Toward a microscopic realization for dS_3

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Based on work with **Scott Collier**, **Lorenz Eberhardt** and **Victor Rodriguez**

dS quantum gravity

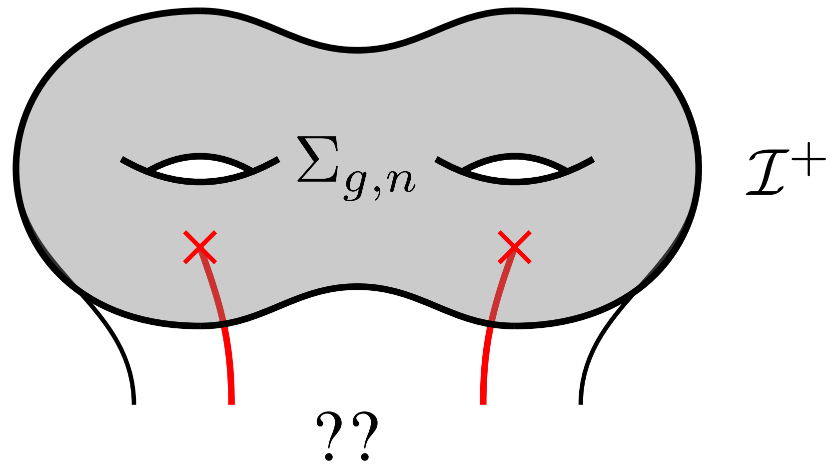
- Dearth of controllable models of QG in expanding spacetimes

- No unifying picture of dS QG that approaches AdS/CFT in scope or rigor

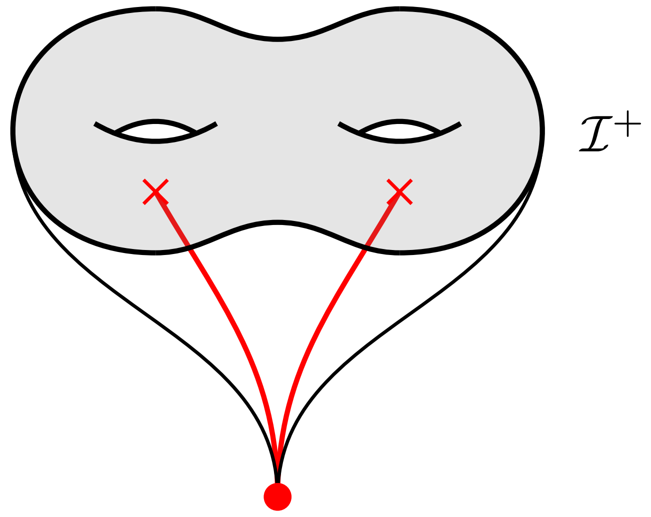
- Low d de Sitter
  [Anninos-Hofman; Maldacena-Turiaci-Yang; Cotler-Jensen-Maloney; Anninos-BM, Anninos-Bautista-BM; Verlinde et al; Susskind; Gorbenko-Silverstein-Torroba; Anninos-Harris;…]

  → Controlled set-up to look for microscopic model

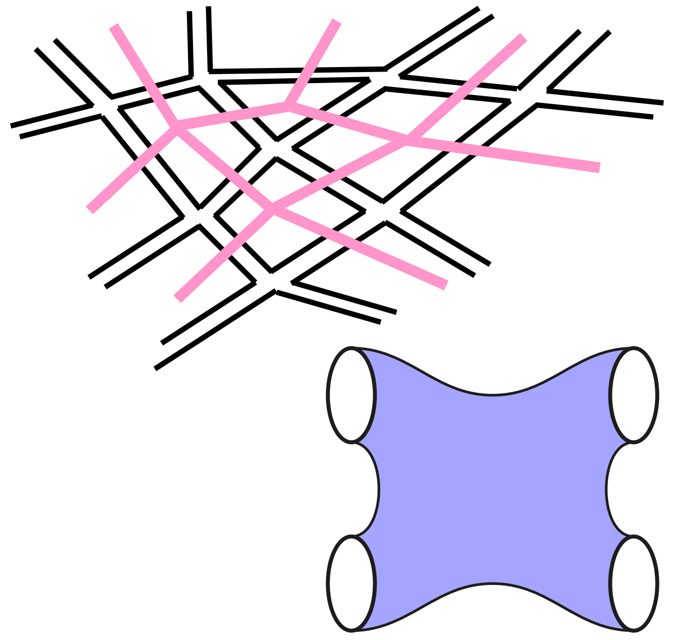
Outline

1 

Canonical quantization of dS_3 gravity

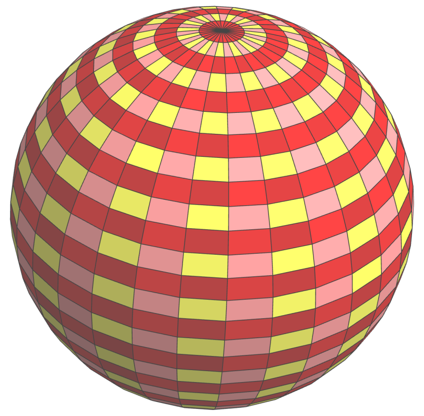
2 

A wavefunction of a universe

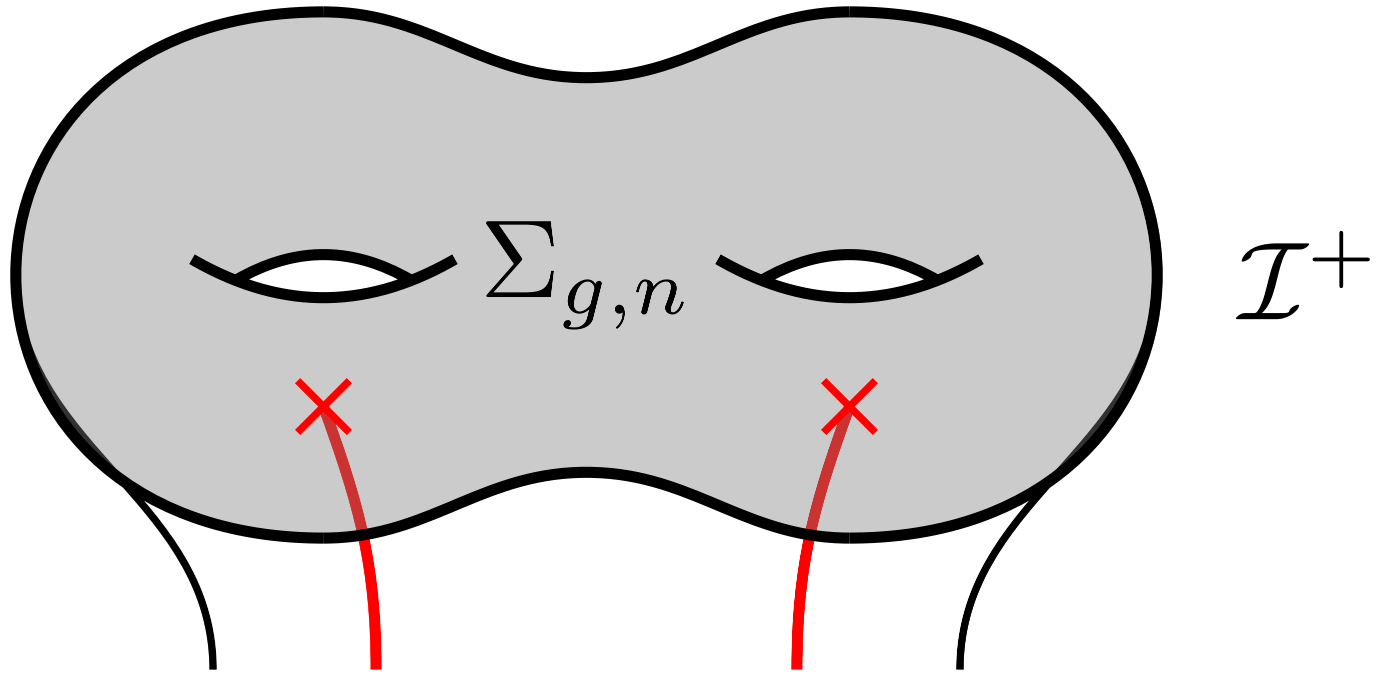
3 

The complex Liouville string

& a matrix integral for cosmological correlators

4 

Gibbons-Hawking dS entropy



Canonical quantization of dS_3 gravity

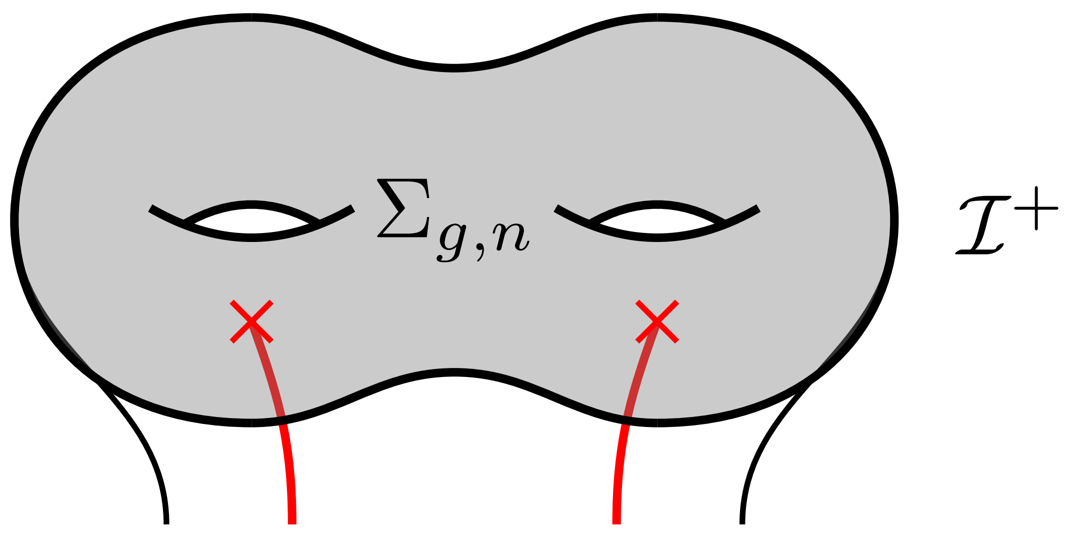
3d de Sitter

- 3d Einstein gravity with positive cosmological constant

$$S = \frac{1}{16\pi G_{\rm N}} \int d^3x \sqrt{-g}(R - 2\Lambda), \quad \Lambda = \frac{1}{\ell_{\rm dS}^2} > 0$$

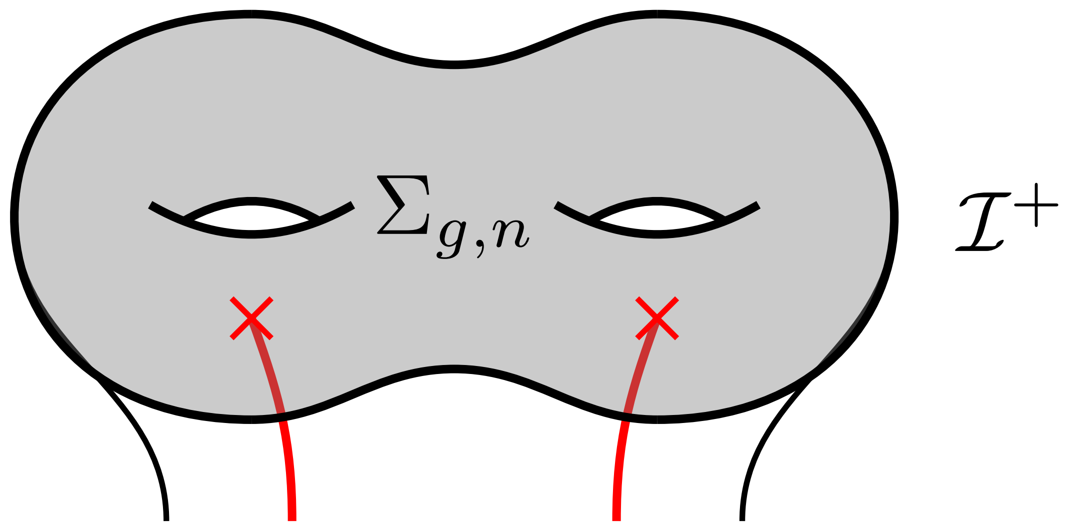
- Loosely related to $\mathrm{SL}(2,\mathbb{C})$ CS theory [Witten;…]

  Caveats: invertibility, global structure, large diffeos,…
  [see also Collier-Eberhardt-Zhang for AdS_3…]



- Consider canonical quantization on an initial value surface $\Sigma_{g,n}$

Canonical quantization



- We mostly consider $\Sigma_{g,n}$ a compact hyperbolic surface ($2g - 2 + n > 0$)

- Phase space $\{g_{ij}, K_{ij}\}$ of $\Sigma_{g,n}$ $\longrightarrow$ $\Psi[g_{ij}]$

- Imposing constraints

  $H\Psi = 0$ WdW equation

  $H_i\Psi = 0$ Momentum constraints

$$\Psi\left[\underbrace{\frac{g_{ij}}{\mathrm{Diff} \times \mathrm{Weyl}}}\right]$$

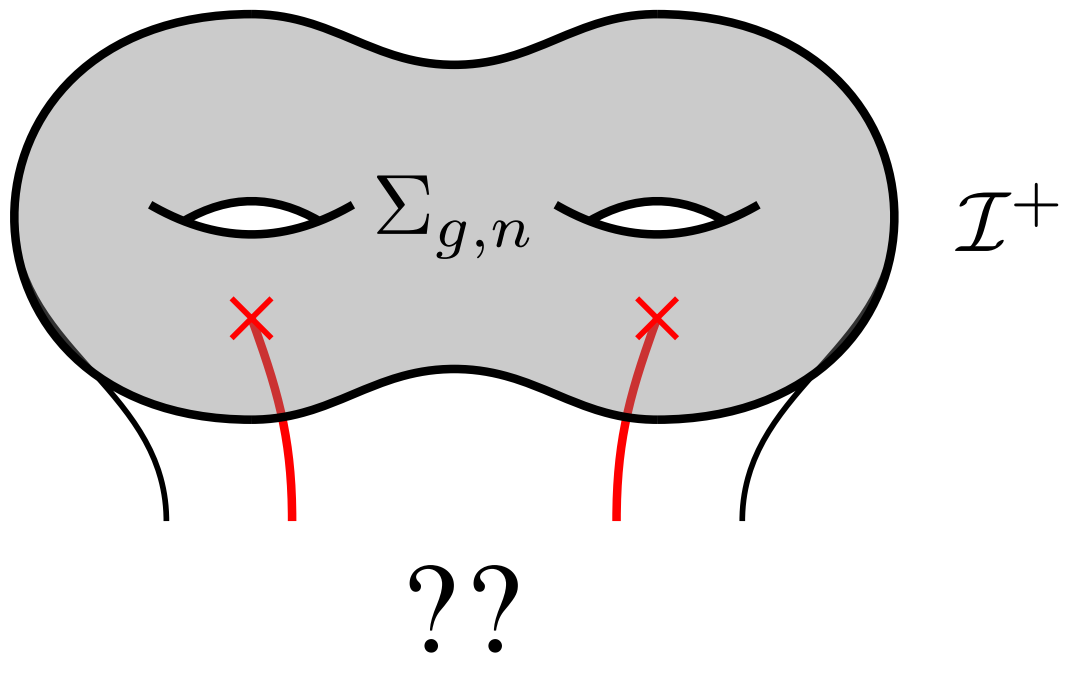
Moduli space $\mathscr{M}_{g,n}$ of $\Sigma_{g,n}$

- Large diffeos are also gauged in gravity

  ➔ gauge $\mathrm{MCG}(M)$ = Diff(M)/Diff$_0$(M)

- Hilbert space $\mathscr{H}_{g,n}$: Ψ is a function* on $\mathscr{M}_{g,n} = \mathscr{T}_{g,n}/\mathrm{MCG}(\Sigma_{g,n})$

Hilbert space



- dS/CFT or proper derivation leads to Virasoro Ward identities [Verlinde;…]

- $\Psi_{g,n} \in \mathscr{H}_{g,n}$ transforms like CFT correlation function with

$$c \sim 13 + \frac{3i\ell_{\mathrm{dS}}}{2G_N}$$

[Brown-Henneaux; Giombi-Maloney-Yin; Strominger;…]

$$\Delta_i = h_i + \tilde{h}_i = 1 \pm \sqrt{1 - m^2\ell_{\mathrm{dS}}^2} \in 1 + i\mathbb{R}$$

$\mathrm{SL}(2,\mathbb{C})$ principal series

$m \gtrsim \ell_{\mathrm{dS}}^{-1}$

- We parametrize

$$c = 13 + 6(b^2 + b^{-2}) \ , \quad b^2 \in i\mathbb{R}$$

Inner product

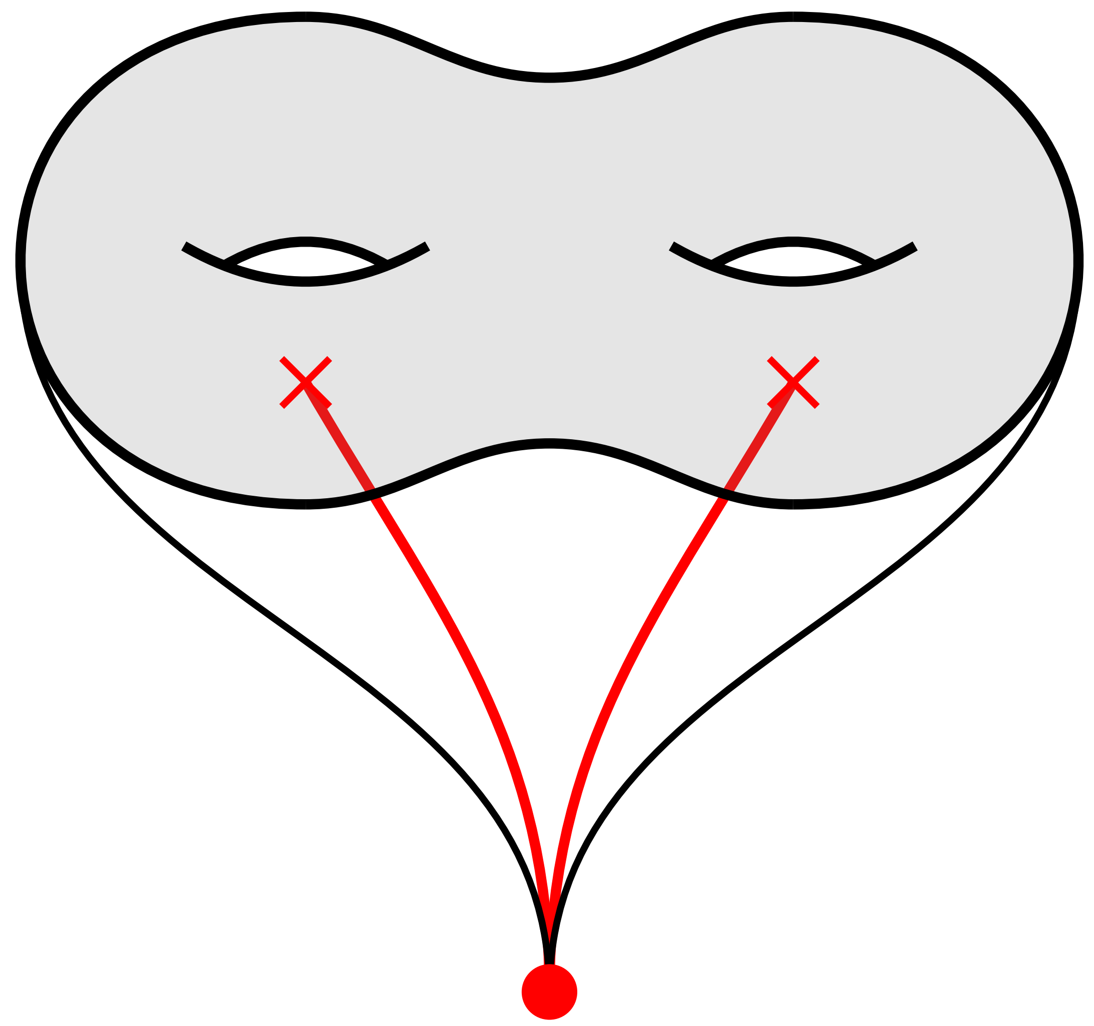
- Inner product: Integrate wave functions over moduli space

$$\langle \Psi_{g,n} | \Psi'_{g,n} \rangle = g_s^{2g-2} \int_{\mathscr{M}_{g,n}} \Psi^*_{g,n} \Psi'_{g,n} \,, \quad \Psi_{g,n}, \Psi'_{g,n} \in \mathscr{H}_{g,n}$$

↑ Normalization that is fixed using TQFT

- Like a string theory path integral

$$c + c^* = 26 \qquad \Delta_i + \Delta_i^* = 2 \qquad s_i - s_i^* = 0$$

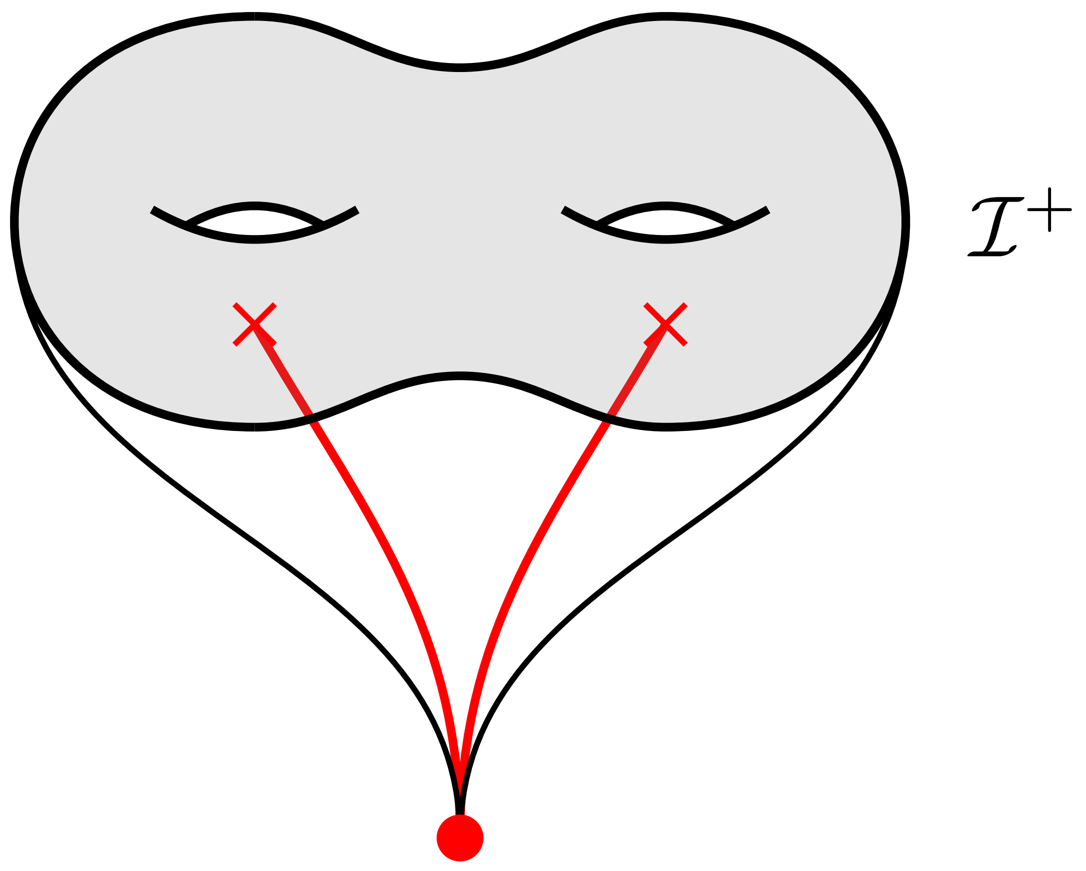


A wavefunction of a universe

Expanding universe

- We now want a wavefunction that describes the state of the universe at $\mathscr{I}^+$
- Consider the following expanding universe (~ dS version of [Maldacena-Maoz] wormhole)

$$ds^2 = -\,dt^2 + \sinh^2(t)\, d\Sigma_{g,n}^2$$



No non-singular on-shell topologies with hyperbolic Cauchy slices

Milne-type "big-bang" singularity

- Differs from the HH proposal

Liouville correlator

- Take the wavefunction on $\mathscr{I}^+$ to be given by the Liouville correlator

Liouville vertex operators

$$\Psi^{(b)}_{g,n}(\boldsymbol{p}) = \left\langle V_{p_1} \cdots V_{p_n} \right\rangle^{(b)}_g \in \mathscr{H}^{(b)}_{g,n}(\boldsymbol{p})$$

$$\Delta_i = 1 + \frac{c-13}{12} - 2p_i^2$$

- This wavefunction has many desirable properties:

→ It **factorizes** as expected under degenerations of $\mathscr{I}^+$

→ It is **crossing symmetric**

→ It is **normalizable**

Integrated cosmological correlators

- Gauge-invariant observables of the theory are **integrated cosmological correlators**

$$\text{“}\ \left\langle \prod_{i=1}^{n} \mathcal{O}_i(z_i) \right\rangle = \int_{\text{metrics on } \mathcal{I}^+} [\mathcal{D}g] \left| \Psi(g) \right|^2 \prod_{i=1}^{n} \mathcal{O}_i(z_i) \text{”}$$

[Maldacena;…]

- Taking the Liouville correlator

$$\sum_{g=0}^{\infty} g_s^{2g-2} \int_{\text{metrics on } \mathcal{I}^+} \frac{[\mathcal{D}g]}{\text{Diff} \times \text{Weyl}} \, | \Psi^{(b)}_{g,n}(p_1, \ldots, p_n) |^2$$

$$= \sum_{g=0}^{\infty} \mathrm{e}^{-S_0(2g-2+n)} \mathsf{A}^{(b)}_{g,n}(p_1, \ldots, p_n)$$

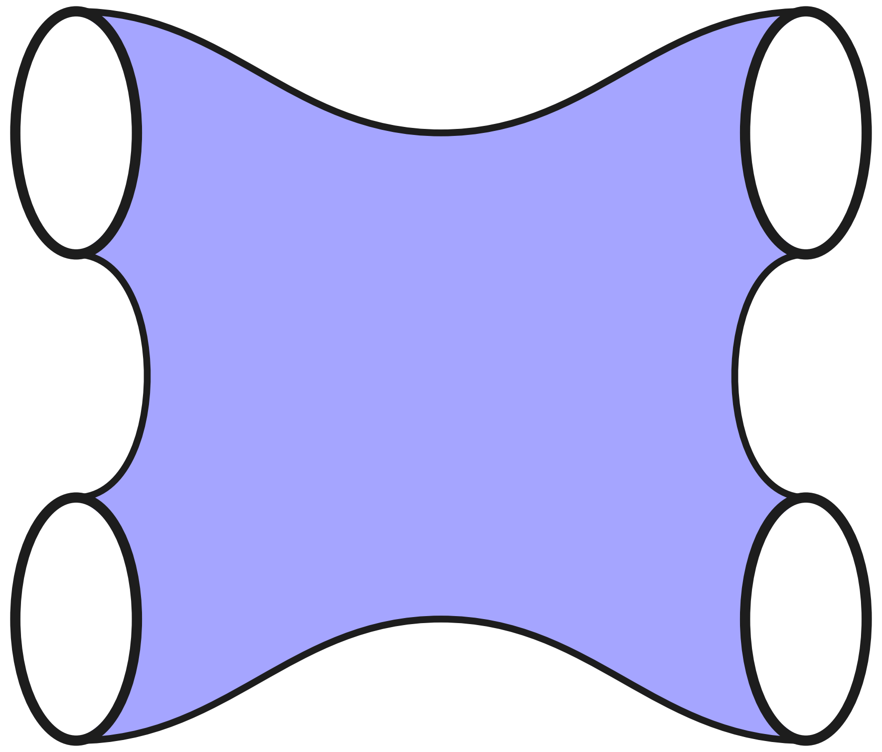
➞ $\Delta_i = h_i + \tilde{h}_i = 1 \pm \sqrt{1 - m^2 \ell_{\text{dS}}^2} \in 1 + i\mathbb{R}$

$\Delta_i = 1 + \dfrac{c-13}{12} - 2p_i^2 \Rightarrow p_i^2 \in i\mathbb{R}$

➞ Massive particles backreact to modify the wavefunction itself

➞ String amplitudes

➞ $g_s^{-2} \sim e^{2S_0} C^{(b)}_{S^2}$



The complex Liouville string

The complex Liouville string

- $\mathsf{A}_{g,n}^{(b)}$ are the string amplitudes of the complex Liouville string (CLS)

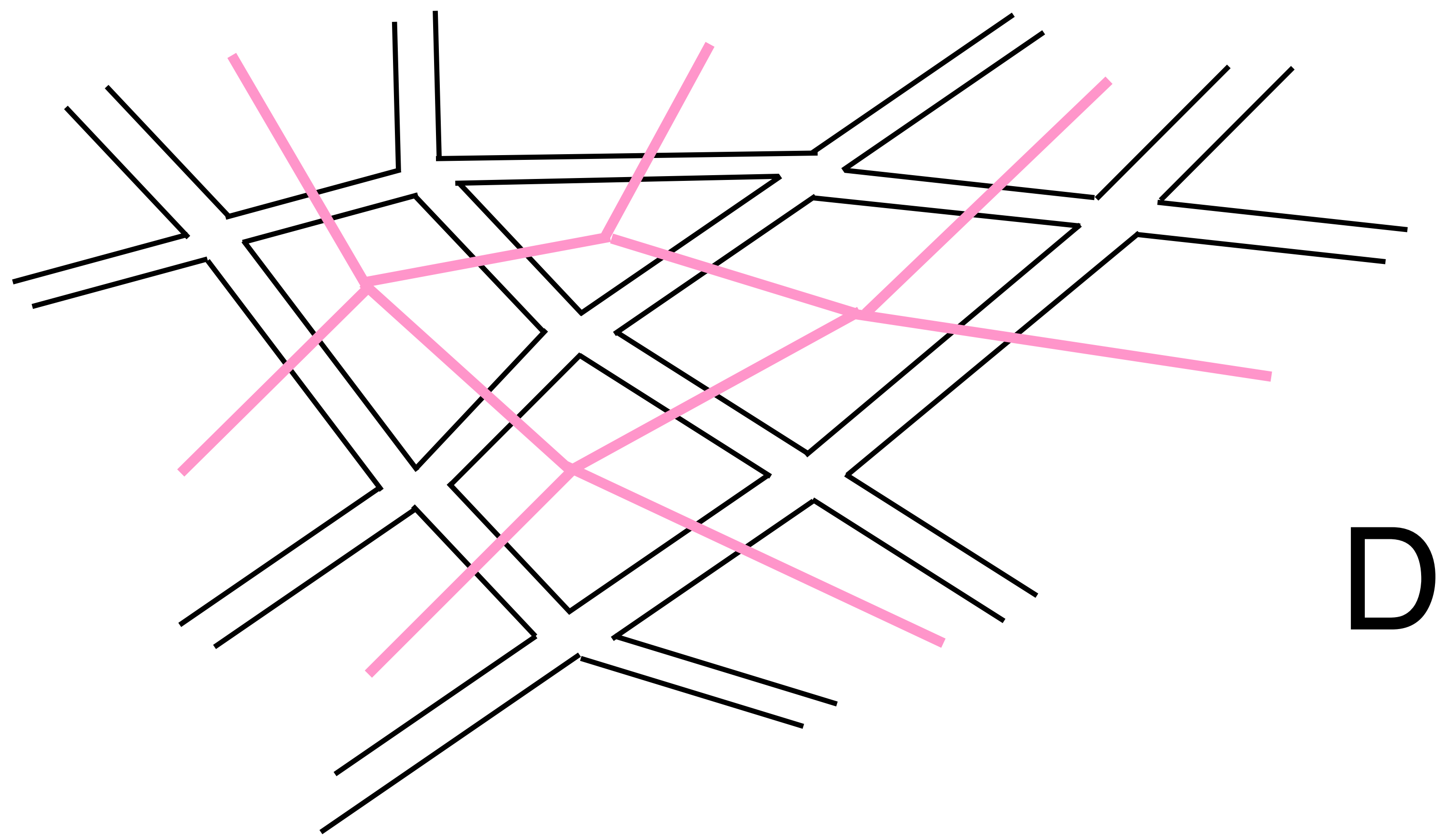
Liouville CFT	$\oplus$	(Liouville CFT)*	$\oplus$	ghosts
$c^+ = c = 13 + i\lambda$		$c^- = c^* = 13 - i\lambda$		$c_{\text{gh}} = -26$

- Mass shell condition: $h + h^- = 1 \longrightarrow p^- = ip$ & $h = (h^-)^* \longrightarrow p^2 \in i\mathbb{R}$

$$\Delta = 1 + i\mathbb{R}$$

- String amplitudes: $\mathsf{A}_{g,n}^{(b)}(\mathbf{p}) = \int_{\mathcal{M}_{g,n}} \left| \underbrace{\left\langle V_{p_1} \ldots V_{p_n} \right\rangle_g^{(b)}}_{\hat{=}\, \Psi_{g,n}^{(b)}} \right|^2 \times \text{ghosts}$

$$b^2 \in i\mathbb{R}$$



Complex Liouville string

⇕

Double scaled matrix integral

2-matrix integral

- M_1 and M_2 Hermitian matrices $\int_{\mathbb{R}^{2N^2}} \mathrm{d}M_1 \mathrm{d}M_2 \, \mathrm{e}^{-N\,\mathrm{tr}(V_1(M_1)+V_2(M_2)-M_1M_2)}$

- Observables are the resolvents $R(x) = \mathrm{tr}\dfrac{1}{x - M_1}$

$$\left\langle \prod_{j=1}^{n} R(x_j) \right\rangle_{\mathrm{c}} = \sum_{g=0}^{\infty} \mathrm{e}^{-S_0(2g-2+n)} R_{g,n}(x_1, \ldots, x_n)$$

- Resolvent is a multivalued function on the complex plane with branch cuts on eigenvalue distribution

- The spectral curve defines a Riemann surface on which the resolvent is single valued

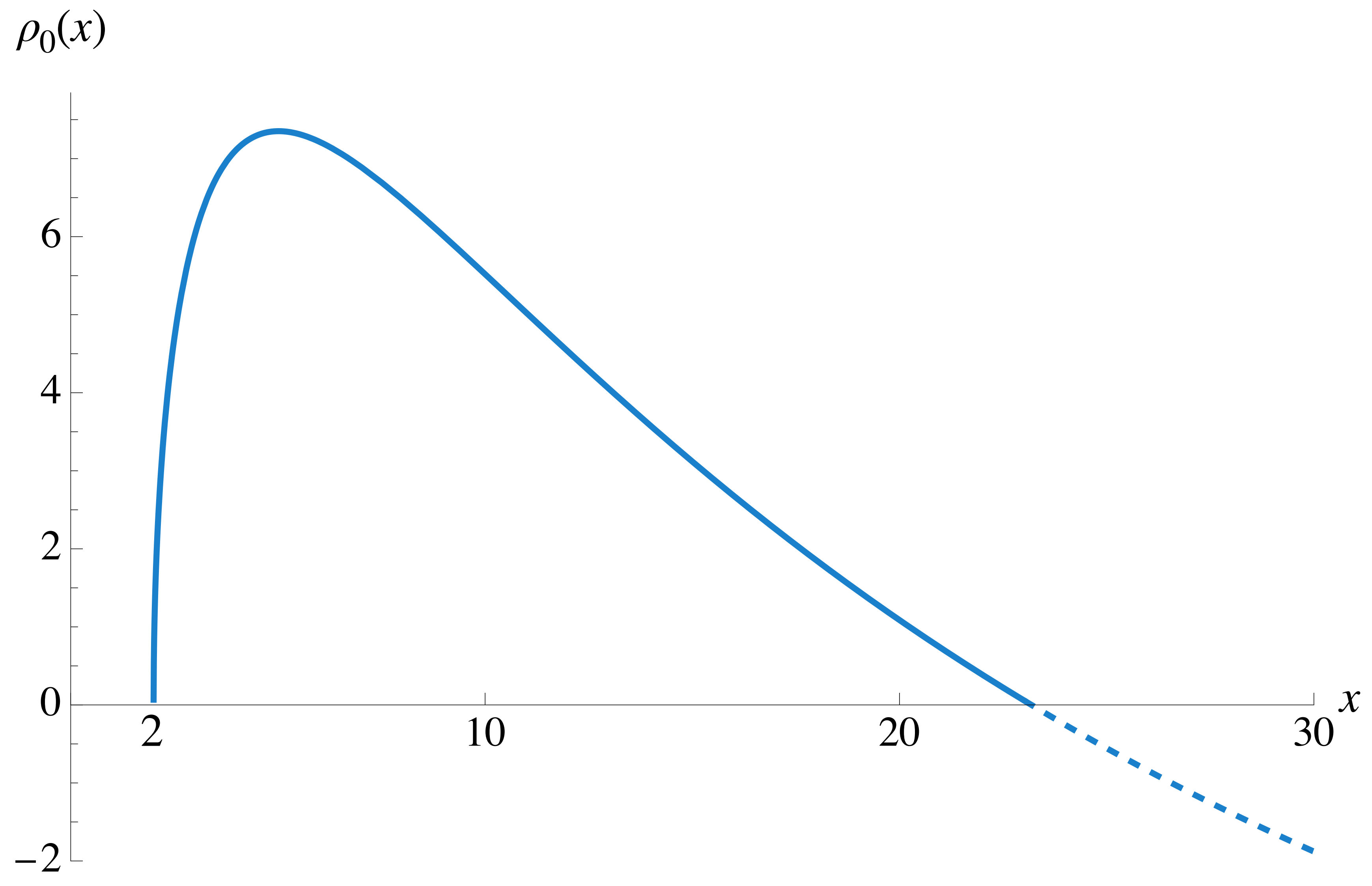
Spectral curve of CLS

- The spectral curve of the matrix integral dual to the CLS is

$$\mathsf{x}(z) = -\,2\cos(\pi b^{-1}\sqrt{z})\ ,\quad \mathsf{y}(z) = 2\cos(\pi b\sqrt{z})$$

- Discontinuity of spectral curve = eigenvalue distributions

$$\rho_0(x) = \frac{2}{\pi}\sinh(-i\pi b^2)\sin\left(-ib^2\text{arccosh}\left(\frac{x}{2}\right)\right)$$



Topological recursion

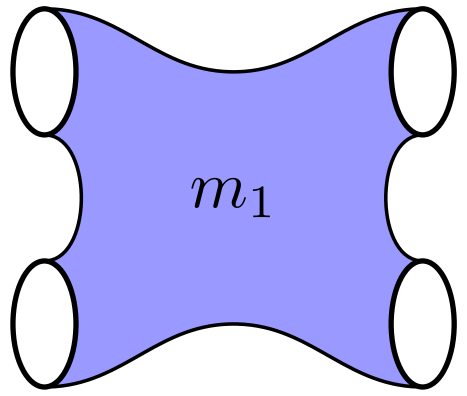
- Use topological recursion for $\omega_{g,n}$ [Eynard-Orantin;…]

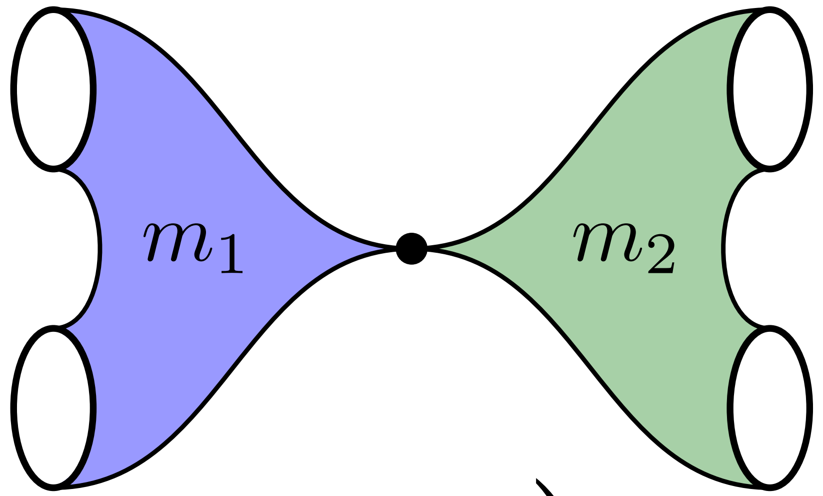
$$\omega_{0,1}(z) = -\,\mathsf{y}(z)\mathrm{d}\mathsf{x}(z)\ , \quad \omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}\ , \quad \omega_{g,n}(z_1,\ldots,z_n) \equiv R_{g,n}(\mathsf{x}(z_1),\ldots,\mathsf{x}(z_n))\,\mathrm{d}\mathsf{x}(z_1)\cdots\mathrm{d}\mathsf{x}(z_n)$$

- **Relation** between $\omega_{g,n}$ and $\mathsf{A}_{g,n}^{(b)}$

- This leads to “Feynman-rules” for $\mathsf{A}_{g,n}^{(b)}$ aka cosmological correlators for arbitrary g, n

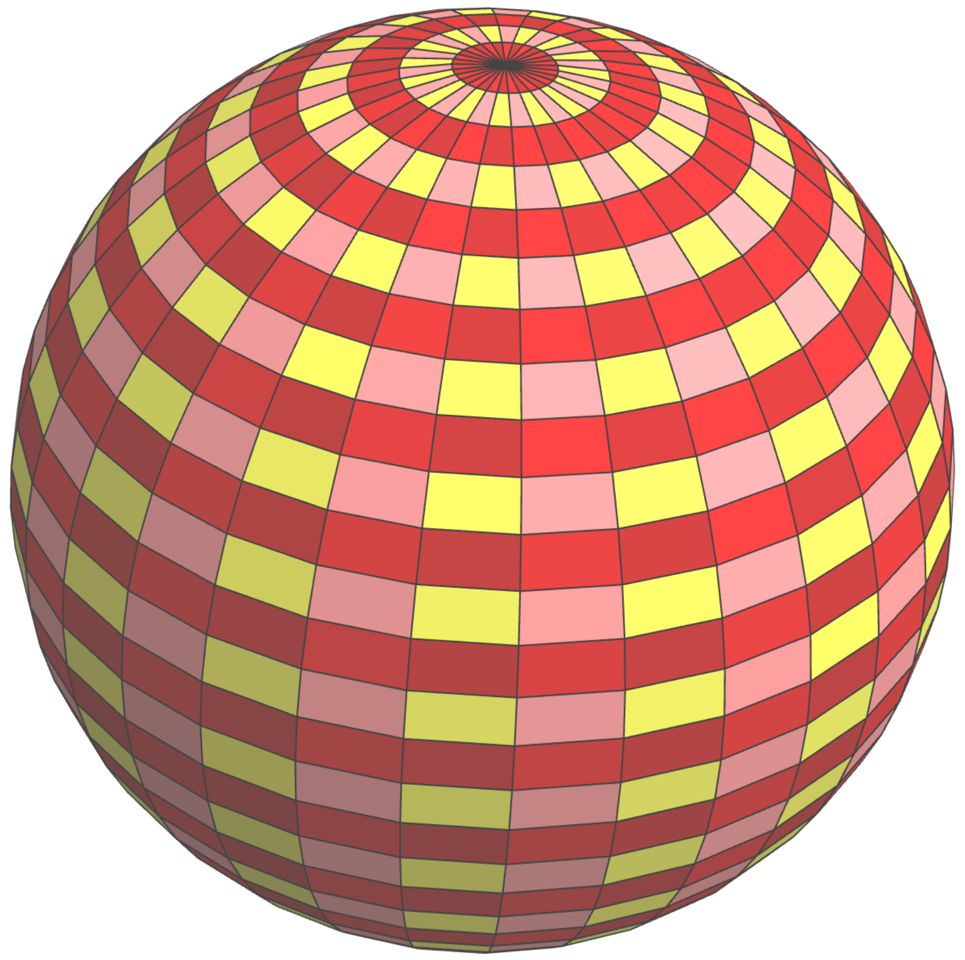
$$\mathsf{A}_{0,4}^{(b)}(\mathbf{p}) = \sum_{m_1=1}^{\infty} \frac{2b^2\mathsf{V}_{0,4}^{(b)}(ip_1,ip_2,ip_3,ip_4)\prod_{j=1}^{4}\sin(2\pi m_1 b p_j)}{\sin(\pi m_1 b^2)^2}$$

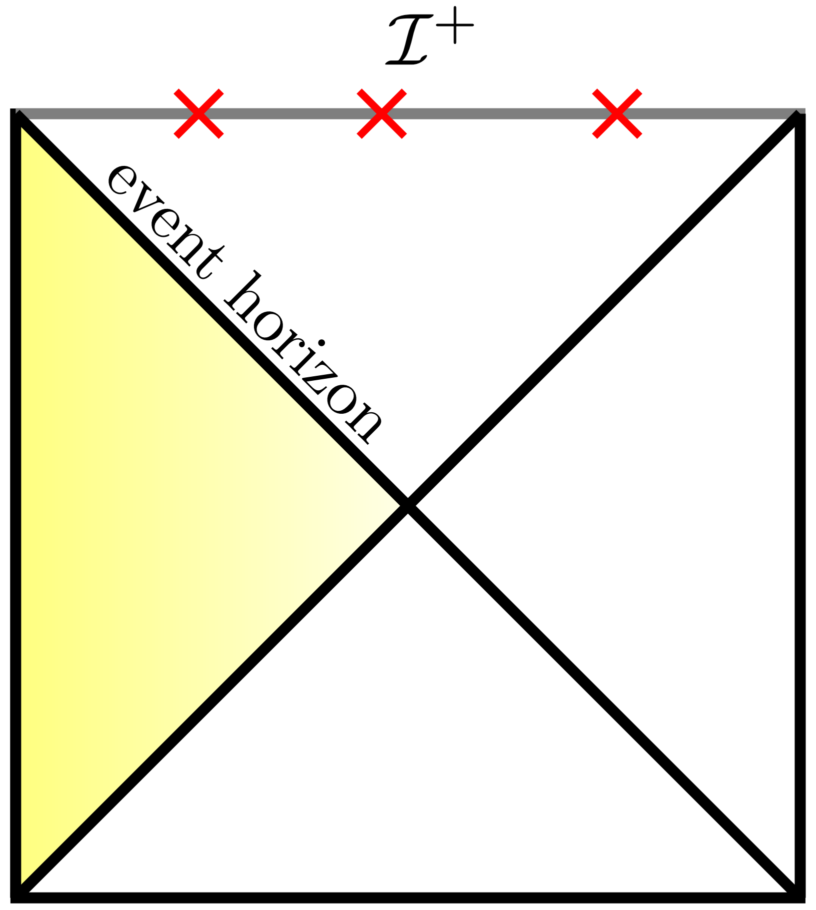




$$-\sum_{m_1,m_2=1}^{\infty} \frac{(-1)^{m_1+m_2}\sin(2\pi m_1 b p_1)\sin(2\pi m_1 b p_2)\sin(2\pi m_2 b p_3)\sin(2\pi m_2 b p_4)}{\pi^2\sin(\pi m_1 b^2)\sin(\pi m_2 b^2)} \times \left(\frac{1}{(m_1+m_2)^2} - \frac{\delta_{m_1\neq m_2}}{(m_1-m_2)^2}\right) + 2\text{ perms}$$

$$\mathsf{V}_{0,4}^{(b)}(p_1,p_2,p_3,p_4) = -\sum_{i=1}^{4} p_i^2 + \frac{b^2+b^{-2}}{4}$$





Gibbons-Hawking dS entropy

Sphere partition function

- Gibbons-Hawking conjecture that the cosmological horizon encodes an entropy captured by the sphere partition function (= Euclidean dS)

$$\mathrm{S}_{\mathrm{GH}} = \log \mathcal{Z}_{\mathrm{grav}}(S^3)$$

- To one-loop order can obtain this from the Euclidean Einstein Hilbert action
  [Anninos-Denef-Law-Zun;…]

$$S_{\mathrm{GH}} = \log \mathcal{Z}_{\mathrm{grav}}^{S^3} \sim \frac{\pi \ell_{\mathrm{dS}}}{2G_N} - 3 \log \frac{\pi \ell_{\mathrm{dS}}}{2G_N} + 5 \log(2\pi) \pm \frac{5\pi i}{2} + \mathcal{O}(G_N^0)$$

$$\frac{A_h}{4G_N}$$

$$3 = \frac{1}{2}\mathrm{dim}(SO(4))$$

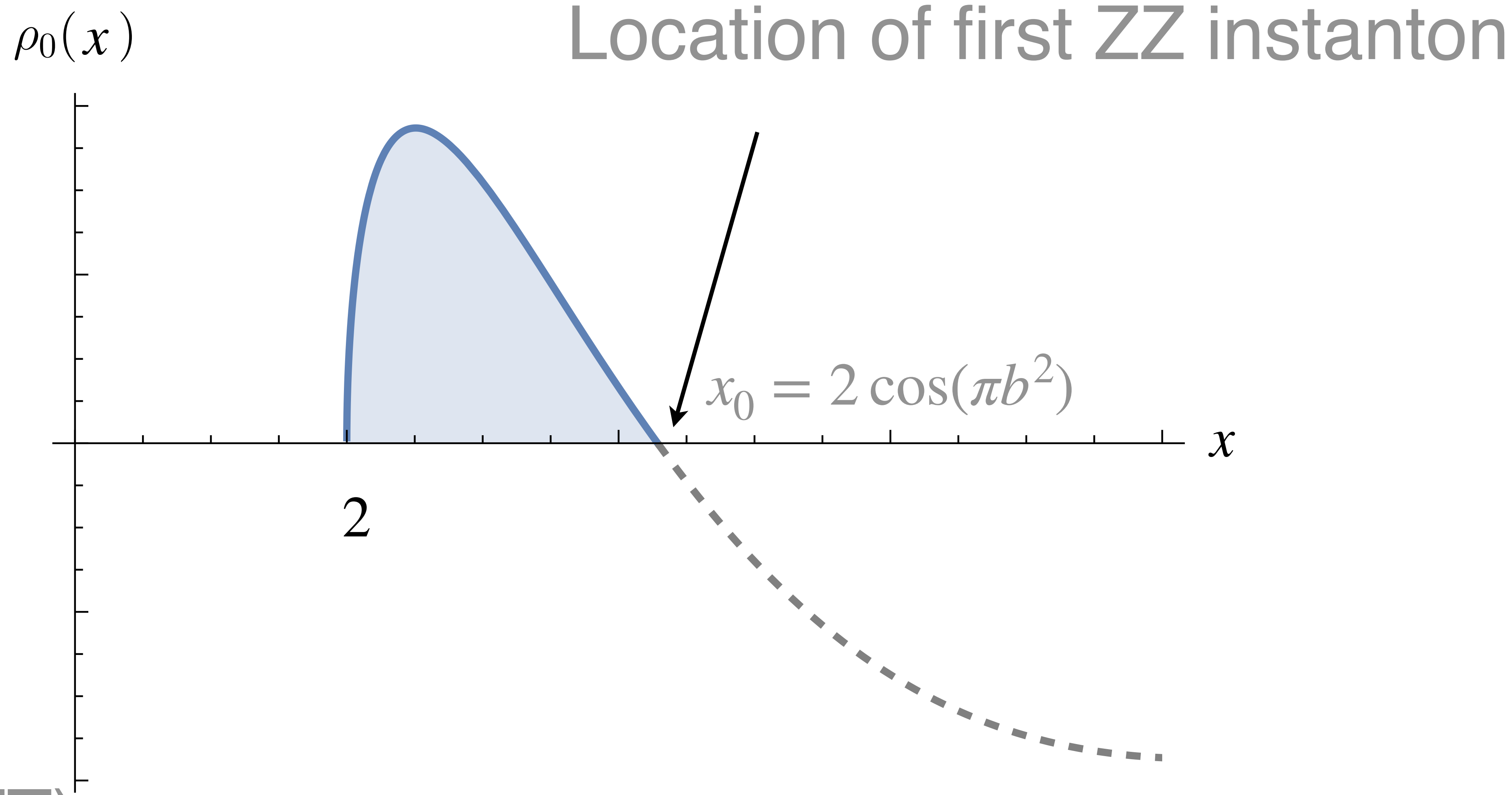
Gibbons-Hawking-Perry phase for Gaussian unsuppressed Euclidean path integral

$S_{\rm GH} = S_{\rm MM}$

- Count instead the effective number $N_{\rm eff}$ until first zero x_0

$$N_{\rm eff} \equiv \int_2^{x_0} \mathrm{d}x\, \mathrm{e}^{S_0} \rho_0(x) \sim \frac{b \sin(\pi b^2) \sin(\pi b^{-2})}{1-b^4}$$

Fix $e^{S_0} \sim b^{-1}$
(consistency of the 3d TQFT)



- **Leads to an entropy that reproduces the sphere partition function**

$$S_{\rm MM} = \log N_{\rm eff}^2 \sim \frac{\pi \ell_{\rm dS}}{2G_N} - 3\log\frac{\pi\ell_{\rm dS}}{2G_N} + 5\log(2\pi) + \frac{\pi i}{2} + \mathcal{O}(G_N^0)$$

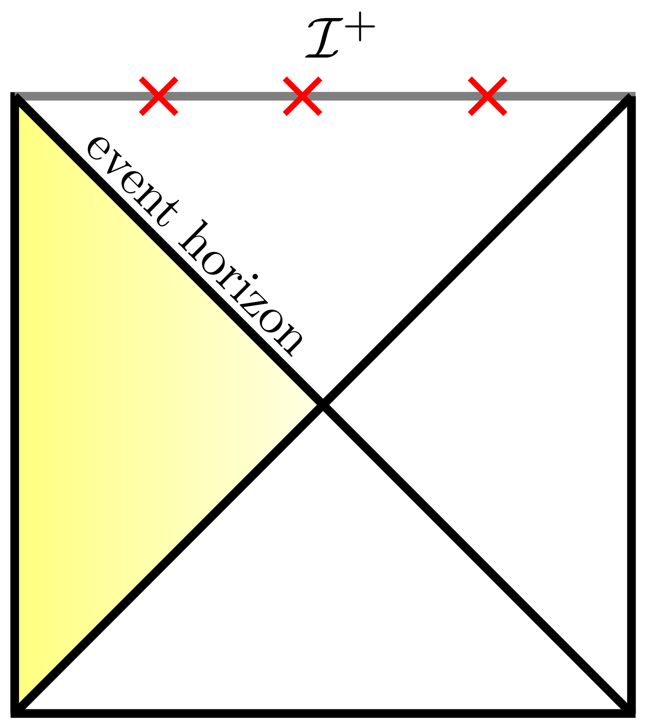
$$= S_{\rm GH} = \log \mathcal{Z}_{\rm grav}^{S^3}$$

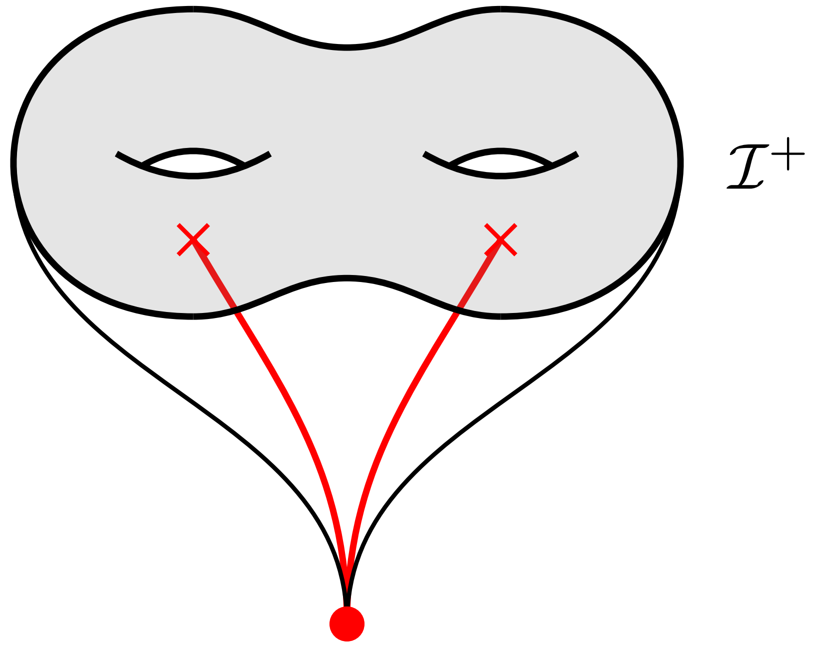
Remember:

$$c = 13 + 6(b^{-2} + b^2)$$
$$\approx 13 + \frac{3i\ell_{\rm dS}}{2G_N}$$

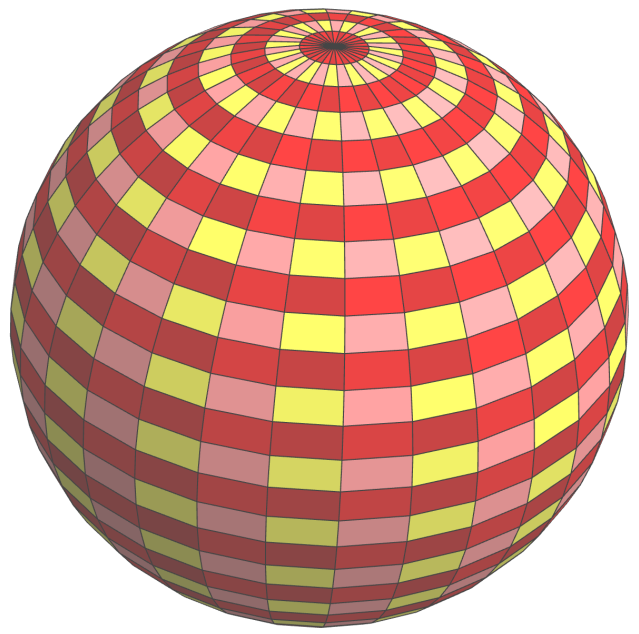
Summary

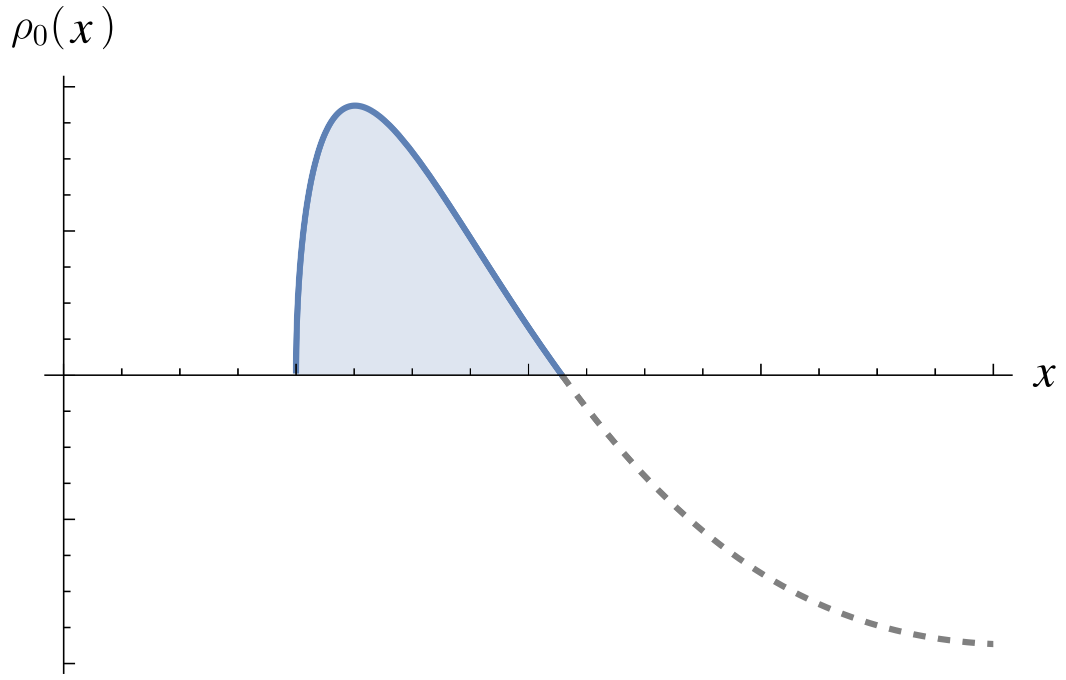
- Relation between integrated cosmological correlators and a double scaled matrix integral
- The wavefunction is the correlator of the CLS, which leads to an expanding spacetime





- $\log N_{\text{eff}}^2$ reproduces the Gibbons-Hawking dS entropy





Open questions

- Since we anyway integrate over the metric at $\mathscr{I}^+$, it seems reasonable to sum over topologies of $\mathscr{I}^+$

→ This sum is asymptotic and requires non-perturbative corrections (eigenvalue instantons/ZZ-instantons). What is their dS_3 interpretation?

- What is a complete basis of normalizable states?
- What about other spacetime topologies? Lens spaces?
- Matrix model as the edge mode theory? TEE?
- What happens if we insert an observer?
- What is the relation to DSSYK?
- Would supersymmetry help?

Thank you!

Stable graphs

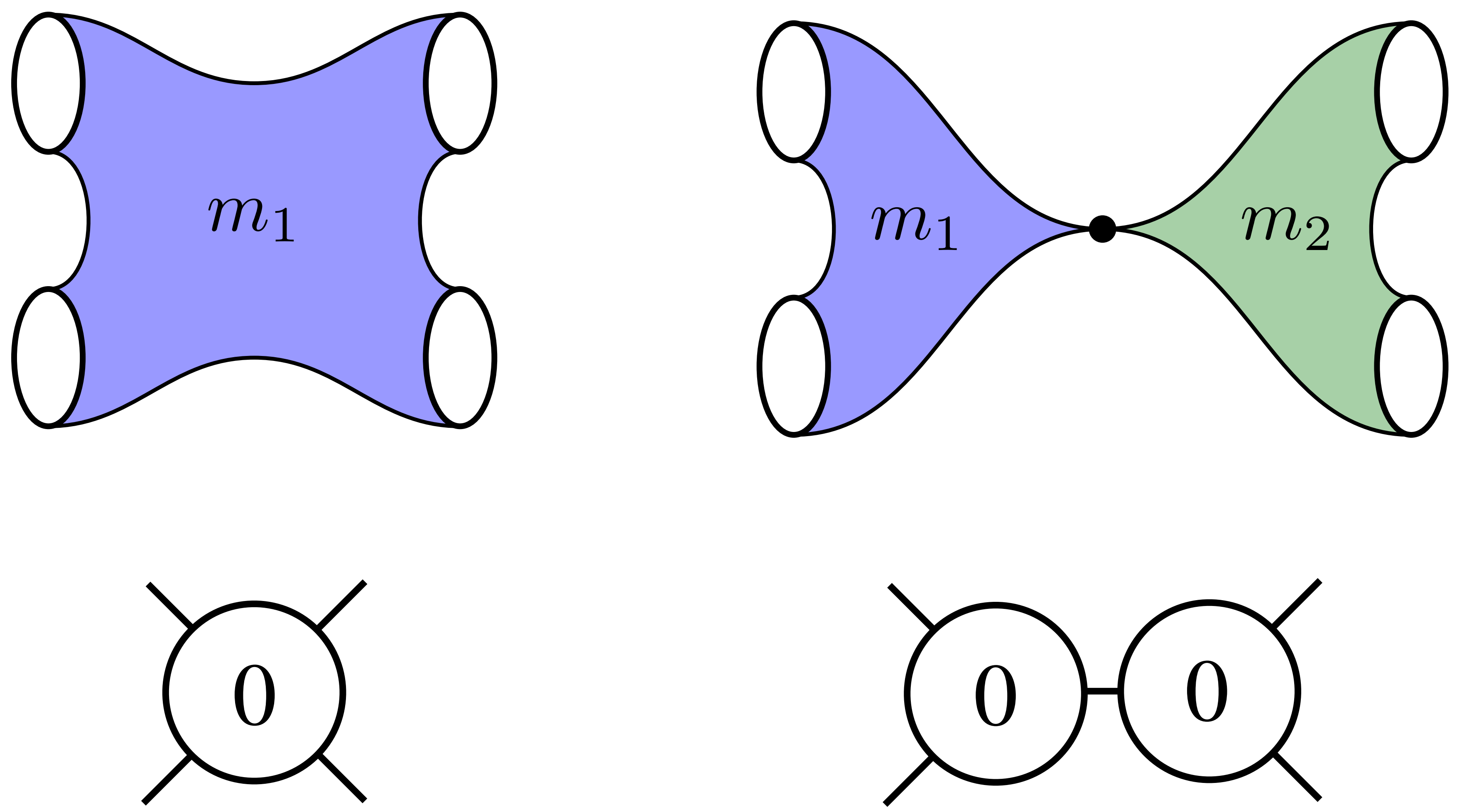
- Matrix model gives an all g, n expression for $\mathsf{A}^{(b)}_{g,n}$

$$\mathsf{A}^{(b)}_{g,n}(p_1,\ldots,p_n) = \sum_{\Gamma\in\mathscr{G}^{\infty}_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \int' \prod_{e\in\mathscr{E}_\Gamma}(-2p_e\,\mathrm{d}p_e) \prod_{v\in\mathscr{V}_\Gamma}\left(\frac{b(-1)^{m_v}}{\sqrt{2}\sin(\pi m_v b^2)}\right)^{2g_v-2+n_v} \times \prod_{j\in I_v}\sqrt{2}\sin(2\pi m_v b p_j)\mathsf{V}^{(b)}_{g_v,n_v}(i\mathbf{p}_v)$$

- Stable graphs $\mathscr{G}^{\infty}_{g,n}$ expansion: **Feynman rules for the Matrix integral**

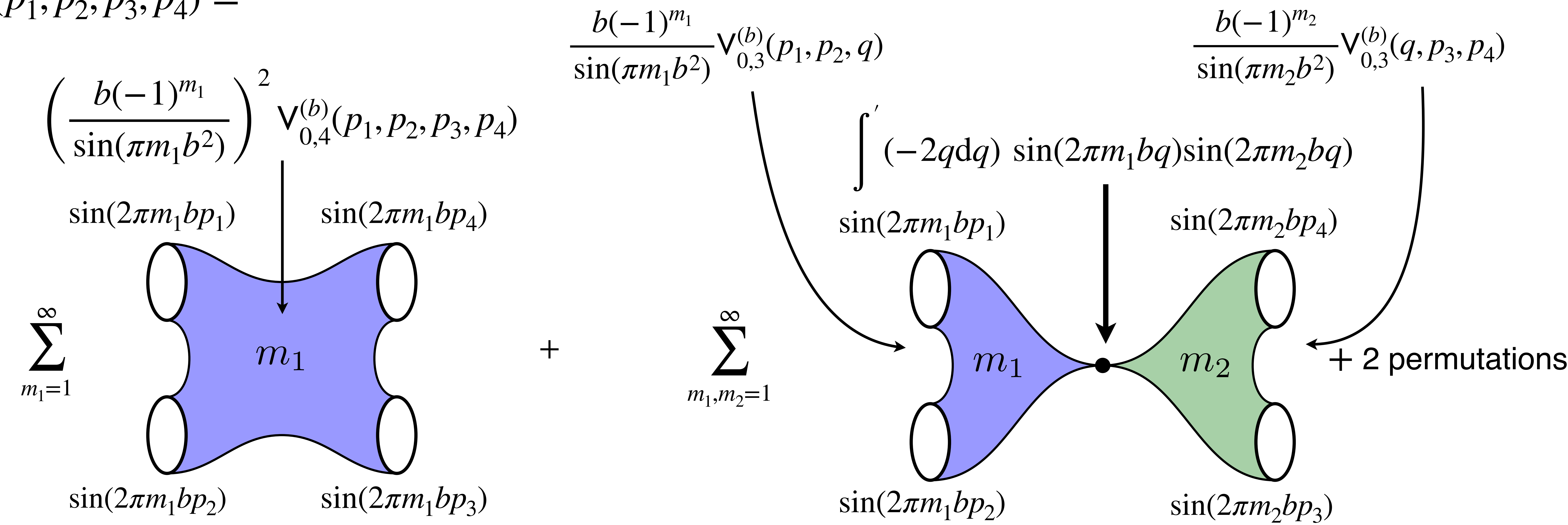
  - $m \in \mathbb{Z}_{\geq 1}$ are the branch points of the spectral curve
  - expansion in colored Riemann surfaces
  - analytically continued quantum volumes $\mathsf{V}^{(b)}_{g,n}(i\mathbf{p})$

degenerations of a Riemann surface



“Feynman rules”

$$\mathsf{A}^{(b)}_{0,4}(p_1,p_2,p_3,p_4) =$$

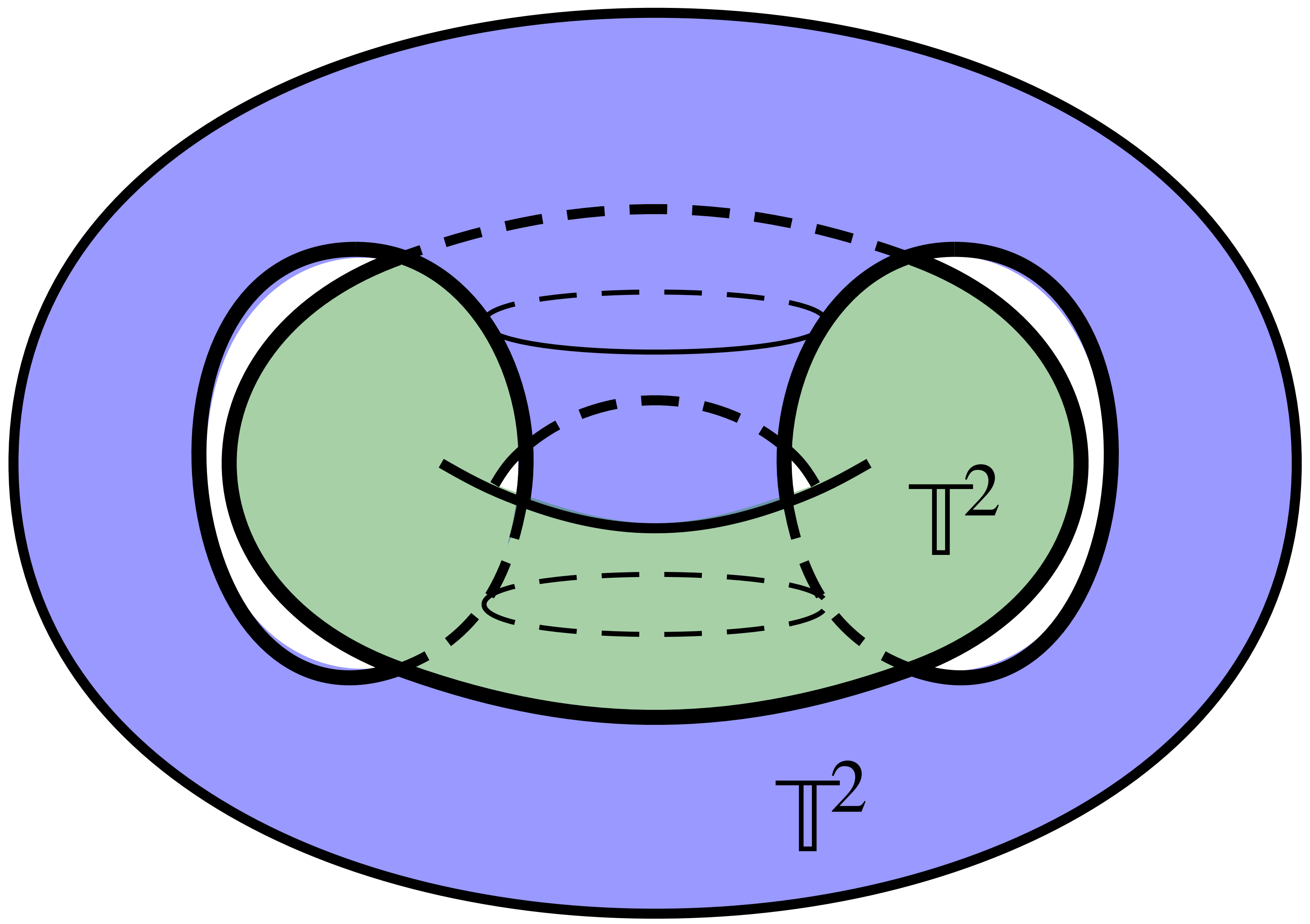


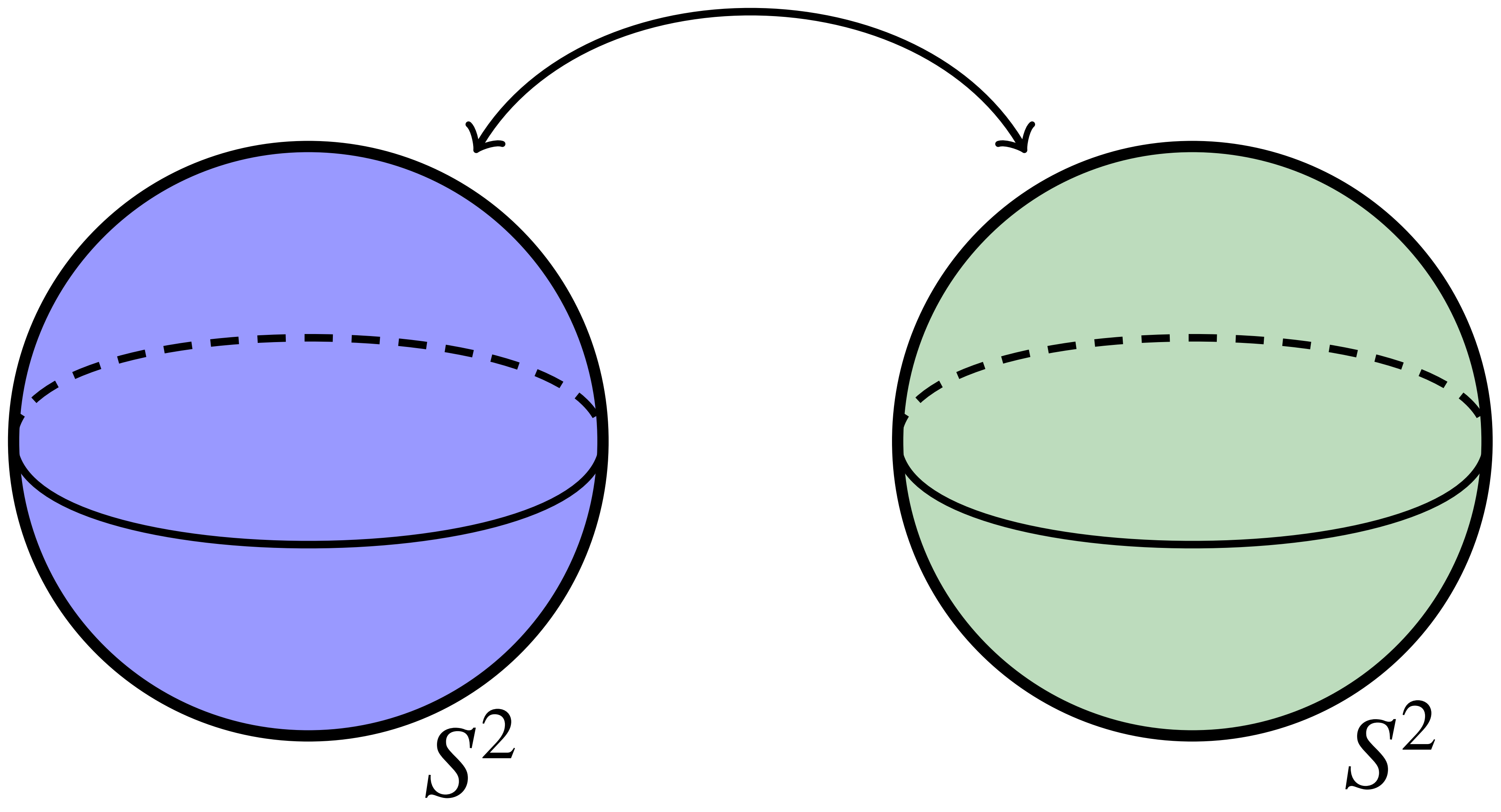
- $\mathsf{V}^{(b)}_{0,3}(\mathbf{p}) = 1 \qquad \mathsf{V}^{(b)}_{0,4}(p_1,p_2,p_3,p_4) = -\sum_{i=1}^{4} p_i^2 + \frac{b^2+b^{-2}}{4}$

[Collier-Eberhardt-BM-Rodriguez]

Heegard splitting

- Using TQFT surgery techniques we fix g_s





$$\mathcal{Z}^{S^3}_{\text{TQFT}} = \langle \mathbb{1}^{(b)}_{1,0} | \mathbb{S} | \mathbb{1}^{(b)}_{1,0} \rangle = \mathbb{S}_{\mathbb{1},\mathbb{1}}$$

$$\mathcal{Z}^{S^3}_{\text{TQFT}} = \langle \mathbb{1}^{(b)}_{0,0} | \mathbb{1}^{(b)}_{0,0} \rangle = g_s^{-2}$$

Virasoro vacuum character   modular S-transformation

Hartle Hawking

- We now want a wavefunction that describes the state of the universe at $\mathscr{I}^+$
- Hartle-Hawking? Take three manifolds with boundary $\mathscr{I}^+$, simplest such choice are handlebodies $S\Sigma_g$

$$|\mathrm{HH}\rangle = \sum_{\text{handlebodies } \mathrm{S}\Sigma_\mathrm{g}} |\underbrace{\mathscr{F}^{(b)}_{\mathbb{1}}(S\Sigma_g)}_{\text{Virasoro vacuum conformal block}}|^2 + \dots$$

- For $g \geq 1$ does not solve Einstein equations but involves continuation to (-,-,-) signature (sum over topologies in -AdS$_3$)
- Each term in the sum is non-normalizable [Castro-Maloney for $\mathbb{T}^2$…]
- The sum over topologies is completely uncontrolled (no small parameter suppressing topology fluctuation)