Boundary Energy-Momentum Tensors for Asymptotically Flat Spacetimes

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Introduction

- It is clear that in AdS/CFT the notion of a boundary energy-momentum tensor (EMT) is crucial.
- This is usually defined in terms of holographic renormalisation methods (see e.g. [de Haro, Solodukhin, Skenderis, 2000]).
- The goal is to do the same for asymptotically flat spacetimes near future null infinity.
- Null infinity is a Carroll manifold [Duval, Gibbons, Horvathy, 2014].
- This requires understanding solutions with arbitrary Carroll data at $\mathcal{I}^+$.
- This work builds on [Hartong, 2015] where this was done in 3D. For similar work see [Freidel, Riello, 2024].

Outline

- Carroll geometry near future null infinity
- Solving the Einstein equations with arbitrary Carroll geometry on the boundary
- Holographic renormalisation and energy-momentum-news Ward identities
- Outlook

Carroll geometry at $\mathcal{I}^+$

$$ds^2 = -2UV + E^a E^a\,, \qquad a = 1, \ldots, d$$

- U and V are null and the E^a are spacelike.
- Treat null infinity as a Penrose boundary (defining function).
- Split: $x^M = (r, x^\mu)$ with r the defining function. Partially fixing local Lorentz transformations and bulk diffeos:

$$g_{rr} = 0\,, \qquad g_{r\mu} = -V_\mu\,, \qquad g_{\mu\nu} = -SV_\mu V_\nu + \Pi_{\mu\nu}$$
$$g^{rr} = S\,, \qquad g^{r\mu} = U^\mu\,, \qquad g^{\mu\nu} = \Pi^{\mu\nu}$$

  $\Pi^{\mu\nu}$ and $\Pi_{\mu\nu}$ have signature $(0, 1, \ldots, 1)$.

- Penrose boundary construction fixes boundary condition:

$$V_\mu\big|_{r=\infty} = \tau_\mu\,, \qquad r^{-2}\Pi_{\mu\nu}\big|_{r=\infty} = h_{\mu\nu}\,, \qquad U^\mu\big|_{r=\infty} = v^\mu\,, \qquad r^2\Pi^{\mu\nu}\big|_{r=\infty} = h^{\mu\nu}$$

- Carroll covariant Bondi–Sachs gauge:

$$g_{rr} = 0\,, \qquad \Gamma^{\rho}_{rr} = 0\,, \qquad \Gamma^{\mu}_{\mu r} = dr^{-1}$$

Constant x^μ curves (tangent $\frac{\partial}{\partial r}$) are null geodesics ending at $\mathcal{I}^+$.

- This fixes $V_\mu = e^\beta \tau_\mu$ and $h^{\mu\nu}\Pi_{\mu\nu}$.
- Often $\tau_\mu dx^\mu = du$ (retarded time) and $h_{\mu\nu} =$ celestial sphere.
- We want the boundary geometry $\tau_\mu, h_{\mu\nu}$ (and also the shear) to be arbitrary so we can vary it freely in the on shell action.
- However at leading order the EOM fix

$$K_{\mu\nu} := -\frac{1}{2}\mathcal{L}_v h_{\mu\nu} = \frac{1}{d} K h_{\mu\nu}$$

This is a constraint in $d \geq 2$. Not a problem though, more later.

- Furthermore we learn that

$$\Pi_{\mu\nu} = r^2 h_{\mu\nu} + r\left(C_{\mu\nu} - 2\tau_{(\mu}a_{\nu)}\right) + \mathcal{O}(1)\,, \quad S = \frac{2}{d}Kr + \mathcal{O}(1)\,, \quad \beta = \mathcal{O}(r^{-2})$$

  $C_{\mu\nu}$ is the shear (spatial and STF) and $a_\mu = \mathcal{L}_v \tau_\mu$.

- Residual gauge transformations: $\xi^\mu = \chi^\mu + r^{-1}h^{\mu\nu}\lambda_\nu + \mathcal{O}(r^{-2})$ and $\xi^r = r\Lambda_D + \mathcal{O}(1)$ (bdry diffeos, Weyl and local Carroll boosts)

$$\begin{aligned} \delta\tau_\mu &= \mathcal{L}_\chi \tau_\mu + \Lambda_D \tau_\mu + \lambda_\mu \\ \delta h_{\mu\nu} &= \mathcal{L}_\chi h_{\mu\nu} + 2\Lambda_D h_{\mu\nu} \\ \delta C_{\mu\nu} &= \mathcal{L}_\chi C_{\mu\nu} + \Lambda_D C_{\mu\nu} + 2P^\rho_{\langle\mu}P^\sigma_{\nu\rangle}\left(D_\rho\lambda_\sigma + a_\rho\lambda_\sigma\right) \end{aligned}$$

  D_ρ is some Carroll covariant derivative.

- The boundary geometry and the shear sit inside one ‘multiplet’ related to the ‘gauging’ of the conformal Carroll algebra.

Solving the Einstein equations

- $\overset{(n)}{X}$ is the coefficient of r^{-n} in expansion of X. Here $d = 1, 2$.
- $R_{rr} = 0$ determines $\overset{(n)}{\beta}$
- $U^\mu R_{\mu r} = 0$ and $r^d \Pi^{\mu\nu} R_{\mu\nu}\big|_{r=\infty} = 0$ give $(n - d + 1)\overset{(n)}{S}$. For $n = d - 1$ the equations are identically satisfied.
- $\Pi^\rho_\mu R_{\rho r} = 0$ gives $(n - d + 1) P^\rho_\mu v^\sigma \overset{(n)}{\Pi}_{\rho\sigma}$. For $n = d - 1$ the equation is not identically satisfied. Need a $r^{-1} \log r$ term in $\Pi_{\mu\nu}$.
- $\Pi^\rho_{\langle\mu} \Pi^\sigma_{\nu\rangle} R_{\rho\sigma} = 0$: gives $\mathcal{L}_v \big($STF part of $\overset{(n)}{\Pi}_{\mu\nu}\big) = \cdots$
- $r^d U^\mu R_{\mu\nu}\big|_{r=\infty} = 0$: PDE for $\overset{(d-1)}{S}$ and $P^\rho_\mu v^\sigma \overset{(d-1)}{\Pi}_{\rho\sigma}$ (energy-momentum-news conservation)
- Solutions agree with [Barnich, Troesaert, 2010] and [Geiller, Zwikel, 2022].

Holographic Renormalisation

- Variation of the bulk action:

$$\delta S_{\text{EH}} = \cdots + \int_{r=\Lambda} d^{d+1}x\, E\, J^r \,, \qquad J^r = \Pi^{\mu\nu}\delta\Gamma^r_{\mu\nu} + 2U^\mu \delta\Gamma^r_{\mu r} - U^\mu \partial_\mu \left(E^{-1}\delta E\right)$$

- IN BS gauge: $\sqrt{-g} = E := \det\left(-V_\mu V_\nu + \Pi_{\mu\nu}\right) = e r^d e^\beta$.
- Cutoff surface $r = \Lambda$ has no definite character. The generalisation of the GHY extrinsic counterterm is

$$S_{\text{ext}} = 2\int d^{d+1}x E \left(\delta^M_P + V^M N_P\right) \nabla_M N^P$$

$V^M = -(\partial_r)^M$ and $N_P = \partial_P r$ [Parattu, Chakraborty, Padmanabhan, 2016].

- This removes the radial derivatives of δS and $\delta\beta$ and leads to:

$$\delta\left(S_{\text{EH}} + S_{\text{ext}}\right) = \cdots + \int_{r=\Lambda} d^{d+1}x\, E \left(\mathcal{T}^\mu \delta V_\mu + \frac{1}{2}\mathcal{T}^{\mu\nu}\delta\Pi_{\mu\nu} + dr^{-1}E^{-1}\delta\left(ES\right)\right)$$

- $dr^{-1}E^{-1}\delta\left(ES\right)$ dominates at LO in r and leads to a variation of an EMT tensor component at $\mathcal{O}(1)$. We cancel it by adding

$$\tilde{S}_{\text{ext}} = -d\int_{r=\Lambda} d^{d+1}x E r^{-1} S$$

- $S_{\text{EH}} + S_{\text{ext}} + \tilde{S}_{\text{ext}}$ for $d=1$ gives a finite Dirichlet problem. We can add a finite intrinsic counterterm to make the EMT traceless:

$$S_{\text{finite}} = -\int d^2x E r \Pi^{\mu\nu} \mathcal{L}_U V_\mu \mathcal{L}_U V_\nu$$

- For $d=1$ the on shell variation gives

$$\begin{aligned} \delta S_{\text{tot}}\Big|_{\text{os}} &= \int d^2x e\left(T^\mu \delta\tau_\mu + \frac{1}{2}T^{\mu\nu}\delta h_{\mu\nu}\right) \\ T^\mu &= M v^\mu + h^{\mu\nu}\left(\partial_\nu K + a_\nu K + \mathcal{L}_v a_\nu\right) \\ T^{\mu\nu} &= -2P_\rho h^{\rho(\mu}v^{\nu)} + M h^{\mu\nu} \end{aligned}$$

Bondi mass $2M = -\overset{(0)}{S} + a^2$ and angular mom. $P_\mu = P_\mu^\rho v^\sigma \overset{(0)}{\Pi}_{\rho\sigma}$.

- Recall the asymptotic gauge transformations:

$$\delta\tau_\mu = \mathcal{L}_\chi \tau_\mu + \Lambda_D \tau_\mu + \lambda_\mu \,, \qquad \delta h_{\mu\nu} = \mathcal{L}_\chi h_{\mu\nu} + 2\Lambda_D h_{\mu\nu}$$

- The Carroll boosts are anomalous. Associated Ward identities:

$$\begin{aligned} 0 &= -e^{-1}\partial_\mu \left(e\left[T^\mu \tau_\nu + T^{\mu\rho} h_{\rho\nu}\right]\right) + T^\mu \partial_\nu \tau_\mu + \frac{1}{2} T^{\mu\rho}\partial_\nu h_{\mu\rho} \\ 0 &= T^\mu \tau_\mu + T^{\mu\nu} h_{\mu\nu} \\ T^\mu h_{\mu\nu} &= P^\mu_\nu \left(\partial_\mu K + a_\mu K + \mathcal{L}_v a_\mu\right) \neq 0 \end{aligned}$$

  are equivalent to what we get from the Einstein equations.
- There is a conserved current (up to an anomaly) for every Carroll conformal Killing vector $\chi^\mu = K^\mu$ which solves $0 = \delta\tau_\mu = \delta h_{\mu\nu}$

$$e^{-1}\partial_\mu \left(e\left[T^\mu \tau_\nu K^\nu + T^{\mu\rho} h_{\rho\nu} K^\nu\right]\right) = \lambda_\mu T^\mu$$

- The anomaly corresponds to the c_M central extension of BMS_3 [Barnich, Compère, 2006].

- A magnetic Carroll Liouville theory whose energy momentum tensor obeys the same properties as the holographic one:

$$S = \int d^2xe\left[-\frac{1}{2}h^{\mu\nu}\left(\partial_\mu\phi + \frac{2}{b}a_\mu\right)\left(\partial_\nu\phi + \frac{2}{b}a_\nu\right) + \chi\left(v^\mu\partial_\mu\phi - \frac{2}{b}K\right) - \mu e^{b\phi}\right]$$

- On flat space this was studied in [Barnich, Gomberoff, González, 2012].
- ϕ is a real scalar that is inert under local Carroll boosts and transforms as $\delta\phi = -\frac{2}{b}\omega$ under Weyl transformations.
- χ transforms under Weyl and local Carroll boosts as

$$\delta\chi = -\omega\chi + \lambda e^\mu\partial_\mu\phi + \frac{2}{b}e^\mu\partial_\mu\lambda + \frac{2}{b}\lambda L$$

- The magnetic theory is Weyl invariant and transforms under local Carroll boosts as

$$\delta S = \int d^2xe\frac{4}{b^2}\lambda_\mu h^{\mu\nu}\left(\partial_\nu K + a_\nu K + \mathcal{L}_v a_\nu\right)$$

$$\delta\left(S_{\mathsf{EH}}+S_{\mathsf{ext}}+\tilde{S}_{\mathsf{ext}}\right)=\cdots+\int_{r=\Lambda}d^{d+1}x\,E\left(\mathcal{T}^{\mu}\delta V_{\mu}+\frac{1}{2}\mathcal{T}^{\mu\nu}\delta\Pi_{\mu\nu}\right)$$

- Now we consider $d=2$. In this case there is a divergence at order r. We remove this by adding an intrinsic counterterm

$$S_{\mathsf{int}}=-\int d^3xEr\left(R-\frac{1}{4}\Pi^{\mu\nu}\mathcal{L}_UV_\mu\mathcal{L}_UV_\nu\right)$$

- We obtain

$$\delta S_{\mathsf{tot}}\Big|_{\mathsf{os}}=\int d^3xe\left(\mathcal{T}^{\rho}\delta\tau_{\rho}+\frac{1}{2}\mathcal{T}^{\rho\sigma}\delta h_{\rho\sigma}+\frac{1}{2}S^{\mu\nu}\delta C_{\mu\nu}\right)$$

Here $S^{\mu\nu}$ is spatial and STF.

- Geometrically: shear is part of the conformal Carroll geometry. Physically: shear is a source [Donnay, Fiorucci, Herfray, Ruzziconi, 2022].

- Getting the shear variation is nontrivial. It means that at $\mathcal{O}(r)$

$$S^{\mu\nu}\delta h_{\mu\nu}=\text{total derivative}$$

- What about the constraint $K_{\mu\nu} = \frac{1}{2} K h_{\mu\nu}$? This can be solved

$$h_{\mu\nu} dx^\mu dx^\nu = M^2 \left(dX^2 + dY^2 \right)$$

where M, X and Y are fully unrestricted scalar fields.

- Either we vary M, X, Y are we use a Lagrange multiplier. Either way we cannot distinguish between $T^{\mu\nu}$ and $T^{\mu\nu} + t^{\mu\nu}$ where

$$t^{\mu\nu} = -K\chi^{\mu\nu} + \frac{1}{2}\mathcal{L}_v \chi^{\mu\nu} - \frac{1}{2} a_\rho \chi^{\rho(\mu} v^{\nu)} - v^{(\mu} h^{\nu)\sigma} \overset{(0)}{D}_\rho \chi^\rho{}_\sigma$$

is an improvement transformation for some spatial STF tensor $\chi^{\mu\nu}$ (the Lagrange multiplier).

- It is convenient to add a finite counterterm such that

$$S^{\mu\nu} = \frac{1}{2} N^{\mu\nu} = -\frac{1}{2} h^{\mu\rho} h^{\nu\sigma} \left(\mathcal{L}_v C_{\rho\sigma} + \frac{1}{2} K C_{\rho\sigma} \right)$$

which is the news tensor with a definite Weyl weight.

- On shell action is diffeo invariant:

$$-e^{-1}\partial_\mu\left(e\left[T^\mu\tau_\nu + T^{\mu\rho}h_{\rho\nu} + \frac{1}{2}N^{\mu\rho}C_{\rho\nu}\right]\right) + T^\mu\partial_\nu\tau_\mu + \frac{1}{2}T^{\mu\rho}\partial_\nu h_{\mu\rho} + \frac{1}{4}N^{\mu\rho}\partial_\nu C_{\mu\rho} = 0$$

agrees with the Bondi mass and angular mom. loss equations.

- Schematic form of the renormalised EMT/News complex:

$$\begin{aligned}
\tau_\mu T^\mu &= \overset{(1)}{S} + C^2\partial + C\partial^2 + \partial^3 \\
\tau_\mu h_{\nu\alpha}T^{\mu\nu} &= v^\mu P^\nu_\alpha \overset{(0)}{\Pi}_{\mu\nu} + C^2\partial + C\partial^2 + \partial^3 + \tau_\mu h_{\nu\alpha}t^{\mu\nu} \\
\text{STF part of } T^{\mu\nu} &= \text{STF part of}\left(\overset{(0)}{S}C^{\mu\nu} + K\overset{(0)}{\Pi}{}^{\mu\nu} + C\partial^2 + t^{\mu\nu}\right)
\end{aligned}$$

- Weyl invariance but no Carroll boost invariance:

$$\begin{aligned}
0 &= \tau_\mu T^\mu + h_{\mu\nu}T^{\mu\nu} + \frac{1}{4}C_{\mu\nu}N^{\mu\nu} \\
P^\mu_\rho T^\rho &= D_\rho N^{\rho\mu} - \frac{1}{2}a_\rho N^{\rho\mu} + \partial^3
\end{aligned}$$

Outlook

- Charges and their algebra
- Anomalies in 4D?
- Covariant notions of soft and hard sectors
- Effective theory for the soft sector