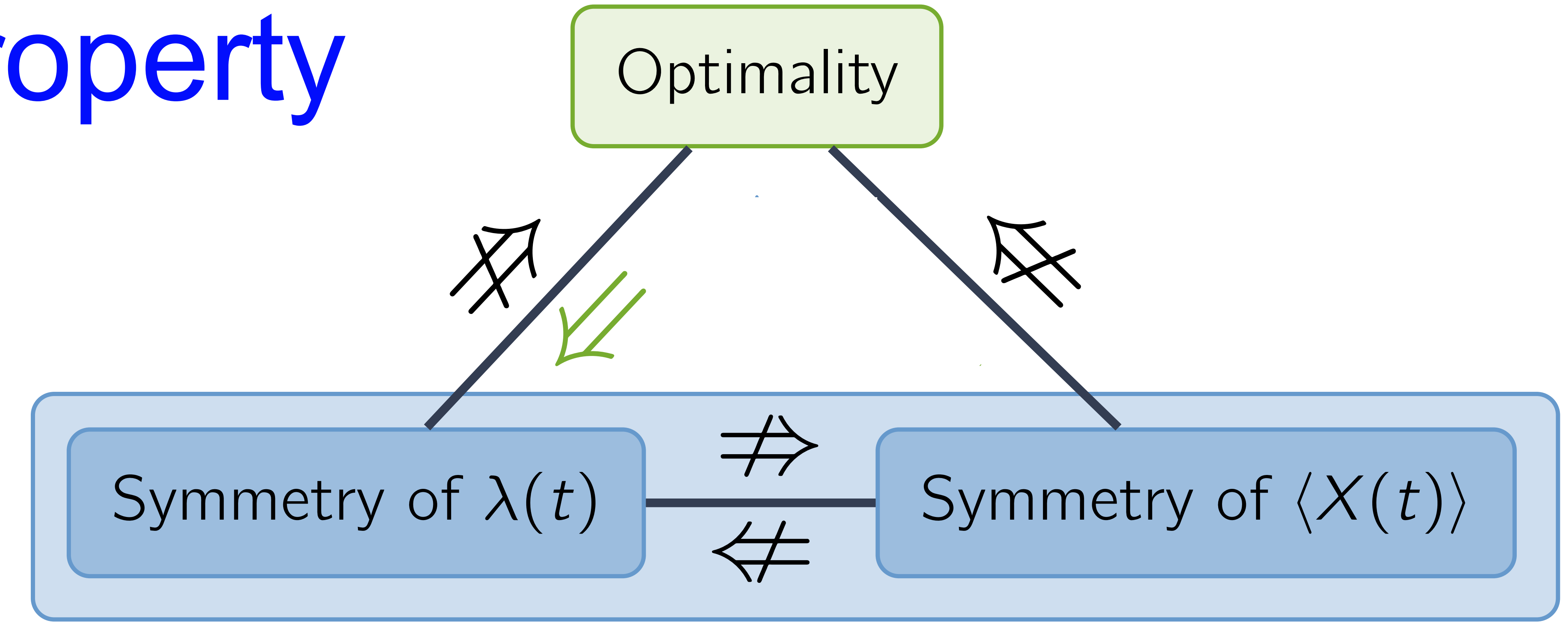
$$\hat{x}(s) = \hat{\Phi}(s) s \hat{\lambda}(s),$$

+ causality assumption

$$\hat{\Phi}(s) := \frac{\kappa}{ms^3 + \hat{\Gamma}(s)s^2 + \kappa s}$$

$$\boxed{\langle W \rangle [\lambda] = \frac{\kappa \lambda_\mathrm{f}^2}{2} + \kappa \int_0^{t_\mathrm{f}} \int_0^{t} \Phi(t-t')\dot{\lambda}(t)\dot{\lambda}(t')\, \mathrm{d}t'\mathrm{d}t}$$

Proof of symmetry property



Key Idea: rewrite mean work as functional of $\dot{\lambda}$ (or $\dot{x}$) only!

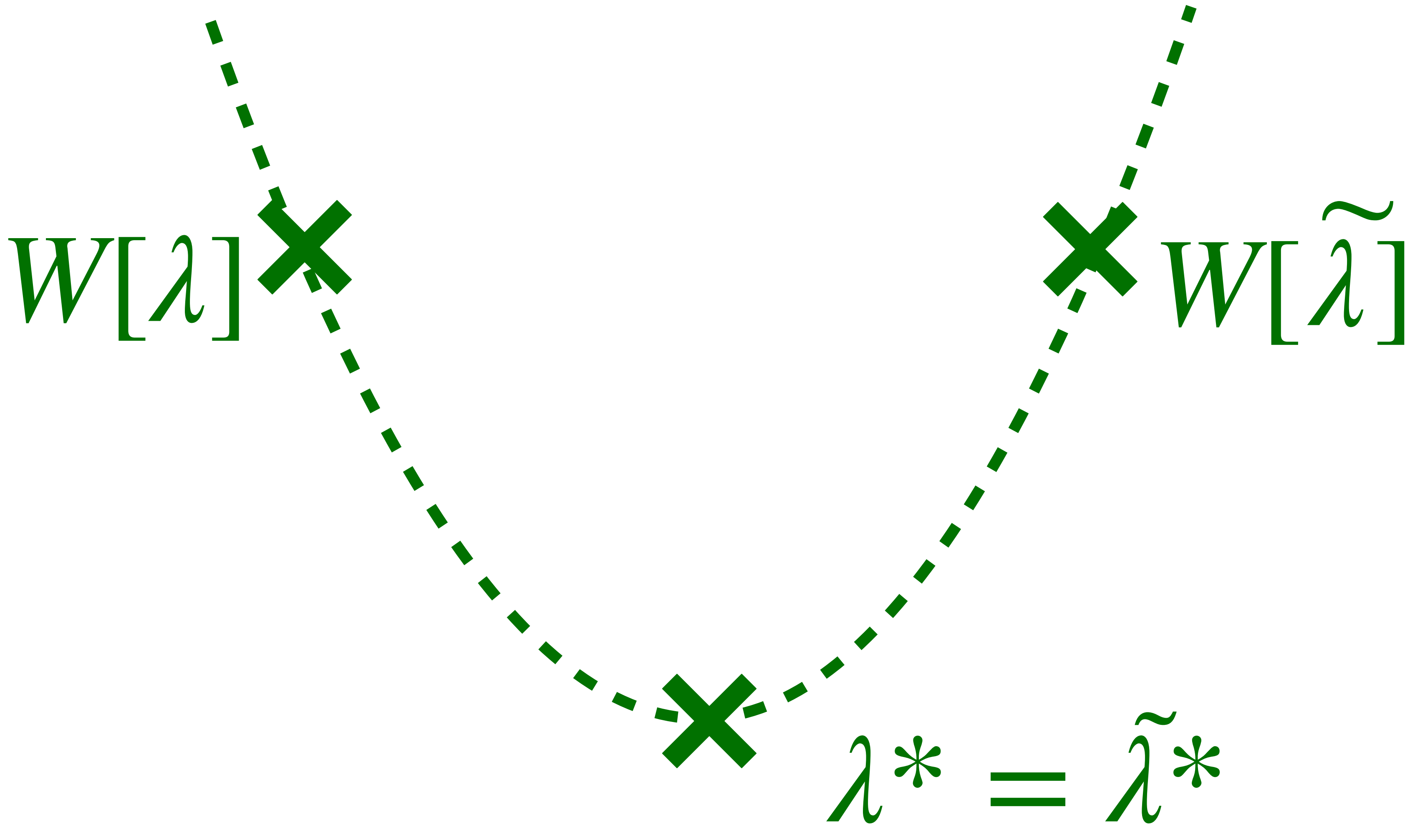
$$\boxed{\langle W\rangle\,[\lambda] = \frac{\kappa\lambda_{\mathrm{f}}^2}{2} + \kappa\int_0^{t_{\mathrm{f}}}\int_0^{t}\Phi(t-t')\dot{\lambda}(t)\dot{\lambda}(t')\,\mathrm{d}t'\mathrm{d}t}$$

<u>Quadratic</u> functional of $\dot{\lambda}$ only:

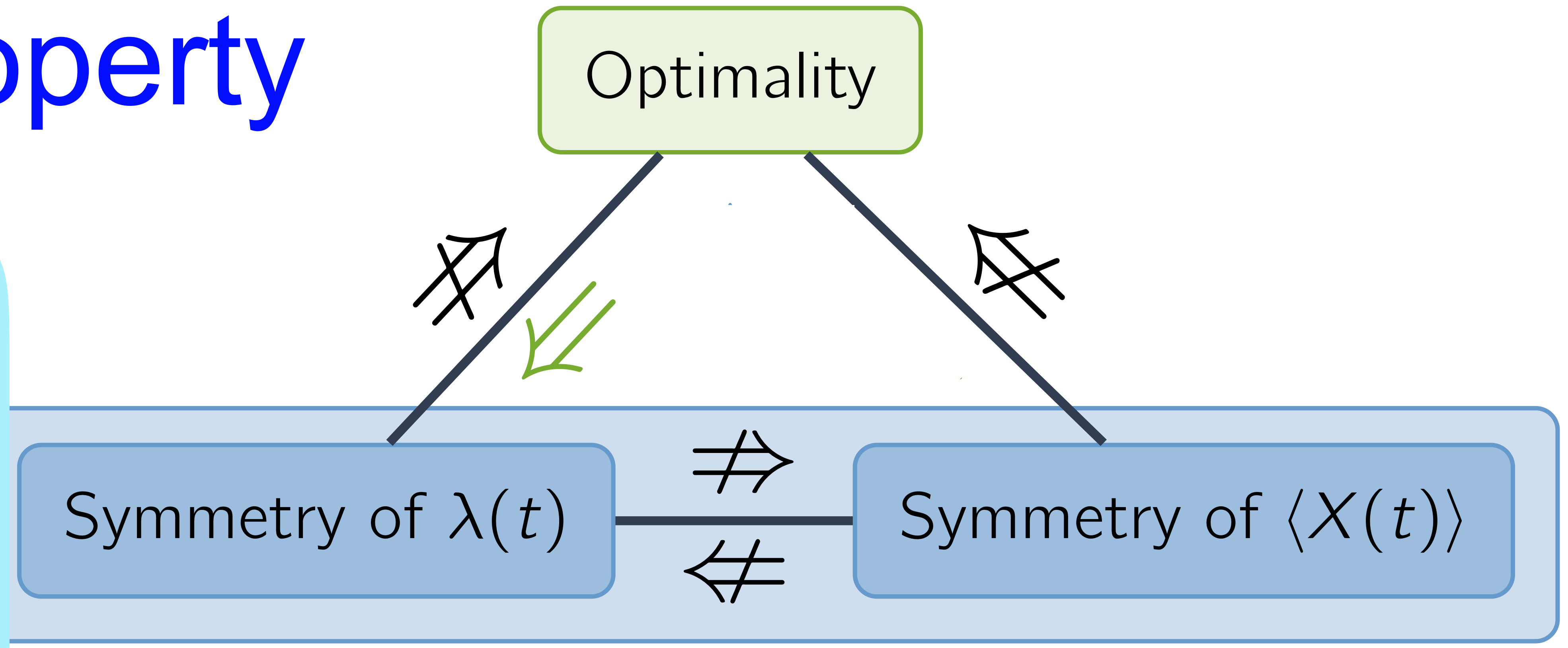
→ invariant under time reversal: $\dot{\lambda}(t) \to -\dot{\lambda}(t_{\mathrm{f}}-t)$

→ $\tilde{\lambda}(t) = -\lambda(t_{\mathrm{f}}-t) + \lambda(t_{\mathrm{f}})$ and $\lambda(t)$ give same work

→ quadratic functional: can only have unique optimum



Proof of symmetry property



Same argument for $\dot{x} = \langle \dot{X} \rangle$!

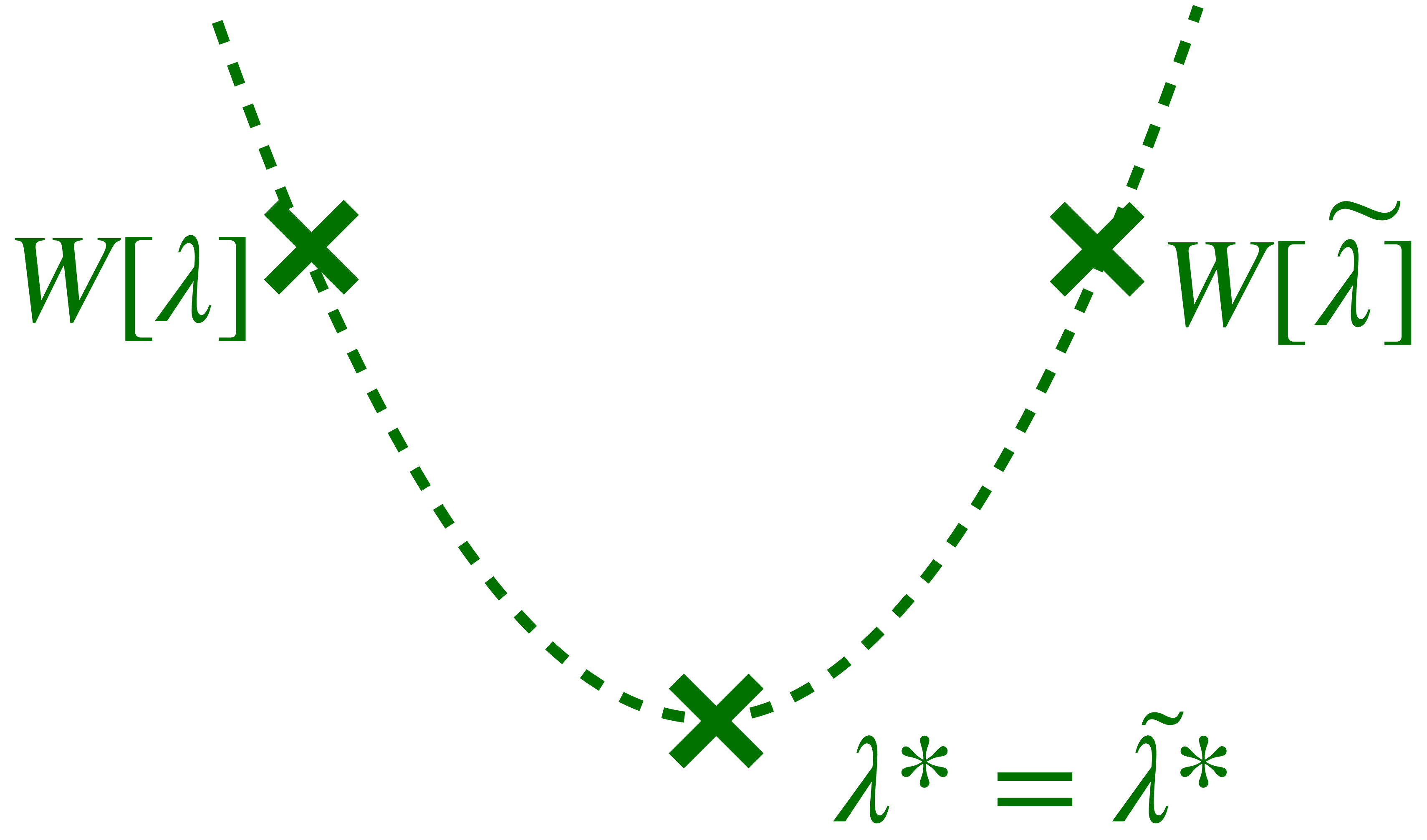
$$\langle W\rangle[x] = \int_0^{t_\mathrm{f}} \int_0^{t} \Gamma(t-t')\dot{x}(t)\dot{x}(t')dt'dt + m[\dot{x}^2(t_\mathrm{f})]/2 + \kappa[x(t_\mathrm{f}) - \lambda_\mathrm{f}]^2/2 + \mathcal{C},$$

<u>Quadratic</u> functional of $\dot{\lambda}$ only:

→ invariant under time reversal: $\dot{\lambda}(t) \to -\dot{\lambda}(t_\mathrm{f} - t)$

→ $\tilde{\lambda}(t) = -\lambda(t_\mathrm{f} - t) + \lambda(t_\mathrm{f})$ and $\lambda(t)$ give same work

→ quadratic functional: can only have unique optimum



Proof of symmetry property

Notation : $\langle X \rangle = x$

$$m\ddot{X} + \int_{-\infty}^{t} \Gamma(t-t')\dot{X}(t')\mathrm{d}t' = -\kappa[X - \lambda(t)] + \eta(t)$$

Optimality

$\not\Rightarrow$ $\Uparrow$ $\not\Leftarrow$

Symmetry of $\lambda(t)$ $\not\Rightarrow$ $\not\Leftarrow$ Symmetry of $\langle X(t) \rangle$

$\Downarrow$ $\Downarrow$

$$\langle W\rangle[x] = \int_0^{t_\mathrm{f}}\int_0^{t} \Gamma(t-t')\dot{x}(t)\dot{x}(t')dt'\,dt + m[\dot{x}^2(t_\mathrm{f})]/2 + \kappa[x(t_\mathrm{f}) - \lambda_\mathrm{f}]^2/2 + \mathcal{C},$$

$$\int_0^{t_\mathrm{f}} \Gamma(|t-t'|)\dot{x}(t')\mathrm{d}t' = \kappa[\lambda_\mathrm{f} - x(t_\mathrm{f})]\,,$$

$\Downarrow$

Optimality!

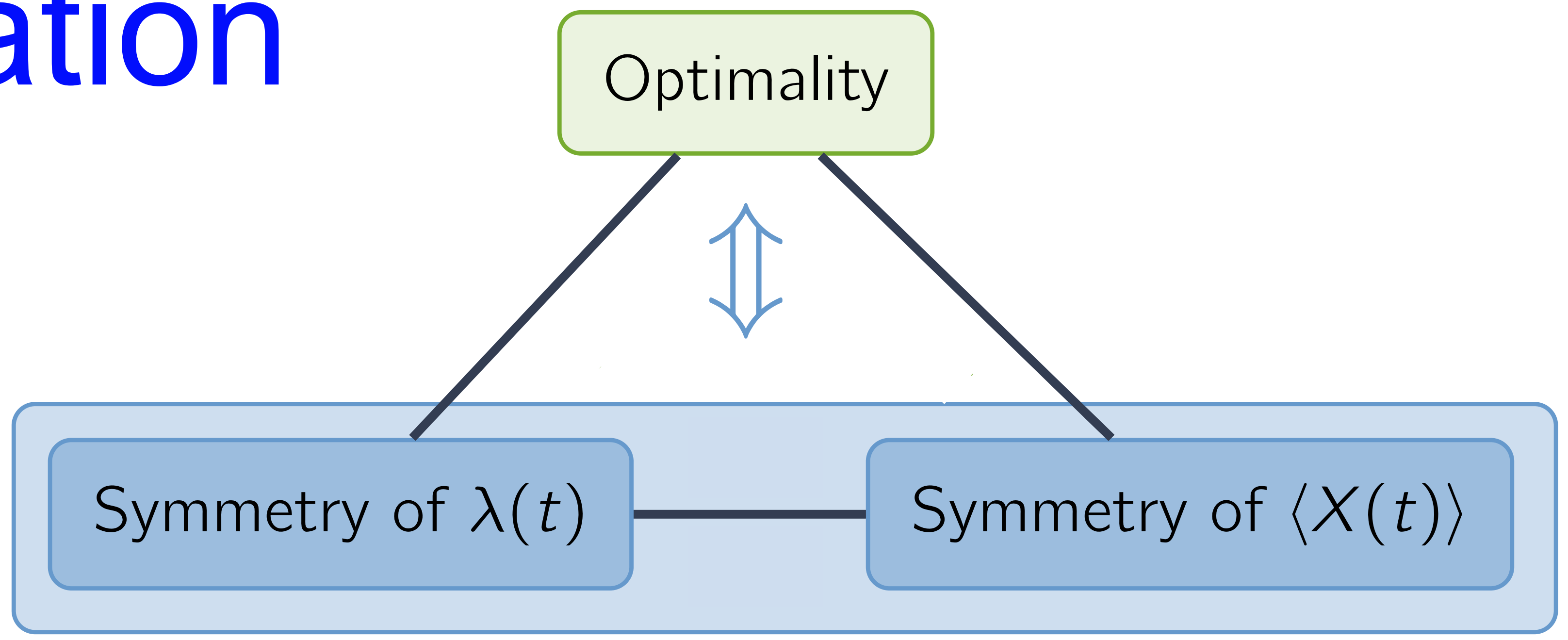
$\Uparrow$

$$\delta x(0) = \delta x(t_\mathrm{f}) = 0,$$

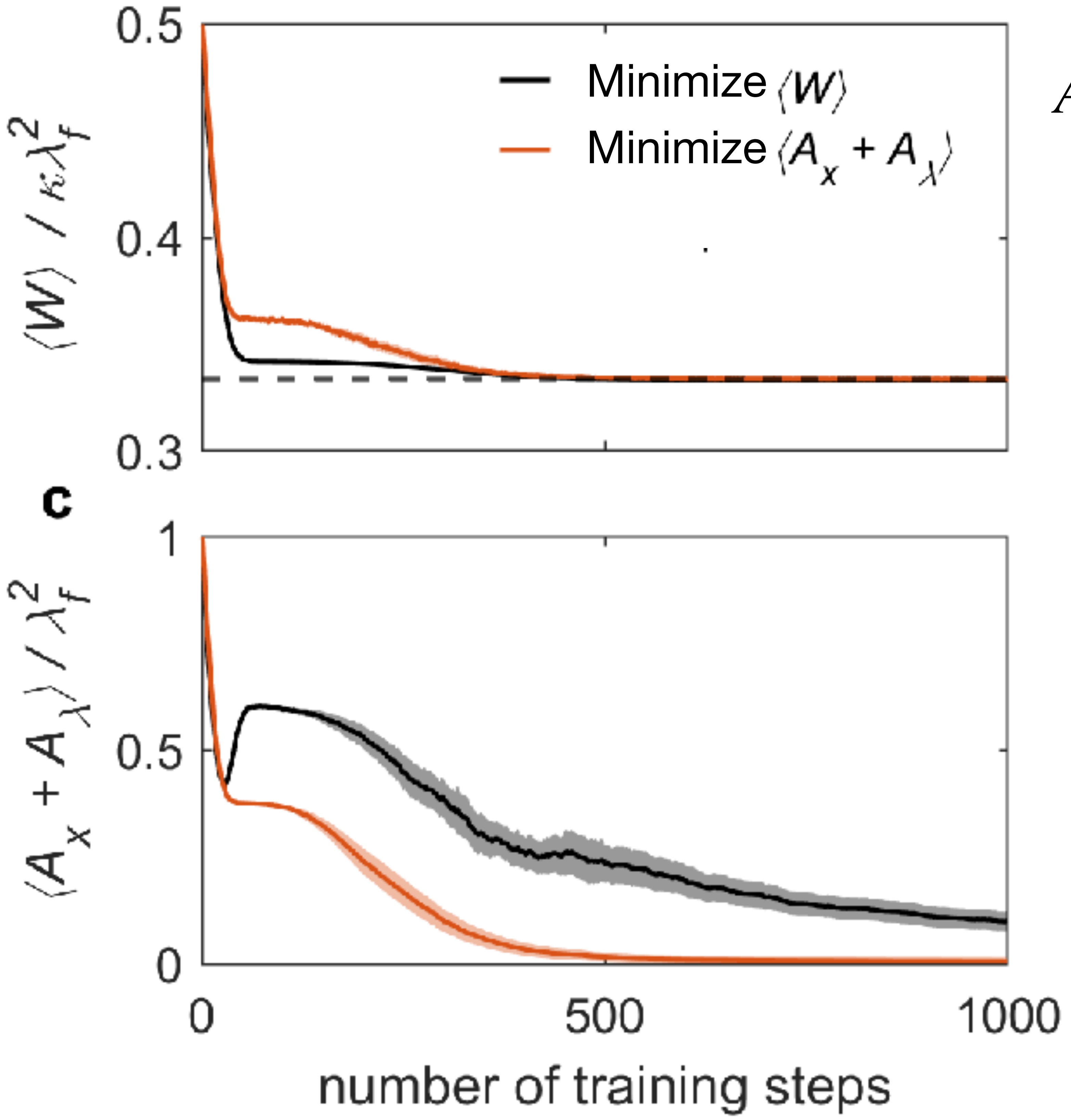
$$\delta \langle W \rangle [x(t), \delta x(t)] = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^{t_\mathrm{f}} \Gamma(|t-t'|)\dot{x}(t')\mathrm{d}t'\right] = 0, \quad \forall t \in [0, t_\mathrm{f}]$$

Symmetry criterion — Application

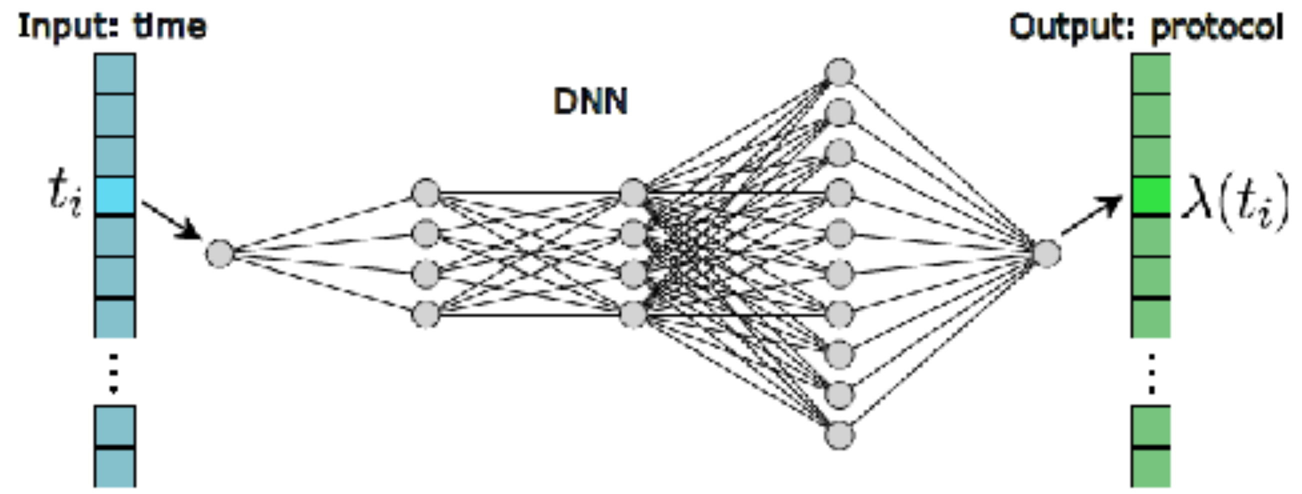
1) "⇔" : Symmetry exclusive to optimal case
→ Criterion to find optimal processes



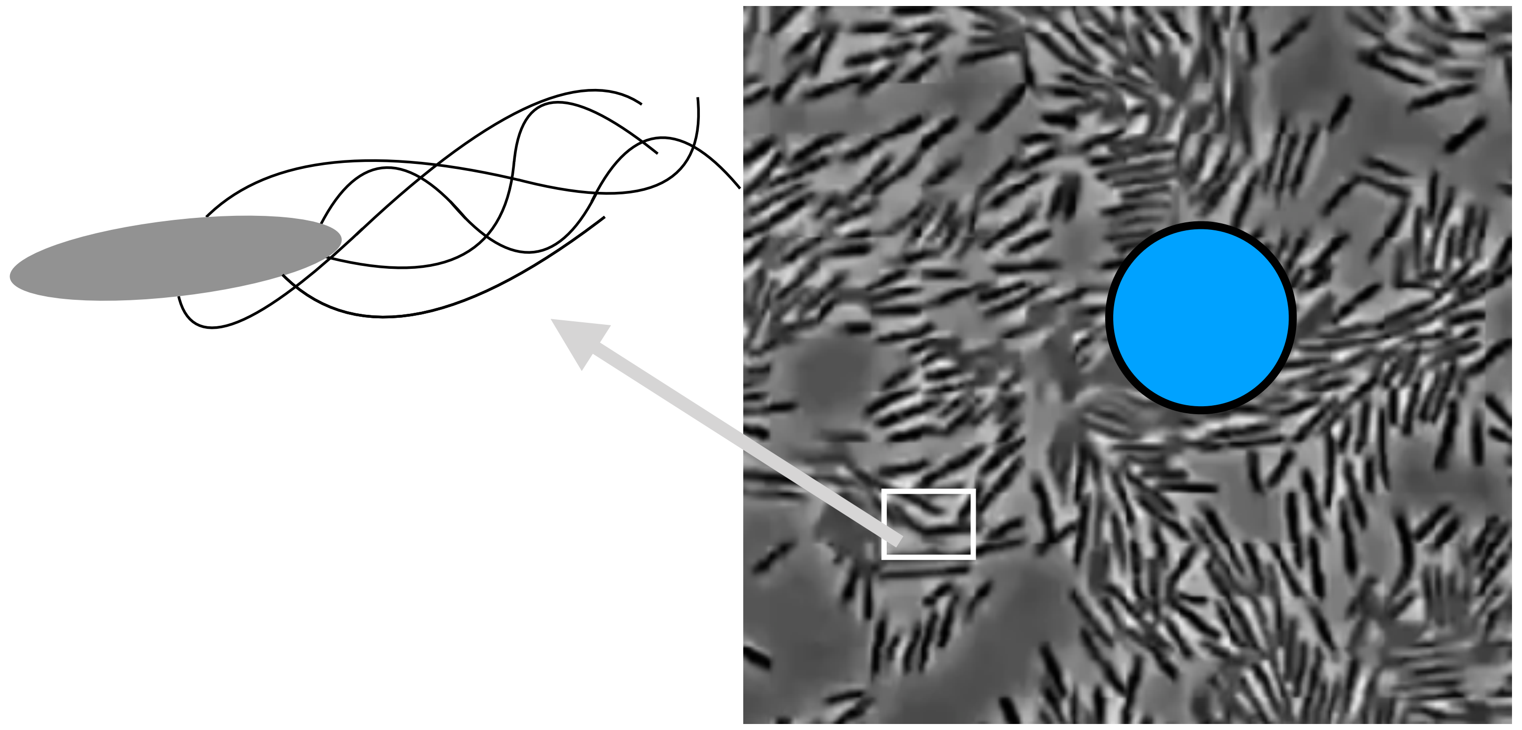
Asymmetry as alternative cost functional
Value at minimum a priori known!



$$A_f = \frac{1}{t_\mathrm{f}} \int_0^{t_\mathrm{f}} \left[f(t) + f(t_\mathrm{f} - t) - 2f(t_\mathrm{f}/2)\right]^2 \mathrm{d}t$$

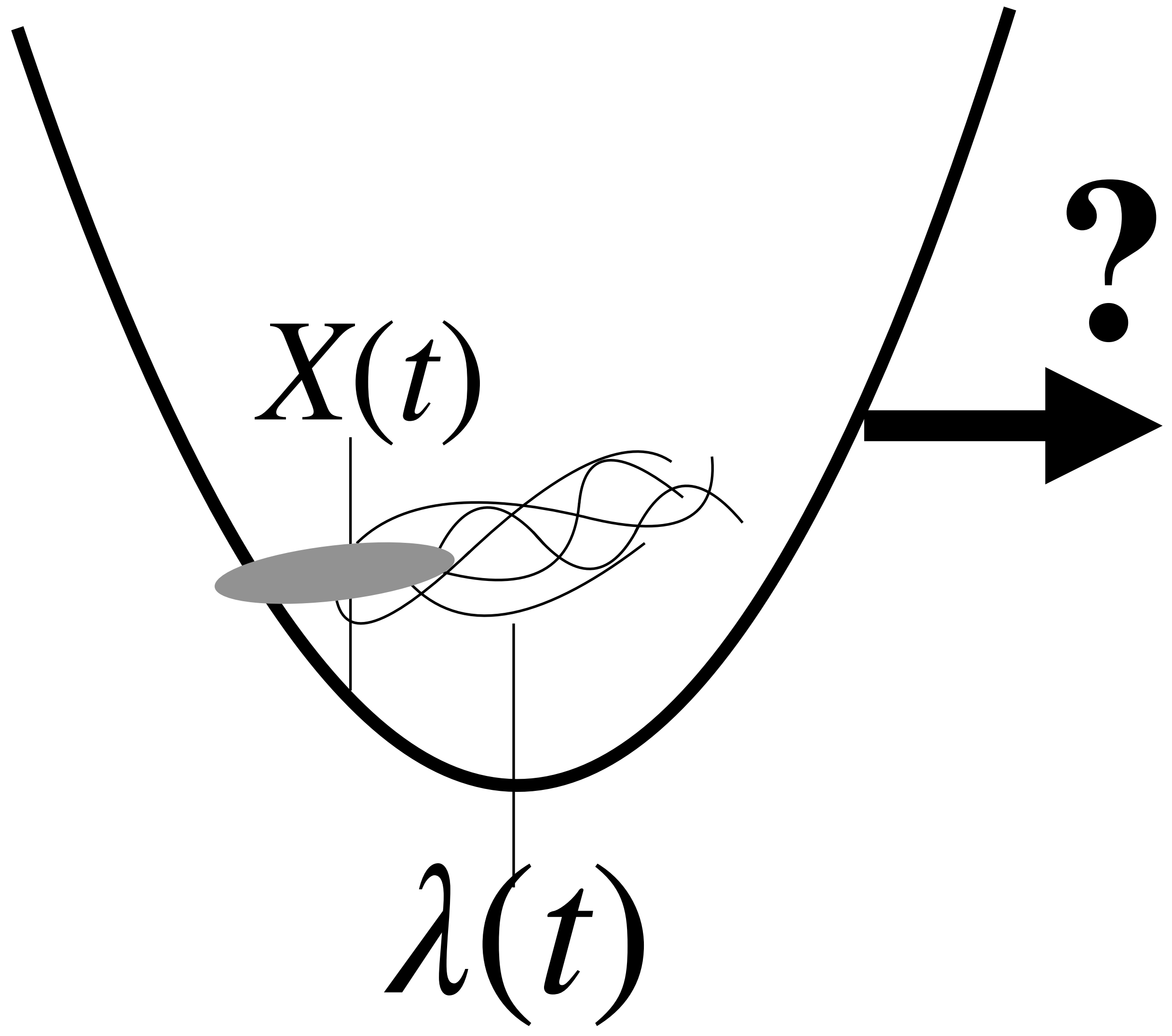


Proof using linearity of GGLE, causality, and time-translation invariance of response and memory kernel Γ …



2) Symmetry for GGLEs : universal* property
→ *also for e.g. granular or glassy media, and active baths*

Outlook (I): Optimal dragging of active particles



Generalized Langevin equation:

$$\tau_0 \dot{X}(t) = -\,[X - \lambda(t)] + \xi(t) + \eta(t)$$

White noise (viscous medium)

$$\langle \xi(t)\xi(t') \rangle \propto \delta(t - t')$$

Coloured noise (activity)

$$\langle \eta(t)\eta(t') \rangle \propto e^{-|t-t'|/\tau_b}$$

Active Ornstein-Uhlenbeck particle (AOUP)/
Run-and-Tumble particle (RTP)

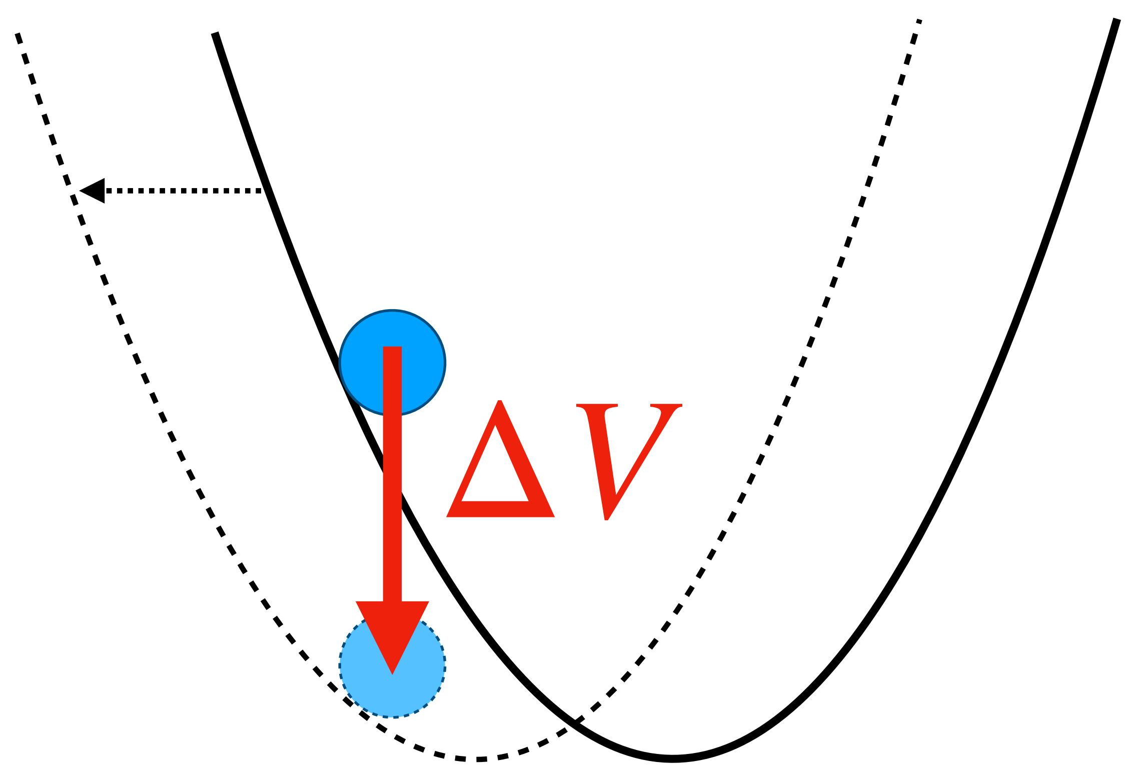
[D Gupta, SHL Klapp, DA Sivak, PRE 108 (2), 024117 (2023)]
[L. K. Davis, K. Proesmans, and É. Fodor, PRX (2024)]
[Garcia-Millan*, Schuettler*, Cates, **Loos**, ArXiv:2407.18542 (2024)]
[Schuettler, Garcia-Millan, Cates, **Loos**, ArXiv:2501.18613 (2025)]

Optimal closed-loop (feedback) control

Measurement of system state

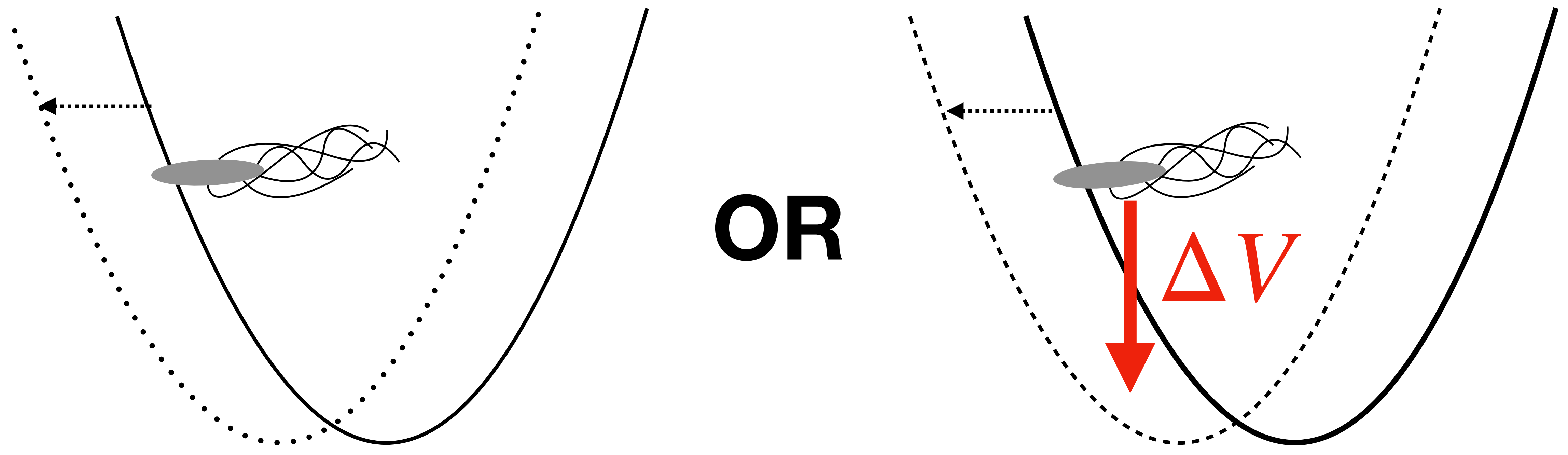
→ Allows to extract work (on average)

X measurement allows instantaneous work extraction from potential energy



[D. Abreu and U. Seifert, EPL (2011).]

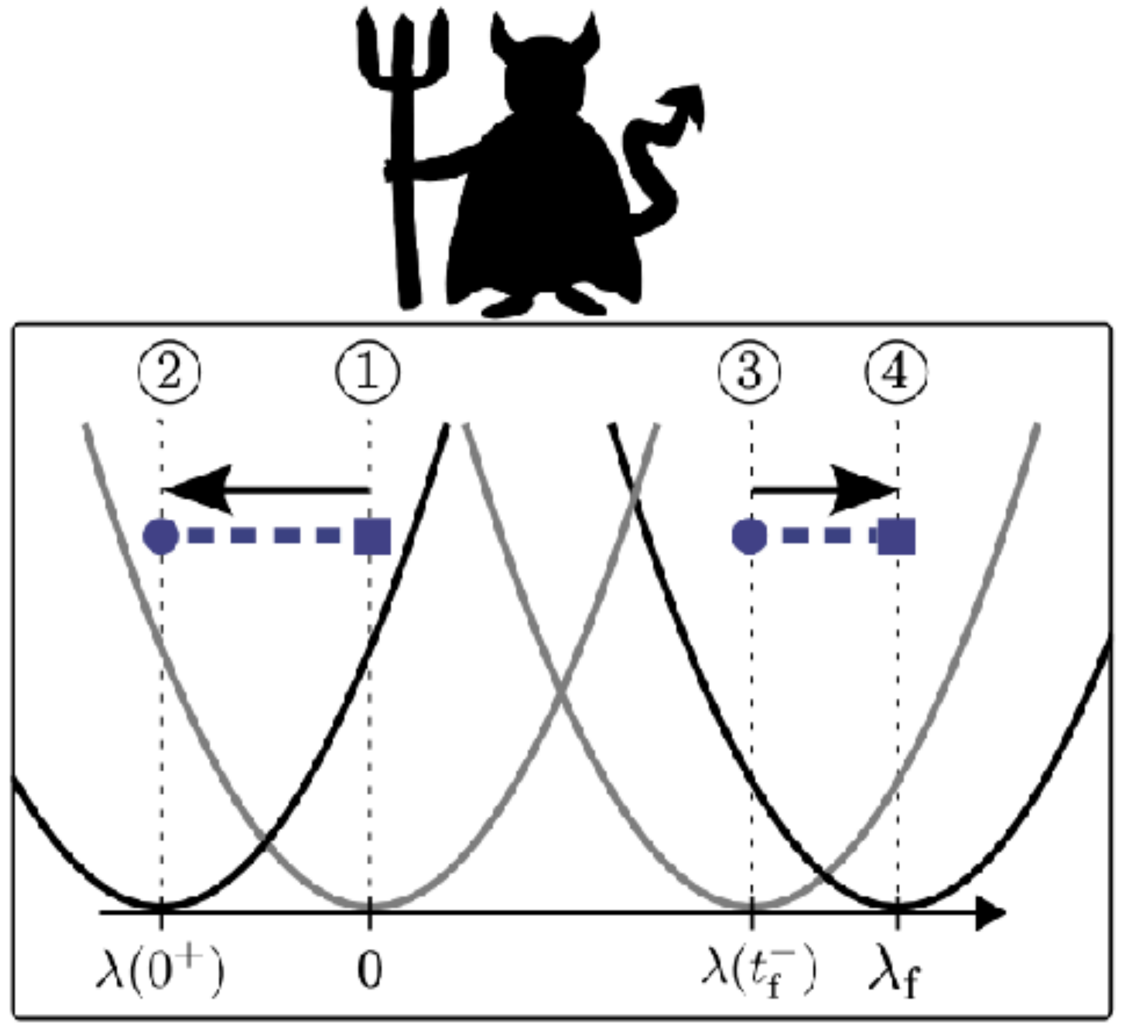
η measurement allow finite-duration work extraction from activity AND potential energy

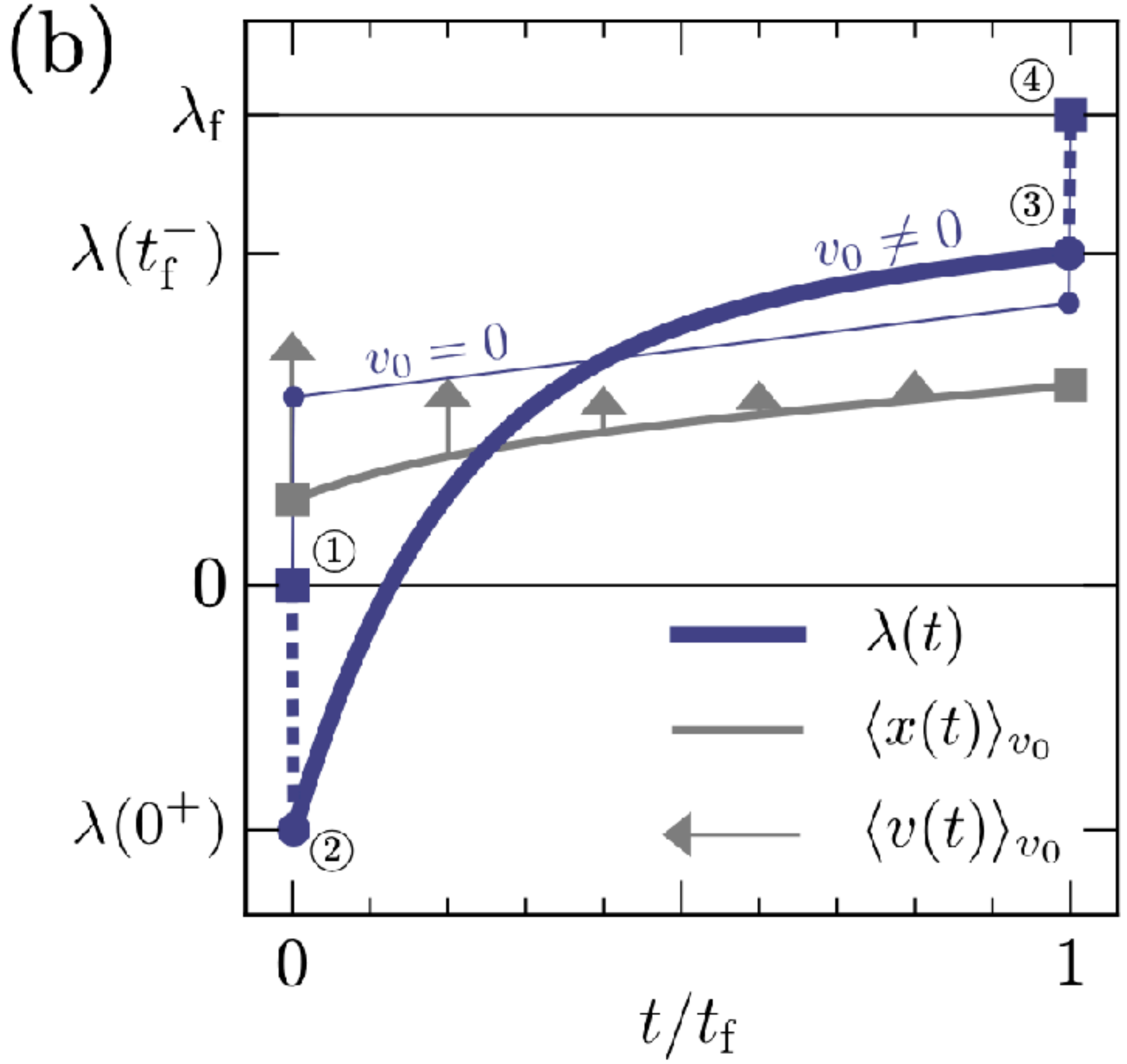


$$S(\langle X \rangle \,|\, \eta) < S(\langle X \rangle)$$

Optimal closed-loop (feedback) control

Allow one measurement at the beginning:
Is the particle heading left or right?

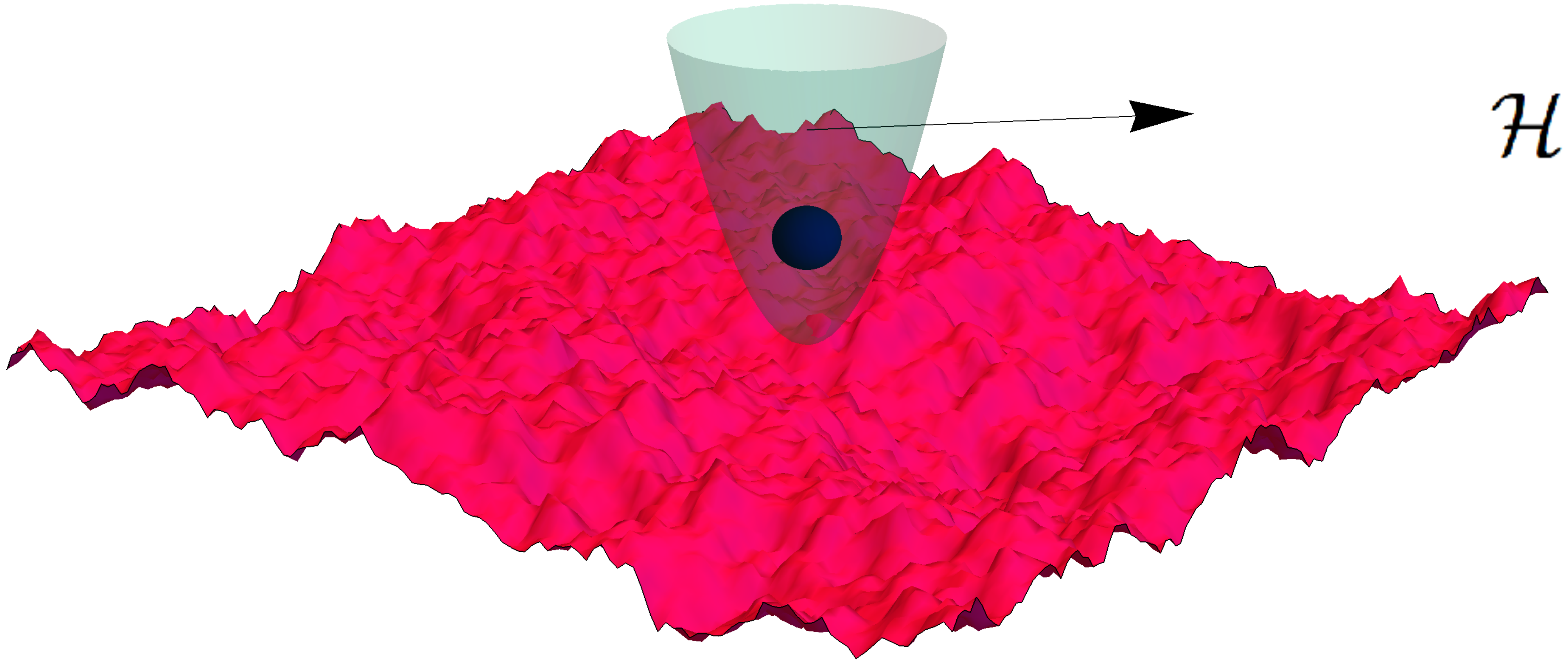




Framework of information thermodynamics (Maxwell demon):
Information-to-work conversion

[D. Abreu and U. Seifert, EPL (2011).]

Outlook (II): Optimal dragging through critical medium



$$\mathcal{H}[\phi, \mathbf{Y}, t] = \mathcal{H}_\phi[\phi] + \mathcal{H}^{\mathrm{int}}[\phi, \mathbf{Y}] + \mathcal{U}(\mathbf{Y}, t)$$

$$\gamma_y \dot{\mathbf{Y}} = -\boldsymbol{\nabla}_{\mathbf{Y}} \mathcal{H} + \mathbf{F}_{\mathrm{ext}} + \boldsymbol{\nu}$$

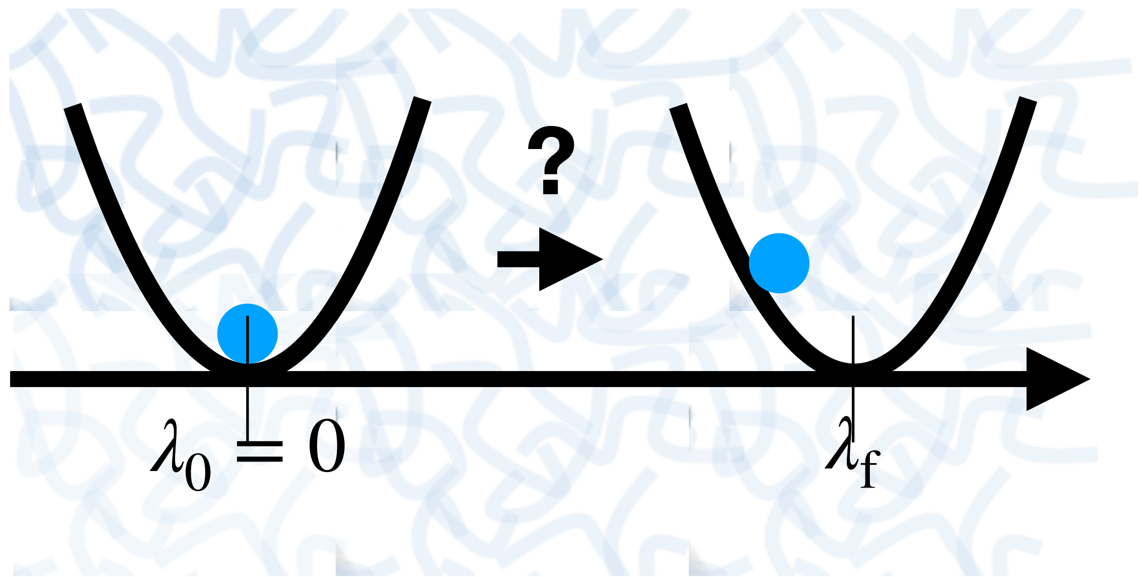
$$\gamma_\phi \dot{\phi} = -(-\nabla^2)^{\mathfrak{a}} \frac{\delta \mathcal{H}}{\delta \phi} + \eta^{(\mathfrak{a})}$$

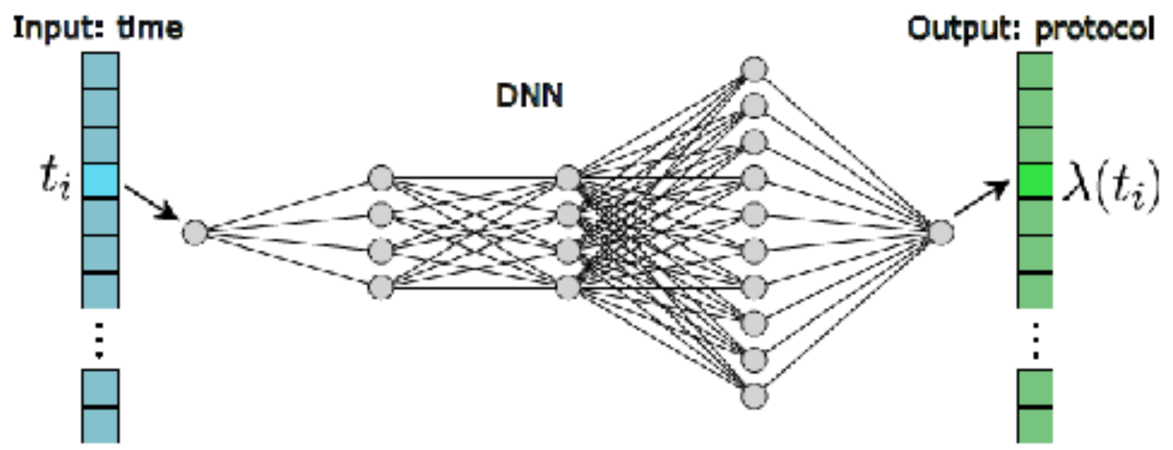
$$\mathcal{H}_{\mathrm{int}} = -\lambda \int \mathrm{d}^d \mathbf{x}\, \phi(\mathbf{x}) V(\mathbf{x} - \mathbf{Y})$$

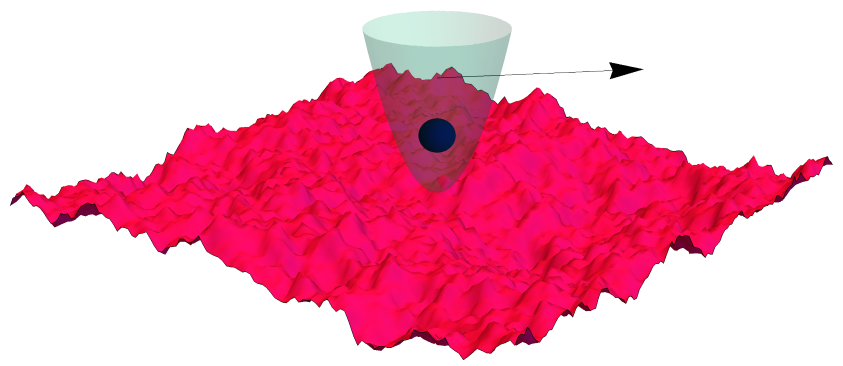
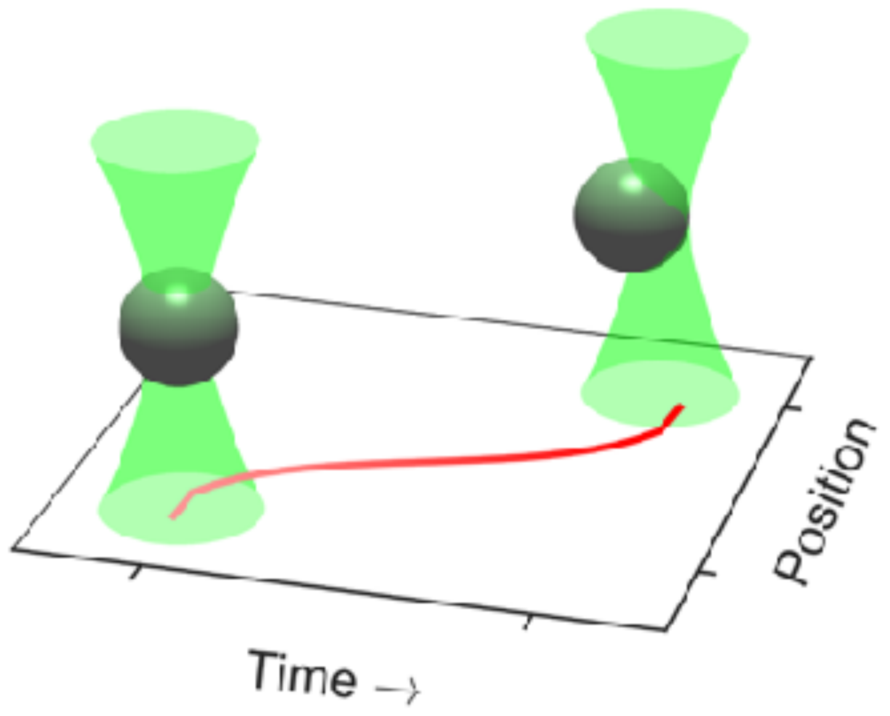
Demery, Dean. PRL (2010), PRE (2011).

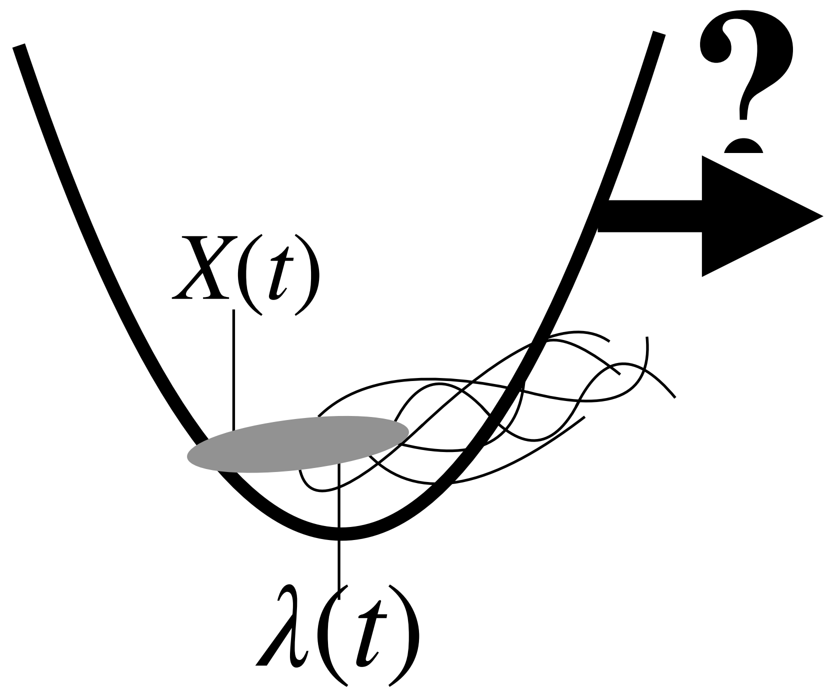
Steady state driving (quasi static regime):

[Venturelli*, Walter*, **Loos***, Roldan, Gambassi, EPL 146, 27001 (2024)]







Conclusions

- Optimal dragging — simple problem to study thermodynamically optimal processes
- Memory effects (correlations in environment) strongly affect optimal dragging strategy
- Symmetry as universal feature of dragging problem (for linear processes)

Loos et al., Physical Review X 14, 021032 (2024)
Garcia-Millan*, Schuettler*, Cates, Loos, ArXiv:2407.18542 (2024)
Schuettler, Garcia-Millan, Cates, Loos, ArXiv:2501.18613 (2025)

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