Searching for new physics with kaons

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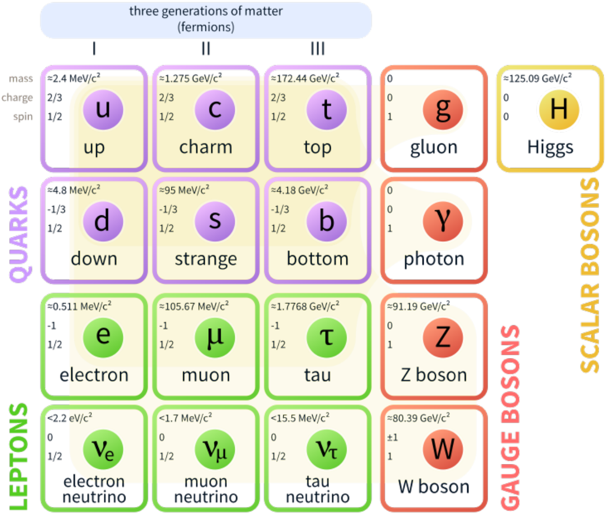
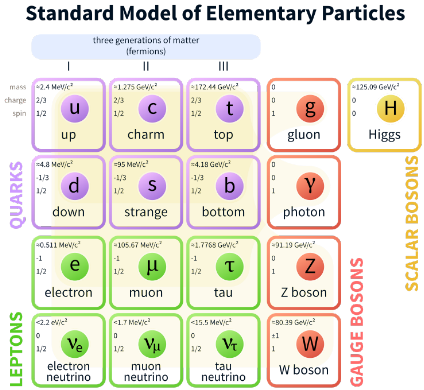
Higgs Centre Colloquium
and
First forum on rare kaon decays
Edinburgh, February 21st - 23rd 2018

Standard Model of Elementary Particles



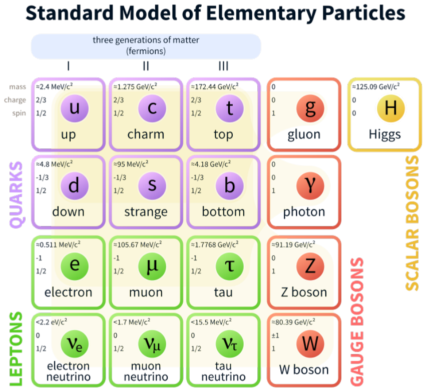
Why 3 generations?



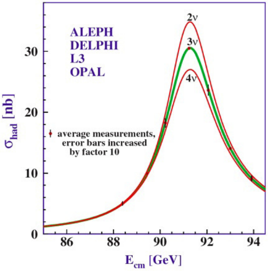
- Who ordered that?

I.I.Rabi, 1936

Discovery of the muon



- Z_0 width



$$N_\nu = 2.9840 \pm 0.0082$$

PDG 2016

The SM is incomplete

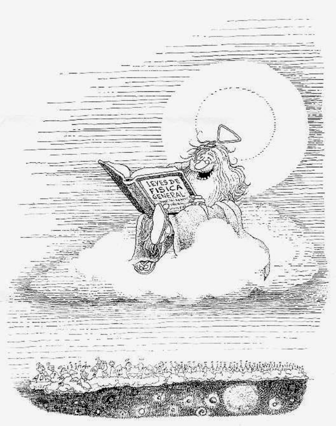
There are many reasons to believe that the Standard Model is incomplete:

- Why are the charges of the proton and electron equal and opposite:

$$\frac{Q_p + Q_e}{e} < 1 \times 10^{-21}\,.$$

  - Unification of forces?
  - Cancellation of anomalies?

- nature of dark matter and dark energy;
- naturalness and mass hierarchies;
- strong CP-problem;
- origin of neutrino masses;
- gravity, $\cdots$



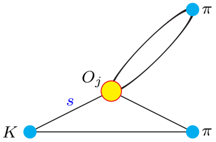
- The focus of this talk will be on a number of important applications to kaon physics which go beyond standard lattice computations.

1. Introduction and general motivation for precision flavour physics.
2. Isospin Breaking Corrections to $K \to \ell\bar{\nu}$ and $K \to \pi\ell\bar{\nu}$ decays.
3. Long-distance contributions to Flavour Changing Neutral Current Processes:
   - 3.1 Δm_K
   - 3.2 ϵ_K
   - 3.3 Rare Kaon Decays $K \to \pi\nu\bar{\nu}$
   - 3.4 Rare Kaon Decays $K \to \pi\ell^+\ell^-$.
4. $K \to \pi\pi$ decays.
5. Summary and Prospects.

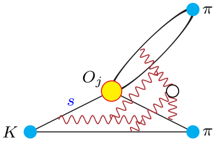
- Warm thanks to my collaborators from the RBC-UKQCD collaborations and from Rome with whom the ideas presented in this talk have been developed.

Introduction (cont.)

- Precision Flavour Physics is a key tool, complementary to the large E_T searches at the LHC, in exploring the limits of the Standard Model of Particle Physics and in searches for new physics.
  - If the LHC experiments discover new elementary particles BSM, then precision flavour physics will be necessary to understand the underlying framework.
  - The discovery potential of precision flavour physics should also not be underestimated. (In principle, the reach is about two-orders of magnitude deeper than the LHC!)
  - Precision flavour physics requires control of hadronic effects for which lattice QCD simulations are essential.



means



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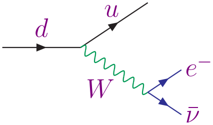
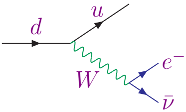
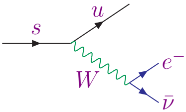
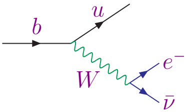
- At the level of quarks we understand nuclear β decay in terms of the fundamental process:



- With the 3 generations of quarks and leptons in the standard model this is generalized to other *charged current* processes, e.g.:







- Weak interaction eigenstates $\neq$ mass eigenstates.

Weak interaction eigenstates $\neq$ mass eigenstates:

$$U_W = \begin{pmatrix} u_W \\ c_W \\ t_W \end{pmatrix} = U_u \begin{pmatrix} u \\ c \\ t \end{pmatrix} = U_u\, U \quad \text{and} \quad D_W = \begin{pmatrix} d_W \\ s_W \\ b_W \end{pmatrix} = U_d \begin{pmatrix} d \\ s \\ b \end{pmatrix} = U_d D$$

where U_u and U_d are unitary matrices.

- For neutral currents:

$$\bar{U}_W \cdots U_W = \bar{U} \cdots U \quad \text{and} \quad \bar{D}_W \cdots D_W = \bar{D} \cdots D$$

  and no FCNC are induced. The $\cdots$ represent Dirac Matrices, but the identity in flavour.
  Flavour Changing Neutral Current (FCNC) Processes are therefore excellent ones with which to search for new physics.

- For much of this talk I will consider $s \leftrightarrow d$ transitions.

- For charged currents:

$$J_W^{\mu\,+} = \frac{1}{\sqrt{2}}\, \bar{U}_W \gamma_L^\mu D_W = \frac{1}{\sqrt{2}}\, \bar{U}_L \gamma^\mu (U_u^\dagger U_d) D_L \equiv \frac{1}{\sqrt{2}}\, \bar{U} \gamma_L^\mu V_{\text{CKM}} D$$

- The charged-current interactions are of the form

$$J^+_\mu = (\bar u, \bar c, \bar t\,)_L \gamma_\mu V_{\rm CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L ,$$

- 2016 Particle Data Group summary for the magnitudes of the entries:

$$\begin{pmatrix} 0.97434^{+0.00011}_{-0.00012} & 0.22506 \pm 0.00050 & 0.00357 \pm 0.00015 \\ 0.22492 \pm 0.00050 & 0.97351 \pm 0.00013 & 0.0411 \pm 0.0013 \\ 0.00875^{+0.00032}_{-0.00033} & 0.0403 \pm 0.0013 & 0.99915 \pm 0.00005 \end{pmatrix}$$

$\Rightarrow$ we can write (Wolfenstein parametrisation)

$$V_{\rm CKM} \simeq \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4)\,.$$

- A, ρ and η are real numbers that a priori were intended to be of order unity.

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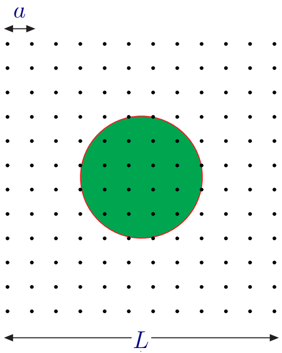
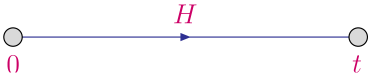


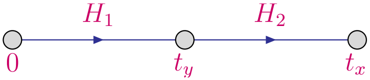
- Lattice phenomenology starts with the evaluation of correlation functions of the form:

$$\langle 0|\, O(x_1,x_2,\cdots,x_n)\,|0\rangle \;=\; \frac{1}{Z}\int [dA_\mu]\,[d\psi]\,[d\bar\psi]\,e^{-S}\,O(x_1,x_2,\cdots,x_n)\;,$$

where $O(x_1,x_2,\cdots,x_n)$ is a multilocal operator composed of quark and gluon fields and Z is the partition function.

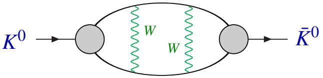
- The physics which can be studied depends on the choice of the multilocal operator O.





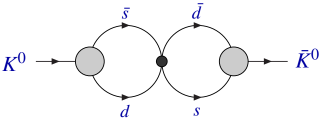
- The functional integral is performed by discretising Euclidean space-time and using Monte-Carlo Integration.

- The mission of lattice calculations is to evaluate hadronic effects.
- "Standard" lattice calculations in flavour physics are of matrix elements of local operators between single hadron states $\langle h_2(p_2)|O(0)|h_1(p_1)\rangle$ (or $\langle 0|O(0)|h(p)\rangle$).
- For example, in the evaluation of ϵ_K, we need to calculate (schematically)



(gluons and quark loops not shown.)

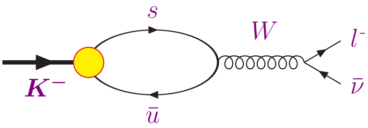
- The process is short-distance dominated and so we can approximate the above by a perturbatively calculable (Wilson) coefficient C times



where the black dot represents the insertion of the local operator $(\bar{s}\gamma_\mu(1-\gamma^5)d)\,(\bar{s}\gamma_\mu(1-\gamma^5)d)$.

  - In the standard model only this single operator contributes.
  - In generic BSM theories there are 5 possible $\Delta S = 2$ operators contributing.

Consider $K_{\ell 2}$ decays in pure QCD:



- All QCD effects are contained in a single constant, f_K, the kaon's *(leptonic) decay constant.*

$$\langle 0|\,\bar{s}\gamma^\mu\gamma^5 u\,|K(p)\rangle = if_K\,p^\mu\ . \qquad\qquad (f_\pi \simeq 132\,\text{MeV})$$

- In pure QCD

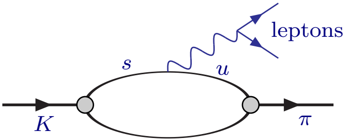
$$\Gamma(K^- \to \ell^-\bar{\nu}_\ell) = \frac{G_F^2\,|V_{us}|^2 f_K^2}{8\pi}\,m_K\,m_\ell^2\left(1-\frac{m_\ell^2}{m_K^2}\right)^2\ .$$

- From the experimental ratio of the $\pi_{\ell 2}$ and $K_{\ell 2}$ widths we get:

$$\frac{|V_{us}|}{|V_{ud}|}\,\frac{f_K}{f_\pi} = 0.2760(4)\ ,$$ M.Moulson, arXiv:1411.5252, J.Rosner, S.Stone & R.Van de Water, arXiv:1509.02220

  so that a precise determination of f_K/f_π will yield V_{us}/V_{ud} .

- Every collaboration calculates f_K and f_π (or uses f_π for calibration).



$$\langle\, \pi(p_\pi)\,|\bar{s}\gamma_\mu u\,|\,K(p_K)\,\rangle = f_0(q^2)\,\frac{M_K^2-M_\pi^2}{q^2}q_\mu + f_+(q^2)\,\left[(p_\pi+p_K)_\mu - \frac{M_K^2-M_\pi^2}{q^2}q_\mu\right]$$

where $q \equiv p_K - p_\pi$.

$$\Gamma_{K\to\pi\ell\nu} = C_K^2\,\frac{G_F^2 m_K^5}{192\pi^3}\,I\,S_{\rm EW}[1+2\Delta_{\rm SU(2)}+\Delta_{EM}]\,|V_{us}|^2\,|f_+(0)|^2$$

From the experimental measurement of the width we get:

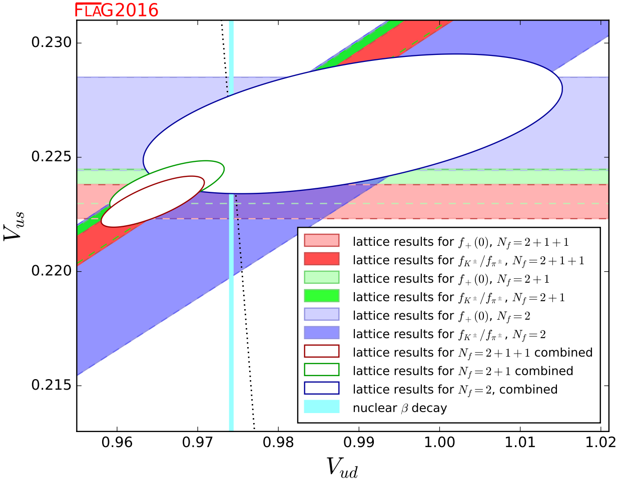
$$|V_{us}|\,f_+(0) = 0.2165(4)\,, \qquad \text{M.Moulson, arXiv:1411.5252}$$

so that a precise determination of $f_+(0)$ will yield V_{us}.

FLAG summary in light-quark physics

Flavour Physics Lattice Averaging Group - arXiv:1607.00299

Quantity	■	$N_f = 2+1+1$	■	$N_f = 2+1$	■	$N_f = 2$
m_s(MeV)	2	93.9(1.1)	5	92.0(2.1)	2	101(3)
m_{ud}(MeV)	1	3.70(17)	5	3.373(80)	1	3.6(2)
m_s/m_{ud}	2	27.30(34)	4	27.43(31)	1	27.3(9)
m_d(MeV)	1	5.03(26)	Flag(4)	4.68(14)(7)	1	4.8(23)
m_u(MeV)	1	2.36(24)	Flag(4)	2.16(9)(7)	1	2.40(23)
m_u/m_d	1	0.470(56)	Flag(4)	0.46(2)(2)	1	0.50(4)
m_c/m_s	3	11.70(6)	2	11.82	1	11.74
$f_+^{K\pi}(0)$	1	0.9704(24)(22)	2	0.9667(27)	1	0.9560(57)(62)
f_{K^+}/f_{π^+}	3	1.193(3)	4	1.192(5)	1	1.205(6)(17)
f_K(MeV)	3	155.6(4)	3	155.9(9)	1	157.5(2.4)
f_π(MeV)			3	130.2(1.4)		
$\Sigma^{\frac{1}{3}}$(MeV)	1	280(8)(15)	4	274(3)	4	266(10)
F_π/F	1	1.076(2)(2)	5	1.064(7)	4	1.073(15)
$\bar{\ell}_3$	1	3.70(7)(26)	5	2.81(64)	3	3.41(82)
$\bar{\ell}_4$	1	4.67(3)(10)	5	4.10(45)	2	4.51(26)
$\hat{B}_K$	1	0.717(18)(16)	4	0.7625(97)	1	0.727(22)(12)



Flavianet Lattice Averaging Group - arXiv:1607.00299

- The precision of “standard" isosymmetric QCD calculations is now such that in order to improve the precision still further isospin breaking (IB) effects (including electromagnetism) need to be included.
- These are

$$O\left(\frac{m_u - m_d}{\Lambda_{\rm QCD}}\right) \quad \text{and} \quad O(\alpha)\,,$$

  i.e. $O(1\%)$ or so.
- {The separation of IB corrections into those due to $m_u \neq m_d$ and those due to electromagnetism requires a convention. It is only the sum which is physical.}
- Such calculations for the spectrum have been performed for a few years now, with perhaps the most noteworthy result being BMW Collaboration, arXiv:1406.4088

$$m_n - m_p = 1.51(16)(23)\,\text{MeV}$$

  to be compared to the experimental value of 1.2933322(4) MeV.
- I stress that including electromagnetic effects, where the photon is massless of course, required considerable theoretical progress, e.g.

$$\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2}\cdots \Rightarrow \frac{1}{L^3T}\sum_k \frac{1}{k^2}\cdots$$

  and we have to control the contribution of the zero mode in the sum.

- Calculating electromagnetic corrections to decay amplitudes has an added major complication, not present in computations of the spectrum,

the presence of infrared divergences

- This implies that when studying weak decays, such as e.g. $K^+ \to \ell^+ \nu$ the physical observable must include soft photons in the final state

$$\Gamma(K^+ \to \ell^+ \nu_\ell(\gamma)) = \Gamma(K^+ \to \ell^+ \nu_\ell) + \Gamma(K^+ \to \ell^+ \nu_\ell \gamma)\,.$$

F.Bloch and A.Nordsieck, PR 52 (1937) 54

- The question for the lattice community is how best to combine this understanding with lattice calculations of non-perturbative hadronic effects.
- This is a generic problem if em corrections are to be included in the evaluation of a decay process.
- In 2015 we proposed a method for including electromagnetic corrections in decay amplitudes.

N.Carrasco, V.Lubicz, G.Martinelli, CTS, N.Tantalo, C.Tarantino & M.Testa, arXiv:1502.00257

- I stress (and will explain) that in order to implement this method successfully, it will be be necessary to work with the experimental community to ensure that we are calculating quantities which correspond to the experimental measurements.

$$\begin{aligned}\Gamma(\pi^+ \to \ell^+\nu_\ell(\gamma)) &= \Gamma(\pi^+ \to \ell^+\nu_\ell) + \Gamma(\pi^+ \to \ell^+\nu_\ell\gamma) \\ &\equiv \Gamma_0 + \Gamma_1\end{aligned}$$

- In principle, it is possible to compute Γ_1 nonperturbatively over a larger range of photon energies.
- At present we do not propose to compute Γ_1 nonperturbatively. Rather we consider only photons which are sufficiently soft for the point-like (pt) approximation to be valid.
  - For pions and kaons at least, a cut-off ΔE of $O(10\,\text{-}\,20\,\text{MeV})$ appears to be appropriate both experimentally and theoretically.

    F.Ambrosino et al., KLOE collaboration, hep-ex/0509045. arXiv:0907.3594, NA62 this workshop
  - Question: What is the best way to translate the photon energy and angular resolutions at LHC, Belle II etc. into the rest frame of the decaying mesons?

- We now write

$$\Gamma_0 + \Gamma_1(\Delta E) = \lim_{V\to\infty}(\Gamma_0 - \Gamma_0^{\rm pt}) + \lim_{V\to\infty}(\Gamma_0^{\rm pt} + \Gamma_1(\Delta E))\,.$$

  - pt stands for *point-like*.
  - The second term on the rhs can be calculated in perturbation theory. It is infrared convergent, but does contain a term proportional to $\log\Delta E$.
  - The first term is also free of infrared divergences.
  - Γ_0 is calculated non-perturbatively and $\Gamma_0^{\rm pt}$ in perturbation theory.

- Finite-volume effects take the form:

$$\Gamma_0^{\rm pt}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\log(m_\pi L) + \frac{C_1(r_\ell)}{m_\pi L} + \dots\,,$$

where $r_\ell = m_\ell/m_\pi$ and m_ℓ is the mass of the final-state charged lepton.

The exhibited terms are *universal*, i.e. independent of the structure of the meson!

  - We have calculated the coefficients (using the $\mathrm{QED_L}$ regulator of the zero mode).

- The leading structure-dependent FV effects in $\Gamma_0 - \Gamma_0^{\rm pt}$ are of $O(1/L^2)$.

V.Lubicz, G.Martinelli, CTS, F.Sanfilippo, S.Simula, N.Tantalo, arXiv:1611.08497

- Writing

$$\frac{\Gamma(K_{\mu 2})}{\Gamma(\pi_{\mu 2})} = \left|\frac{V_{us}}{V_{ud}}\frac{f_K^{(0)}}{f_\pi^{(0)}}\right|^2 \frac{m_\pi^3}{m_K^3}\left(\frac{m_K^2 - m_\mu^2}{m_\pi^2 - m_\mu^2}\right)^2 (1 + \delta R_{K\pi})$$

where $m_{K,\pi}$ are the physical masses, we find

$$\delta R_{K\pi} = -0.0122(16)\,.$$

D.Giusti et al., arXiv:1711.06537

This first calculation can certainly be improved.

- $f_P^{(0)}$ are the decay constants obtained in iso-symmetric QCD with the renormalized $\overline{\mathrm{MS}}$ masses and coupling equal to those in the full QCD+QED theory extrapolated to infinite volume and to the continuum limit.
- This result can be compared to the PDG value, based on ChPT, is $\delta R_{K\pi} = -0.0112(21)$.
- Our result, together with $V_{ud} = 0.97417(21)$ from super-allowed nuclear β-decays gives $V_{us} = 0.22544(58)$ and

$$V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 0.99985(49)\,.$$

- We are now expanding this framework to semileptonic decays, such as $K \to \pi\bar{\nu}$, where several new features arise, such as the dependence of the $1/L$ corrections on $df_\pm/dq^2$, which however are physical quantities.

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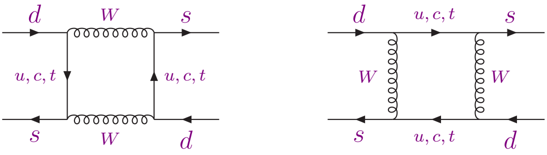
Illustrative Example: the K_L-K_S mass difference

- Consider the neutral-kaon system:
  - Strong interaction eigenstates: $|K_0\rangle = |\bar{s}d\rangle$ and $|\bar{K}_0\rangle = |s\bar{d}\rangle$.
  - CP-eigenstates: $|K_{1,2}\rangle = \frac{1}{\sqrt{2}}(|K_0\rangle \pm |\bar{K}_0\rangle)$.
  - Mass eigenstates: $|K_S\rangle \propto (|K_1\rangle + \epsilon|K_2\rangle)$ and $|K_L\rangle \propto (|K_2\rangle + \epsilon|K_1\rangle)$.

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 3.483(6) \times 10^{-12}\,\text{MeV} \ll \Lambda_{\text{QCD}}.$$



$$\Delta m_K = 3.483(6) \times 10^{-12}\,\text{MeV}$$

- It is frequently said that Flavour Physics can probe scales which are unreachable in colliders.
  - Here, if we could reproduce the experimental value of Δm_K in the SM to 10% accuracy and if we imagine an effective new-physics $\Delta S = 2$ contribution $\frac{1}{\Lambda^2}(\bar{s}\cdots d)(\bar{s}\cdots d)$ then $\Lambda \gtrsim (10^3 - 10^4)$ TeV.
- Below I will show that the RBC-UKQCD collaborations are well on the way to an *ab initio* calculation of Δm_K.

The RBC & UKQCD collaborations

BNL and RBRC

Mattia Bruno
Tomomi Ishikawa
Taku Izubuchi
Luchang Jin
Chulwoo Jung
Christoph Lehner
Meifeng Lin
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni
Sergey Syritsyn

Columbia University

Ziyuan Bai
Norman Christ
Duo Guo
Christopher Kelly
Bob Mawhinney
David Murphy
Masaaki Tomii
Jiqun Tu
Bigeng Wang
Tianle Wang

University of Connecticut

Tom Blum
Dan Hoying
Cheng Tu

Edinburgh University

Peter Boyle
Guido Cossu
Luigi Del Debbio
Richard Kenway
Julia Kettle
Ava Khamseh
Brian Pendleton
Antonin Portelli
Tobias Tsang
Oliver Witzel
Azusa Yamaguchi

KEK

Julien Frison

University of Liverpool

Nicolas Garron

Peking University

Xu Feng

University of Southampton

Jonathan Flynn
Vera Guelpers
James Harrison
Andreas Juettner
Andrew Lawson
Edwin Lizarazo
Chris Sachrajda

York University (Toronto)

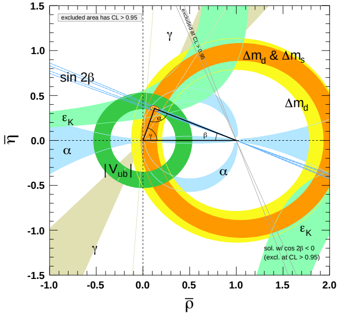
Renwick Hudspith

Physics Motivation: ϵ_K

- ϵ_K is one of the standard inputs into the unitarity triangle analysis.
- SD dominance $\Rightarrow$ the non-perturbative QCD effects are contained in $\langle \bar{K}^0 | O_{LL}^{\Delta S=2} | K^0 \rangle$ which is given by B_K and f_K.
  - Now known to $O(2\%)$ precision.
- Currently the dominant uncertainty is due to that in V_{cb}^4. ($V_{cb} = (40.5 \pm 1.5) \times 10^{-3}$)

  PDG 2016
- LD effects are estimated to be of $O(5\text{-}10\%)$.

  A.Buras, D.Guadagnoli and G.Isidori, arXiv:1002.3612



PDG 2016

- The aim of our work is to compute the LD effects with controlled uncertainties.

Physics Motivation: $K \to \pi\nu\bar{\nu}$ Decays

- NA62 ($K^+ \to \pi^+\nu\bar{\nu}$) and KOTO ($K_L \to \pi^0\nu\bar{\nu}$) are beginning their experimental programme to study these decays. These FCNC processes provide ideal probes for the observation of new physics effects.
- The dominant contributions from the top quark $\Rightarrow$ they are also very sensitive to V_{ts} and V_{td}.
- Experimental results and bounds:

$$\mathrm{Br}(K^+ \to \pi^+\nu\bar{\nu})_{\text{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10}$$

A.Artamonov et al. (E949), arXiv:0808.2459

$$\mathrm{Br}(K_L \to \pi^0\nu\bar{\nu}) \leq 2.6 \times 10^{-8} \ \text{at 90\% confidence level},$$

J.Ahn et al. (E291a), arXiv:0911.4789

- Sample recent theoretical predictions:

$$\mathrm{Br}(K^+ \to \pi^+\nu\bar{\nu})_{\text{SM}} = (9.11 \pm 0.72) \times 10^{-11}$$
$$\mathrm{Br}(K_L \to \pi^0\nu\bar{\nu})_{\text{SM}} = (3.00 \pm 0.30) \times 10^{-11},$$

A.Buras, D.Buttazzo, J.Girrbach-Noe, R.Knejgens, arXiv:1503.02693

- To what extent can lattice calculations reduce the theoretical uncertainty?

- To what extent can lattice calculations reduce the theoretical uncertainty?
- $K \to \pi\nu\bar{\nu}$ decays are SD dominated and the hadronic effects can be determined from CC semileptonic decays such as $K^+ \to \pi^0 e^+ \nu$.
  - Lattice calculations of the $K_{\ell 3}$ form factors are well advanced,

    FLAG review, S.Aoki et al, arXiv:1607.00299
- LD contributions, i.e. contributions from distances greater than $1/m_c$ are negligible for K_L decays and are expected to be $O(5\%)$ for K^+ decays.
  - K_L decays are therefore one of the cleanest places to search for the effects of new physics.
  - The aim of our study is to compute the LD effects in K^+ decays.
    These provide a significant, if probably still subdominant, contribution to the theoretical uncertainty (which is dominated by the uncertainties in CKM matrix elements).
  - A phenomenological estimate of the long distance effects, estimated these to enhance the branching fraction by 6% with an uncertainty of 3%.

    G.Isidori, F.Mescia and C.Smith, hep-ph/0503107
- Lattice QCD can provide a first-principles determination of the LD contribution with controlled errors.
  - Given the NA62 experiment, it is timely to perform a lattice QCD calculation of these effects.

There are three main contributions to the amplitude:

1. Short distance contributions: F.Mescia, C,Smith, S.Trine hep-ph/0606081

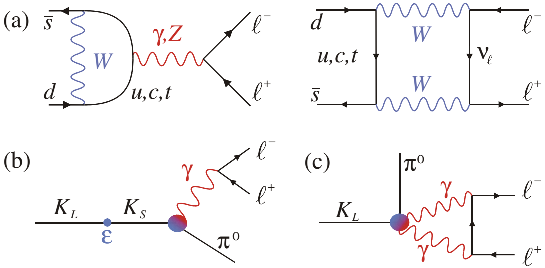
$$H_{\rm eff} = -\frac{G_F\alpha}{\sqrt{2}}\, V_{ts}^* V_{td}\{y_{7V}(\bar{s}\gamma_\mu d)\,(\bar\ell\gamma^\mu\ell) + y_{7A}(\bar{s}\gamma_\mu d)\,(\bar\ell\gamma^\mu\gamma_5\ell)\} + \text{h.c.}$$

   - Direct CP-violating contribution.
   - In BSM theories other effective interactions are possible.

2. Long-distance indirect CP-violating contribution

$$A_{ICPV}(K_L \to \pi^0\ell^+\ell^-) = \epsilon A(K_1 \to \pi^0\ell^+\ell^-) \simeq \epsilon A(K_S \to \pi^0\ell^+\ell^-).$$

3. The two-photon CP-conserving contribution $K_L \to \pi^0(\gamma^*\gamma^* \to \ell^+\ell^-)$.



- The current phenomenological status for the SM predictions is nicely summarised by: V.Cirigliano et al., arXiv1107.6001

$$\mathrm{Br}(K_L \to \pi^0 e^+ e^-)_{\mathrm{CPV}} = 10^{-12} \times \left\{ 15.7|a_S|^2 \pm 6.2|a_S| \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right) + 2.4 \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)^2 \right\}$$

$$\mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\mathrm{CPV}} = 10^{-12} \times \left\{ 3.7|a_S|^2 \pm 1.6|a_S| \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right) + 1.0 \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)^2 \right\}$$

  - $\lambda_t = V_{td} V_{ts}^*$ and Im $\lambda_t \simeq 1.35 \times 10^{-4}$.
  - $|a_S|$, the amplitude for $K_S \to \pi^0 \ell^+ \ell^-$ at $q^2 = 0$ as defined below, is expected to be $O(1)$ but the sign of a_S is unknown. $|a_S| = 1.06^{+0.26}_{-0.21}$.
  - For $\ell = e$ the two-photon contribution is negligible.
  - Taking the positive sign (?) the prediction is

$$\begin{aligned} \mathrm{Br}(K_L \to \pi^0 e^+ e^-)_{\mathrm{CPV}} &= (3.1 \pm 0.9) \times 10^{-11} \\ \mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\mathrm{CPV}} &= (1.4 \pm 0.5) \times 10^{-11} \\ \mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\mathrm{CPC}} &= (5.2 \pm 1.6) \times 10^{-12}\,. \end{aligned}$$

  - The current experimental limits (KTeV) are:

$$\mathrm{Br}(K_L \to \pi^0 e^+ e^-) < 2.8 \times 10^{-10} \quad \text{and} \quad \mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-) < 3.8 \times 10^{-10}\,.$$

G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026

- We now turn to the CPC decays $K_S \to \pi^0\ell^+\ell^-$ and $K^+ \to \pi^+\ell^+\ell^-$ and consider

$$T_i^\mu = \int d^4x\, e^{-iq\cdot x}\, \langle \pi(p)\,|\,\mathrm{T}\{J^\mu_{\mathrm{em}}(x)\,Q_i(0)\,\}\,|\,K(k)\rangle\,,$$

  where Q_i is an operator from the $\Delta S = 1$ effective weak Hamiltonian.

- EM gauge invariance implies that

$$T_i^\mu = \frac{\omega_i(q^2)}{(4\pi)^2}\,\left\{q^2(p+k)^\mu - (m_K^2 - m_\pi^2)\,q^\mu\right\}\,.$$

- Within ChPT the low energy constants a_+ and a_S are defined by

$$a = \frac{1}{\sqrt{2}}\,V_{us}^*V_{ud}\left\{C_1\omega_1(0) + C_2\omega_2(0) + \frac{2N}{\sin^2\theta_W} f_+(0)C_{7V}\right\}$$

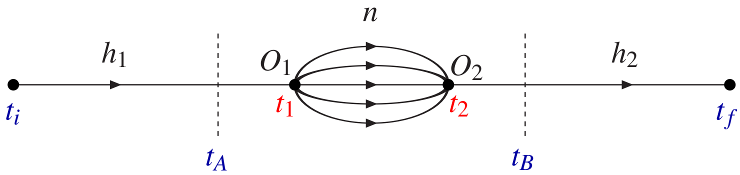
  where $Q_{1,2}$ are the two current-current GIM subtracted operators and the C_i are the Wilson coefficients. (C_{7V} is proportional to y_{7V} above).

G.D'Ambrosio, G.Ecker, G.Isidori and J.Portoles, hep-ph/9808289

- Phenomenological values: $a_+ = -0.578 \pm 0.016$ and $|a_S| = 1.06^{+0.26}_{-0.21}$.

What can we achieve in lattice simulations?

(a) The fiducial volume

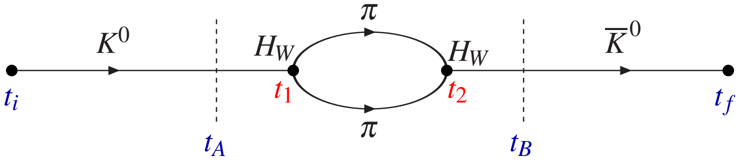


(b) Unphysical exponentially growing contributions

(c) Finite-volume corrections – ✓

N.H. Christ, X. Feng, G. Martinelli and C.T. Sachrajda, arXiv:1504.01170

(d) Renormalization – ✓



- Δm_K is given by

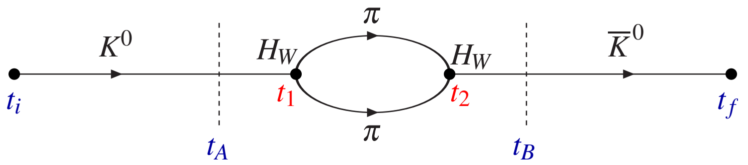
$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathcal{P}\sum_\alpha \frac{\langle \bar{K}^0 \,|\, \mathcal{H}_W \,|\, \alpha\rangle \,\langle \alpha \,|\, \mathcal{H}_W \,|\, K^0\rangle}{m_K - E_\alpha} = 3.483(6)\times 10^{-12}\,\mathrm{MeV}.$$

- The above correlation function gives ($T = t_B - t_A + 1$)

$$C_4(t_A, t_B; t_i, t_f) = |Z_K|^2 e^{-m_K(t_f - t_i)} \sum_n \frac{\langle \bar{K}^0 \,|\, \mathcal{H}_W \,|\, n\rangle \,\langle n \,|\, \mathcal{H}_W \,|\, K^0\rangle}{(m_K - E_n)^2} \times \left\{ e^{(m_K - E_n)T} - (m_K - E_n)T - 1 \right\}.$$

- From the coefficient of T we can therefore obtain

$$\Delta m_K^{\mathrm{FV}} \equiv 2\sum_n \frac{\langle \bar{K}^0 \,|\, \mathcal{H}_W \,|\, n\rangle \,\langle n \,|\, \mathcal{H}_W \,|\, K^0\rangle}{(m_K - E_n)}\,.$$



$$C_4(t_A, t_B; t_i, t_f) = |Z_K|^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0 \,|\, \mathcal{H}_W \,|\, n\rangle \, \langle n \,|\, \mathcal{H}_W \,|\, K^0\rangle}{(m_K - E_n)^2} \times \left\{ e^{(m_K-E_n)T} - (m_K - E_n)T - 1 \right\}.$$

- The presence of terms which (potentially) grow exponentially in T is a generic feature of calculations of matrix elements of bilocal operators.

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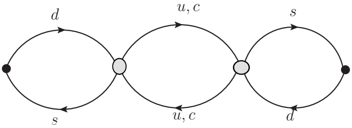
$$\int d^4x \, \langle h_2 \,|\, T\{O_1(x)\, O_2(0)\} \,|\, h_1 \rangle \,,$$

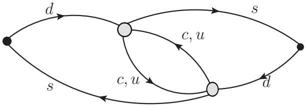
- The local operators $O_{1,2}$ are renormalised in a standard way, e.g. non-perturbatively into a RI-SMOM scheme & then perturbatively into the $\overline{\mathrm{MS}}$ scheme if appropriate.
- However, additional ultraviolet divergences may arise as $x \to 0$.
- This does not happen in two of our cases in the four-flavour theory: i) Δm_K (because of the $V - A$ nature of the weak currents + GIM) and ii) $K \to \pi \ell^+ \ell^-$ decays (because of electromagnetic gauge invariance + GIM).
- For the remaining two processes, ϵ_K and $K \to \pi \nu \bar{\nu}$ decays, additional short-distance divergences do occur (even with GIM) and we have had to develop a variant of the RI-SMOM technique for non-perturbative renormalization to subtract these divergences. N.H.Christ, X.Feng, A.Portelli and CTS, arXiv:1605.04442

RBC-UKQCD, **Z.Bai** et al.

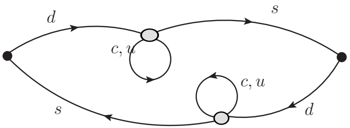
- There are four types of diagram to be evaluated:



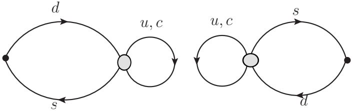
Type 1



Type 2



Type 3



Type 4

- Physical value: $\Delta m_K = 3.483(6) \times 10^{-12}\,\text{MeV}$

1. "Long-distance contribution to the $K_L - K_S$ mass difference,"
   N.H. Christ, T. Izubuchi, CTS, A. Soni and J. Yu,
   Phys.Rev. D88 (2013) 014508 (arXiv:1212.5931)
   Development of techniques and exploratory calculation on a $16^3 \times 32$ lattice with unphysical masses ($m_\pi = 421\,\text{MeV}$) including only connected diagrams $\cdots$ but results were encouragingly in the right ball park.

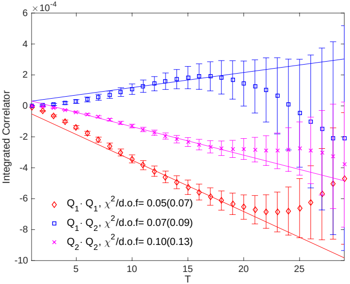
2. "$K_L - K_S$ mass difference from Lattice QCD,"
   Z. Bai, N.H. Christ, T.Izubuchi, CTS, A.Soni and J. Yu,
   Phys.Rev.Lett. 113 (2014) 112003 (arXiv:1406.0916)
   All diagrams included on a $24^3 \times 64$ lattice with $a^{-1} = 1.729(28)\,\text{GeV}$,
   $m_\pi = 330\,\text{MeV}$, $m_K = 575\,\text{MeV}$, $m_c^{\overline{\text{MS}}}(2\,\text{GeV}) = 949\,\text{MeV}$
   $\Rightarrow \Delta m_K = 3.19(41)(96) \times 10^{-12}\,\text{MeV}$.

- At Lattice2017 I presented an update of the computations and results at physical masses!
- Thanks to all my colleagues from RBC-UKQCD, but in particular to Ziyuan Bai for the analysis presented there. Z.Bai, Ph.D. thesis (to be published)

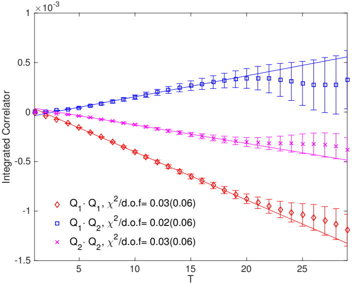
- The calculation was performed on a $64^3 \times 128 \times 12$ lattice with Möbius DWF and the Iwasaki gauge action. The inverse lattice spacing is 2.359(7) GeV, $m_\pi = 135.9(3)$ MeV and $m_K = 496.9(7)$ MeV.

  T.Blum et al., RBC-UKQCD Collabs., arXiv:1411.7017

  Charm-physics studies with this action $\Rightarrow am_c \simeq 0.32 - 0.33$. We have used $am_c \simeq 0.31$ and studied the dependence on m_c.



all diagrams



Type 1 & 2 diagrams only

- Lines here correspond to uncorrelated fits in the range $10 < T < 20$.

- We have performed the first non-perturbative calculations of the $K_L - K_S$ mass difference, now with physical quark masses.
- Our preliminary result based on an analysis of 59 configurations is

$$\Delta m_K = (5.5 \pm 1.7) \times 10^{-12}\,\mathrm{MeV}\,,$$

to be compared to the physical value

$$(\Delta m_K)^{\mathrm{phys}} = 3.483(6) \times 10^{-12}\,\mathrm{MeV}\,.$$

- We plan to finish the present calculation by performing measurements on 160 configurations, aiming to reduce the uncertainty to about 1.0×10^{-12} MeV.
- Longer term, we plan to develop a strategy which will include an improved determination of Δm_K together with other elements of our kaon physics programme.

$$\epsilon_K^{\rm Exp} = 2.228(11)\times 10^{-3}$$

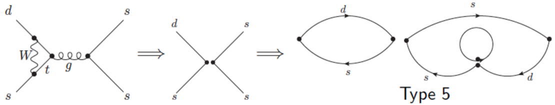
- There has been no journal publication on the long-distance contribution to ϵ_K, although one is in an advanced stage of preparation.
- A number of conference papers have been presented including:

"Long distance part of ϵ_K from lattice QCD", Z.Bai, arXiv:1611.06601

- The preliminary results below were obtained from 200 configurations on a $N_f = 2+1$ flavour ensemble using DWF and Iwasaki gauge action on a $24^3\times 64\times 16$ lattice with $a^{-1} = 1.78$ GeV. C.Allton et al, arXiv:0804.0473
- The quark masses are unphysical, $m_\pi = 339$ MeV, $m_K \simeq 592$ MeV and $m_c^{\overline{\rm MS}}(2\,{\rm GeV}) = 968$ MeV.
- Our preliminary result for the LD contribution at these unphysical masses is

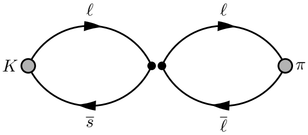
$$\epsilon_K^{\rm LD} = 0.11(0.08)\times 10^{-3}\,.$$

- We need Im $M_{\bar 0 0} \Rightarrow t$-quark contributions not suppressed $\Rightarrow$ QCD penguin operators contribute and we have a Type 5 topology.

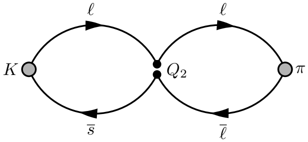


Type 5

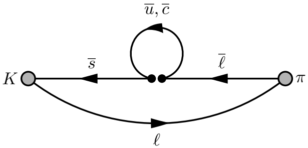
- For example for K^+ decays we need to evaluate the diagrams obtained by inserting the current at all possible locations in the three point function (and adding the disconnected diagrams):



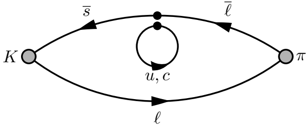
W



C



S



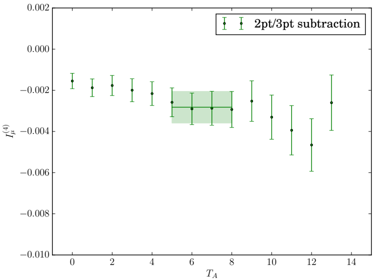
E

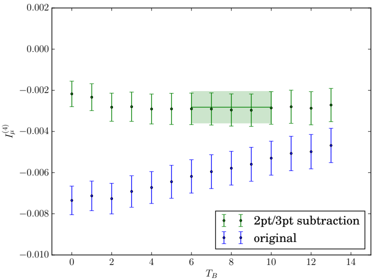
- W=Wing, C=Connected, S=Saucer, E=Eye.
- For K_S decays there is an additional topology with a gluonic intermediate state.

N.Christ, X.Feng, A.Jüttner, A.Lawson, A.Portelli and CTS, arXiv:1608.07585

- The numerical study is performed on the $24^3 \times 64$ DWF+Iwasaki RBC-UKQCD ensembles with $am_l = 0.01$ ($m_\pi \simeq 420$ MeV), $am_s = 0.04$ ($m_K \simeq 625$ MeV), $a^{-1} \simeq 1.78$ fm.
- 128 configurations were used with $\vec{k} = \vec{0}$ and $\vec{p} =$(1,0,0), (1,1,0) and (1,1,1) in units of $2\pi/L$.
- With this kinematics we are in the unphysical region, $q^2 < 0$.
- The charm quark is also lighter than physical $m_c^{\overline{\text{MS}}}(2\,\text{GeV}) \simeq 520$ MeV.
- The calculation is performed using the (5-dimensional) conserved vector current.
- Disconnected diagrams not included.

Method 1 for $\vec{p} = (1,0,0)$

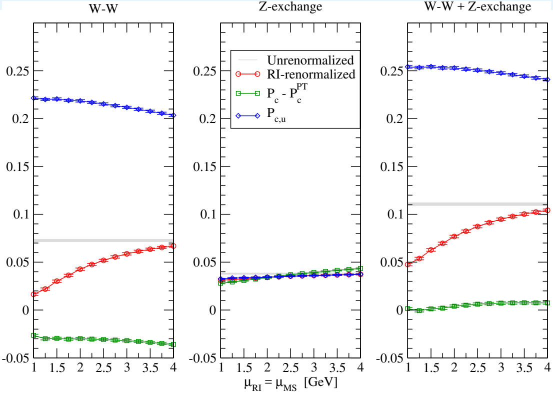




$$\int_{t_J-T_A}^{t_J+8} \tilde{\Gamma}_0^{(4)}\, dt_H \qquad \int_{t_J-6}^{t_J+T_B} \tilde{\Gamma}_0^{(4)}\, dt_H$$

$$A_0(q^2) = -0.0028(6).$$

$K^+ \to \pi^+ \nu\bar{\nu}$ decays - Results from exploratory calculation



- Details of simulation: 800 configs on a $16^3 \times 32$ lattice with $N_f = 2+1$ DWF, $a^{-1} \simeq 1.73\,\text{GeV}$, $m_\pi \simeq 420\,\text{MeV}$, $m_K \simeq 563\,\text{MeV}$ and $m_c^{\overline{\text{MS}}}(2\,\text{GeV}) \simeq 863\,\text{MeV}$.
- For this unphysical kinematics, we find

$$P_c = 0.2529(\pm 13)(\pm 32)(-45) \quad \text{and} \quad \Delta P_c = 0.0040(\pm 13)(\pm 32)(-45)\,.$$

- Large cancellation between WW and Z-exchange contributions.

- For $K^+ \to \pi^+\ell^+\ell^-$, $K_S \to \pi^0\ell^+\ell^-$ and $K^+ \to \pi^+\nu\bar{\nu}$ decays we now have a "complete" theoretical framework with which to perform lattice computations of the amplitudes.
- The results from exploratory calculations are encouraging.
- For computations at physical masses we need to have a large enough volume to accommodate propagating pions and simultaneously a sufficiently fine lattice to accommodate the charm quark.
- To use this framework in a simulation with physical quark masses requires major projects and is part of our agenda:
  - For $K^+ \to \pi^+\nu\bar{\nu}$ decays computations are beginning on the same lattices as being used for Δm_K.
  - For $K \to \pi\ell^+\ell^-$ decays, preparatory work is being done for a project at physical masses.
  - At Lattice2017 Andrew Lawson presented a talk

    "Rare Kaon Decays $K \to \pi\ell^+\ell^-$ with 3 flavours"

    in which the corresponding formalism is developed.
    The GIM mechanism no longer applies and therefore the short-distance divergences must be subtracted, but this approach may be useful as a check (or interim calculation).

- $K \to \pi\pi$ decays are a very important class of processes for standard model phenomenology with a long and noble history.
  - It is in these decays that both indirect and direct CP-violation was discovered.
- Bose Symmetry $\Rightarrow$ the two-pion state has isospin 0 or 2.
- Among the very interesting issues are the origin of the $\Delta I = 1/2$ rule (Re A_0/Re $A_2 \simeq 22.5$) and an understanding of the experimental value of ε'/ε, the parameter which was the first experimental evidence of direct CP-violation.
- The evaluation of $K \to \pi\pi$ matrix elements requires an extension of the standard computations of $\langle 0\,|\,O(0)\,|\,h\rangle$ and $\langle h_2\,|\,O(0)\,|\,h_1\rangle$ matrix elements with a single hadron in the initial and/or final state.
- We plan to update the results with about ≥ 6 times the statistics

- In 2015 RBC-UKQCD published our first result for ϵ'/ϵ computed at physical quark masses and kinematics, albeit still with large relative errors:

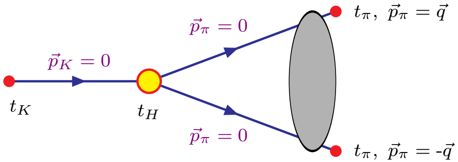
$$\left.\frac{\epsilon'}{\epsilon}\right|_{\text{RBC-UKQCD}} = (1.38 \pm 5.15 \pm 4.59) \times 10^{-4}$$

  to be compared with

$$\left.\frac{\epsilon'}{\epsilon}\right|_{\text{Exp}} = (16.6 \pm 2.3) \times 10^{-4}\,.$$

RBC-UKQCD, arXiv:1505.07863

- This is by far the most complicated project that I have ever been involved with.
- This single result hides much important (and much more precise) information which we have determined along the way.
- We plan to update the result with about 7 times the statistics and some improvements of the systematic errors by the summer.



- $K \to \pi\pi$ correlation function is dominated by lightest state, i.e. the state with two-pions at rest. Maiani and Testa, PL B245 (1990) 585

$$C(t_\pi) = A + B_1 e^{-2m_\pi t_\pi} + B_2 e^{-2E_\pi t_\pi} + \cdots$$

  - Solution 1: Study an excited state. Lellouch and Lüscher, hep-lat/0003023
  - Solution 2: Introduce suitable boundary conditions such that the $\pi\pi$ ground state is $|\pi(\vec{q})\pi(-\vec{q})\rangle$. RBC-UKQCD, C.h.Kim hep-lat/0311003

For B-decays, with so many intermediate states below threshold, this is the main obstacle to producing reliable calculations.

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- The amplitude A_2 is considerably simpler to evaluate that A_0.
- Our first results for A_2 at physical kinematics were obtained at a single, rather coarse, value of the lattice spacing ($a \simeq 0.14$ fm). Estimated discretization errors at 15%. arXiv:1111.1699, arXiv:1206.5142
- Our latest results were obtained on two new ensembles, 48^3 with $a \simeq 0.11$ fm and 64^3 with $a \simeq 0.084$ fm so that we can make a continuum extrapolation:

$$\begin{aligned} \mathrm{Re}(A_2) &= 1.50(4)_{\rm stat}(14)_{\rm syst} \times 10^{-8}\ \mathrm{GeV}. \\ \mathrm{Im}(A_2) &= -6.99(20)_{\rm stat}(84)_{\rm syst} \times 10^{-13}\ \mathrm{GeV}\,. \end{aligned}$$

arXiv:1502.00263

- The experimentally measured value is $\mathrm{Re}(A_2) = 1.479(4) \times 10^{-8}$ GeV.
- Although the precision can still be significantly improved (partly by perturbative calculations), the calculation of A_2 at physical kinematics can now be considered as standard.

- In this talk I have presented some of the exciting physics which is beginning to be done using lattice simulations.
- This builds on the enormous improvement in precision in the evaluation of standard quantities, which has been made in the last 10 years or so.
- This precision is such that isospin breaking effects (including electromagnetism) must be included if further progress in determining the CKM matrix elements is to be made. This is underway for leptonic decays and is being developed for semileptonic decays.
- The theoretical framework for evaluating long-distance contributions has been developed by RBC-UKQCD collaboration and is being applied to the evaluation of Δm_K, ϵ_K and the rare kaon decays $K \to \pi \ell^+ \ell^-$ and $K^+ \to \pi^+ \nu \bar{\nu}$.
- The RBC-UKQCD collaboration has also demonstrated that $K \to \pi\pi$ decays are amenable to lattice computations and have calculated both the real and imaginary parts of A_2 and A_0 (and hence ϵ'/ϵ).
  - The priority now is to reduce the errors on A_0 and to consolidate the result for ϵ'/ϵ.

- So much more to be done!

1. *QED Corrections to Hadronic Processes in Lattice QCD,*
N.Carrasco, V.Lubicz, G.Martinelli, C.T.Sachrajda, N.Tantalo, C.Tarantino and M.Testa,
Phys. Rev. D **91** (2015) no.7, 07450 [arXiv:1502.00257 [hep-lat]].
2. *Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD*,
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