Lessons for Celestial Holography from AdS-CFT

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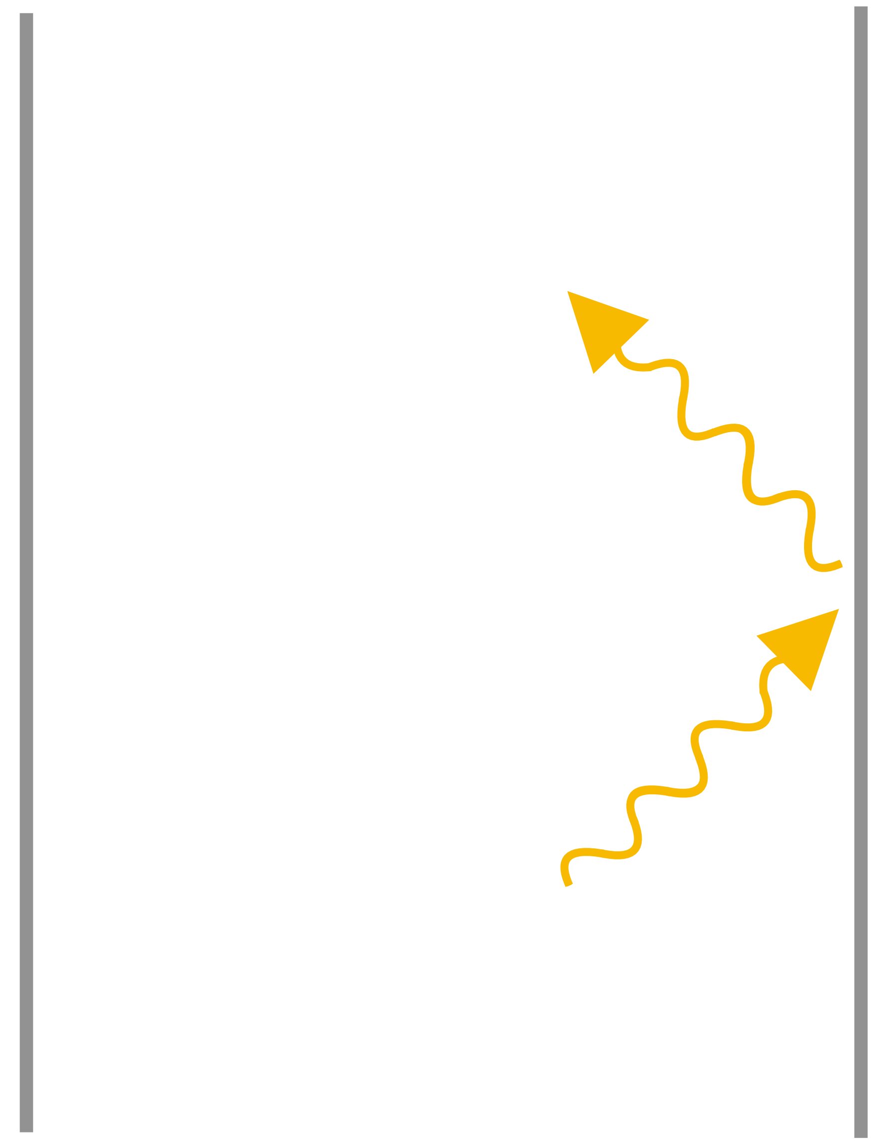
AdS-CFT

Anti-de Sitter

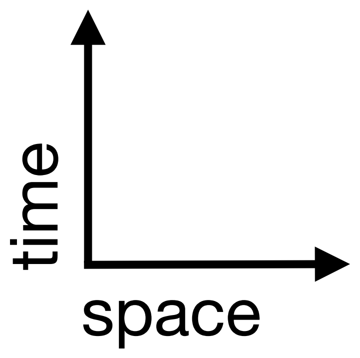


``Quantum Gravity in a box”

Boundary system is a Minkowski CFT

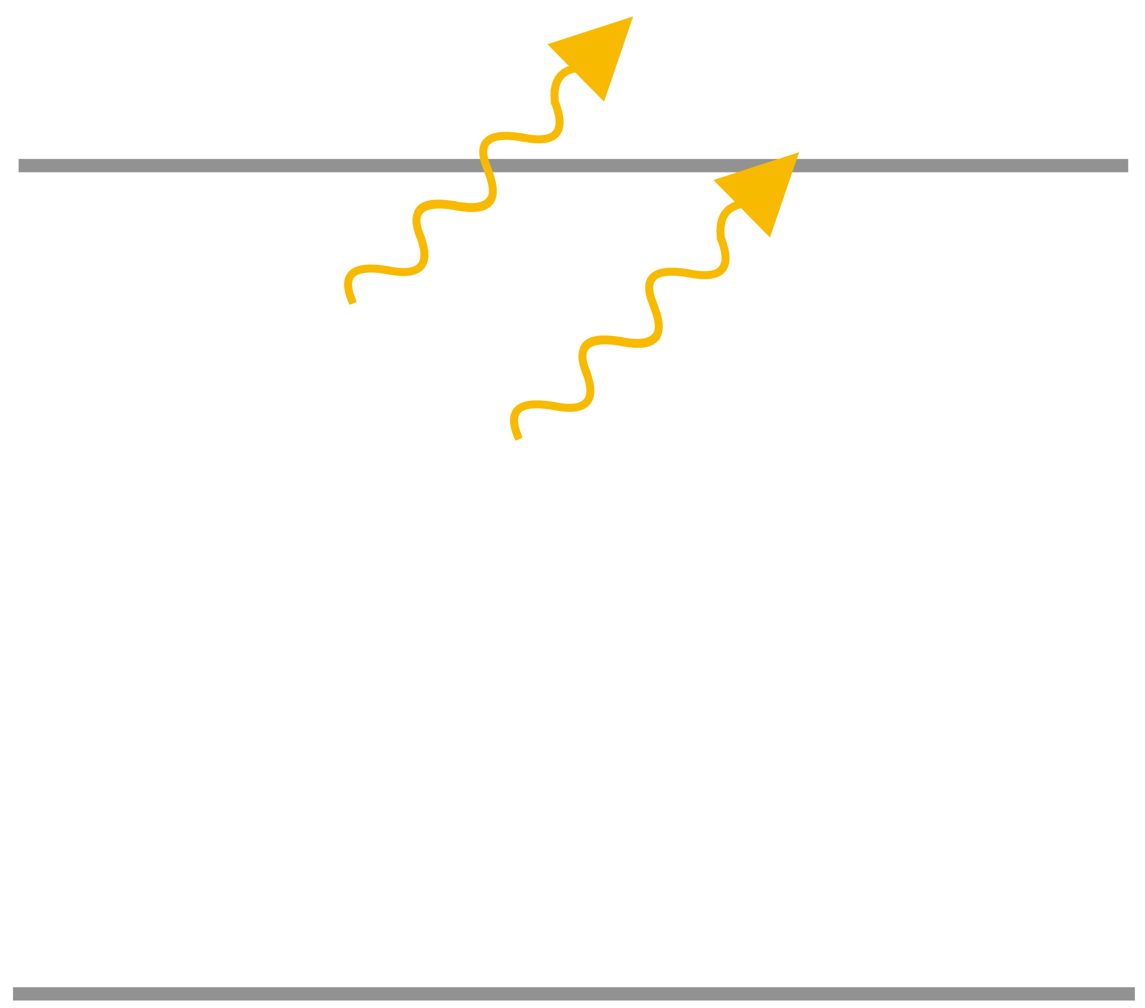
Constrained non-perturbatively by axioms:

- Conformal symmetry
- Unitarity
- Associative OPE

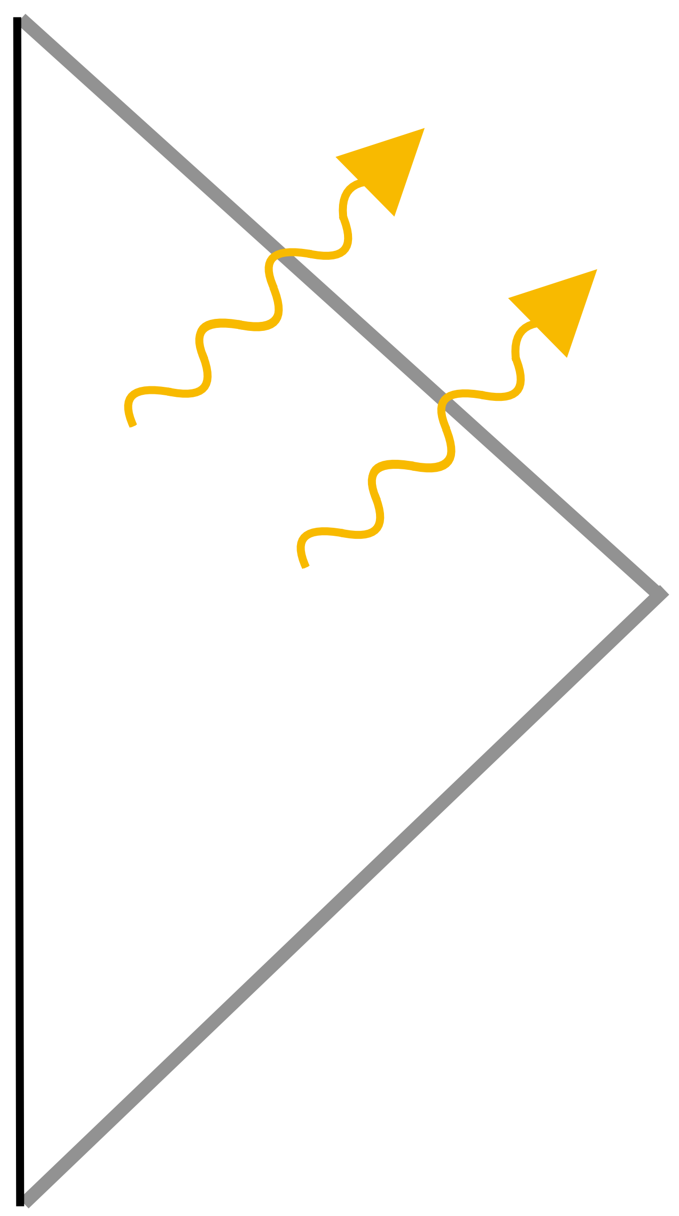


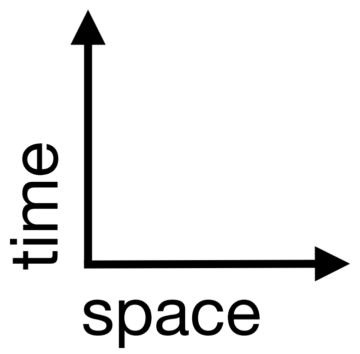
Holograms in the Sky?

de Sitter



Minkowski



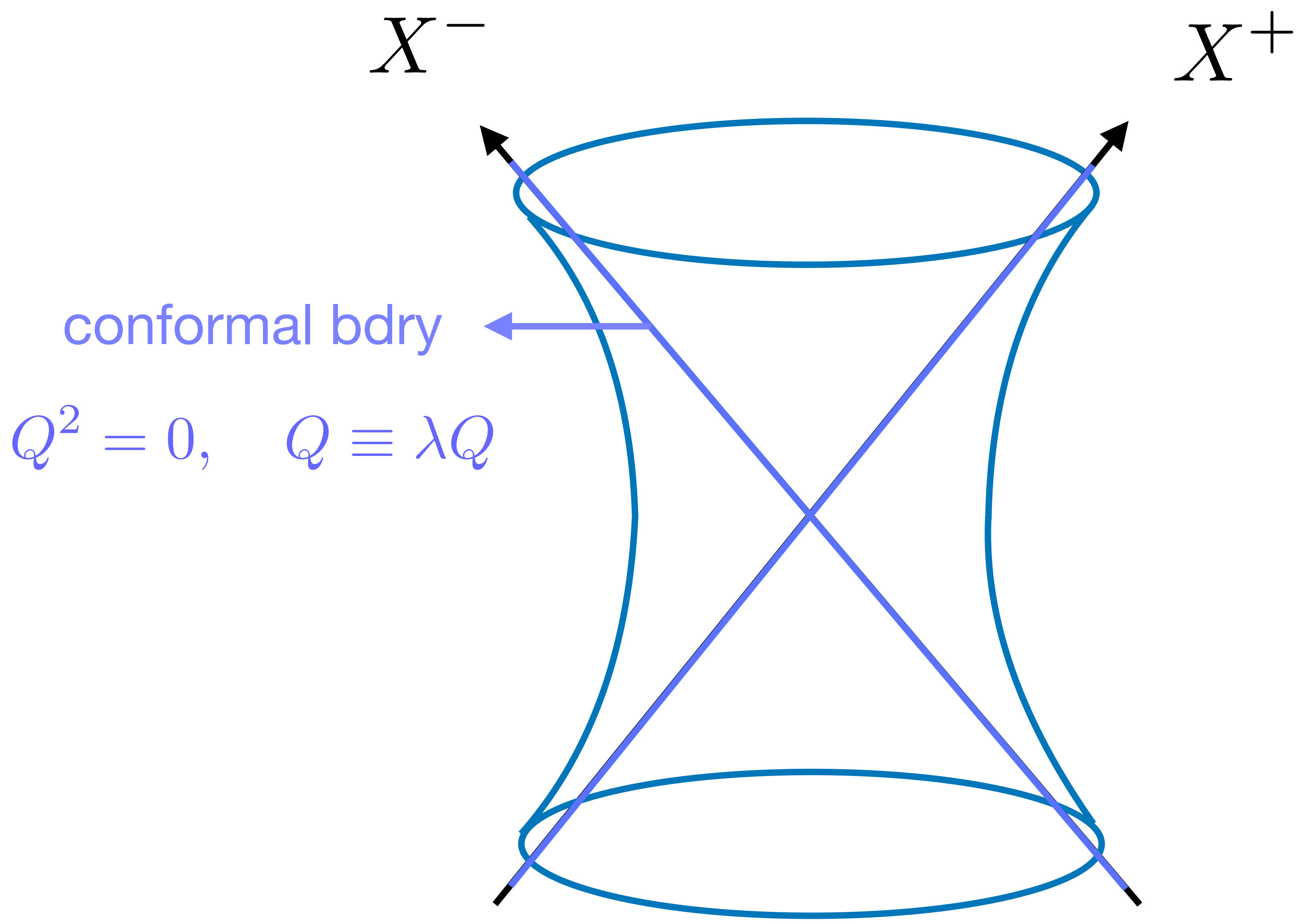


Outgoing radiation. What are the axioms governing the boundary CFT in these cases?

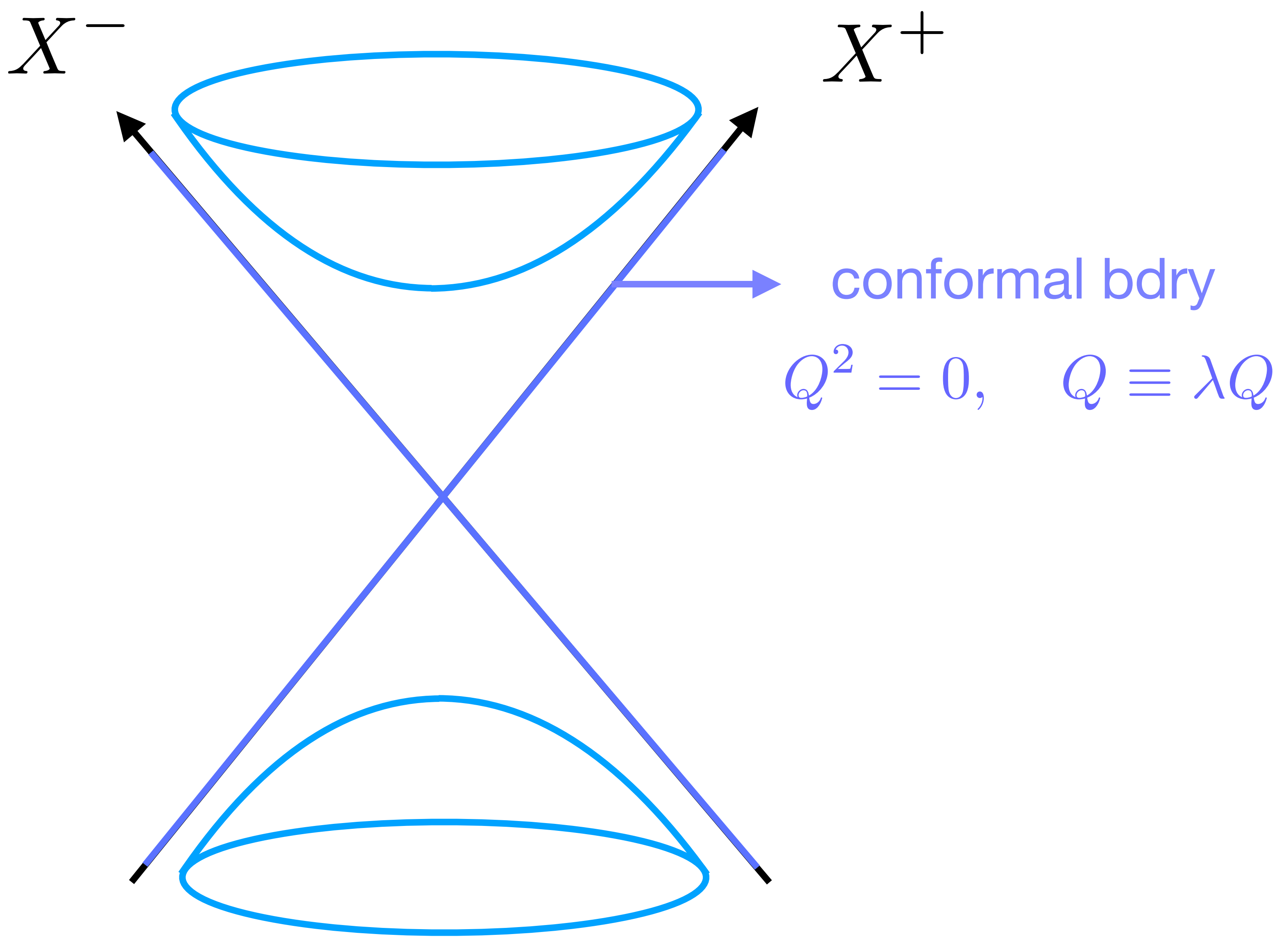
From dS to Euclidean AdS

[C.S. and M. Taronna '20 '21]

$\mathrm{dS}: \; X^2 = +t^2$ $\qquad$ $\mathrm{EAdS}: \; X^2 = -t^2$



$X \to iX$



$$G_F^{\mathrm{dS}}(X,Y) \longrightarrow e^{-i\pi\Delta_+} G^{\mathrm{AdS}}_{\Delta_+}(X,Y) + e^{-i\pi\Delta_-} G^{\mathrm{AdS}}_{\Delta_-}(X,Y)$$

Bunch-Davies vacuum $\qquad m^2 t^2 = \Delta_+ \Delta_- \qquad$ Dirichlet b.c. $\qquad$ Neumann b.c.

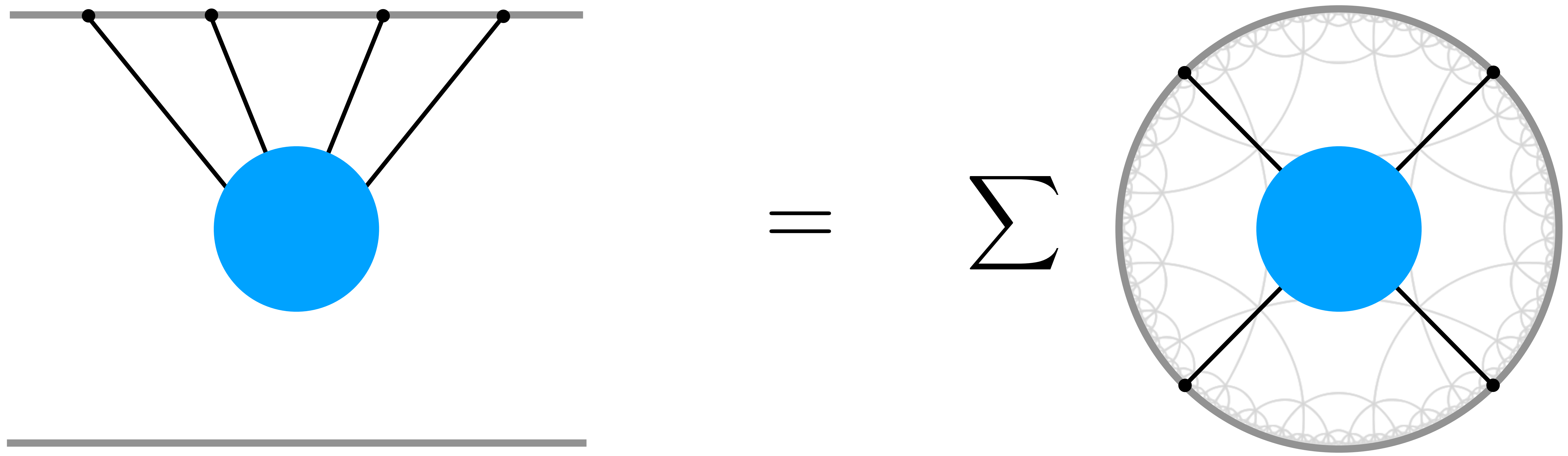
$$K_\Delta^{\mathrm{dS}}(X,Q) = \frac{C_\Delta^{\mathrm{dS}}}{(-2X\cdot Q + i\epsilon)^\Delta} \longrightarrow e^{-\frac{i\pi}{2}\Delta} K_\Delta^{\mathrm{AdS}}(X,Q), \qquad K_\Delta^{\mathrm{AdS}}(X,Q) = \frac{C_\Delta^{\mathrm{AdS}}}{(-2X\cdot Q)^\Delta}$$

From dS to Euclidean AdS

[C.S. and M. Taronna '20 '21]

Bogoliubov initial states: [A. Chopping, C.S. and M. Taronna '23]

dS boundary correlators can be recast as EAdS boundary correlators (at least perturbatively)



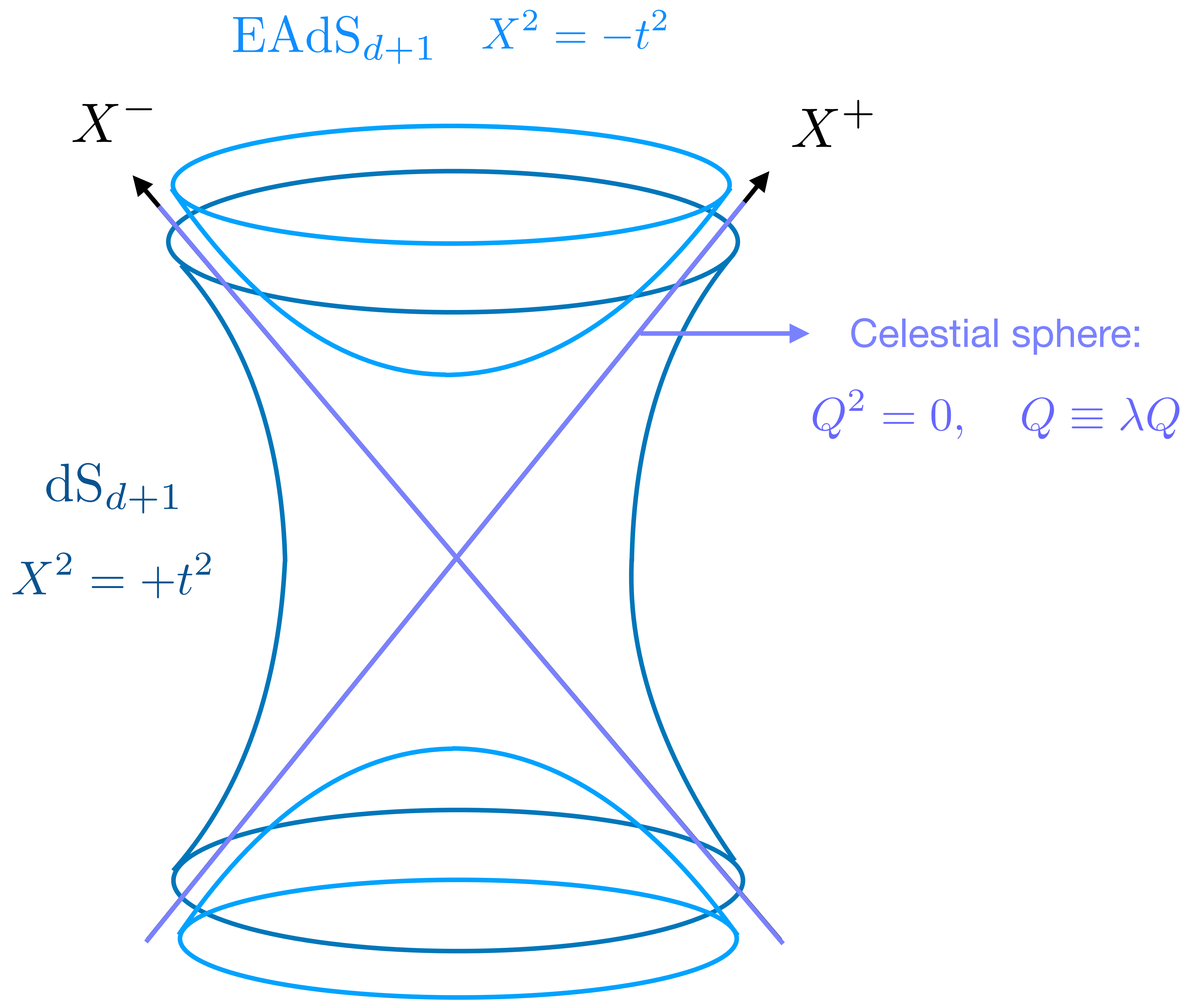
Q: What are the EAdS axioms for dS boundary correlators?

→ dS boundary correlators have a similar analytic structure to AdS ones (in Euclidean regime).

Can import results and techniques from AdS-CFT to boundary correlators in dS!

Hyperbolic foliation of Minkowski space

Hyperbolic foliation of (d+2) dimensional Minkowski space: $X = t\hat{X}$

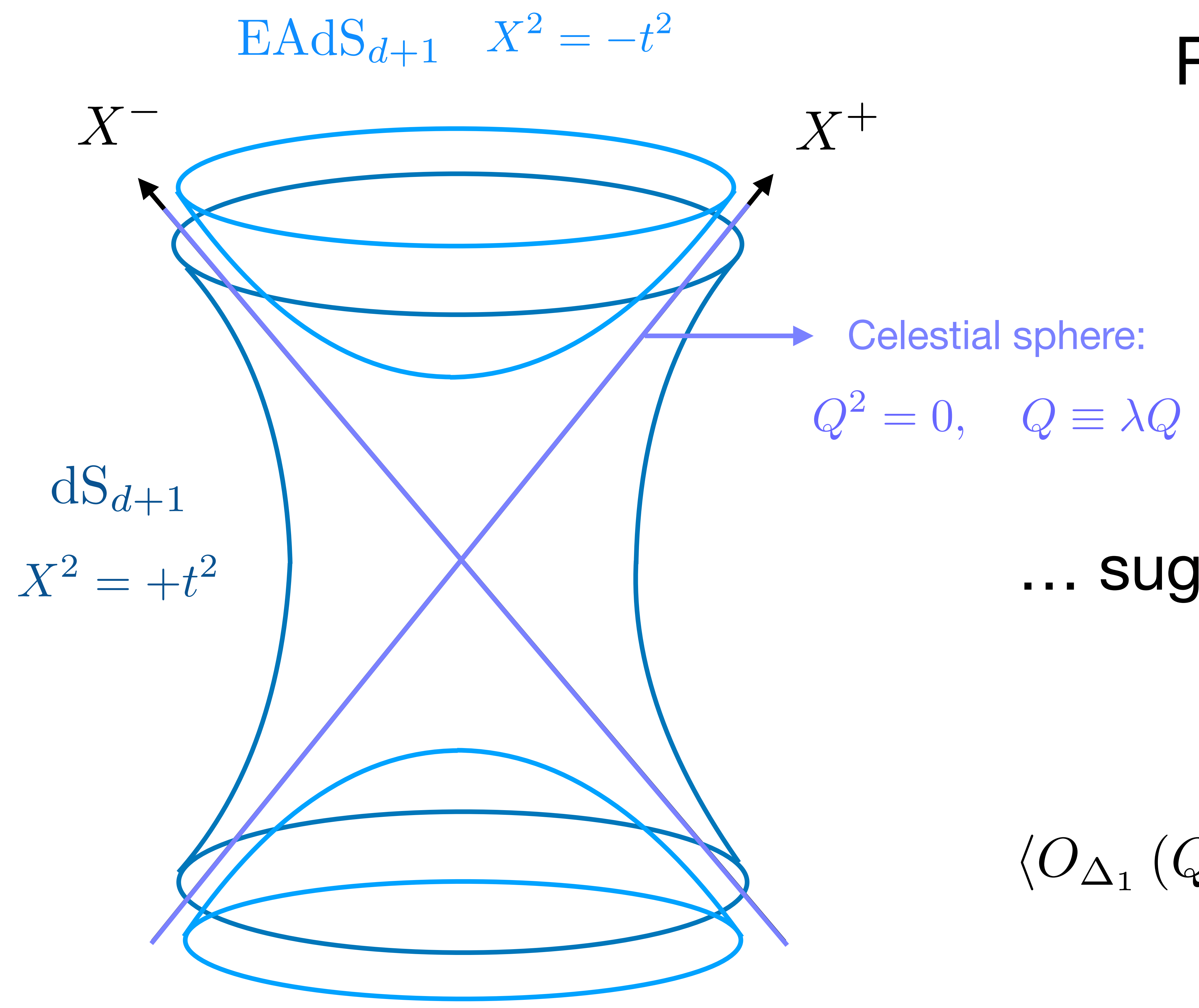


Celestial holography from applying holography to each hyperbolic slice [de Boer and Solodukhin '03]

Taking a leaf from AdS-CFT

[C.S. and M. Taronna '23]

Hyperbolic foliation of (d+2) dimensional Minkowski space: $X = t\hat{X}$



Recall extrapolate holographic correlators in (A)dS:

$$\langle O_{\Delta_1}(Q_1) \dots O_{\Delta_n}(Q_n) \rangle = \prod_i \lim_{\hat{X}_i \to Q_i} \langle \phi_1(t_1 \hat{X}_1) \dots \phi_n(t_n \hat{X}_n) \rangle$$

… suggests an **off-shell** extrapolate prescription for correlators on the celestial sphere:

$$\langle O_{\Delta_1}(Q_1) \dots O_{\Delta_n}(Q_n) \rangle = \prod_i \lim_{\hat{X}_i \to Q_i} \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} \left\langle \phi_1(t_1 \hat{X}_1) \dots \phi_n(t_n \hat{X}_n) \right\rangle$$

Fourier transform on the light cone!

Celestial Correlators

[C.S. and M. Taronna '23]

Define ``Celestial Correlators" as:

$$\langle O_{\Delta_1}(Q_1)\ldots O_{\Delta_n}(Q_n)\rangle = \prod_i \lim_{\hat{X}_i \to Q_i} \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} \left\langle \phi_1(t_1\hat{X}_1)\ldots\phi_n(t_n\hat{X}_n)\right\rangle$$

Fourier transform on the light cone

Celestial Amplitudes ~ LSZ [Celestial Correlators]

Feynman rules:

Bulk-to-bulk propagator (Feynman): $G_F(X,Y)$

Bulk-to-boundary propagator: $K_\Delta(X,Q) = \lim_{\hat{Y}\to Q} \int_0^\infty \frac{dt}{t} t^\Delta G_F\left(X, t\hat{Y}\right)$

[Fields with spin - to appear: S. Malherbe, M. Pannier, Paolo Pergola, C.S. and M. Taronna]

Celestial Correlators

[C.S. and M. Taronna '23]

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Fourier transform on the light cone

Celestial Amplitudes ~ LSZ [Celestial Correlators]

Feynman rules:

For **Celestial Amplitudes**, one extrapolates instead the **Wightman function** - gives rise to the **Conformal Primary Wavefunction**.

Bulk-to-bulk propagator (Feynman): $G_F(X,Y)$

Bulk-to-boundary propagator: $K_\Delta(X,Q) = \lim_{\hat{Y}\to Q}\int_0^\infty \frac{dt}{t}t^\Delta G_F\left(X,t\hat{Y}\right)$

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Massless limit is smooth! $\mathcal{K}_\Delta^{(m)}(R) = \left(\frac{m}{2}\right)^{\frac{d}{2}-\Delta} \frac{2R^{-\frac{d}{2}}}{\Gamma(\frac{d}{2}-\Delta)} K_{\Delta-\frac{d}{2}}(mR) \to R^{\Delta-d}$ as $m \to 0$

Bessel K

Outline

~~I Introduction~~

II Celestial Correlators as EAdS Witten diagrams

2301.01810 C.S. and M. Taronna

2401.16591 L. Iacobacci, C.S. and M. Taronna

III Celestial Mellin Amplitudes

2412.11992 F. Pacifico, P. Pergola and C.S.

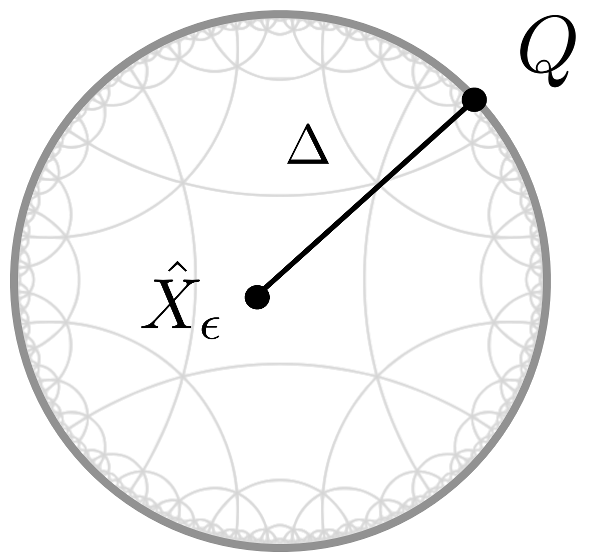
Celestial Correlators as EAdS Witten diagrams

[C.S. and M. Taronna ’23; L. Iacobacci, C.S. and M. Taronna ’24]

$$\langle O_{\Delta_1}(Q_1)\dots O_{\Delta_n}(Q_n)\rangle = \prod_i \lim_{\hat{X}_i\to Q_i}\int_0^\infty \frac{\mathrm{d}t_i}{t_i}\, t_i^{\Delta_i}\left\langle \phi_1(t_1\hat{X}_1)\dots\phi_n(t_n\hat{X}_n)\right\rangle$$

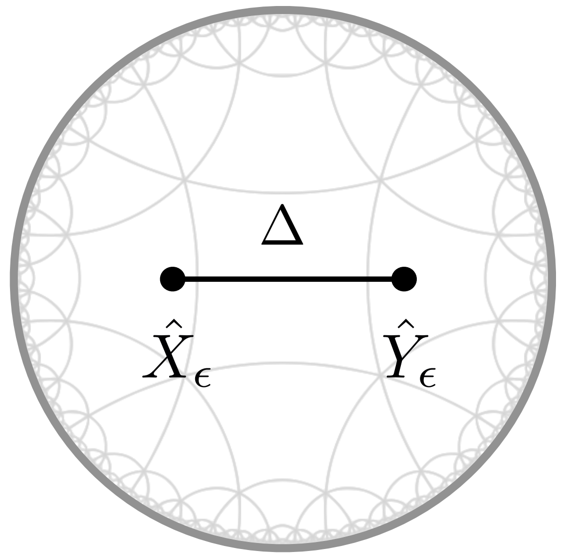
Feynman rules:

Bulk-to-boundary propagator: $K_\Delta\left(X,Q\right) = \mathcal{K}^{(m)}_{\Delta}\left(\sqrt{X^2+i\epsilon}\right) \times$



Kernel of radial reduction

Bulk-to-bulk propagator: $G_F\left(X,Y\right) = \frac{1}{2}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}\,\mathcal{K}^{(m)}_{\Delta}\left(\sqrt{X^2+i\epsilon}\right)\mathcal{K}^{(m)}_{d-\Delta}\left(\sqrt{Y^2+i\epsilon}\right)$



Principal series of SO(d+1,1)

Use to perturbatively recast Celestial Correlators in terms of EAdS Witten diagrams at all orders!

Boundary 2pt function

[C.S. and M. Taronna '23]

Free theory two point function on the celestial sphere:

$$\langle O_{\Delta_1}(Q_1)\, O_{\Delta_2}(Q_2)\rangle = \lim_{\hat{X}\to Q_2} \int_0^\infty \frac{\mathrm{d}t}{t} t^{\Delta_2} K_{\Delta_1}(t\hat{X}, Q_1)$$

Q_i can be null separated

$$= \frac{C^{(m)}_{\Delta_1}}{(-2Q_1\cdot Q_2 + i\epsilon)^{\Delta_1}} (2\pi i)\delta(\Delta_1 - \Delta_2)$$

Standard 2pt conformal structure

$$C^{(m)}_{\Delta} = \left(\frac{m}{2}\right)^{d-2\Delta} \frac{1}{4\pi^{\frac{d+2}{2}}} \Gamma(\Delta)\Gamma(\Delta - \tfrac{d}{2})$$

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Standard 2pt conformal structure

$$C^{(m)}_{\Delta} = \left(\frac{m}{2}\right)^{d-2\Delta} \frac{1}{4\pi^{\frac{d+2}{2}}} \Gamma(\Delta)\Gamma(\Delta - \tfrac{d}{2}) \xrightarrow{m\to 0} \frac{\Gamma(\Delta)}{4\pi^{\frac{d+2}{2}}}(2\pi i)\delta\left(\Delta - \tfrac{d}{2}\right)$$

Celestial Correlators as EAdS Witten diagrams

[C.S. and M. Taronna '23]

Examples: Contact diagrams

Contact diagrams can be expressed as contact Witten diagrams in EAdS!

Consider non-derivative vertex of scalars fields $\mathcal{V}\left(X\right)=g\phi_1\left(X\right)\ldots\phi_n\left(X\right)$

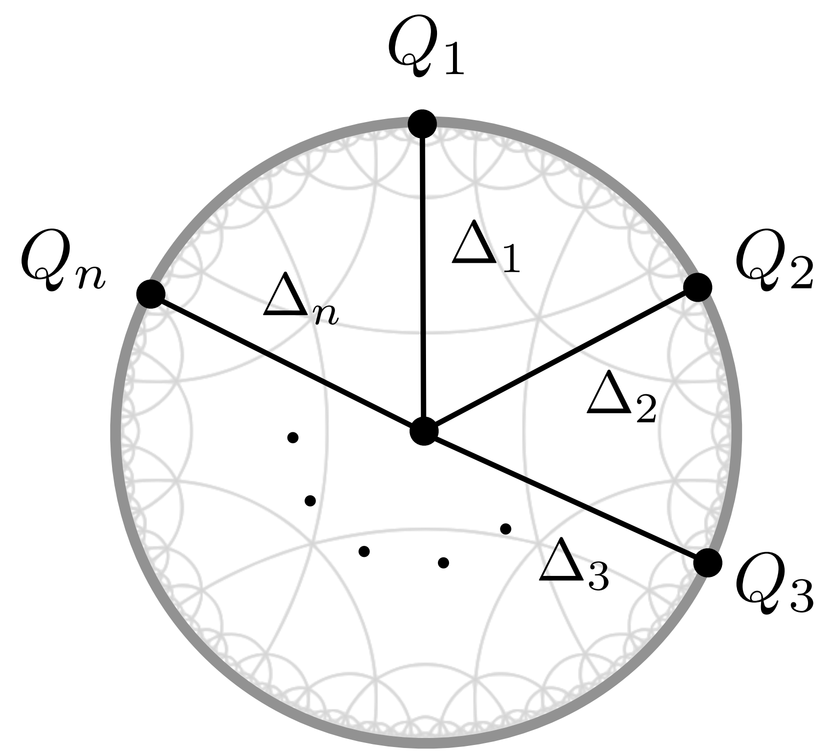
$$\langle O_{\Delta_1}\left(Q_1\right)\ldots O_{\Delta_n}\left(Q_n\right)\rangle=-ig\int \mathrm{d}^{d+2}X\, K_{\Delta_1}\left(X,Q_1\right)\ldots K_{\Delta_n}\left(X,Q_n\right)$$

$$=\quad \sin\left(-\frac{d}{2}+\frac{1}{2}\sum_{i=1}^{n}\Delta_i\right)\pi \quad\times\quad R_{\Delta_1\ldots\Delta_n}\left(m_1,\ldots,m_n\right)$$

Combines contributions from dS and EAdS regions of Minkowski space.

$$=\int_0^\infty dR\, R^{d+1}\prod_{i=1}^{n}\mathcal{K}^{(m_i)}_{\Delta_i}(R)$$

Contribution from radial integral. Encodes all mass dependence.



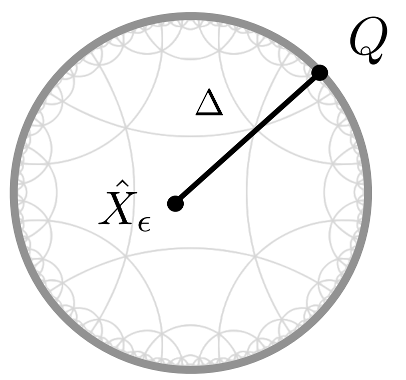
Non-derivative contact Witten diagram

$$D_{\Delta_1\ldots\Delta_n}\left(Q_1,\ldots,Q_n\right)$$

$$Q_i\cdot Q_j\to Q_i\cdot Q_j-i\epsilon$$

Recall bulk-boundary propagator :

$$K_\Delta\left(X,Q\right)=\ \mathcal{K}^{(m)}_\Delta\left(\sqrt{X^2+i\epsilon}\right)\ \times$$



Celestial Correlators as EAdS Witten diagrams

[L. Iacobacci, C.S. and M. Taronna ’24]

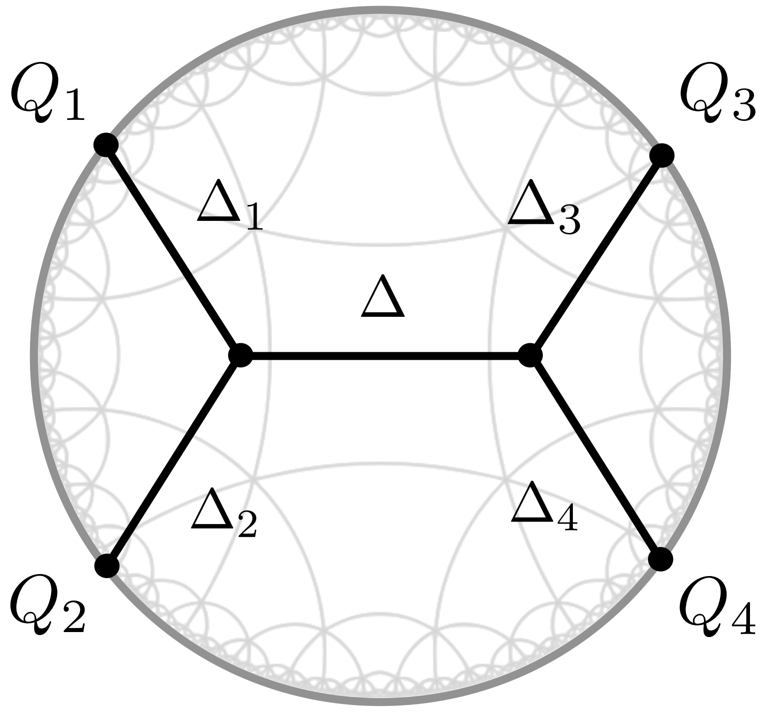
Examples: Exchange diagram

E.g. Four-point tree level direct channel exchange of mass m scalar between scalar fields of mass m_i

$$\langle O_{\Delta_1}(Q_1)\, O_{\Delta_2}(Q_2)\, O_{\Delta_3}(Q_3)\, O_{\Delta_4}(Q_4)\rangle$$

$$= (-ig_{12})(-ig_{34}) \int \mathrm{d}^{d+2}X \mathrm{d}^{d+2}Y\, K_{\Delta_1}(X,Q_1)\, K_{\Delta_2}(X,Q_2)\, G_F(X,Y)\, K_{\Delta_3}(Y,Q_3)\, K_{\Delta_4}(Y,Q_4)$$

$$= \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{\mathrm{d}\Delta}{2\pi i} \frac{c^{\text{flat-AdS}}_{\Delta_1\Delta_2\Delta}(m_1,m_2,m)\, c^{\text{flat-AdS}}_{\Delta\Delta_3\Delta_4}(m,m_3,m_4)}{c^{\text{flat-AdS}}_{\Delta}(m)}$$



Principal series of SO(d+1,1)

Change in 3 and 2 point coefficients from AdS to Minkowski

Outline

~~I Introduction~~

~~II Celestial Correlators as EAdS Witten diagrams~~

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III Celestial Mellin Amplitudes

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Mellin Amplitudes

Mellin amplitude $M\left(\delta_{ij}\right)$ for a conformal correlator [Mack '09]

$$\langle O_{\Delta_1}\left(Q_1\right)\ldots O_{\Delta_n}\left(Q_n\right)\rangle = \int_{-i\infty}^{i\infty}\frac{\mathrm{d}\delta_{ij}}{2\pi i}\, M\left(\delta_{ij}\right)\prod_{i<j}\Gamma\left(\delta_{ij}\right)\left(-2Q_i\cdot Q_j\right)^{-\delta_{ij}}$$

Conformal symmetry imposes: $\delta_{ij}=\delta_{ji}, \qquad \delta_{ii}=-\Delta_i, \qquad \sum_{j=1}^{n}\delta_{ij}=0$

Contributions to the OPE $O_{\Delta_i}O_{\Delta_j}\sim O_{\Delta_k}+\ldots$ are encoded in poles in δ_{ij} at:

$$\delta_{ij}=\frac{\Delta_i+\Delta_j-\left(\Delta_k+2n\right)}{2}, \qquad n\in\mathbb{N}$$

($n>0$ are descendant contributions)

Mellin Amplitudes for AdS Witten diagrams

Mellin amplitude $M(\delta_{ij})$ for a conformal correlator at large N

[Mack '09]

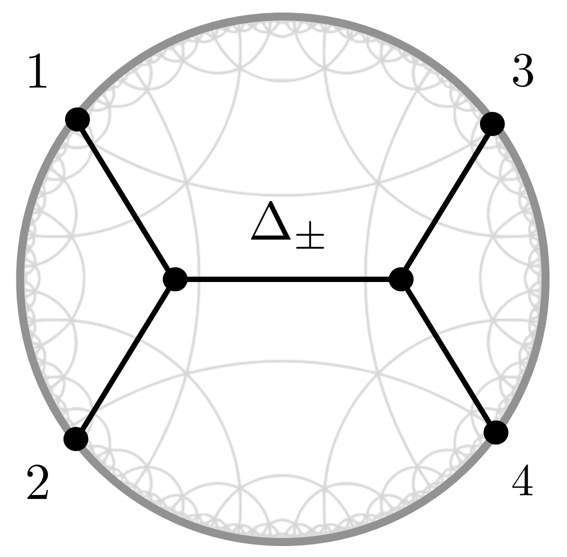
multi-trace operator contributions

$$\langle O_{\Delta_1}(Q_1)\ldots O_{\Delta_n}(Q_n)\rangle = \int_{-i\infty}^{i\infty}\frac{\mathrm{d}\delta_{ij}}{2\pi i} M(\delta_{ij}) \prod_{i<j}\Gamma(\delta_{ij})(-2Q_i\cdot Q_j)^{-\delta_{ij}}$$

[Penedones '10]

Witten diagrams in AdS space		**Mellin amplitude**
Contact Witten diagrams	→	Polynomial in δ_{ij}
Tree-level exchange Witten diagrams	→	Simple poles

e.g.



→ $m^2R^2 = -\Delta_+\Delta_-$

at $\delta_{12} = \frac{\Delta_1+\Delta_2-(\Delta_\pm+2n)}{2}$ $n\in\mathbb{N}$

``single trace” operator $O_{\Delta_\pm}$

Celestial Mellin Amplitudes

[F. Pacifico, P. Pergola and C.S. '24]

Mellin amplitude $M\left(\delta_{ij}\right)$ for celestial correlators

$$\left\langle O_{\Delta_1}\left(Q_1\right)\ldots O_{\Delta_n}\left(Q_n\right)\right\rangle = \prod_i \lim_{\hat{X}_i \to Q_i} \int_0^\infty \frac{dt_i}{t_i} t_i^{\Delta_i} \left\langle \phi_1(t_1\hat{X}_1)\ldots\phi_n(t_n\hat{X}_n)\right\rangle$$

$$= \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\delta_{ij}}{2\pi i} M\left(\delta_{ij}\right) \prod_{i<j} \Gamma\left(\delta_{ij}\right) \left(-2Q_i\cdot Q_j\right)^{-\delta_{ij}}$$

$\delta_{ij} = \delta_{ji}, \qquad \delta_{ii} = -\Delta_i, \qquad \sum_{j=1}^{n} \delta_{ij} = 0$

$Q_i \cdot Q_j \to Q_i \cdot Q_j - i\epsilon$

Celestial Mellin amplitudes are straightforwardly computed perturbatively using Schwinger param etc.

We already know that:

Contact diagrams $\longrightarrow$ Polynomial Celestial Mellin amplitudes

Celestial Mellin Amplitudes

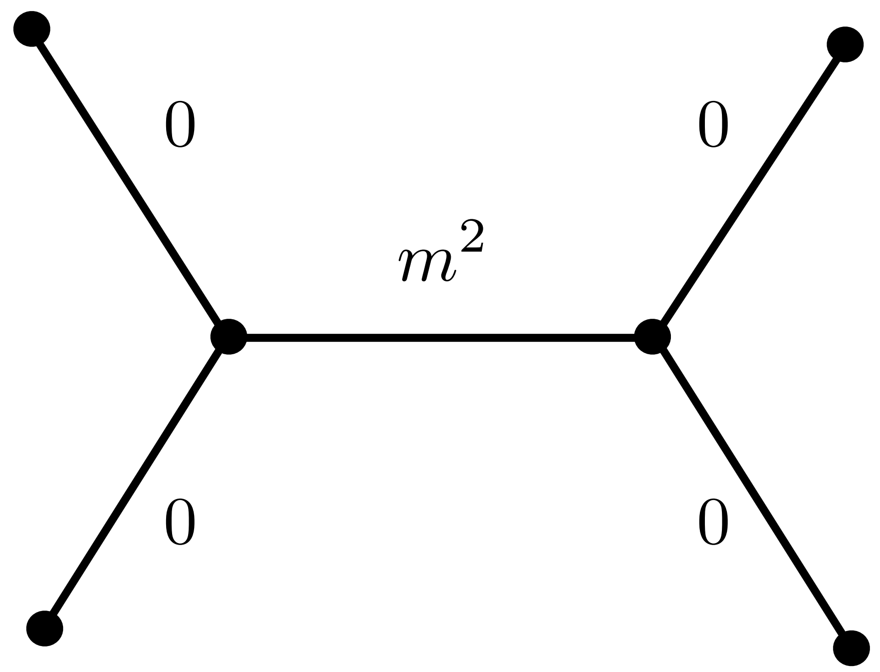
[F. Pacifico, P. Pergola and C.S. '24]

4pt tree-level exchange of a scalar particle of **mass m** between massless scalars:

$$M(\delta_{ij}) \sim \left(\frac{m}{2}\right)^{3d-4-\sum_i \Delta_i} \Gamma\left(\frac{-3d+4+\sum_i \Delta_i}{2}\right)$$

$$\times \frac{\Gamma\left(\delta_{12}+1-\frac{d}{2}\right)\Gamma\left(\delta_{34}+1-\frac{d}{2}\right)}{\Gamma\left(\delta_{12}+\frac{1}{2}\left(d+2+\sum_i \Delta_i\right)\right)\Gamma\left(\delta_{34}+\frac{1}{2}\left(d+2+\sum_i \Delta_i\right)\right)}$$

$$\times\, {}_3F_2\left(\begin{matrix}\delta_{12}+1-\frac{d}{2}, \delta_{34}+1-\frac{d}{2}, \frac{d}{2}+1-\frac{1}{2}\sum_i \Delta_i \\ \delta_{12}+\frac{d}{2}+1+\frac{1}{2}\sum_i \Delta_i, \delta_{34}+\frac{d}{2}+1+\frac{1}{2}\sum_i \Delta_i\end{matrix}; 1\right)$$



Poles in $\delta_{12} = \frac{\Delta_1+\Delta_2-(\Delta+2n)}{2}$, $n \in \mathbb{N}$. Two infinite families:

$$\Delta = \Delta_1+\Delta_2+(2-d)+2n', \qquad n' \in \mathbb{N}$$

$$\Delta = \Delta_3+\Delta_4+(2-d)+2m', \qquad m' \in \mathbb{N}$$

⟶ Exchanged single particle state is encoded by two **infinite** families of operators:

$O_{\Delta_1+\Delta_2+(2-d)+2n'}$ and $O_{\Delta_3+\Delta_4+(2-d)+2m'}$ + their descendants

Celestial Mellin Amplitudes

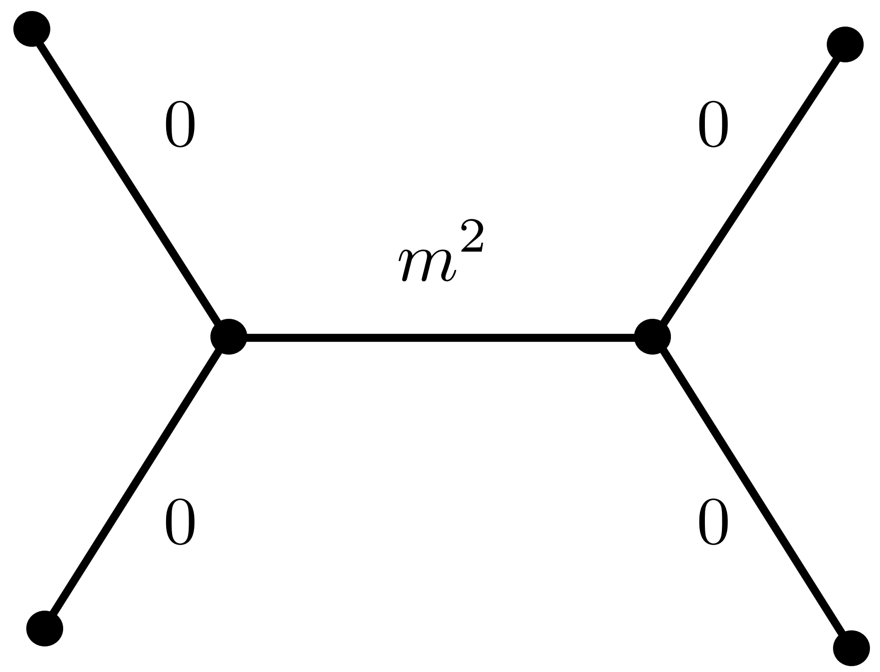
[F. Pacifico, P. Pergola and C.S. ’24]

4pt tree-level exchange of a **massless** scalar particle between massless scalars:

$$M(\delta_{ij}) \sim \left(\frac{m}{2}\right)^{3d-4-\sum_i \Delta_i} \Gamma\left(\frac{-3d+4+\sum_i \Delta_i}{2}\right) \longrightarrow 2\pi i\, \delta\left(\frac{-3d+4+\sum_i \Delta_i}{2}\right)$$

$$\times \frac{\Gamma\left(\delta_{12}+1-\frac{d}{2}\right)\Gamma\left(\delta_{34}+1-\frac{d}{2}\right)}{\Gamma\left(\delta_{12}+\frac{1}{2}\left(d+2+\sum_i \Delta_i\right)\right)\Gamma\left(\delta_{34}+\frac{1}{2}\left(d+2+\sum_i \Delta_i\right)\right)}$$

$$\times\, {}_3F_2\left(\begin{matrix}\delta_{12}+1-\frac{d}{2}, \delta_{34}+1-\frac{d}{2}, \frac{d}{2}+1-\frac{1}{2}\sum_i \Delta_i \\ \delta_{12}+\frac{d}{2}+1+\frac{1}{2}\sum_i \Delta_i, \delta_{34}+\frac{d}{2}+1+\frac{1}{2}\sum_i \Delta_i\end{matrix}; 1\right)$$



Poles in $\delta_{12} = \frac{\Delta_1+\Delta_2-(\Delta+2n)}{2}\,,\quad n \in \mathbb{N}\,.$ Two **shadow** families:

$\Delta_+ + \Delta_- = d$

$$\Delta = \Delta_+ = \Delta_1 + \Delta_2 + (2-d)$$

$$\Delta = \Delta_- = \Delta_3 + \Delta_4 + (2-d)$$

⟶ Exchanged single particle state is encoded by two **shadow** families of operators:

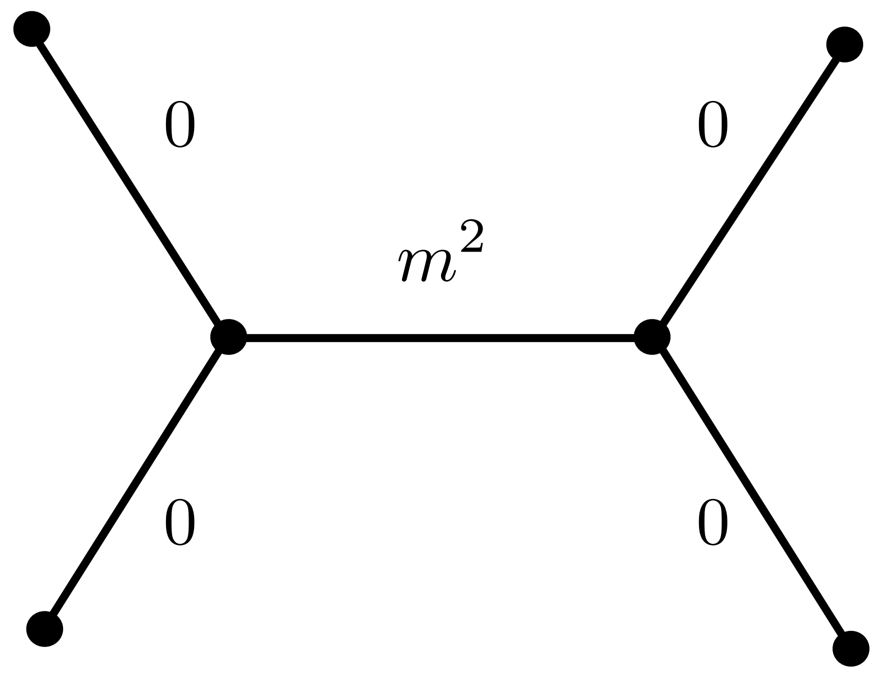
O_{Δ_+} and O_{Δ_-} + their descendants

Celestial Mellin Amplitudes

[F. Pacifico, P. Pergola and C.S. '24]

4pt tree-level exchange of a **massless** scalar particle between massless scalars:

$$M(\delta_{ij}) \sim \left(\frac{m}{2}\right)^{3d-4-\sum_i \Delta_i} \Gamma\left(\frac{-3d+4+\sum_i \Delta_i}{2}\right) \rightarrow 2\pi i\, \delta\left(\frac{-3d+4+\sum_i \Delta_i}{2}\right)$$



$$\times \frac{\mathcal{F}_{\Delta_+,0}(\delta_{ij})}{\prod_{i<j} \Gamma(\delta_{ij})}$$

Mellin amplitude for conformal partial wave $\mathcal{F}_{\Delta_+,0}$

Poles in $\delta_{12} = \frac{\Delta_1 + \Delta_2 - (\Delta + 2n)}{2}$, $n \in \mathbb{N}$. Two **shadow** families:

$\Delta_+ + \Delta_- = d$

$$\Delta = \Delta_+ = \Delta_1 + \Delta_2 + (2-d)$$

$$\Delta = \Delta_- = \Delta_3 + \Delta_4 + (2-d)$$

⟶ Exchanged single particle state is encoded by two **shadow** families of operators:

O_{Δ_+} and O_{Δ_-} + their descendants

Other tools for Celestial Correlators

- **Conformal Partial Wave expansion**

Spectral density, meromorphic in Δ

$$\langle \mathcal{O}(\mathbf{x}_1)\,\mathcal{O}(\mathbf{x}_2)\,\mathcal{O}(\mathbf{x}_3)\,\mathcal{O}(\mathbf{x}_4)\rangle = \sum_{J=0}^{\infty}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}\rho_J(\Delta)\,\underbrace{\mathcal{F}_{\Delta,J}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)}_{\text{Conformal Partial Wave}}$$

Celestial correlators also admit a conformal partial wave decomposition (at least perturbatively)

Harmonic analysis in AdS can be extended to celestial correlators via hyperbolic foliation of Minkowski

2502.03087 F. Pacifico, C.S. and M. Taronna

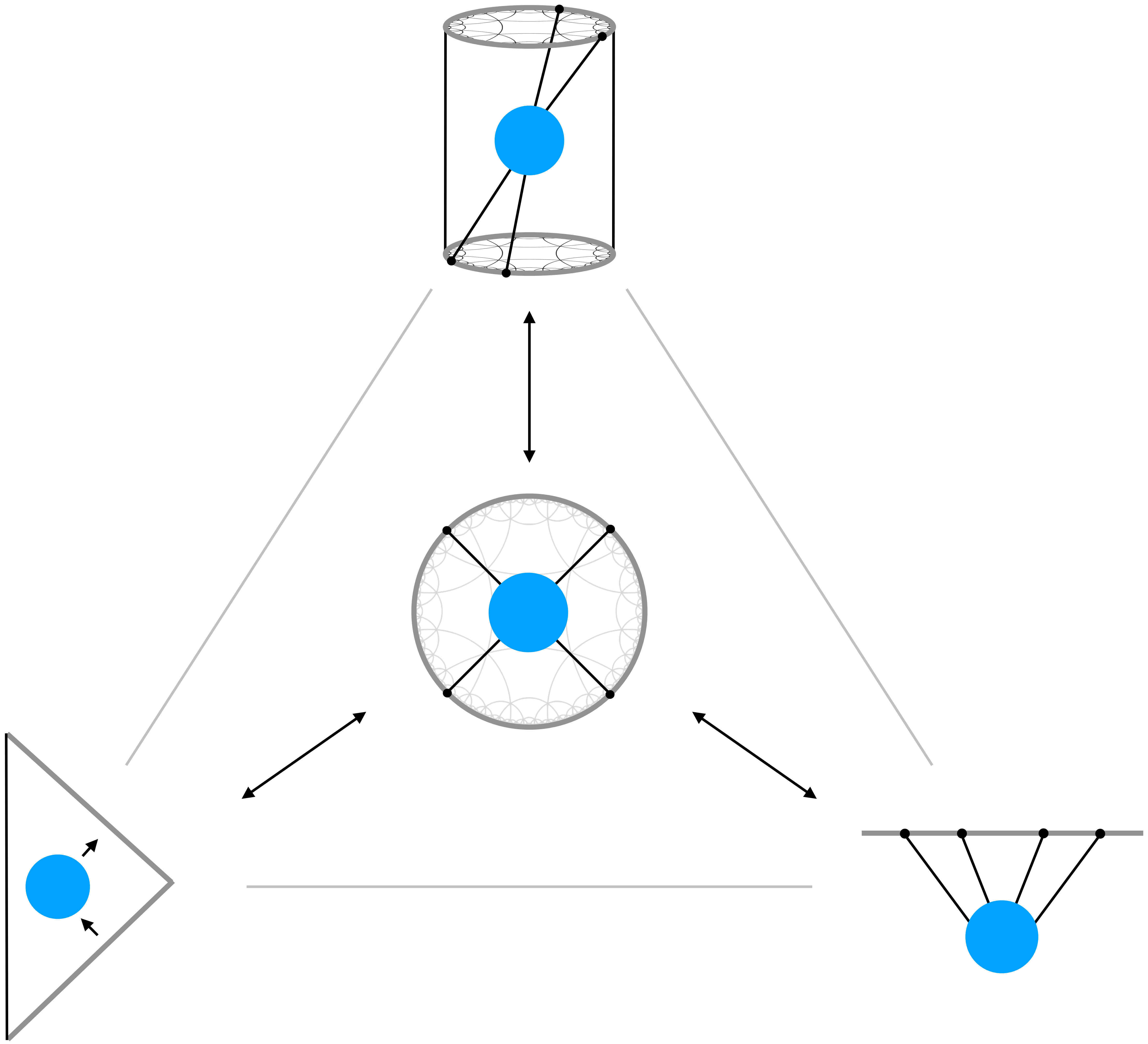
Unitarity: $\rho_J(\Delta) \geq 0$

2401.16591 L. Iacobacci, C.S. and M. Taronna

2412.11992 F. Pacifico, P. Pergola and C.S.

- **Källén-Lehmann spectral decomposition**

$$\int_0^\infty \mathrm{d}\mu^2\,\rho(\mu^2)\,\ldots \quad\to\quad \mathcal{M}[\rho](\Delta)\,\ldots\,, \qquad \mathcal{M}[\rho](\Delta) = \int_0^\infty \frac{\mathrm{d}z}{z}\,\rho(z)\,z^{\Delta}$$



Thank you.