# Quasinormal Corrections to Near-Extremal Black Hole Thermodynamics

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# Outline

- Review: old puzzles about cold black holes
- Low temperature quantum corrections to Kerr thermodynamics
- Questions about the calculation
- Rotating BTZ: $T^{3/2}$ from the full determinant, lessons for Kerr

First half based on [2310.00848] with Sheta, Strominger, Toldo

Second half based on [2409.14928] with Albert Law, Chiara Toldo

See also: interesting work [2409.16248] by [Kolanowski, Marolf, Rakic, Rangamani, Turiaci]

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$$S(T, J) = S_0 + 8\pi^2 J^{3/2} T + O(T^2) \, , \qquad C = T \frac{\partial S}{\partial T} \sim 8\pi^2 J^{3/2} T$$

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So at temperatures $T \sim J^{-3/2}$ the emission of a single Hawking quantum can lead to relatively large fluctuations in temperature.

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**Do quantum corrections lift the ground states?**

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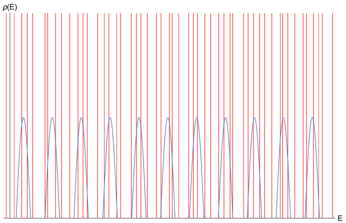


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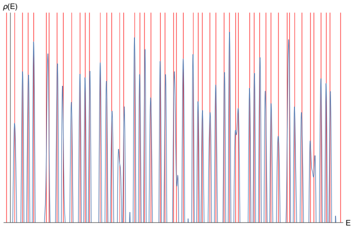


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Saddle point means solution to the Einstein equation subject to the boundary condition, and the solution is NHEK (near-horizon extreme Kerr)

$$ds^2 = J(1+\cos^2\theta)(-\sinh^2\eta dt^2 + d\eta^2 + d\theta^2) + \frac{4J\sin^2\theta}{1+\cos^2\theta}(d\phi + [\cosh\eta - 1]dt)^2$$

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The first correction comes from integrating over fluctuations about the saddle. Write $g = \bar{g}_{NHEK} + h$ and expand the action to quadratic order

$$Z \sim e^{2\pi J} \int [Dh] e^{-\int h(x)\mathcal{D}h(x)}$$

where

$$h_{\alpha\beta} D^{\alpha\beta,\mu\nu}_{\textsf{NHEK}} h_{\mu\nu} = -\frac{1}{16\pi} h_{\alpha\beta} \left( \frac{1}{4} \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} \bar{\Box} - \frac{1}{8} \bar{g}^{\alpha\beta} \bar{g}^{\mu\nu} \bar{\Box} + \frac{1}{2} \bar{R}^{\alpha\mu\beta\nu} \right) h_{\mu\nu}$$

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$\mathcal{D}$ is a $2^{nd}$-order linear differential operator, an infinite dimensional matrix.

$$\int [Dh] e^{-\int h(x)\mathcal{D}h(x)} \sim \frac{1}{[\text{det}\mathcal{D}]^{1/2}}$$

There is some universal information in this 1-loop correction [Sen, many others].

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Treated quantum mechanically, this mode dramatically alters the low temperature thermo. Recent history starting with [Maldacena, Stanford, Yang]

In the framework of log corrections and eigenvalue perturbation theory: Reissner-Nordstrom: [Larsen; Iliesiu, Murthy, Turiaci; Banerjee, Saha]. Kerr: [Kapec, Sheta, Strominger, Toldo; Rakic, Rangamani, Turiaci]. More general cases: [Maulik, Pando Zayas, Ray, Zhang]

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The eigenvalues are lifted because

$$h^{(n)} = \mathcal{L}_{\xi^{(n)}}g_{\text{NHEK}} \qquad h^{(n)} \neq \mathcal{L}_{\zeta}g_{\text{not-NHEK}}$$

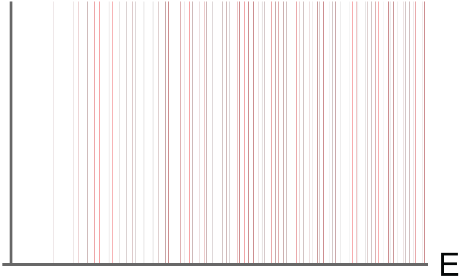
So the finite temperature lifts the eigenvalues and regulates the partition function. Computing the determinant gives

$$\delta\log Z \sim \log\prod_{n=2}^{\infty}\frac{1}{nT} \sim \frac{3}{2}\log T$$

So $Z[T]$ is becoming small at low temperatures, not exponentially large: the ground state degeneracy has been lifted.

$$Z[T] \sim T^{3/2} e^{S_0} \qquad \text{as} \qquad T \to 0$$

Instead the states fill out a dense energy band above the vacuum




We expect the eigenvalue spacing in this region of the spectrum to be roughly $e^{-S_0} \sim e^{-1/G_N}$ which is non-perturbatively small. Thermodynamics still applies.

# Recap

For many questions, the leading approximation to the black hole density of states, as computed using the Euclidean black hole saddle, is sufficient.

$$Z_{AF}(\beta, \mu, \Omega) = \underbrace{\int [Dg]\, e^{-I_{EH} - I_{GH} - I_{ct}}}_{\substack{\text{Asymptotically flat metrics with} \\ (\beta,\mu,\Omega) \text{ boundary conditions at } i^0}} \sim \frac{1}{\sqrt{\det -\nabla^2}} \exp\left[-I_{\text{on-shell}}\right]$$

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**Recent observation**: the gas of gravitons at low temperatures in a black hole background becomes important even when curvatures are small.

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In the meantime we learned that for low temperatures and certain black brane observables, we can replace $Z_{AF}(\beta)$ with a throat path integral

$$Z_{\text{throat}}(\beta,\mu,\Omega) \quad = \quad \underbrace{\int [Dg]\, e^{-I_{EH}-I_{GH}-I_{ct}}}_{\substack{\text{Asymptotically AdS}_{d+1}\text{ metrics with} \\ (\beta,\mu,\Omega)\text{ boundary conditions at }\partial\text{AdS}}}$$

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That is basically the AdS/CFT duality, but it is subtle for $\text{AdS}_2$. Sen found

$$Z_{\text{throat}}(\beta=\infty,Q,J) \quad = \quad \underbrace{\int [Dg]\, e^{-I_{EH}-I_{GH}-I_{ct}}}_{\substack{\text{Asymptotically AdS}_{2}\text{ metrics with} \\ (\beta,Q,J)\text{ boundary conditions at }\partial\text{AdS}}} \quad \sim \infty \times e^{S_0 + c\log S_0}$$

We interpret the infinity as an infrared divergence due to an unsupressed Goldstone mode. We regulate it by turning on an irrelevant deformation.

# Questions and concerns

So the quantity that we actually compute is a regularized partition function in the deformed "not-NHEK" throat.

$$Z_{\text{reg}}(\beta, Q, J) \quad = \quad \underbrace{\int [Dg]\, e^{-I_{EH}-I_{GH}-I_{ct}}}_{\text{Asymptotically ``not-NHEK'' metrics}} \quad \sim \quad T^{3/2} e^{S_0 + c \log S_0}$$

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Modes which are (non)normalizable in the throat might not complete to (non)normalizable modes in the full asymptotically flat geometry.

Example: the source and response terms for the gauge field flip.

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Seems hard: we cannot even perform the full not-NHEK path integral, only the piece responsible for the $T^{3/2}$ behavior.

The calculation in near-extremal Kerr is more complicated.

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There is an interesting formula due to [Denef, Hartnoll, Sachdev '09] which expresses the **Euclidean** determinant in terms of the **Lorentzian** quasinormal modes

$$\frac{1}{\sqrt{\det(-\nabla^2)}} = \prod_{k,l\in\mathbb{Z}} \prod_{z_l} \left(\omega_{|k|,l} + iz_l\right)^{-1/2}$$

Here the $z_l$ are the quasinormal modes of the field whose determinant we are calculating. The $\omega_{k,l}$ are the Matsubara frequencies

$$\omega_{k,l} = \frac{2\pi k}{\beta} - i\Omega l$$

These frequencies are required for periodicity on the Euclidean section.

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However, there is a particular branch of quasinormal modes whose frequencies can be computed analytically and which are closely related to the existence of the throat in the near-extremal Kerr geometry.

These “lightly-damped” modes have real parts that accumulate at the superradiant bound and small imaginary parts spaced evenly in units of $T_H$

$$\omega = m\Omega_H - 2\pi i T_H(n + 1/2)$$

The imaginary parts are small precisely because waves with $\omega = m\Omega_H$ penetrate and spend a long time in the throat region.

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The second incalculable term will correspond to a nonuniversal contribution which does not have a singular limit as $T \to 0$ since it is not really sensitive to the geometry near the throat.

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Their exclusion from the DHS product formula plays the same role as the exact $SL(2,\mathbb{R})$ symmetry in the extremal throat. Should be true in Kerr.

# The full BTZ determinant

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The thermal AdS determinant is simply the identity character in $\text{CFT}_2$

$$Z^{\text{graviton}}_{\text{TAdS}_3}(\tau, \bar{\tau}) = \chi_1(\tau)\chi_1(\bar{\tau}) \; , \qquad \chi_1(\tau) = \frac{(1-q)q^{\frac{1-c}{24}}}{\eta(\tau)}$$

This was argued indirectly in [Maloney, Witten '07], verified in [Giombi, Maloney, Yin '08] using heat kernel techniques and the method of images (thermal AdS and BTZ are $\mathbb{Z}$ quotients of $\mathbb{H}_3$).

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[Ghosh, Maxfield, Turiaci '19] took the low-T limit of the modular transform of this character and got a $T^{3/2}$.

In terms of the left and right temperatures of BTZ

$$\frac{2}{T} = \frac{1}{T_L} + \frac{1}{T_R} \, , \qquad \Omega = \frac{T_R - T_L}{T_R + T_L}$$

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If you only want to see the $T^{3/2}$, there is a faster derivation. Ignoring the tree-level piece

$$Z_{BTZ} \sim \prod_{n=2}^{\infty} \frac{1}{1 - q^n} \prod_{n=2}^{\infty} \frac{1}{1 - \bar{q}^n} \qquad q = e^{-(2\pi)^2 T_L} \,, \quad \bar{q} = e^{-(2\pi)^2 T_R}$$

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That looks like the determinant we calculated in the throat. To see that expand $q \sim 1-(2\pi)^2 T_L$

$$Z_{BTZ} \quad \sim \quad \prod_{n=2}^{\infty} \frac{1}{nT_L} \prod_{n=2}^{\infty} \frac{1}{1-\bar{q}^n} \quad \sim \quad T^{3/2} \times \prod_{other} [\cdots]$$

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We want to understand how to account for this structure using QNMs.

# Graviton determinant from QNM

The graviton determinant in a locally $AdS_3$ geometry can be expressed as the ratio of two determinants

$$Z_{\text{grav}} = \frac{\det\left(-\nabla^2_{(1)} + 2\right)^{1/2}}{\det\left(-\nabla^2_{(2)} - 2\right)^{1/2}}$$

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First, the naive application of the DHS formula. The $(s, \Delta)$ QNM spectrum

$$\omega_{nl}^{\Delta,s,L,\mp} = l - 2\pi i T_L\left(2n + \Delta \mp s\right) \qquad \omega_{nl}^{\Delta,s,R,\mp} = -l - 2\pi i T_R\left(2n + \Delta \pm s\right)$$

A spin-s field has two independent degrees of freedom, so there are 4 branches instead of two for the scalar.

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There are huge cancellations because the QNM of the two fields are related

$$\omega_{nl}^{2,2,L,+} = \omega_{nl}^{3,1,L,+} \,, \qquad \omega_{nl}^{2,2,R,-} = \omega_{nl}^{3,1,R,-} \,, \qquad \forall n = 0, 1, \cdots \,, \quad l \in \mathbb{Z}$$

and

$$\omega_{n+1,l}^{2,2,L,-} = \omega_{n,l}^{3,1,L,-} \,, \qquad \omega_{n+1,l}^{2,2,R,+} = \omega_{n,l}^{3,1,R,+} \,, \qquad \forall n = 0, 1, \cdots \,, \quad l \in \mathbb{Z} \,.$$

The two branches that contribute are actually totally undamped modes

$$\omega_{n=0,l}^{2,2,L,-} = \ell \,, \qquad \omega_{n=0,l}^{2,2,R,+} = -\ell \,.$$

# Excluded modes

Plugging into DHS, we get a formula that is not quite right

$$\log Z_{\mathsf{naive}} = \sum_{p=1}^{\infty} \frac{1}{p} \left( 1 + \sum_{n_L=1}^{\infty} q_L^{n_L p} + \sum_{n_R=1}^{\infty} q_R^{n_R p} \right)$$

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This is always true for scalar fields. It is not always true for spinning fields. The $(L, -)$ and $(R, +)$ modes with $n < s - |k|$ are not normalizable [Datta, David '11; Castro, Keeler, Szepietowski '17; Grewal, Law, Parmentier '22]

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So we have to exclude them from the DHS product, they cannot contribute to the determinant.

Following the usual steps we find we have to subtract the contribution

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- Only a tiny subset of the graviton QNM are needed to calculate the determinant due to cancellations with ghosts (probably true for Kerr)
- The only graviton QNM that contribute are totally undamped $\omega = \pm\ell$
- The exclusion of certain modes for spinning fields is crucial for $T^{3/2}$

# Which QNM contribute the $T^{3/2}$?

We had four branches of QNM for the graviton

$$\omega_{nl}^{\Delta,s,L,\mp} = l - 2\pi i T_L \left(2n + \Delta \mp s\right) \qquad \omega_{nl}^{\Delta,s,R,\mp} = -l - 2\pi i T_R \left(2n + \Delta \pm s\right)$$

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The ones that do have no imaginary part. You can check that it is actually the "right branch" that is responsible for the $T^{3/2}$.

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Since we know which modes account for the scaling, we can look for a shortcut in the product representation.

The naive product form of DHS using the undamped right moving modes is

$$(Z_{\mathsf{right}})^2 = \prod_{k,l\in\mathbb{Z}} \frac{1}{2\pi|k|T - il\Omega_H - il} = \prod_{k,l\in\mathbb{Z}} \frac{1}{2\pi|k|\frac{2T_L T_R}{T_L+T_R} - \frac{2ilT_R}{T_L+T_R}}$$

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Infinite constants of the form $\prod_{l\in\mathbb{Z}} \frac{1}{A}$ where $A$ is independent of $l$, can be absorbed into field redefinitions/local counterterms. So we have

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The low-$T$ limit of this product exhibits $T^{3/2}$ scaling. To see this explicitly we separate off the $l = 0$ term so that the product becomes

$$Z_{\text{right}} = \left[\prod_{k>1} \frac{1}{2\pi kT_L}\right]\left[\prod_{k>1}\prod_{l>0} \frac{1}{l^2 + (2\pi kT_L)^2}\right]$$

# Wrapping up

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$$Z_{BH} \sim T^{3/2} e^{S_0}$$

The result resolved some old questions about cold black holes, but the source of the effect was subtle.

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Along the way we derived the spectral density for the BTZ black hole, which allowed for an explicit derivation of the DHS formula without assumptions.