Higher Soft Theorems

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Based on work to appear (2504.XXXXX)

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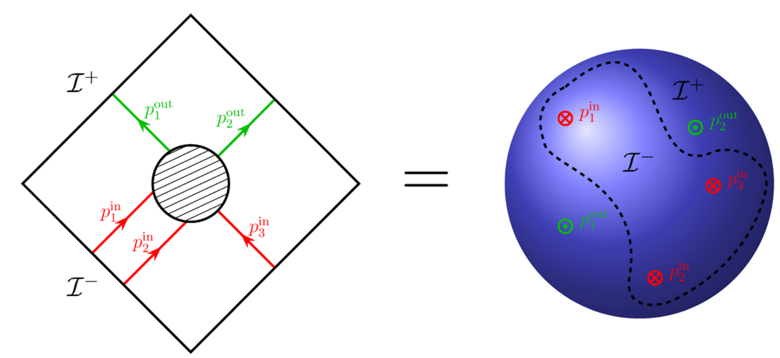
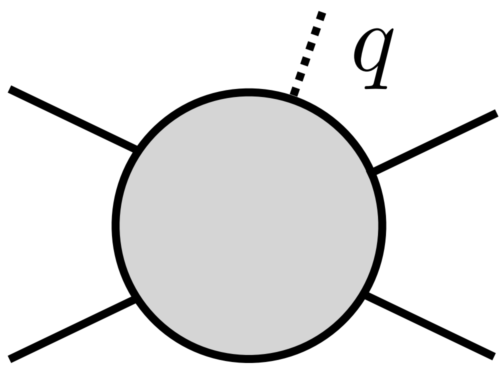


@ Simons Celestial Holography Satellite Meeting, April 2025

The behavior of scattering amplitudes when the momentum of a particle is small is often universal



$$\lim_{q \to 0} A_{n+1} = \mathcal{S} A_n$$

Earliest examples:

- Soft pion theorem

  [Adler]

$$\lim_{q_\pi \to 0} \mathscr{A}_{n+\pi} = 0$$

- Soft photon theorem

  [Low; Burnett, Kroll; Weinberg]

$$\lim_{q_\gamma \to 0} \mathscr{A}_{n+\gamma} = -\sum_a Q_a \frac{\epsilon_\gamma \cdot p_a}{q_\gamma \cdot p_a} \mathscr{A}_n$$

Many perspectives

Symmetry:

“asymptotic (photons, gravitons), spontaneously broken (pions),…”

Geometry

“encode geometry of moduli space of vacua”

Factorization (EFT)

“soft theorems are just leading operators in EFT of soft modes”

Connection between these?

It always bugged me that soft scalars and soft photons required different ideas…

Today I'll try to argue that recent tools of so-called "higher" or "generalized" symmetry can bring these together.

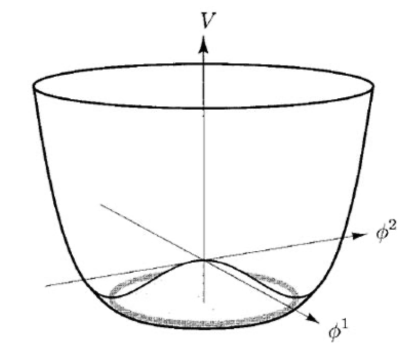
Furthermore, they suggest new structures!

Outline

- Review: soft pions
- Soft photons from 1-form symmetry
- 2-group soft theorem

Review: soft pions

Spontaneously broken symmetry



Familiar situation $\langle \mathcal{O}^a \rangle = v^a$ breaks $G \to H$

$$[T^a, T^b] = i f^{abc}\, T^c$$
$$[T^a, X^b] = i f^{abc}\, X^c$$
$$[X^a, X^b] = i f^{abc}\, T^c$$

T^a generate unbroken subgroup H

X^a broken generators in coset G/H

Goldstone's theorem: SSB implies existence of NGB for each X^a

"NGB parameterize fluctuations of VEV" $O^a \sim v^a e^{i\pi^a X^a}$

Pions as Nambu-Goldstone bosons

More precisely: $\langle 0 \,|\, j^a_\mu(x) \,|\, \pi^b(q) \rangle = f_\pi \, \delta^{ab} q_\mu \, e^{iq\cdot x}$

This implies current linear in the NBG and shift symmetry

$$j^a_\mu = \partial_\mu \pi^a + \mathcal{O}(\pi^3) \qquad \pi^a \to \pi^a + c^a + \mathcal{O}(\pi^2)$$

E.g., QCD in the chiral limit ($m_q = 0$)

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V \qquad \langle \psi^a_L \psi^b_R \rangle = \Lambda^3_{QCD} \, \delta^{ab}$$

$N_f^2 - 1$ pions at low energies $\qquad \mathcal{L} = f_\pi^2 \mathrm{Tr}(\partial U \partial U^\dagger) + \cdots$

Soft pion: Adler Zero

Considering current overlap with multi-particle state $|\alpha\rangle$

$$\langle 0 | j^{a\mu}(q) | \alpha \rangle = i f_\pi \frac{q^\mu}{q^2} \mathscr{A}(\alpha + \pi^a) + \mathscr{N}^\mu$$

Ward identity implies relation

$$0 = \langle 0 | \partial_\mu j^{a\mu}(q) | \alpha \rangle = - f_\pi \mathscr{A}(\alpha + \pi^a) + q_\mu \mathscr{N}^\mu$$

Which gives Adler's zero [Adler]

$$\lim_{q \to 0} \mathscr{A}(\alpha + \pi^a) \sim \lim_{q \to 0} q_\mu \mathscr{N}^\mu \longrightarrow 0$$

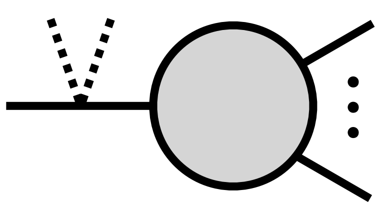
Double-soft pion

Current algebra also implies multi-soft pion theorems

$$\partial \cdot j^a(x) j^{b\mu}(y) \sim i f^{abc} \delta^{(4)}(x-y) j^{c\mu}(x)$$

By considering $\langle 0 | j^{a\mu}(q_a) j^{b\nu}(q_b) | \alpha \rangle$ one finds

$$\lim_{q_{a,b}\to 0} \mathscr{A}_{n+2} = (S^{(0)} + S^{(1)}) \mathscr{A}_n$$



with

$$S^{(0)} = \sum_i \frac{p_i \cdot (q_a - q_b)}{p_i \cdot (q_a + q_b)} [X^a, X^b]$$

[Weinberg; Arkani-Hamed, Cachazo, Kaplan]

$$S^{(1)} = \sum_i \left(\frac{p_i \cdot (q_a - q_b)}{p_i \cdot (q_a + q_b)} \frac{q_a \cdot q_b}{p_i \cdot (q_a + q_b)} + \frac{q_a \cdot J_i \cdot q_b}{p_i \cdot (q_a + q_b)} \right) [X^a, X^b] + \frac{q_a \cdot q_b}{p_i \cdot (q_a + q_b)} \{X^a, X^b\}$$

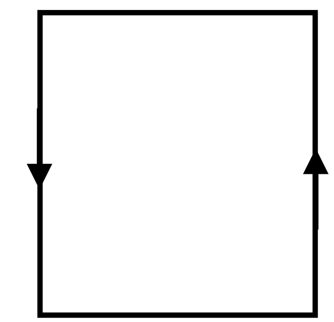
[Cachazo; Low]

Soft photons from 1-form symmetry

1-form symmetry

Charged operators are not local but line operators, get a v.e.v

$$\langle W(\Gamma) \rangle = c e^{-L(\Gamma)} \sim 1$$ “perimeter law”

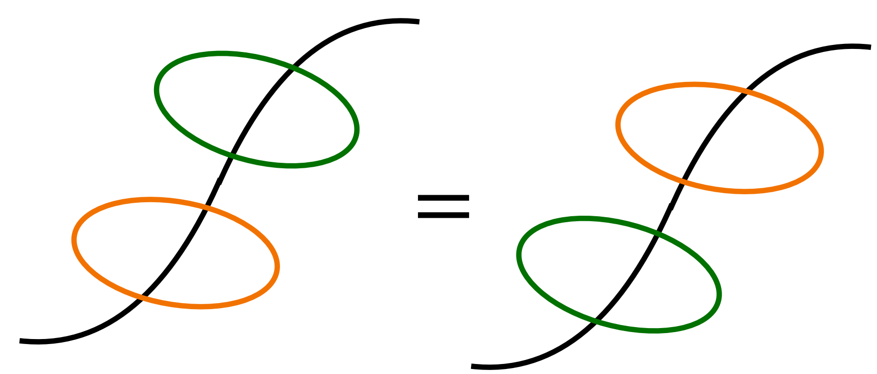


Γ

Conserved currents are rank two tensor $J^{\mu\nu} = J^{[\mu\nu]}$

Charges supported on codimension-two surfaces → Abelian

$$Q = \oint_{\Sigma} J_{\mu\nu} d\Sigma^{\mu}\nu$$



Photon parameterizes fluctuations of v.e.v $W(\Gamma) = e^{i \oint_{\Gamma} A}$

One-form Ward identity

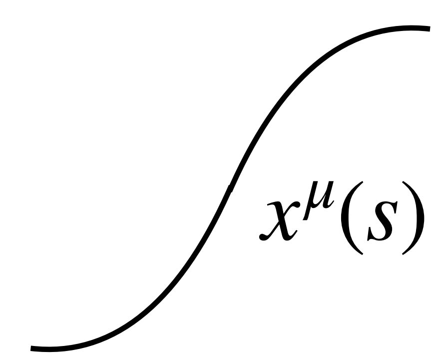
One-form symmetry implies the following operator eq.

$$\partial^\mu J_{\mu\nu} W_Q(\Gamma) \sim i\, Q\, \delta_\nu^{(D-1)}(x_\Gamma)\, W_Q(\Gamma)$$

(c.f., zero-form WI $\partial^\mu j_\mu(x) O(y) \sim i\, q\, \delta^{(D)}(x-y)\, O(x)$)

The codimension-1 delta function is defined as

$$\delta_\nu^{(D-1)}(x_\Gamma) = \int ds\, \dot{x}_\nu(s)\, \delta^{(D)}(x - x(s))$$



Photon as NG boson

More precisely: $$\langle 0 | J_{\mu\nu}(x) | \gamma^h(q) \rangle = g\,(q_\mu \epsilon^*_\nu - q_\nu \epsilon^*_\mu)\, e^{iq\cdot x}$$

This implies current linear in the NBG and shift symmetry

$$J_{\mu\nu} \sim \partial_{[\mu} A_{\nu]} + \mathcal{O}(A^3) \qquad A_\mu \to A_\mu + \lambda_\mu + \mathcal{O}(A^2)$$

with $\partial_{[\mu}\lambda_{\nu]} = 0$

E.g., free Maxwell theory

$$U(1)^{(1)}_e \times U(1)^{(1)}_e \to \varnothing \qquad J^e_{\mu\nu} = \frac{1}{g^2} F_{\mu\nu} \qquad J^m_{\mu\nu} = {*F}_{\mu\nu}$$

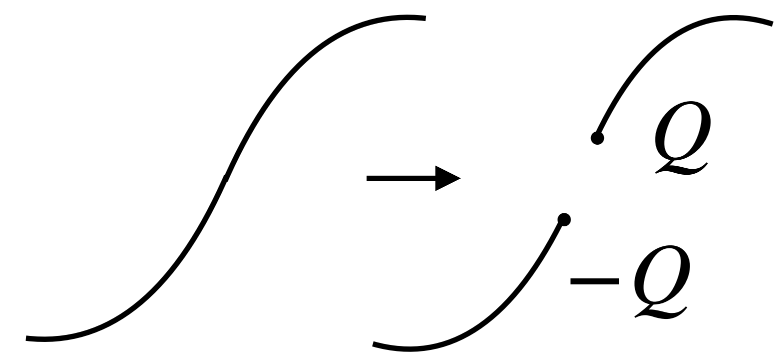
Photon is massless because it's a NGB

Broken one-form symmetry

Electric one-form symmetry is broken by charged matter

$$\partial_\mu J_e^{\mu\nu} = j^\nu \neq 0$$

“screening”



This means it is a priory hard to interpret the soft photon theorem in terms of a 1-form symmetry.

However, 1-form symmetry is “robust”: No charged local operators can break it, so photon remains massless

The magnetic symmetry is unbroken, since it just follows from Bianchi

$$\partial^\mu J^m_{\mu\nu} = \partial^\mu {*F}_{\mu\nu} = 0$$

Emergent one-form symmetry (boring)

For energies $E \ll m_Q$ we can integrate out charged matter

$$\mathscr{L} = -\frac{1}{4g^2}F^2 + \frac{c}{g^2 m_Q^4}F^4 + \cdots$$

(c.f., Euler-Heisenberg)

1-form symmetry emerges <u>exactly</u> $\qquad J \sim \frac{1}{g^2}F + \frac{c}{g^2 m_Q^4}F^3 + \cdots$

Only broken non-perturbatively $\sim e^{-m|x|}$

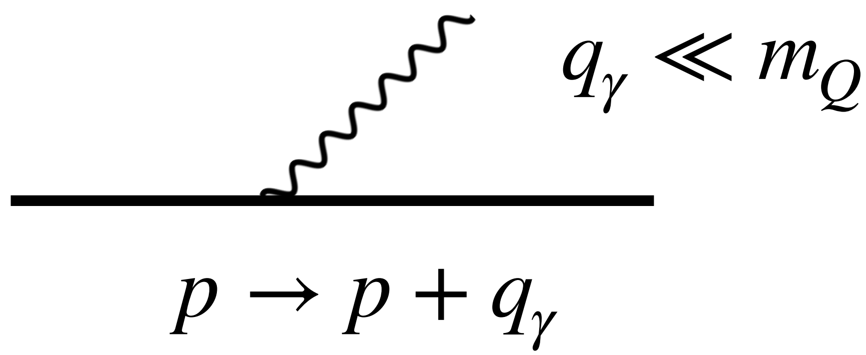
There is indeed an exact "Adler zero" following from WI

$$\lim_{q_\gamma \to 0} \mathscr{A}_{n+\gamma} = 0$$

A bit boring, because higher-form symmetries are Abelian, so there is no interesting double-soft theorem

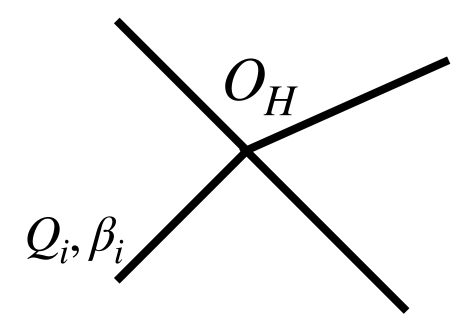
Emergent one-form symmetry

If interested in scattering of matter by soft photons



In this limit amplitude is correlation function of Wilson lines

$$\mathscr{A} = \langle W_{Q_1}^{\beta_1} \cdots W_{Q_n}^{\beta_n} O_H \rangle + \mathcal{O}\left((q/m)^{\#} \right)$$



along paths $x_i^\mu(s) = \beta^\mu s$ with $s \in [0, \infty]$

Soft-photon theorem

Consider the correlator $\mathscr{C} = \langle \epsilon^\mu \partial^\nu J_{\mu\nu} W_{Q_1}^{\beta_1} \cdots W_{Q_n}^{\beta_n} O_H \rangle$

On the one hand creates a photon $\langle \gamma(q) \,|\, W_{Q_1}^{\beta_1} \cdots W_{Q_n}^{\beta_n} O_H \rangle + \epsilon_\mu q_\nu \mathscr{R}^{\mu\nu}$

On the other use WI $i \sum_i Q_i \,\mathsf{FT}[\epsilon^\mu \delta_\mu^{(D-1)}(x_{\Gamma_i})] \, \langle W_{Q_1}^{\beta_1} \cdots W_{Q_n}^{\beta_n} O_H \rangle$

Contact term yields precisely soft factor

$$iQ\,\mathsf{FT}[\delta_\nu^{(D-1)}(x_\Gamma)] = iQ \int_0^\infty ds\, e^{iq\cdot\beta s} \epsilon \cdot \beta = -\,Q \frac{\epsilon \cdot \beta}{q \cdot \beta}$$

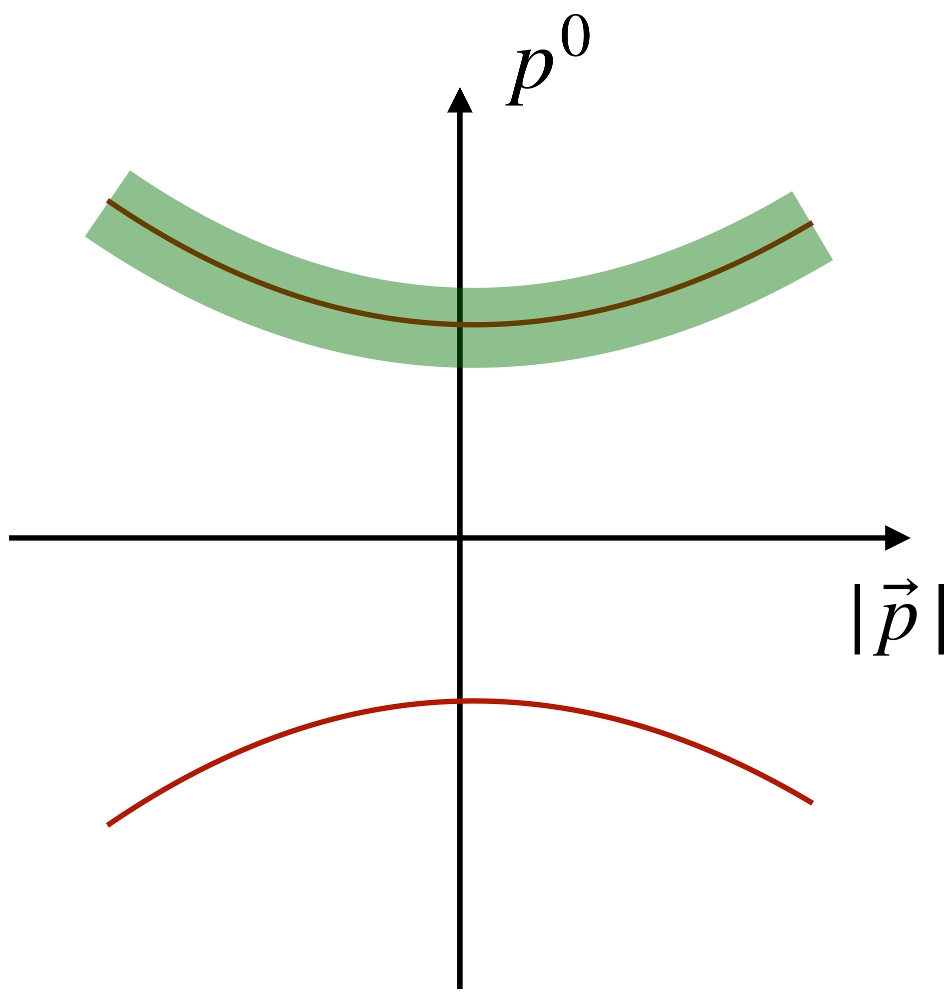
Combined gives soft theorem $\lim_{q_\gamma \to 0} \mathscr{A}_{n+\gamma} = -\sum_a Q_a \frac{\epsilon_\gamma \cdot p_a}{q_\gamma \cdot p_a} \mathscr{A}_n$

Heavy particle EFT (HEFT)

Split field into positive and negative frequency modes

$$\phi = e^{im\beta\cdot x}\left(P_+\phi_\beta + P_-\phi_\beta^*\right)$$

Integrate out negative frequency modes, which have gap $2m$



$$\mathscr{L} = \phi_\beta^*\beta\cdot D\phi_\beta + \phi_\beta^*\frac{D_\perp^2}{2m}\phi_\beta + \cdots$$

This is a theory without antiparticles, no screening of Wilson lines!

1-form symmetry in HEFT

BPS field redefinition
$$\phi_\beta = \exp\left(iQ\int_0^\infty \beta^\mu A_\mu(s\beta) \right)\tilde{\phi}_\beta$$

Makes claimed form of the amplitude manifest

Also makes effective action function of field strength

$$D_\mu \phi_\beta = W_Q \left(\partial_\mu + \frac{F_{\mu\nu}\beta^\nu}{\beta\cdot\partial} \right)\tilde{\phi}_\beta$$

Invariant under shift $A \to A + \lambda$, so 1-form symmetry is an exact symmetry of HEFT

So far no new soft theorem, only alternative perspective on old photon bringing it closer to pion

Is there some kind of symmetry that intertwines them?

2-group soft theorem

Two-group symmetry

Zero-form and one-form symmetries can be intertwined into a two-group, Ward identity modified:

$$\partial \cdot j^a(x)\, j^{b\mu}(y) \sim i f^{abc}\, \delta^{(4)}(x-y)\, j^{c\mu}(x) + \frac{\kappa}{2\pi}\delta^{ab}\, \partial^\nu \delta^{(4)}(x-y) J_{\mu\nu}$$

$$\sim f^{ab\gamma}$$

This is roughly an extension of the 0-form symmetry by the 1-form symmetry

$$G \times_\kappa U(1)^{(1)}$$

Suggests there should be a soft theorem that mixes up photons and pions

Aside: more on 2-group

Full description of 2-group requires more data:

- 0-form symmetry group G
- 1-form symmetry group Γ
- Action $\rho : G \to \text{Aut}(\Gamma)$
- Class $H^3(G, \Gamma) \sim \kappa$

First characterized in thesis by Hoàng Xuân Sính, student by correspondence of Grothendieck while he was at IHES and Vietnam war was raging on

Example

Take QCD with gauged $U(1)_V \sim$ Baryon number

$$\mathscr{L} = -f_\pi^2 \text{Tr}(\partial U \partial U^\dagger) - \frac{1}{4\pi^2} F^2 + A_\mu B^\mu + \cdots$$

with $$B^\mu = i\frac{\kappa}{24\pi^2}\, \varepsilon^{\mu\nu\alpha\beta} \text{Tr}\left[(iU^\dagger \partial_\nu U)(iU^\dagger \partial_\alpha U)(iU^\dagger \partial_\beta U)\right].$$

A mixed $U(1)_V$-Axial-Axial 't Hooft anomaly turns into 2-group after gauging!

$$SU(N_f)_L \times SU(N_f)_R \times_\kappa U(1)_m^{(1)}$$

2-group soft theorem

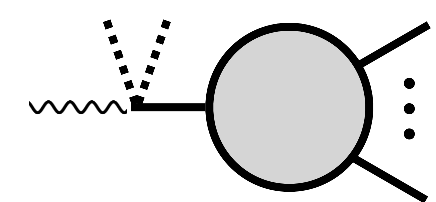
[Berean-Dutcher, Derda, **JPM**]

Double-soft pion theorem modified

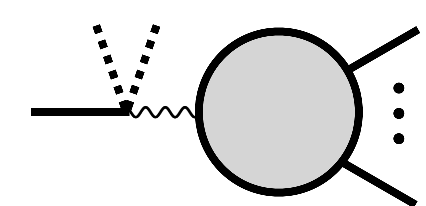
$$\lim_{q^{\pi}_{a,b}\to 0} \mathscr{A}_{(n_\pi+2,\, n_\gamma)} = (S^{(0)} + S^{(1)} + S^{(1)}_{\kappa})\, \mathscr{A}_{(n_\pi,\, n_\gamma)}$$

with

$$S^{(1)}_{\kappa}\mathscr{A}_{(n_\pi, n_\gamma)} = \sum_{i\in\gamma} \frac{\varepsilon(q_a, q_b, p_i, \epsilon_i)}{p_i\cdot(q_a+q_b)} f^{abd} f^{dc\gamma} \mathscr{A}^{\cdots c\cdots}_{(n_\pi+1,\, n_\gamma-1)}$$



$$+\sum_{i\in\pi}\sum_{h} \frac{\varepsilon(q_a, q_b, p_i, \epsilon^{*}_{(h)})}{p_i\cdot(q_a+q_b)} f^{abd} f^{dc\gamma} \mathscr{A}^{\cdots\hat{c}\cdots}_{(n_\pi-1,\, n_\gamma+1)}$$



Proof follows by considering $\langle 0\,|\, j^{a\mu}(q_a) j^{b\nu}(q_b)\,|\,\alpha\rangle$ and using 2-group WI.
We have also checked it explicitly in all tree amplitudes up to 6-point.

Summary and comments

- New angle on soft theorems from 1-form symmetry, in direct analogy to pions
- Also works for soft gluons gravitons, from emergent 1-form symmetry at weak coupling
- Is there a direction connection to the story at $\mathscr{I}$? Or to geometric perspective?
- Emergent higher symmetries in Lorentzian EFTs likely ubiquitous, connection to factorization?
- New soft theorem from 2-group structure. More to discover? Non-invertible?

Thank you!