$K^+ \rightarrow \pi^+ \nu \bar{\nu}$ Decays on the Lattice

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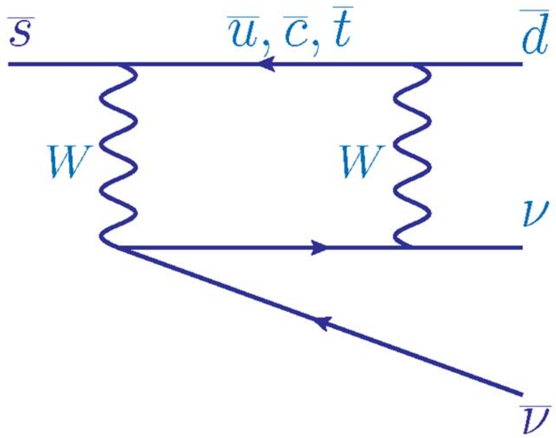
Outline

- $K^+ \rightarrow \pi^+ \nu \bar{\nu}$: The long-distance part – a target for lattice QCD.
- Overview of lattice calculation
  1. Subtraction of exponentially growing terms
  2. Renormalization of bilocal operators
  3. Power-law, finite-volume corrections.
- Lattice studies:
  - 16^3 x 32, m_π = 420 MeV (completed)
  - 32^3 x 64, m_π = 420 MeV (completed)
  - 64^3 x 128, physical masses (underway)
- Outlook

$K^+ \rightarrow \pi^+ \nu \bar{\nu}$

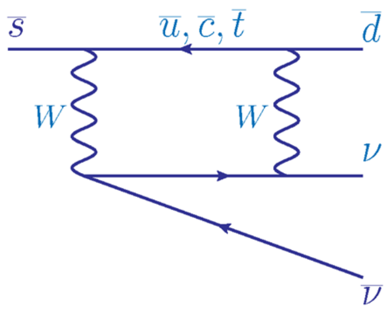
- Flavor changing neutral current
  - Allowed in the Standard Model only in second order
  - Short distance dominated
- Target of NA62 at CERN
  - 100 events by 2018
  - Test Standard Model prediction at 10% level
  - Use lattice QCD for long distance part: 5% effect ?

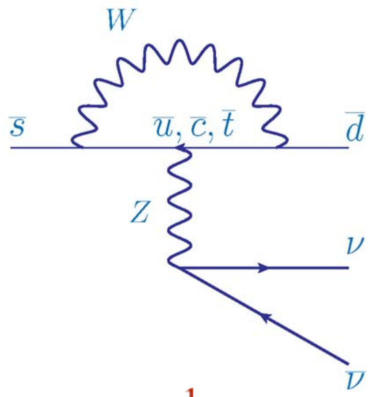


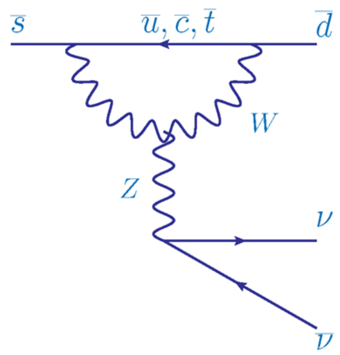
Overview of lattice efforts

- First discussion of rare kaon decays using LGT:
  - G. Isidori, G. Martinelli, and P. Turchetti, Phys.Lett. B633, 75 (2006), arXiv:hep-lat/0506026 [hep-lat].
- Analysis and first calculation of $K \to \pi l^+ l^-$
  - N. H. Christ, X. Feng, A. Portelli, and C. T. Sachrajda (RBC, UKQCD), Phys. Rev. D92, 094512 (2015), arXiv:1507.03094 [hep-lat].
  - N. H. Christ, X. Feng, A. Juttner, A. Lawson, A. Portelli, and C. T. Sachrajda, Phys. Rev. D94, 114516 (2016), arXiv:1608.07585 [hep-lat].
- Analysis and first calculation of $K^+ \to \pi^+ \nu \bar{\nu}$
  - N. H. Christ, X. Feng, A. Portelli, and C. T. Sachrajda (RBC, UKQCD), Phys. Rev. D93, 114517 (2016), arXiv:1605.04442 [hep-lat].
  - Z. Bai, N. H. Christ, X. Feng, A. Lawson, A. Portelli, and C. T. Sachrajda, Phys. Rev. Lett. 118, 252001 (2017), arXiv:1701.02858 [hep-lat]. **[A]**
- $K^+ \to \pi^+ \nu\bar{\nu}$, 32^3 x 64, M_π = 170 MeV, $m_c^{\overline{\text{ms}}=2\text{GeV}}$ = 750 MeV **[B]**
- $K^+ \to \pi^+ \nu\bar{\nu}$, 64^3 x128, M_π = 140 MeV, $m_c^{\overline{\text{ms}}=2\text{GeV}}$ = 1.2 GeV**[C]**

$K^+ \rightarrow \pi^+ \nu \bar{\nu}$ in the Standard Model





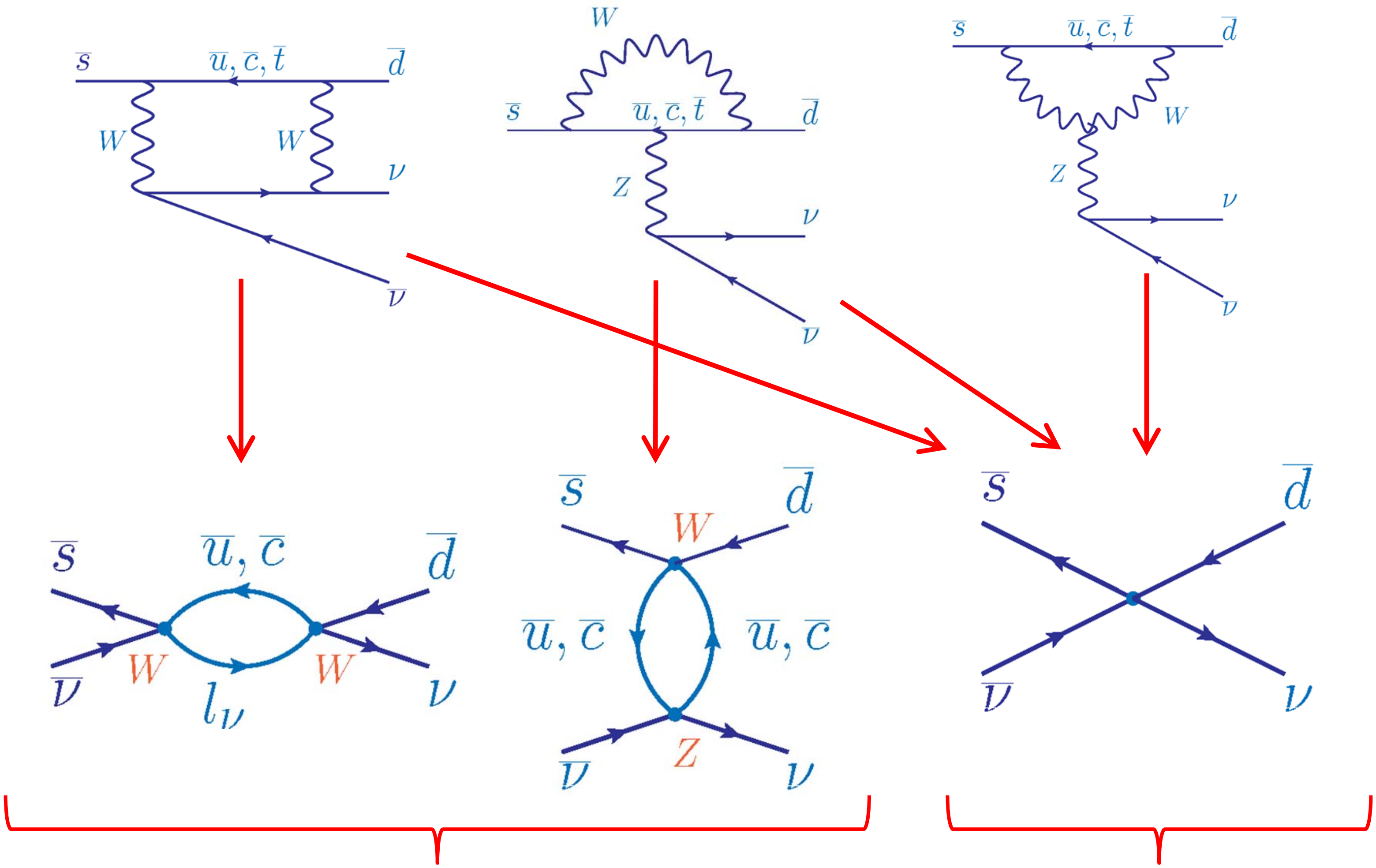


- Factors of $\frac{1}{M_W^4}$ or $\frac{1}{M_W^2 M_Z^2}$ force the largest contribution to come from short distance

Pert. Th. {
- Top quark contribution largest.
- GIM: charm - up ~ $\frac{m_c^2 - m_u^2}{M_W^4} \ln(M_W^2/m_c^2)$

Lattice {
- Long distance part ~ $\frac{m_c^2}{M_W^4}$

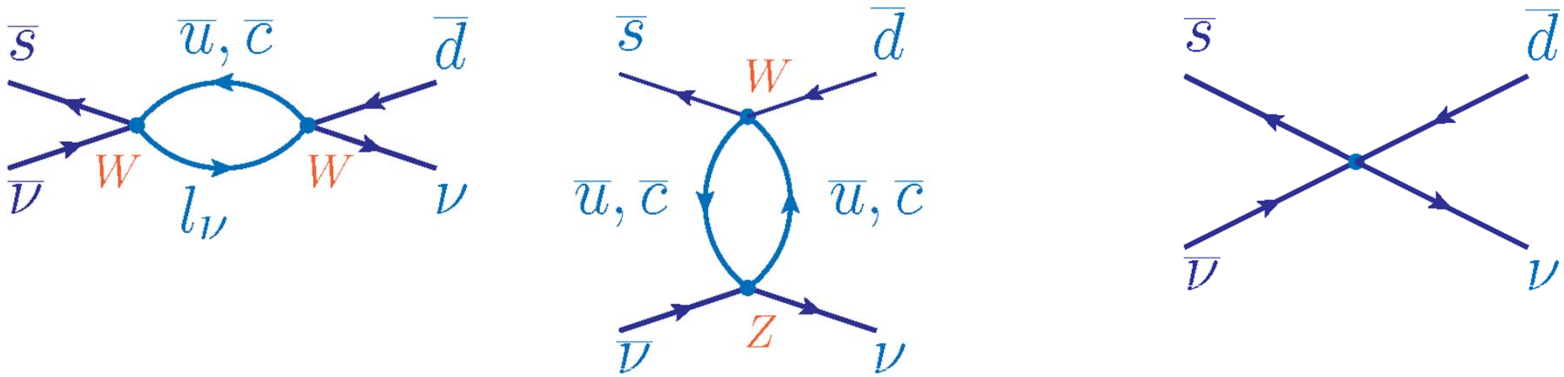
$K^+ \rightarrow \pi^+ \nu \bar{\nu}$ at long distance



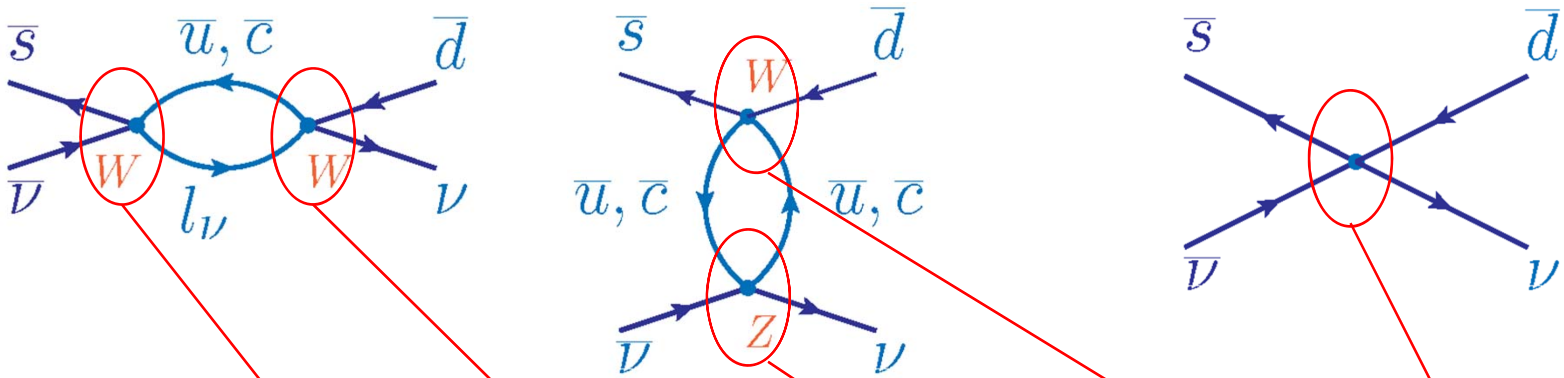
Use lattice QCD

Known $\langle \pi^+ | V_\mu | K^+ \rangle$

H_{eff} for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$

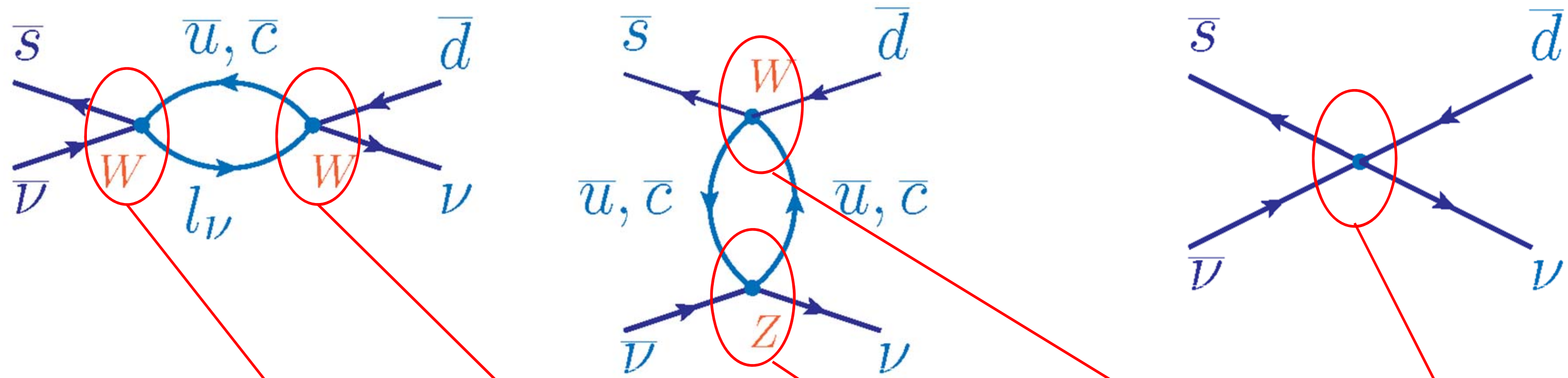


H_{eff} for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$



$$\mathcal{H}_{\text{eff}} = +\frac{G_F}{\sqrt{2}} \left\{ \sum_{\substack{q=u,c \\ \ell=e,\mu,\tau}} \left(V_{qs}^* O_{q\ell}^{\Delta S=1} + V_{qd} O_{q\ell}^{\Delta S=0} \right) + \sum_{\ell=e,\mu,\tau} O_\ell^Z + \sum_{q=u,c} \lambda_q O_q^W \right\} + O_0$$

H_{eff} for $K^+ \to \pi^+ \nu \bar{\nu}$



$$\mathcal{H}_{\text{eff}} = +\frac{G_F}{\sqrt{2}} \left\{ \sum_{\substack{q=u,c \\ \ell=e,\mu,\tau}} \left(V_{qs}^* O_{q\ell}^{\Delta S=1} + V_{qd} O_{q\ell}^{\Delta S=0} \right) + \sum_{\ell=e,\mu,\tau} O_\ell^Z + \sum_{q=u,c} \lambda_q O_q^W \right\} + O_0$$

$$O^{\Delta S=1} = C_{\Delta S=1} (\bar{s}q)_{V-A} (\bar{\ell}\nu_\ell)_{V-A}$$

$$O_q^W = C_1 (\bar{s}_a q_b)_{V-A} (\bar{q}_b d_a)_{V-A} + C_2 (\bar{s}_a q_a)_{V-A} (\bar{q}_b d_b)_{V-A}$$

$$O^{\Delta S=0} = C_{\Delta S=0} (\bar{q}d)_{V-A} (\bar{\ell}\nu_\ell)_{V-A}$$

$$O_0 = C_0 \sum_{\ell=e,\mu,\tau} (\bar{s}d)_{V-A} (\bar{\nu}_\ell \nu_\ell)_{V-A}$$

$$O_\ell^Z = C_Z \sum_{q=u,c,d,s} \left(T_3^q \bar{q} \gamma_\mu (1-\gamma_5) q - Q_{\text{em},q} \sin^2\theta_W \bar{q} \gamma_\mu q \right) \overline{\nu_\ell} \gamma_\mu (1-\gamma_5) \ell$$

$K^+ \to \pi^+ \nu \bar{\nu}$: 2nd order effective theory

Bilocal

Local

$$\mathcal{A}(K^+ \to \pi^+ \nu\bar{\nu}) = \langle \pi^+ \nu\bar{\nu} |\, T\left\{\int d^4x \mathcal{H}'_{\rm eff}(x)\mathcal{H}'_{\rm eff}(0)\right\} + O_0(0)\ | K^+ \rangle$$

$$\mathcal{H}_{\rm eff} = +\frac{G_F}{\sqrt{2}}\left\{ \sum_{\substack{q=u,c\\ \ell=e,\mu,\tau}} \left(V_{qs}^* O_{q\ell}^{\Delta S=1} + V_{qd} O_{q\ell}^{\Delta S=0}\right) + \sum_{\ell=e,\mu,\tau} O_\ell^Z + \sum_{q=u,c} \lambda_q O_q^W \right\} + O_0$$

$$O^{\Delta S=1} = C_{\Delta S=1}(\bar{s}q)_{V-A}(\bar{\ell}\nu_\ell)_{V-A}$$

$$O_q^W = C_1(\bar{s}_a q_b)_{V-A}(\bar{q}_b d_a)_{V-A} + C_2(\bar{s}_a q_a)_{V-A}(\bar{q}_b d_b)_{V-A}$$

$$O^{\Delta S=0} = C_{\Delta S=0}(\bar{q}d)_{V-A}(\bar{\ell}\nu_\ell)_{V-A}$$

$$O_0 = C_0 \sum_{\ell=e,\mu,\tau} (\bar{s}d)_{V-A}(\bar{\nu}_\ell \nu_\ell)_{V-A}$$

$$O_\ell^Z = C_Z \sum_{q=u,c,d,s} \left(T_3^q \bar{q}\gamma_\mu(1-\gamma_5)q - Q_{{\rm em},q}\sin^2\theta_W \bar{q}\gamma_\mu q\right) \overline{\nu_\ell}\gamma_\mu(1-\gamma_5)\ell$$

$K^+ \rightarrow \pi^+ \nu \bar{\nu}$: Effect of bilocal operator

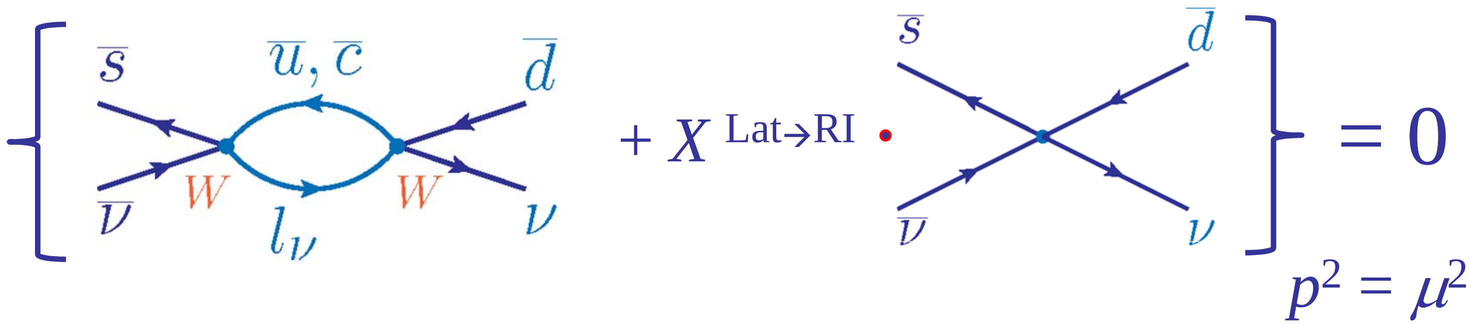
Bilocal   Local

$$\mathcal{A}(K^+ \to \pi^+ \nu\overline{\nu}) = \langle \pi^+ \nu\overline{\nu} |\, T\left\{\int d^4x \mathcal{H}'_{\text{eff}}(x)\mathcal{H}'_{\text{eff}}(0)\right\} + O_0(0)\; | K^+ \rangle$$

- Standard continuum treatment
  - Replace bilocal term with (perturbative coefficient) x (local operator)
- Lattice treatment: Evaluate $H_{eff}(x)$ $H_{eff}(0)$ product
  - Revolve logarithmic divergence as $x \rightarrow 0$
  - Deal with intermediate states with $E \leq M_K$
    - Exponential Euclidean time dependence
    - Power-law finite volume corrections
    - Exploit methods from M_{K_L}-M_{K_S} calculation

New short-distance divergence

- Second-order effective theory requires new counter terms



- Use NPR for bilocal operator

$$\left\{\int d^4x\, T\left(Q_A^{\overline{\text{MS}}}(x)\, Q_B^{\overline{\text{MS}}}(0)\right)\right\}^{\overline{\text{MS}}}$$

$$= Z_A Z_B \left\{\int d^4x\, T\left(Q_A^{\text{Lat}}(x)\, Q_B^{\text{Lat}}(0)\right)\right\}^{\text{Lat}} + \left(Z_A Z_B X_{AB}^{\text{Lat}\to\text{RI}} + Y_{AB}^{\text{RI}\to\overline{\text{MS}}}\right) Q_0(0)$$

Unphysical terms growing exponentially with time

- Encountered previously for $M_{K_L} - M_{K_S}$

$$\int_{-T}^{T} dt \langle \pi \nu \overline{\nu} | T\Big(O_A(t) O_B(0)\Big) | K \rangle$$

$$= \sum_n \left\{ \frac{\langle \pi \nu \overline{\nu} | O_A | n \rangle \langle n | O_B | K \rangle}{M_K - E_n} + \frac{\langle \pi \nu \overline{\nu} | O_B | n \rangle \langle n | O_A | K \rangle}{M_K - E_n} \right\} \left(1 - e^{(M_K - E_n)T}\right)$$

- $- e^{(M_K - E_n)T}$ term with $M_K > E_n$ must be removed.
- Principal part has been replaced by a finite volume sum: possibly large finite volume corrections. (N.H. Christ, X. Feng, G. Martinelli, C.T. Sachrajda, arXiv:1504.01170)

Possibly large finite volume corrections

- Encountered previously for $M_{K_L} - M_{K_S}$

$$\int_{-T}^{T} dt \langle \pi \nu \overline{\nu} | T\Big(O_A(t) O_B(0)\Big) | K\rangle$$

$$= \sum_n \left\{ \frac{\langle \pi \nu \overline{\nu} | O_A | n\rangle \langle n | O_B | K\rangle}{M_K - E_n} + \frac{\langle \pi \nu \overline{\nu} | O_B | n\rangle \langle n | O_A | K\rangle}{M_K - E_n} \right\} \left(1 - e^{(M_K - E_n)T}\right)$$

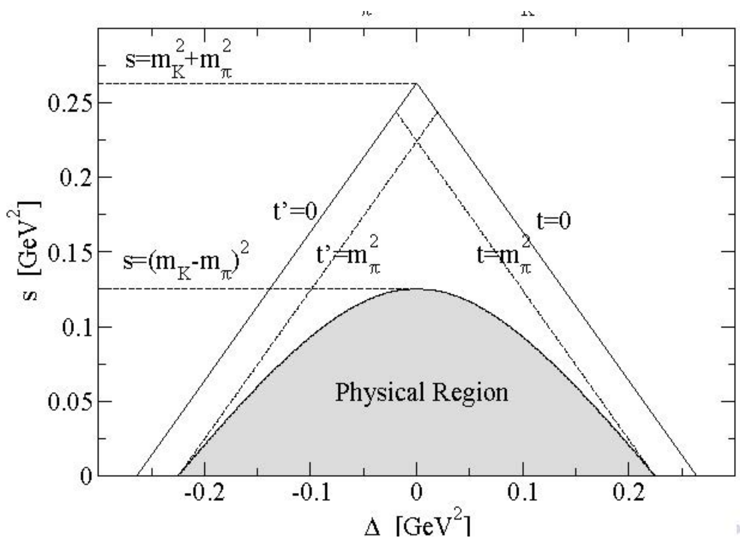
- Large finite-volume distortion if $E_n \rightarrow M_K$
- Apply a known correction, involving free particle kinematics for $|n\rangle = |\pi^0 l^+ \nu\rangle$ and the $I = 2$, s-wave $\pi\pi$ phase shift for $|n\rangle = |\pi^+ \pi^0\rangle$ (N.H. Christ, X. Feng, G. Martinelli, C.T. Sachrajda, arXiv:1504.01170)

Exploratory Lattice Calculation **[A]**

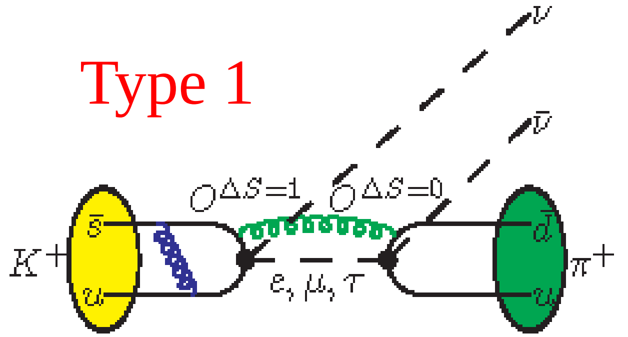
- 16^3 x 32, RBC-UKQCD ensemble
  - 2+1 flavor DWF, $1/a$ = 1.73 GeV
  - M_π = 420 MeV, M_K = 540 MeV,
  - $m_c(2\text{ GeV})^{\overline{MS}}$ = 863 GeV
- Calculate all diagrams
- 800 configurations
- Low-mode deflation with 100 modes
- Place sources on all 32 time slices
- Treat internal lepton as an overlap fermion moving in ∞ time.

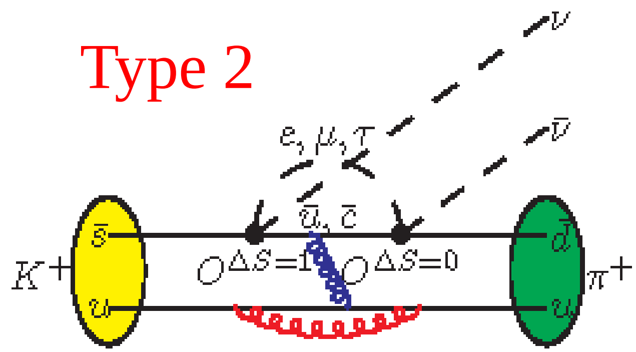
Exploratory Lattice Calculation

- All results given as scalar amplitudes
  - *W*-exchange diagram determines $F_{WW}(s, \Delta)$ for Dalitz variables:
    $s = (p_K - p_\pi)^2, \quad \Delta = (p_K - p_\nu)^2 - (p_K - p_{\bar{\nu}})^2$
  - Vector and axial from *Z*-exchange determine familiar *Kl*3 $f_+(s)$
  - Treat these as constants
  - Evaluate at $\vec{p}_K = 0$ and $\vec{p}_\pi = (0.0414, 0.0414, 0.0414)$
  - For vector *Z*-exchange also use $\vec{p}_\pi = 0$



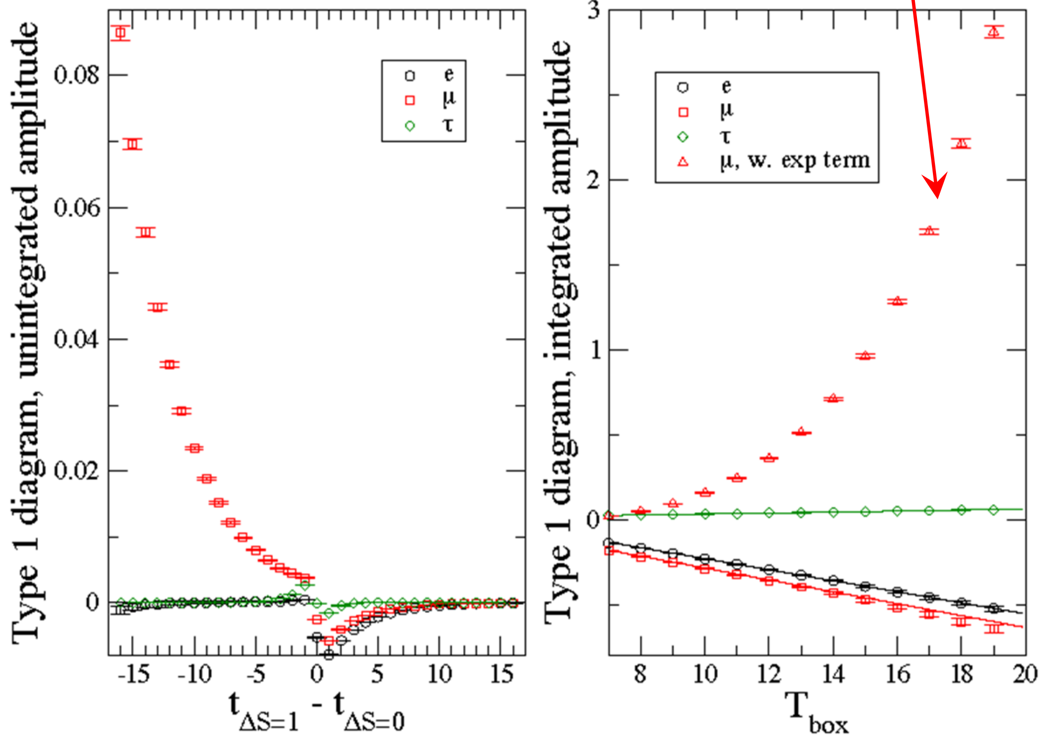
W W diagrams





Type 1

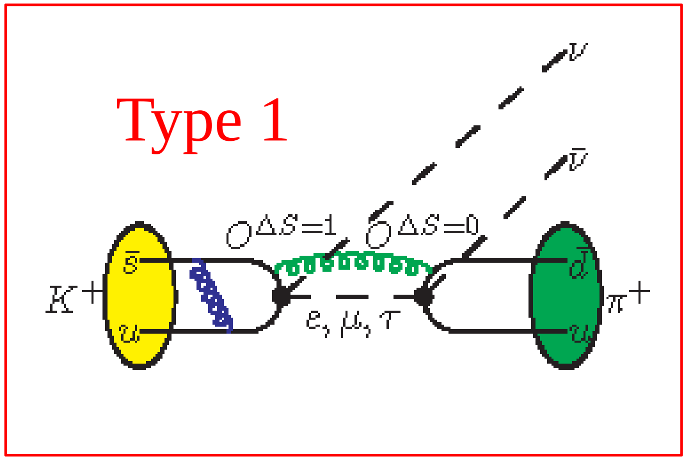
Exponentially growing term before subtraction



F_{WW}	Type 1	model
e	$-1.685(47) \times 10^{-2}$	$-1.740(6) \times 10^{-2}$
μ	$-1.818(40) \times 10^{-2}$	$-1.822(6) \times 10^{-2}$
τ	$1.491(36) \times 10^{-3}$	$1.471(5) \times 10^{-3}$

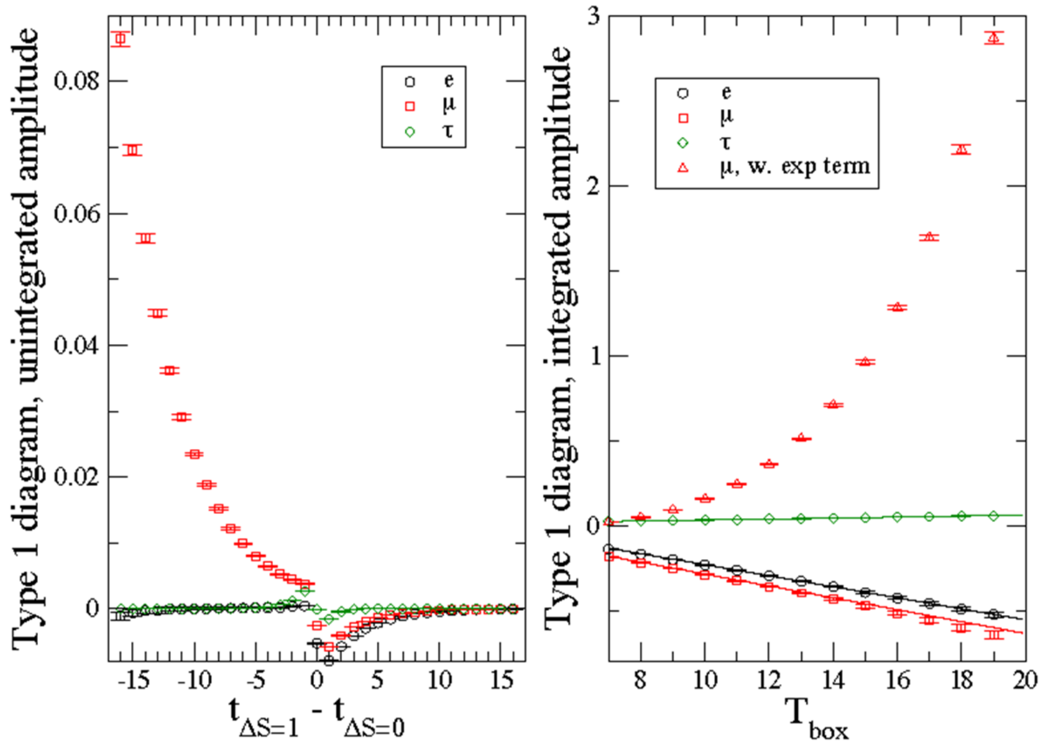
Exponentially growing term comes from $\mu^+\nu$ state

W W diagrams



- Require:
  - $|t_K - t_{\mathrm{op}}| \geq 6$
  - $|t_\pi - t_{\mathrm{op}}| \geq 6$
- Sum periodic and anti-periodic propagators to double the volume: $T = 32 \rightarrow 64$
- Require: $30 \geq |t_\pi - t_{\mathrm{op}}|$

Type 1



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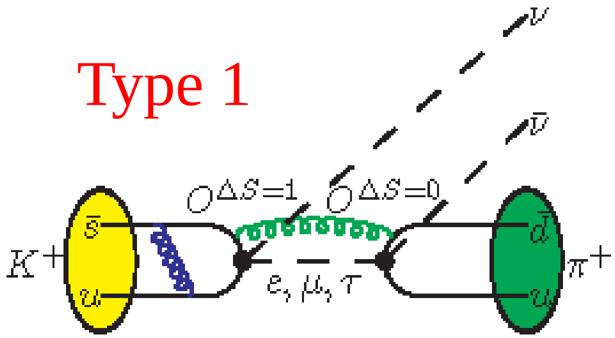
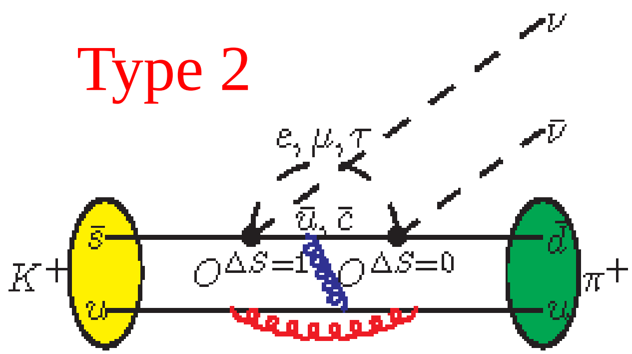
W W diagrams

Type 1

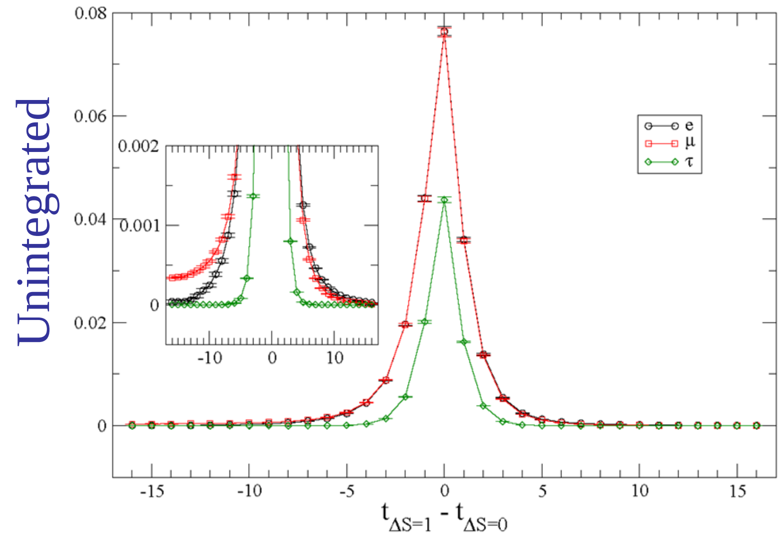


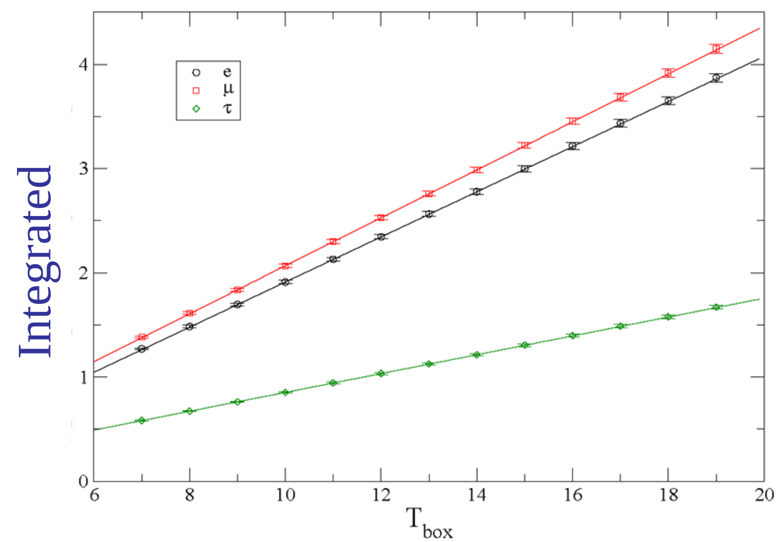
Type 2



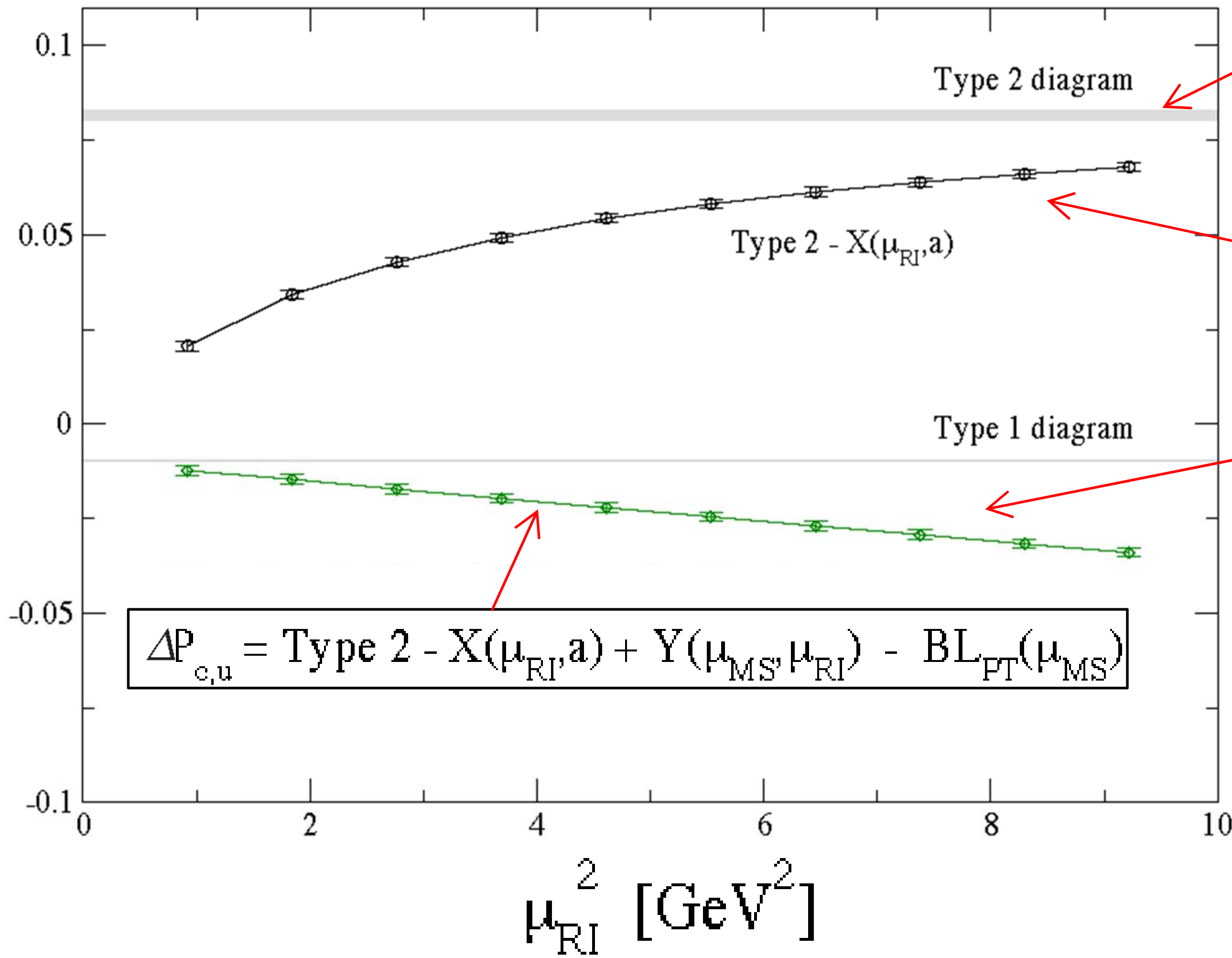
Now the $|\pi\,\mu\,\nu\rangle$ state is 16% of the result. The divergent SD part must be subtracted.

Type 2





W W diagrams

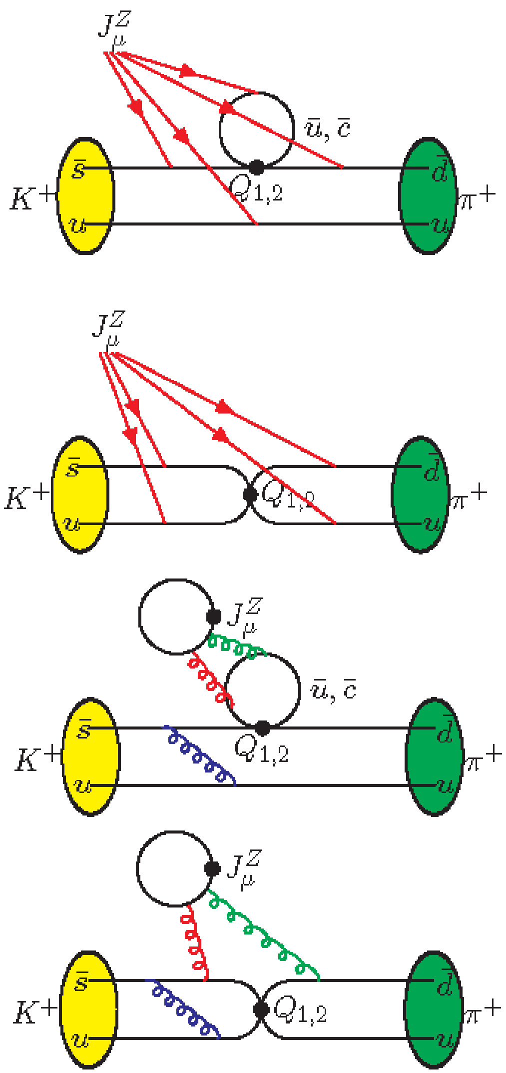


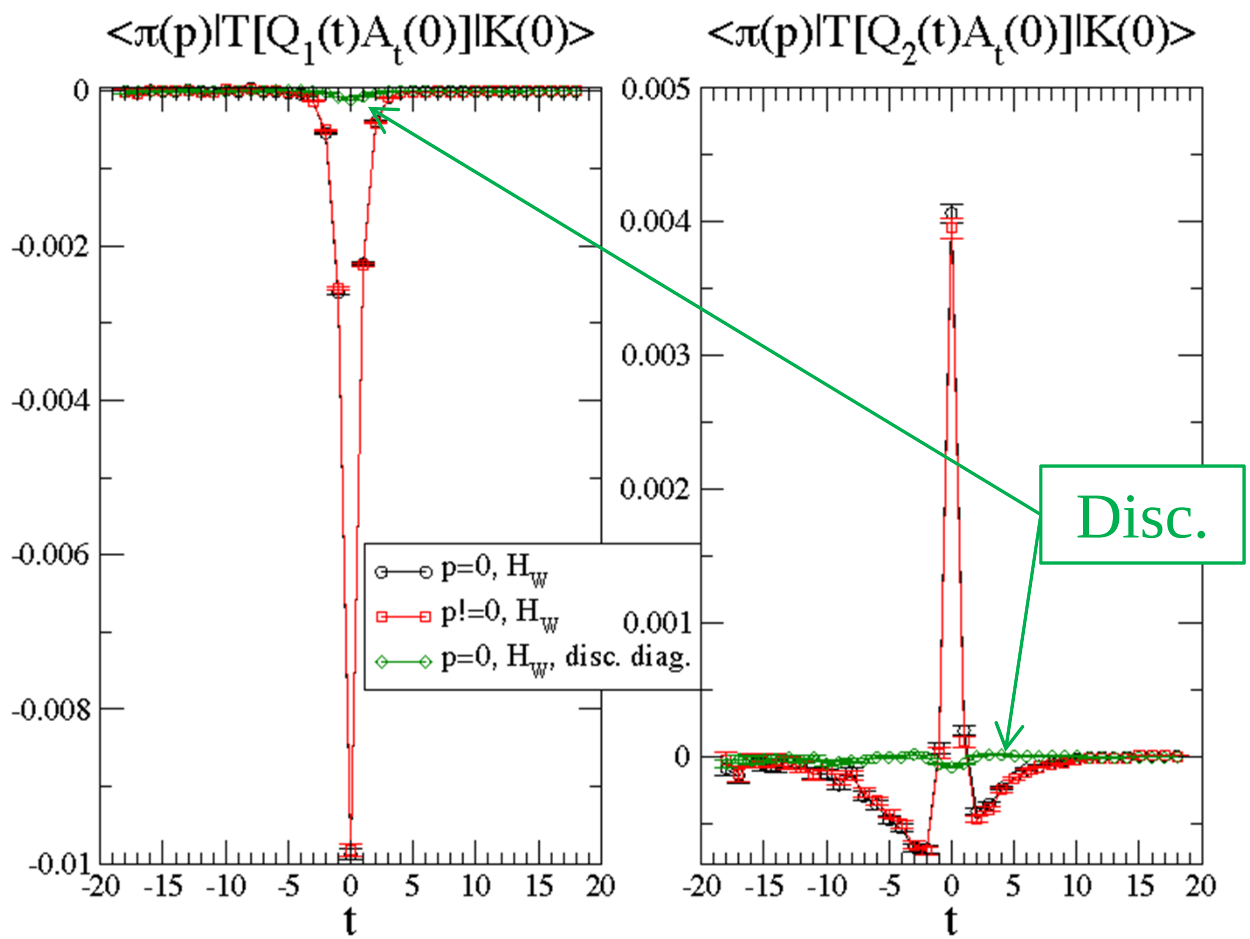
“Bare” lattice

RI - normalized

Difference between lattice and perturbative $\overline{\text{MS}}$ operators, $\mu_{\overline{\text{MS}}}$=2GeV

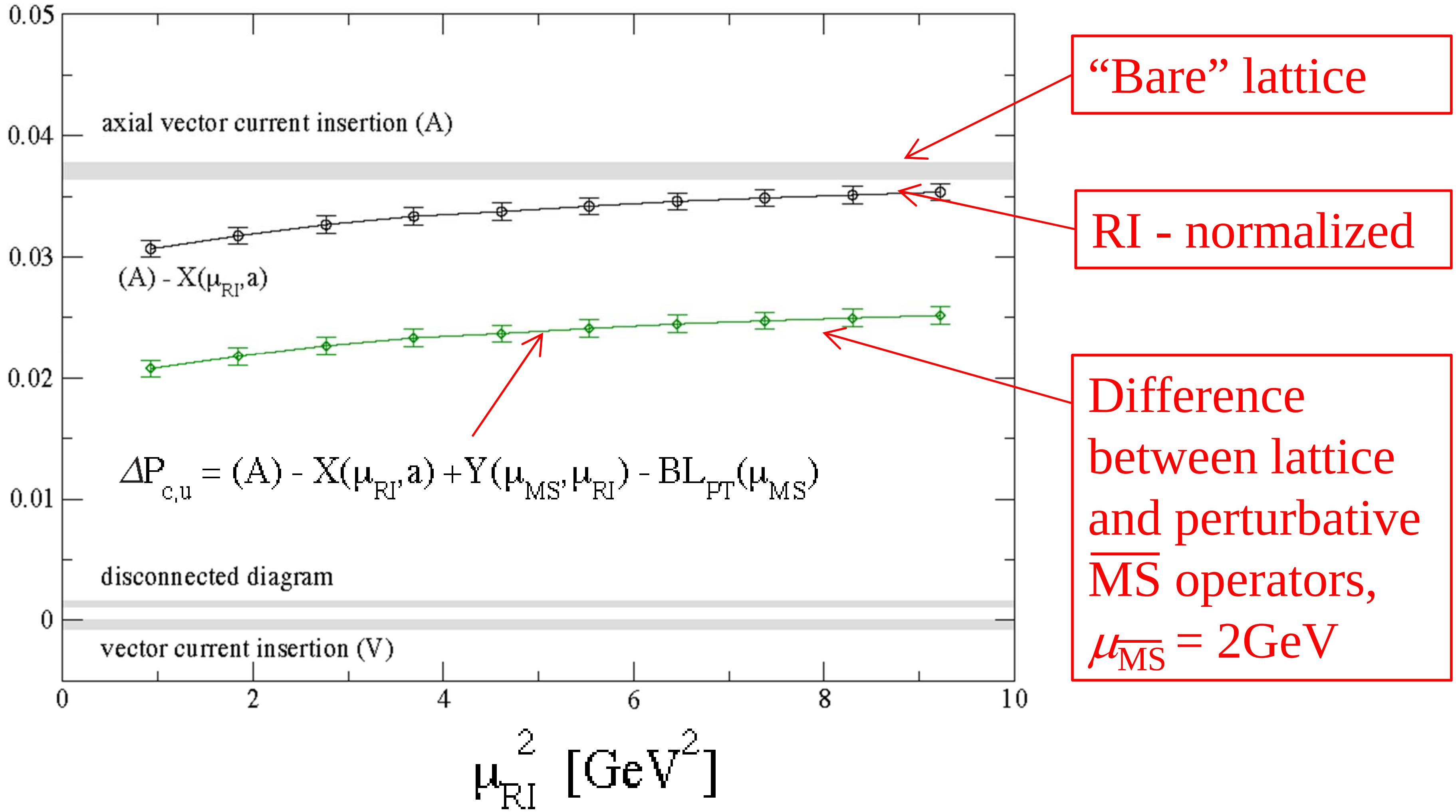
Z – Exchange Diagrams





Unintegrated results from axial *Z* coupling

Z – Exchange diagrams



Connect with conventional result for $K^+ \to \pi^+ \nu \bar{\nu}$ decay

- Decay rate is short distance dominated:

$$\mathrm{Br} = \kappa_+(1+\Delta_{\mathrm{EM}})\left[\left(\underbrace{\frac{\mathrm{Im}\lambda_t}{\lambda^4}X(x_t)}_{0.270\,\times 1.481}\right)^2 + \left(\underbrace{\frac{\mathrm{Re}\lambda_c}{\lambda}P_c}_{-0.974\,\times\,0.405} + \underbrace{\frac{\mathrm{Re}\lambda_t}{\lambda^5}X(x_t)}_{-0.533\,\times 1.481}\right)^2\right]$$

$$P_c^{\mathrm{SD}} = \frac{1}{\lambda^4}\frac{X_c^e + X_c^\mu + X_c^\tau}{3} \qquad \lambda = |V_{us}|$$

- Charm contribution is less than top but is significant (removing charm lowers BR by 50%)

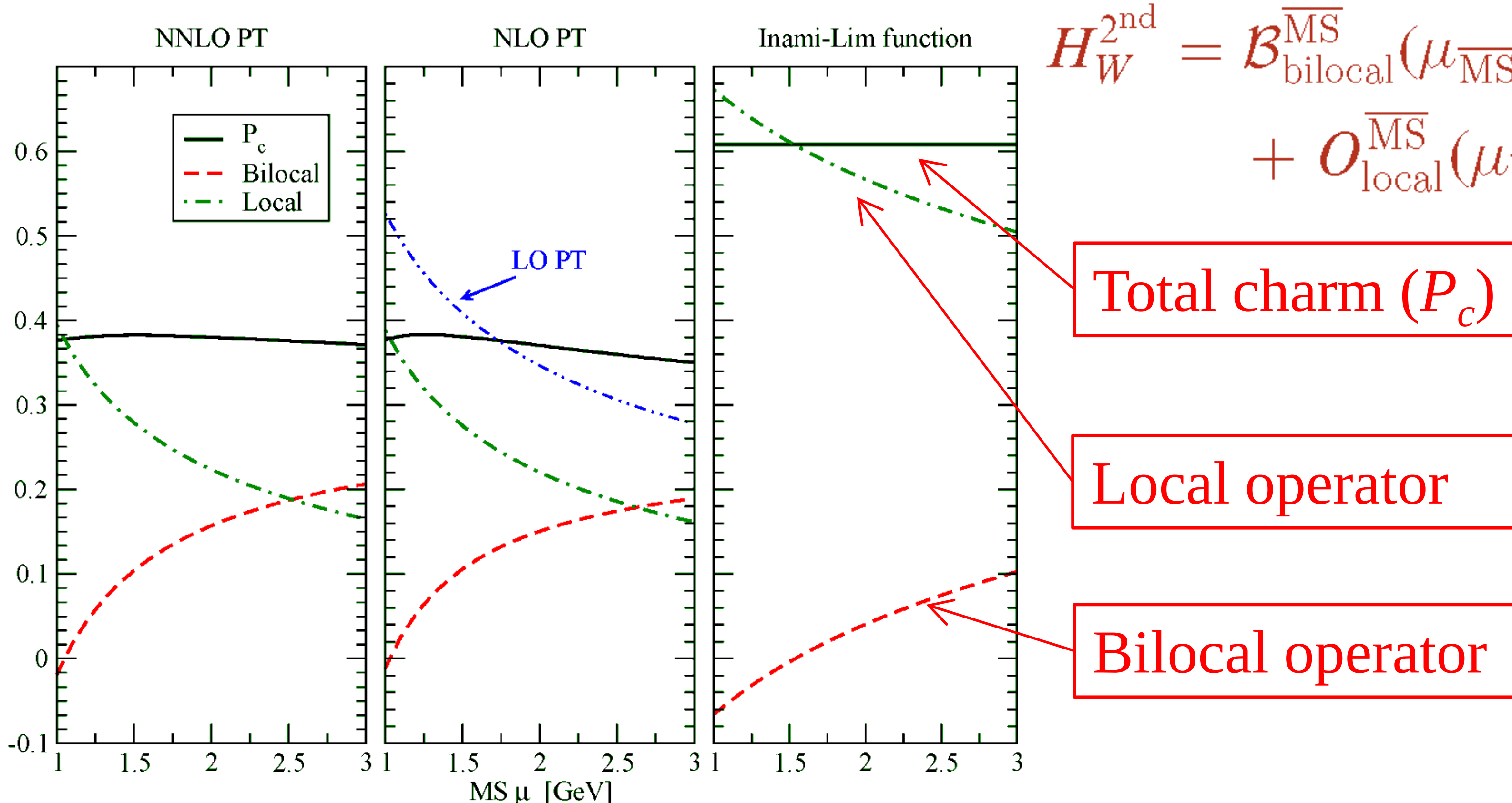
$$\lambda_t X_t(x_t) \;:\; \lambda_c X_c^\ell \quad :: \quad \lambda_t \frac{m_t^2}{M_W^2} \;:\; \lambda_c \frac{m_c^2}{M_W^2}\ln\frac{M_W}{m_c}$$

[A. J. Buras, M. Gorbahn, U. Haisch, and U. Nierste, JHEP 0611, 002 (2006), arXiv:hep-ph/0603079]

Importance of charm energy scale

- Presence of $\ln(M_W^2/m_c^2) = 8.4$ suggests that 12% of charm contribution comes from charm scale?
- However, the log term is suppressed at NLO:

$$H_W^{\text{2nd}} = \mathcal{B}^{\overline{\text{MS}}}_{\text{bilocal}}(\mu_{\overline{\text{MS}}}) + O^{\overline{\text{MS}}}_{\text{local}}(\mu_{\overline{\text{MS}}})$$

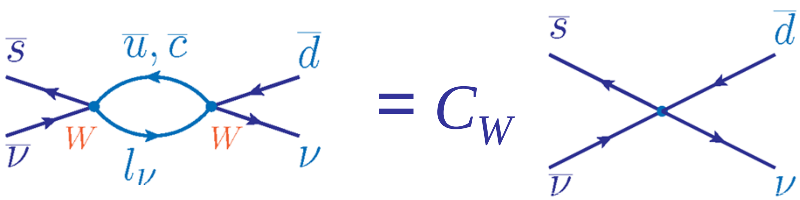


Conventional treatment of $p \leq m_c$

- Electroweak and QCD perturbation theory provides:

$$H_W^{2^{\text{nd}}} = \mathcal{B}_{\text{bilocal}}^{\overline{\text{MS}}}(\mu_{\overline{\text{MS}}}) + O_{\text{local}}^{\overline{\text{MS}}}(\mu_{\overline{\text{MS}}})$$

- Integrate out charm:



$$\mathcal{B}_{\text{bilocal}}^{\overline{\text{MS}}}(\mu_{\overline{\text{MS}}}) \approx C_W(\mu_{\overline{\text{MS}}}) \cdot Q_0^{\overline{\text{MS}}}(\mu_{\overline{\text{MS}}}) \qquad Q_0 = (\overline{s}d)_{V-A}(\overline{\nu}\nu)_{V-A}$$

- Long distance effect of up quark is missing, represented by δP_{cu}: $\quad P_c = P_c^{\text{SD}} + \delta P_{cu}$
  - $P_c^{\text{SD}} = 0.365(12)$
  - $\delta P_{cu} = 0.040(20)$ [Isidori *et. al*, hep-ph/0503107]

Lattice result (unphysical kinematics)

- P_c = 0.2529 $(\pm 13)_{stat}(\pm 32)_{scale}(-45)_{FV}$
- Difference with standard charm treatment:

Bilocal operator

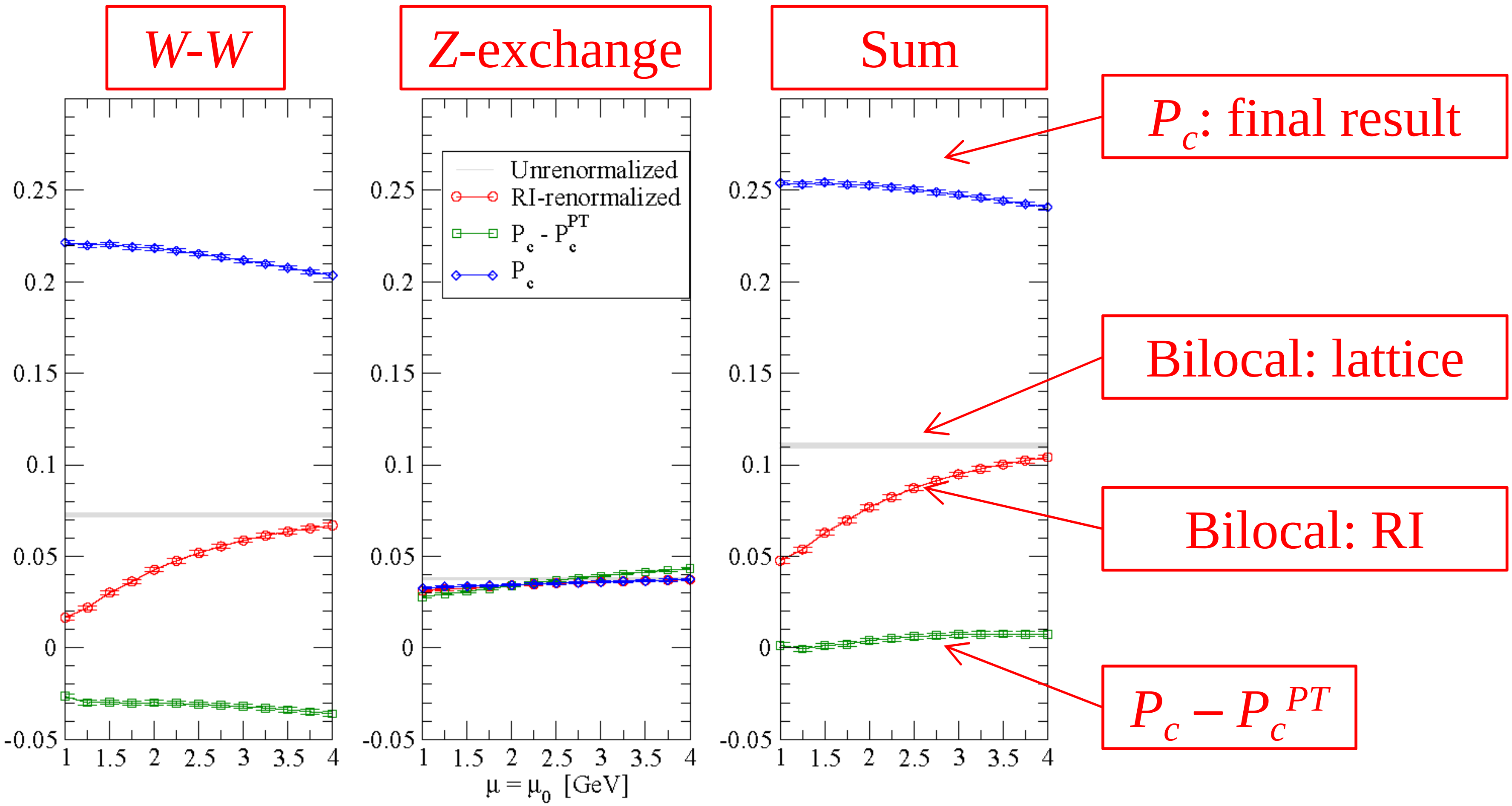
$$P_c(\mu_{\overline{\rm MS}}) - P^{\rm PT}(\mu_{\overline{\rm MS}}) \propto \langle \pi \nu \bar{\nu} | \left\{ \mathcal{B}^{\overline{\rm MS}}_{\rm bilocal}(\mu_{\overline{\rm MS}}) - C_W(\mu_{\overline{\rm MS}}) \cdot Q_0^{\overline{\rm MS}}(\mu_{\overline{\rm MS}}) \right\} | K^+ \rangle$$

Perturbative approximation to bilocal operator

$$P_c(\mu_{\overline{\rm MS}}) - P^{\rm PT}(\mu_{\overline{\rm MS}}) = 0.0040\,(\pm 13)_{\rm stat}\,(\pm 32)_{\rm scale}\,(-45)_{\rm FV}$$

- Small difference because of large W-W - Z-exchange cancellation.

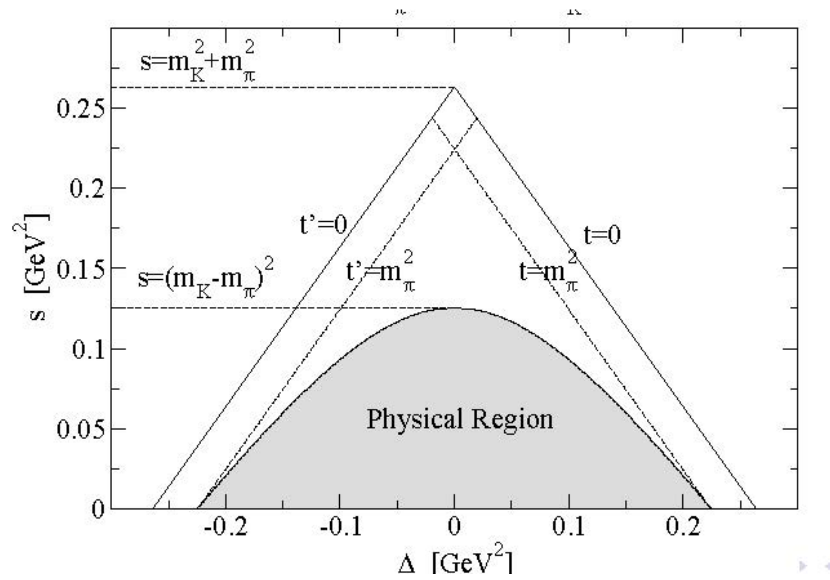
Details of W-W – Z-exchange cancellation



Increase volume – decrease M_π **[B]**

- 32^3 x 64, RBC-UKQCD ensemble
  - 2+1 flavor DWF, 1/*a* = 1.38 GeV
  - Iwasaki+DSDR action, Mobius valence fermions
  - M_π = 172 MeV, M_K = 493 MeV,
  - $m_c(2\ \text{GeV})^{\overline{\text{MS}}}$ = 750 GeV
- 100 configurations separated by 16 MD time units.
- Low-mode deflation with 560 modes
- Study two principal questions:
  - What is the dependence on the Dalitz variables?
  - How large is the contribution of the lowest $\pi\pi$ intermediate state?

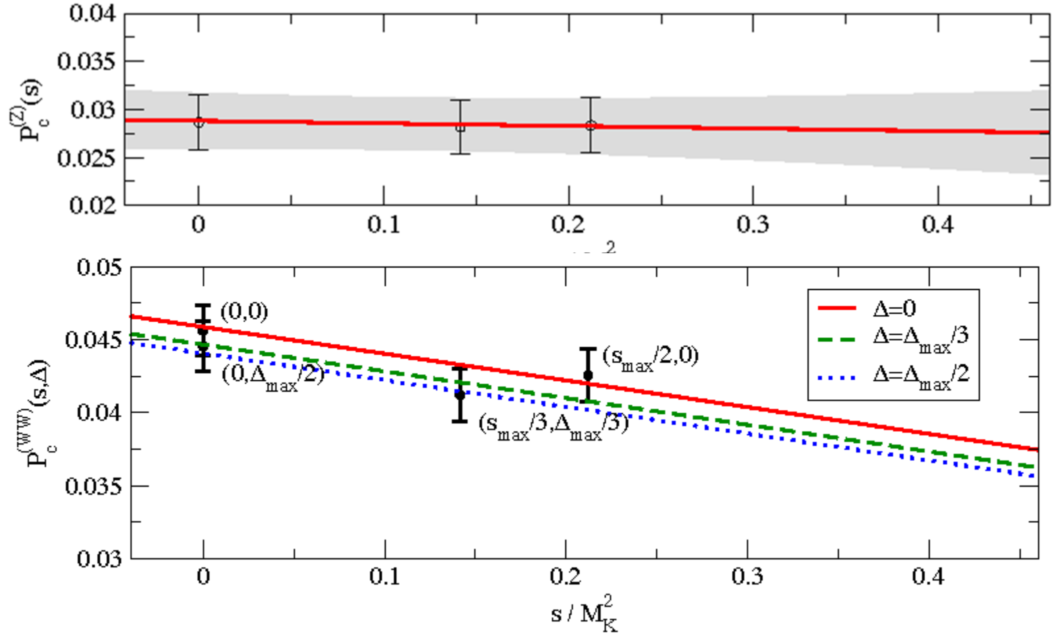
Dependence on Dalitz variables



- Vary $s = (p_K - p_\pi)^2$ and $\Delta = (p_K - p_\nu)^2 - (p_K - p_{\bar{\nu}})^2$
- $s_{max} = (M_K - M_\pi)^2$ $\Delta_{max} = M_K^2 - M_\pi^2$
- Evaluate for four pairs:

(s, Δ) = $(0,0)$ $(0, \Delta_{max}/2)$ $(s_{max}/2, 0)$ $(s_{max}/3, \Delta_{max}/3)$

Dependence on Dalitz variables



- Parameterize and find effect on branching ratio:

$$P_c^{(Z)}(s) = P_c^{(Z)}(0) + b_s^{(Z)} \frac{s}{M_K^2} \qquad P_c^{(WW)}(s,\Delta) = P_c^{(WW)}(0,\Delta) + b_s^{(WW)} \frac{s}{M_K^2} + b_\Delta^{(WW)} \frac{\Delta}{M_K^2}$$

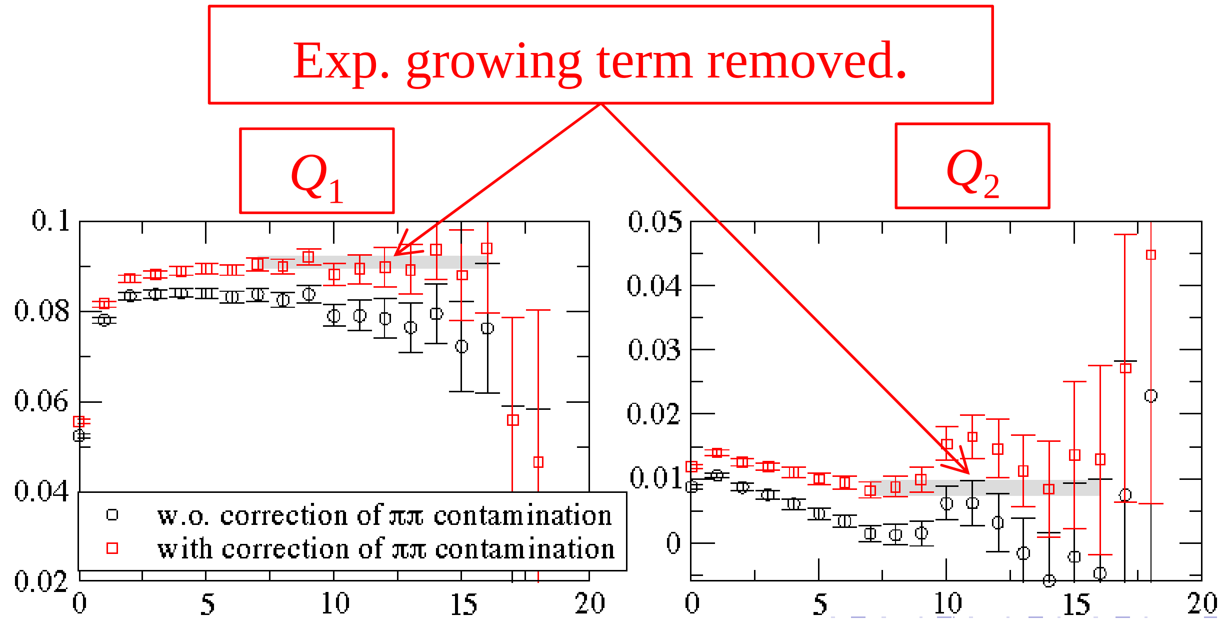
$$b_s^{(Z)} = -1.8(9.7)\times 10^{-3} \qquad b_s^{(WW)} = -1.8(0.9)\times 10^{-1} \qquad b_\Delta^{(WW)} = -4.1(7)\times 10^{-3}$$

$$BR(K^+ \to \pi^+ \nu\bar{\nu}) \propto 1 + 0.071 \cdot b_\Delta^2 + 0.202 \cdot \left(b_s^{(Z)} + b_s^{(WW)}\right)$$

- Neglect *Δ* and *s* dependence.

Effects of $\pi\pi$ intermediate state

- Explore effects of a two pion state with $E_{\pi\pi}$ = 346 MeV, lighter than M_K = 493 MeV
- Intermediate $\pi\pi$ state contributes 7.5% to F_0^Z but exponentially growing term must be removed:



- Finite volume correction from discrete $\pi\pi$ state: 2.1%

Use physical kinematics [C]

- $64^3 \times 128$, RBC-UKQCD ensemble
  - 2+1 flavor DWF, $1/a = 2.38$ GeV
  - Iwasaki action, Mobius fermions
  - $M_\pi = 139.2(5)$ MeV, $M_K = 493$ MeV,
  - $m_c(2\text{ GeV})^{MS} = 1.2$ GeV
- 20 configurations separated by 40 MD time units.
- Low-mode deflation with 2000 modes
- Now almost Running on 8 racks of Mira at the ALCF (Argonne).
- Expect 10% accurate result. [Errors order $(m_c a)^2$ are the largest uncertainty.]

Conclusion

- Lattice methods can be used to compute the long distance contribution to $K^+ \to \pi^+ \nu \overline{\nu}$ from $E \leq m_c$
- Exponentially growing terms and bilinear operator normalization can be controlled.
- Demonstrated by a 16^3 x 32 exploratory lattice calculation with $m_\pi = 420$ MeV
- Larger volume, lighter pion calculation shows weak dependence on Dalitz variables and controlled $\pi\pi$ effects.
- Calculation with physical kinematics underway.